MATHEMATICS NOTES

FORM 4

MATRIX AND TRANSFORMATION

Specific Objectives

By the end of the topic the learner should be able to:

(a) Relate image and object under a given transformation on the Cartesian

Plane;

(b) Determine the matrix of a transformation;

(c) Perform successive transformations;

(d) Determine and identify a single matrix for successive transformation;

(e) Relate identity matrix and transformation;

(f) Determine the inverse of a transformation;

(g) Establish and use the relationship between area scale factor and determinant of a matrix;

(h) Determine shear and stretch transformations;

(i) Define and distinguish isometric and non-isometric transformation;

(j) Apply transformation to real life situations.

Content

(a) Transformation on the Cartesian plane

(b) Identification of transformation matrix

(c) Successive transformations

(d) Single matrix of transformation for successive transformations

(e) Identity matrix and transformation

(f) Inverse of a transformation

(g) Area scale factor and determinant of a matrix

(h) Shear and stretch (include their matrices)

(i) Isometric and non-isometric transformations

(j) Application of transformation to real life situations.

Matrices of transformation

A transformation change the shape, position or size of an object as discussed in book two.

Pre –multiplication of any 2 x 1 column vector by a 2 x 2 matrix results in a 2 x 1 column vector

Example

$$\begin{bmatrix}3 & 4\\-1 & 2\end{bmatrix}\begin{bmatrix}7\\-1\end{bmatrix}=\begin{bmatrix}17\\-9\end{bmatrix}$$

If the vector $\begin{bmatrix}7\\-1\end{bmatrix}$ is thought of as apposition vector that is to mean that it is representing the points with coordinates (7, -1) to the point (17, -9).

Note;

The transformation matrix has an effect on each point of the plan. Let's make T a transformation matrix T$\begin{bmatrix}7\\-1\end{bmatrix}$ Then T maps points (x, y) onto image points x^1 ,y^1

$$T\begin{bmatrix}x\\y\end{bmatrix}\rightarrow\begin{bmatrix}x^1\\y^1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}\begin{bmatrix}3 & 4\\-1 & 2\end{bmatrix}=\begin{bmatrix}x^1\\y^1\end{bmatrix}$$

$$=\begin{bmatrix}3x+4y\\-1x+2y\end{bmatrix}$$

Finding the Matrix of transformation

The objective is to find the matrix of given transformation.

Examples

Find the matrix of transformation of triangle PQR with vertices P (1, 3) Q (3, 3) and R (2, 5).The vertices of the image of the triangle sis$P^1(1,-3)$,$Q^1(3,-3)$ and $R^1(2,-5)$.

Solution

Let the matrix of the transformation be $\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}P & Q & R\\1 & 3 & 2\\3 & 3 & 5\end{pmatrix}=\begin{pmatrix}P^1 & Q^1 & R^1\\1 & 3 & 2\\-3 & -3 & -5\end{pmatrix}$$

$$\begin{pmatrix} a+3b & 3a+3b & 2a+5b \\ c+3d & 3c+3d & 2c+5d \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ -3 & -3 & -5 \end{pmatrix}$$

Equating the corresponding elements and solving simultaneously

$$a + 3b = 1$$

$$\underline{3a + 3b = 3}$$

$$2a = 2$$

$$a = 1 \text{ and } b = 0$$

$$c + 3d = -3$$

$$\underline{3c + 3d = -3}$$

$$2c = 0$$

$$c = 0 \text{ and } d = -1$$

Therefore the transformation matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example

A trapezium with vertices A (1 ,4) B(3,1) C (5,1) and D(7,4) is mapped onto a trapezium whose vertices are $A^1(-4,1)$,$B^1(-1,3)$,$C^1(-1,5)$,$D^1(-4,7)$.Describe the transformation and find its matrix

Solution

Let the matrix of the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B & C D \\ 1 & 3 & 5\,7 \\ 4 & 1 & 1\,4 \end{pmatrix} = \begin{pmatrix} A^1 & B^1 & C^1 D^1 \\ -4 & -1 & -1\,-4 \\ 1 & 3 & 5\;\;7 \end{pmatrix}$$

Equating the corresponding elements we get;

$$a + 4b = -4 \qquad\qquad c + 4d = 1$$

$$3a + b = -1 \qquad\qquad 3c + d = 3$$

Solve the equations simulteneously

$$3a + 12b = -12$$

$$\underline{3a + b = -1}$$

$$11b = -11 \qquad \text{hence } b = -1 \text{ or } a = 0$$

$$3c + 12d = 3$$

$$3c + d = 3$$

$$11d = 0$$

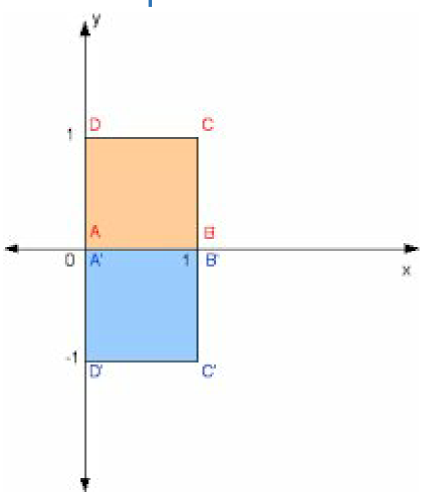
$$d = 0 \qquad c = 1$$

The matrix of the transformation is therefore $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The transformation is positive quarter turn about the origin

Note;
Under any transformation represented by a 2 x 2 matrix, the origin is invariant, meaning it does not change its position.Therefore if the transformtion is a rotation it must be about the origin or if the transformation is reflection it must be on a mirror line which passses through the origin.

The unit square



The unit square ABCD with vertices A $(0,0)$,$B(1,0)$,$C(1,1)$ and $D(0,1)$ helps us to get the transformation of a given matrix and also to identify what trasformation a given matrix represent.

Example

Find the images of I and J under the trasformation whose matrix is;

a) $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 6 \\ 4 & 5 \end{pmatrix}$

Solution

a) $\begin{pmatrix}2 & 3\\5 & 4\end{pmatrix}\begin{pmatrix}I & J\\1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}I^1 & J^1\\2 & 3\\5 & 4\end{pmatrix}$

b) $\begin{pmatrix}-1 & 6\\4 & 5\end{pmatrix}\begin{pmatrix}I & J\\1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}I^1 & J^1\\-1 & 6\\4 & 5\end{pmatrix}$

NOTE;

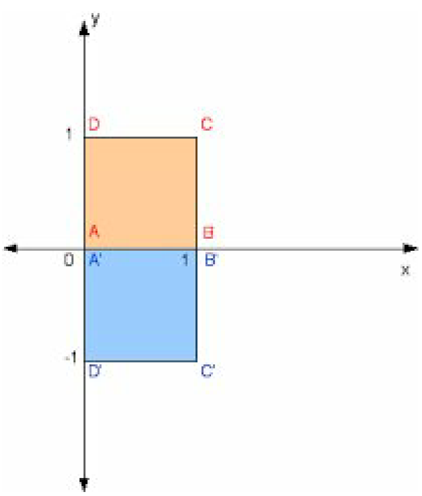
The images of I and J under transformation represented by any 2 x 2 matrix i.e., $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ are $I^1(a,c)$ and$J^1(b,d)$

Example

Find the matrix of reflection in the line y = 0 or x axis.

Solution

Using a unit square the image of B is (1, 0) and D is (0 , -1) .Therefore , the matrix of the transformation is $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$



Example

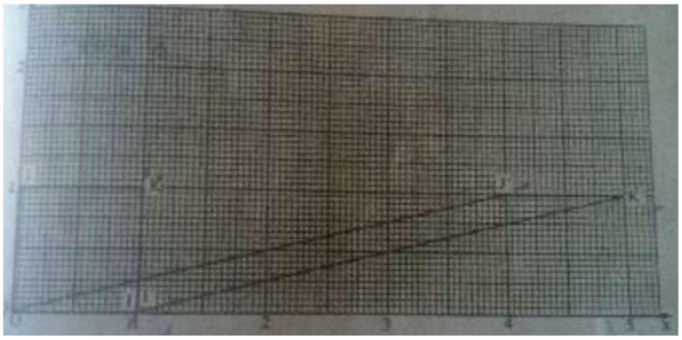
Show on a diagram the unit square and it image under the transformation represented by the matrix$\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}$

Solution

Using a unit square, the image of I is (1 ,0), the image of J is (4 , 1),the image of O is (0,0) and that of

K is $\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}\begin{pmatrix}k\\1\\1\end{pmatrix} = \begin{pmatrix}K^1\\5\\1\end{pmatrix}$

Therefore ,K^1,the image of K is (5 ,1)



Successive transformations

The process of performing two or more transformations in order is called successive transformation eg performing transformation H followed by transformation Y is written as follows YH or if A , b and C are transformations ; then ABC means perform C first ,then B and finally A , in that order.

The matrices listed below all perform different rotations/reflections:

This transformation matrix is the identity matrix. When multiplying by this matrix, the point matrix is unaffected and the new matrix is exactly the same as the point matrix.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}(4 \text{ x } 1) + (3 \text{ x} 0)\\(4 \text{ x } 0)+(3 \text{ x} 1)\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$

This transformation matrix creates a reflection in the x-axis. When multiplying by this matrix, the x co-ordinate remains unchanged, but the y co-ordinate changes sign.

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}(4 \text{ x } 1) + (3 \text{ x} 0)\\(4 \text{ x } 0)+(3 \text{ x} -1)\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}$$

This transformation matrix creates a reflection in the y-axis. When multiplying by this matrix, the y co-ordinate remains unchanged, but the x co-ordinate changes sign.

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}(4 \text{ x} -1) + (3 \text{ x} 0)\\(4 \text{ x } 0)+(3 \text{ x} 1)\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$$

This transformation matrix creates a rotation of 180 degrees. When multiplying by this matrix, the point matrix is rotated 180 degrees around (0, 0). This changes the sign of both the x and y co-ordinates.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 \text{ x} -1) + (3 \text{ x} 0) \\ (4 \text{ x } 0) + (3 \text{ x} 1) \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

This transformation matrix creates a reflection in the line y=x. When multiplying by this matrix, the x co-ordinate becomes the y co-ordinate and the y-ordinate becomes the x co-ordinate.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 \text{ x } 0) + (3 \text{ x} 1) \\ (4 \text{ x } 1) + (3 \text{ x} 0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

This transformation matrix rotates the point matrix 90 degrees clockwise. When multiplying by this matrix, the point matrix is rotated 90 degrees clockwise around (0, 0).

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 \text{ x } 0) + (3 \text{ x} 1) \\ (4 \text{ x} -1) + (3 \text{ x} 0) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

This transformation matrix rotates the point matrix 90 degrees anti-clockwise. When multiplying by this matrix, the point matrix is rotated 90 degrees anti-clockwise around (0, 0).

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 \text{ x } 0) + (3 \text{ x} 1) \\ (4 \text{ x} -1) + (3 \text{ x} 0) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

This transformation matrix creates a reflection in the line y=-x. When multiplying by this matrix, the point matrix is reflected in the line y=-x changing the signs of both co-ordinates and swapping their values.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (4 \text{ x } 0) + (3 \text{ x} -1) \\ (4 \text{ x} -1) + (3 \text{ x} 0) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Inverse matrix transformation

A transformation matrix that maps an image back to the object is called an inverse of matrix.

Note;

If A is a transformation which maps an object T onto an image T^1,then a transformation that can map T^1 back to T is called the inverse of the transformation A , written as image A^{-1}.

If R is a positive quarter turn about the origin the matrix for R is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the matrix for R^{-1} is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ hence $R^{-1}R = R^{-1}R = 1$

Example

T is a triangle with vertices A (2, 4), B (1, 2) and C (4, 2).S is a transformation represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$

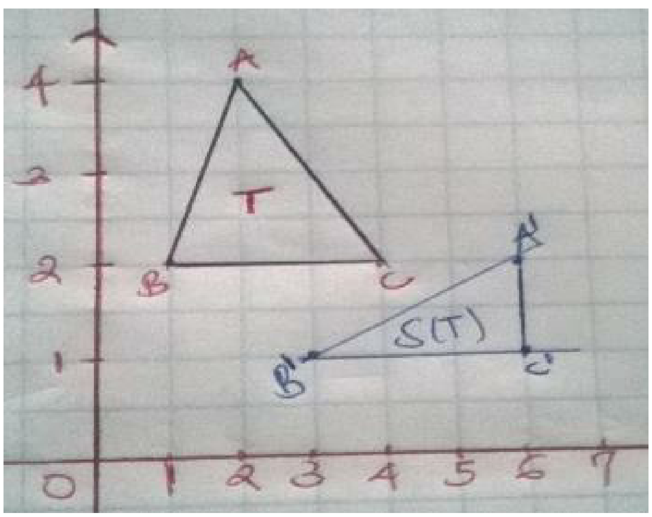
a) Draw T and its image T^1 under the transformation S

b) Find the matrix of the inverse of the transformation S

Solution

a) Using transformation matrix S = $\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} A & B & C \\ 2 & 1 & 4 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} A^1 & B^1 & C^1 \\ 6 & 3 & 6 \\ 2 & 1 & 1 \end{pmatrix}$$



b) Let the inverse of the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This can be done in the following ways

I. $S^{-1}S = 1$

Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equating corresponding elements and solving simultaneously;

$$a = 1, b = -2, c = 0 \text{ and } d = 2$$

Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$

$$S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

II. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A^1 & B^1 & C^1 \\ 6 & 3 & 6 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & B & C \\ 2 & 1 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

$$a = 1, b = -2, c = 0 \text{ and } d = 2$$

$$S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

Area Scale Factor and Determinant of Matrix

The ratio of area of image to area object is the area scale factor (A.S.F)

Are scale factor $= \dfrac{\text{area of image}}{\text{area of object}}$

Area scale factor is numerically equal to the determinant. If the determinant is negative you simply ignore the negative sign.

Example

Area of the object is 4 cm and that of image is 36 cm find the area scale factor.

Solution

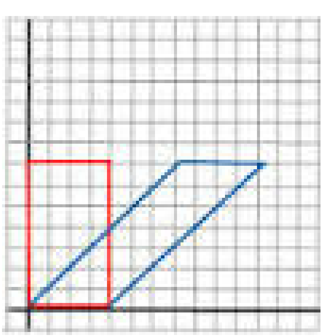
$$\frac{36}{4} = 9$$

If it has a matrix of $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ the determinant is $9 - 0 = 9$ hence equal to A.S.F

Shear and stretch

Shear

The transformation that maps an object (in orange) to its image (in blue) is called a shear



The object has same base and equal heights. Therefore, their areas are equal. Under any shear, area is always invariant (fixed)

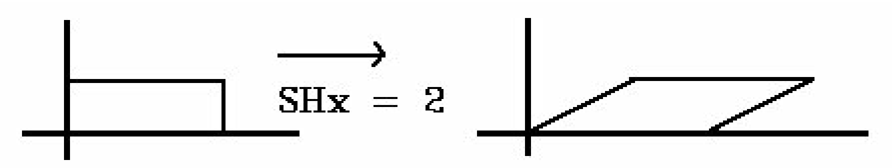
A shear is fully described by giving;

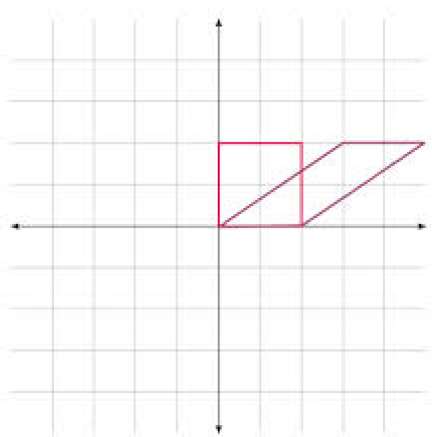
a.) The invariant line

b.) A point not on the invariant line, and its image.

Example

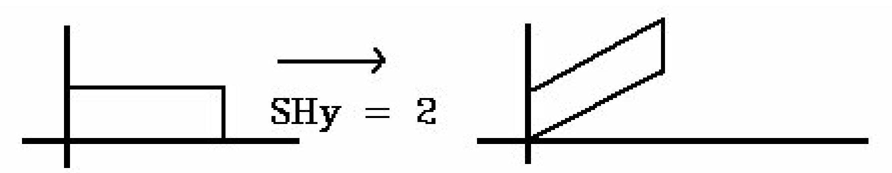
A shear X axis invariant

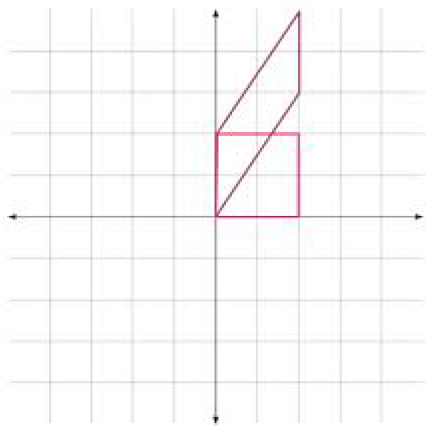




Example

A shear Y axis invariant





Note;

Shear with x axis invariant is represented by a matrix of the form $\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}$ under this trasnsformation, J (0, 1) is mapped onto $J^1(k,1)$.

Likewise a shear with y – axis invariant is represented by a matrix of the form $\begin{pmatrix}1 & 0\\k & 1\end{pmatrix}$. Under this transformation, I (0,1) is mapped onto $I^1(1,k)$.

Stretch

A stretch is a transformation which enlarges all distance in a particular direction by a constant factor. A stretch is described fully by giving;

- The scale factor
- The invariant line

Note;

i.)If K is greater than 1, then this really is a stretch.

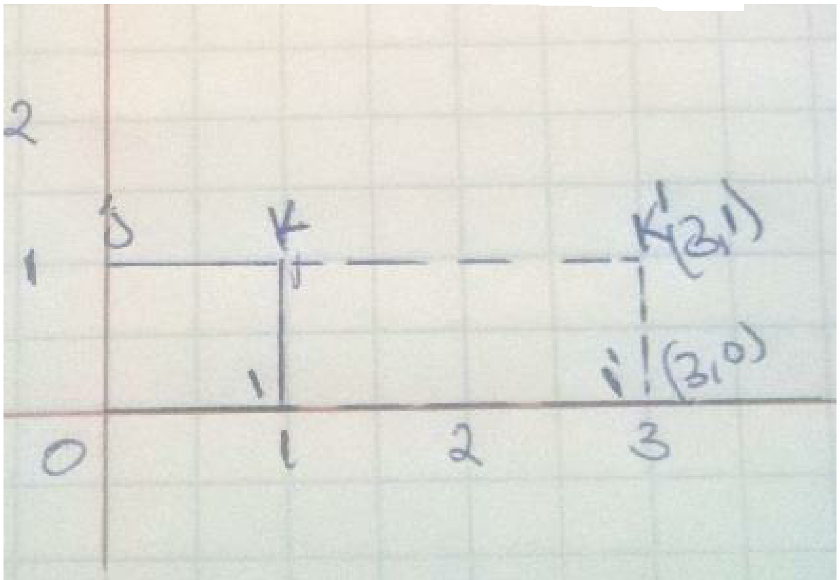
ii.) If k is less than one 1, it is a squish but we still call it a stretch

iii.)If k = 1, then this transformation is really the identity i.e. it has no effect.

Example

Using a unit square, find the matrix of the stretch with y axis invariant ad scale factor 3

Solution



The image of I is $I^1(1, 0)$ and the image of J is $(0,1)$ therefore the matrix of the stretch is $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Note;

The matrix of the stretch with the y-axis invariant and scale factor k is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ and the matrix of a stretch with x – axis invariant and scale factor k is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

Isometric and Non- Isometric Transformation

Isometric transformations are those in which the object and the image have the same shape and size (congruent) e.g. rotation, reflection and translation

Non- isometric transformations are those in which the object and the image are not congruent e.g., shear stretch and enlargement

End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. Matrix p is given by $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

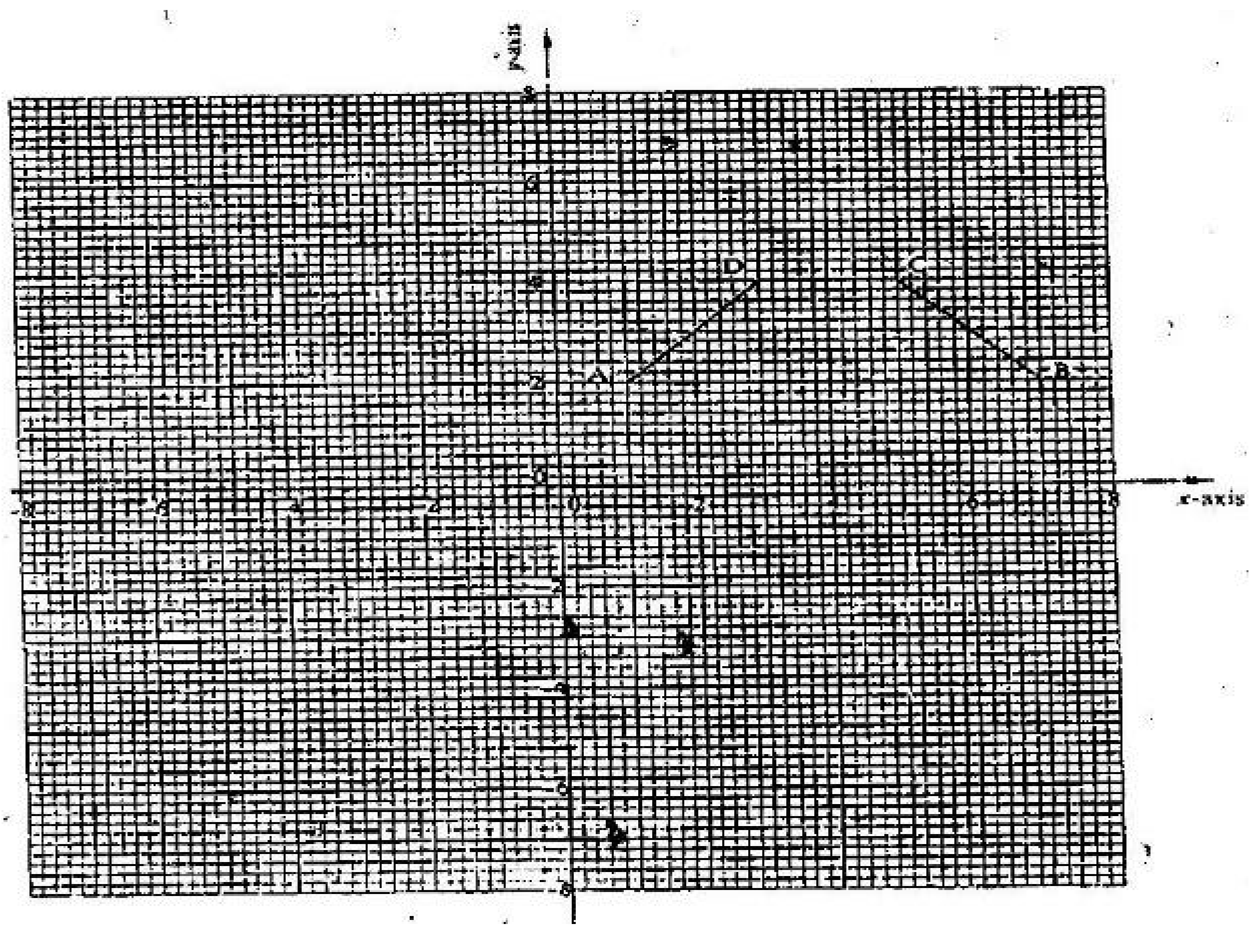
(a) Find P^{-1}

(b) Two institutions, Elimu and Somo, purchase beans at Kshs. B per bag and maize at Kshs m per bag. Elimu purchased 8 bags of beans and 14 bags of maize for Kshs 47,600. Somo purchased 10 bags of beans and 16 of maize for Kshs. 57,400

(c) The price of beans later went up by 5% and that of maize remained constant. Elimu bought the same quantity of beans but spent the same total amount of money as before on the two items. State the new ratio of beans to maize.

2. A triangle is formed by the coordinates A (2, 1) B (4, 1) and C (1, 6). It is rotated clockwise through 90^0 about the origin. Find the coordinates of this image.

3. On the grid provided on the opposite page A (1, 2) B (7, 2) C (4, 4) D (3, 4) is a trapezium



(a) ABCD is mapped onto A'B'C'D' by a positive quarter turn. Draw the image A'B'C'D on the grid

(b) A transformation $\begin{pmatrix}-2 & -1\\ 0 & 1\end{pmatrix}$ maps A'B'C'D onto A"B" C"D" Find the coordinates of A"B"C"D"

4. A triangle T whose vertices are A (2, 3) B (5, 3) and C (4, 1) is mapped onto triangle T^1 whose vertices are A^1 (-4, 3) B^1 (-1, 3) and C^1 (x, y) by a

Transformation M = $\begin{pmatrix} a & b\\ b & N\end{pmatrix}$

c d

a) Find the: (i) Matrix M of the transformation

(ii) Coordinates of C_1

b) Triangle T^2 is the image of triangle T^1 under a reflection in the line y = x.

Find a single matrix that maps T and T_2

5. Triangles ABC is such that A is (2, 0), B (2, 4), C (4, 4) and A”B”C” is such that A” is (0, 2), B” (-4 – 10) and C “is (-4, -12) are drawn on the Cartesian plane

Triangle ABC is mapped onto A”B”C” by two successive transformations

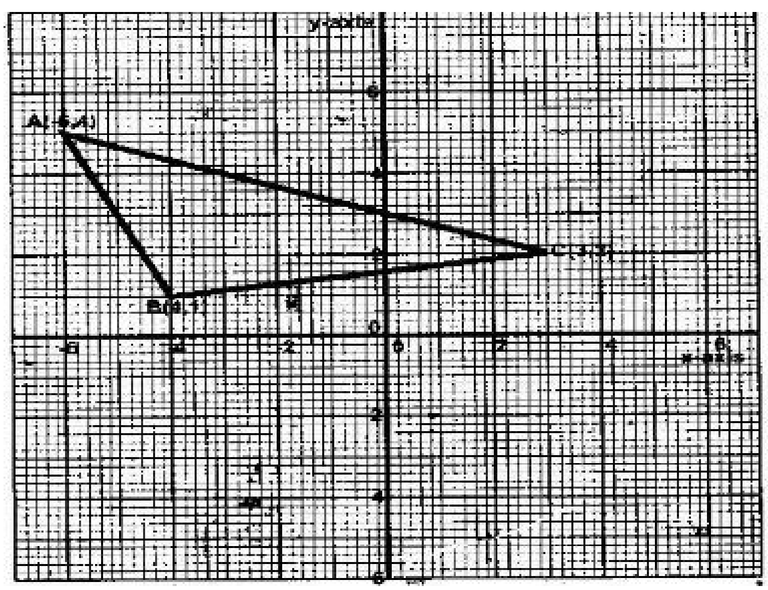
R = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Followed by P = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(a) Find R

(b) Using the same scale and axes, draw triangles A’B’C’, the image of triangle ABC under transformation R

Describe fully, the transformation represented by matrix R

6. Triangle ABC is shown on the coordinate’s plane below



(a) Given that A (-6, 5) is mapped onto A (6,-4) by a shear with y- axis invariant

(i) Draw triangle A’B’C’, the image of triangle ABC under the shear

(ii) Determine the matrix representing this shear

(b) Triangle A B C is mapped on to A” B” C” by a transformation defined by the matrix -1 0

$\begin{pmatrix} 1½ & -1 \end{pmatrix}$

(i) Draw triangle A” B” C”

(ii) Describe fully a single transformation that maps ABC onto A”B” C”

7. Determine the inverse T^{-1} of the matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

Hence find the coordinates to the point at which the two lines

x + 2y = 7 and x - y =1

8. Given that A = $\begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 0 \\ 2 & -4 \end{pmatrix}$

Find the value of x if

(i) A- 2x = 2B

(ii) 3x – 2A = 3B

(iii) 2A - 3B = 2x

9. The transformation R given by the matrix

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps $\begin{pmatrix} 17 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 15 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 17 \end{pmatrix}$ to $\begin{pmatrix} -8 \\ 15 \end{pmatrix}$

(a) Determine the matrix A giving a, b, c and d as fractions

(b) Given that A represents a rotation through the origin determine the angle of rotation.

(c) S is a rotation through 180 about the point (2, 3). Determine the image of (1, 0) under S followed by R.

CHAPTER FIFTY SEVEN

STATISTICS II

Specific Objectives

By the end of the topic the learner should be able to:

(a) State the measures of central t e n d e n c y;

(b) Calculate the mean using the assumed mean method;

(c) Make cumulative frequency table,

(d) Estimate the median and the quartiles b y

- Calculation and
- Using ogive;

(e) Define and calculate the measures of dispersion: range, quartiles,interquartile range, quartile deviation, variance and standard deviation

(f) Interpret measures of dispersion

Content

(a) Mean from assumed mean:

(b) Cumulative frequency table

(c) Ogive

(d) Meadian

(e) Quartiles

(f) Range

(g) Interquartile range

(h) Quartile deviation

(i) Variance

(j) Standard deviation

These statistical measures are called measures of central tendency and they are mean, mode and median.

Mean using working (Assumed) Mean

Assumed mean is a method of calculating the arithmetic men and standard deviation of a data set. It simplifies calculation.

Example

The masses to the nearest kilogram of 40 students in the form 3 class were measured and recorded in the table below. Calculate the mean mass

Mass kg	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Number of employees	2	0	1	2	3	2	5	6	7	5	3	2	1	1

Solution

We are using assumed mean of 53

Mass x kg	t= x - 53	f	*ft*
47	-6	2	-12
48	-5	0	0
49	-4	1	-4
50	-3	2	-6
51	-2	3	-6
52	-1	2	-2
53	0	5	0
54	1	6	6
55	2	7	14
56	3	5	15
57	4	3	12
58	5	2	10
60	7	1	6
		1	7

		$\Sigma f = 40$	$\Sigma ft = 40$

Mean of t = $\frac{\sum f t}{\sum f} = \frac{40}{40} = 1$

Mean of x = 53 + mean of t

= 53 + 1

= 54

Mean of grouped data

The masses to the nearest gram of 100 eggs were as follows

Marks	100- 103	104- 107	108- 111	112-115	116-119	120-123
Frequency	1	15	42	31	8	3

Find the mean mass

Solution

Let use a working mean of 109.5.

class	Mid-point x	t= x - 109.5	f	f t
100-103	101.5	-8	1	− 8
104-107	105.5	-4	15	− 60
108-111	109.5	0	42	0
112-115	113.5	4	31	124
116- 119	117.5	8	8	64
120 -123	121.5	12	3	36
			$\Sigma f = 100$	$\Sigma ft = 156$

Mean of t = $\frac{156}{100} = 1.56$

Therefore,mean of x = 109.5 + mean of t

= 109.5 + 1.56

= 111.06 g

To get the mean of a grouped data easily, we divide each figure by the class width after substracting the assumed mean.Inorder to obtain the mean of the original data from the mean of the new set of data, we will have to reverse the steps in the following order;

- Multiply the mean by the class width and then add the working mean.

Example

The example above to be used to demonstrate the steps

class	Mid-point x	$t=\frac{x-109.5}{4}$	f	f t
100-103	101.5	-2	1	− 2
104-107	105.5	-1	15	− 15
108-111	109.5	0	42	0
112-115	113.5	1	31	31
116- 119	117.5	2	8	16
120 -123	121.5	3	3	9
			$\Sigma f = 100$	$\Sigma ft = 39$

$$t = \frac{\sum ft}{\sum f} = \frac{39}{100}$$

= 0.39

Therefore $\bar{x} = 0.39 \times 4 + 109.5$

$= 1.56 + 109.5$

$= 111.06$ g

Quartiles, Deciles and Percentiles

A median divides a set of data into two equal part with equal number of items.

Quartiles divides a set of data into four equal parts.The lower quartile is the median of the bottom half.The upper quartile is the median of the top half and the middle coincides with the median of the whole set od data

Deciles divides a set of data into ten equal parts.Percentiles divides a set of data into hundred equal parts.

Note;

For percentiles deciles and quartiles the data is arranged in order of size.

Example

Height in cm	145- 149	150-154	155-159	160-164	165-169	170-174	175-179
frquency	2	5	16	9	5	2	1

Calculate the ;

a.) Median height

b.) I.)Lower quartile

ii) Upper quartile

c.) 80^{th} percentile

Solution

I. There are 40 students. Therefore, the median height is the average of the heights of the 20^{th} and 21^{st} students.

class	frequency	Cumulative frequency
145-149	2	2
150 - 154	5	7
155 - 159	16	23
160 - 164	9	32
165 - 169	5	37
170 - 174	2	39
175 - 179	1	40

Both the 20th and 21st students falls in the 155 -159 class. This class is called the median class.

Using the formula $m = L + \frac{\left(\frac{n}{2}-C\right)i}{f}$

Where L is the lower class limit of the median class

N is the total frequency

C is the cumulative frequency above the median class

I is the class interval

F is the frequency of the median class

Therefor;

Height of the 20th student = $154.5 + \frac{13}{16} \times 5$

= 154.5 + 4.0625

=158.5625

Height of the 21st = $154.5 + \frac{14}{16} \times 5$

= 154.5 + 4.375

=158.875

Therefore median height = $\frac{158.5625+158.875}{2}$

= 158.7 cm

b.) (I) lower quartile $Q_1 = L + \frac{\left(\frac{n}{4}-C\right)i}{f}$

The 10th student fall in the in 155 – 159 class

$Q_1 = 154.5 + \frac{\left(\frac{40}{4}-7\right)5}{16}$

= 154.5 + 0.9375

= 155.4375

(ii) Upper quartile $Q_3 = L + \frac{\left(\frac{3}{4}n-C\right)i}{f}$

The 10th student fall in the in 155 – 159 class

$$Q_3 = 159.5 + \frac{\left(\frac{3}{4} \times 40 - 23\right)5}{9}$$

$= 159.5 + 3.888$

$= 163.3889$

Note;

The median corresponds to the middle quartile Q_2 or the 50th percentile

c.) $\frac{80}{100} \times 40 = 32$ the 32nd student falls in the 160 -164 class

$$\text{The 80th percentile} = L + \frac{\left(\frac{80}{100}n - C\right)i}{f}$$

$= 159.5 + \frac{(32 - 23)5}{9}$

$= 159.5 + 5$

$= 164.5$

Example

Determine the upper quartile and the lower quartile for the following set of numbers

5, 10 ,6 ,5 ,8, 7 ,3 ,2 ,7 , 8 ,9

Solution

Arranging in ascending order

2, 3, 5,5,6, 7,7,8,8,9,10

The median is 7

The lower quartile is the median of the first half, which is 5.

The upper quartile is the median of the second half, which is 8.

Median from cumulative frequency curve

Graph for cumulative frequency is called an ogive. We plot a graph of cumulative frequency against the upper class limit.

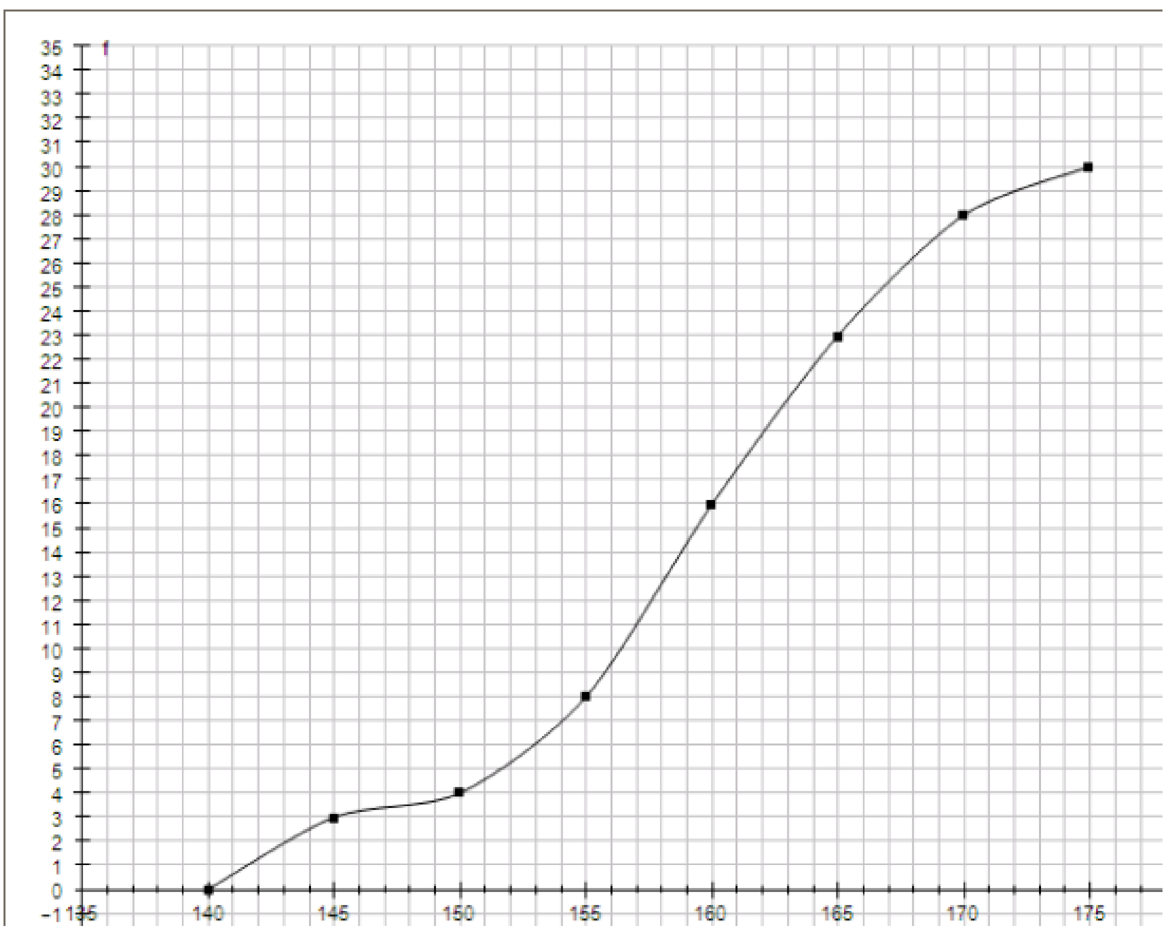
Example

Given the class interval of the measurement and the frequency, we first find the cumulative frequency as shown below.

Then draw the graph of cumulative frequency against upper class limit

Arm Span (cm)	Frequency (f)	Cumulative Frequency
140 ≤ x ‹ 145	3	3
145 ≤ x ‹ 150	1	4
150 ≤ x ‹ 155	4	8
155 ≤ x ‹ 160	8	16
160 ≤ x ‹ 165	7	23
165 ≤ x ‹ 170	5	28
170 ≤ x ‹ 175	2	30
Total:	30	

Solution



Example

The table below shows marks of 100 candidates in an examination

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
FRCY	4	9	16	24	18	12	8	5	3	1

a.) Determine the median and the quartiles

b.) If 55 marks is the pass mark, estimate how many students passed

c.) Find the pass mark if 70% of the students are to pass

d.) Determine the range of marks obtained by

(I) The middle 50 % of the students

(ii) The middle 80% of the students

Solution

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frqcy	4	9	16	24	18	12	8	5	3	1

Cumulative 4 13 29 53
71 83 91 96 99 100

Frequency

Solution

a.) Reading from the graph

The median = 39.5

The Lower quartile Q_1 = 28.5

The upper quartile Q_2 = 53.5

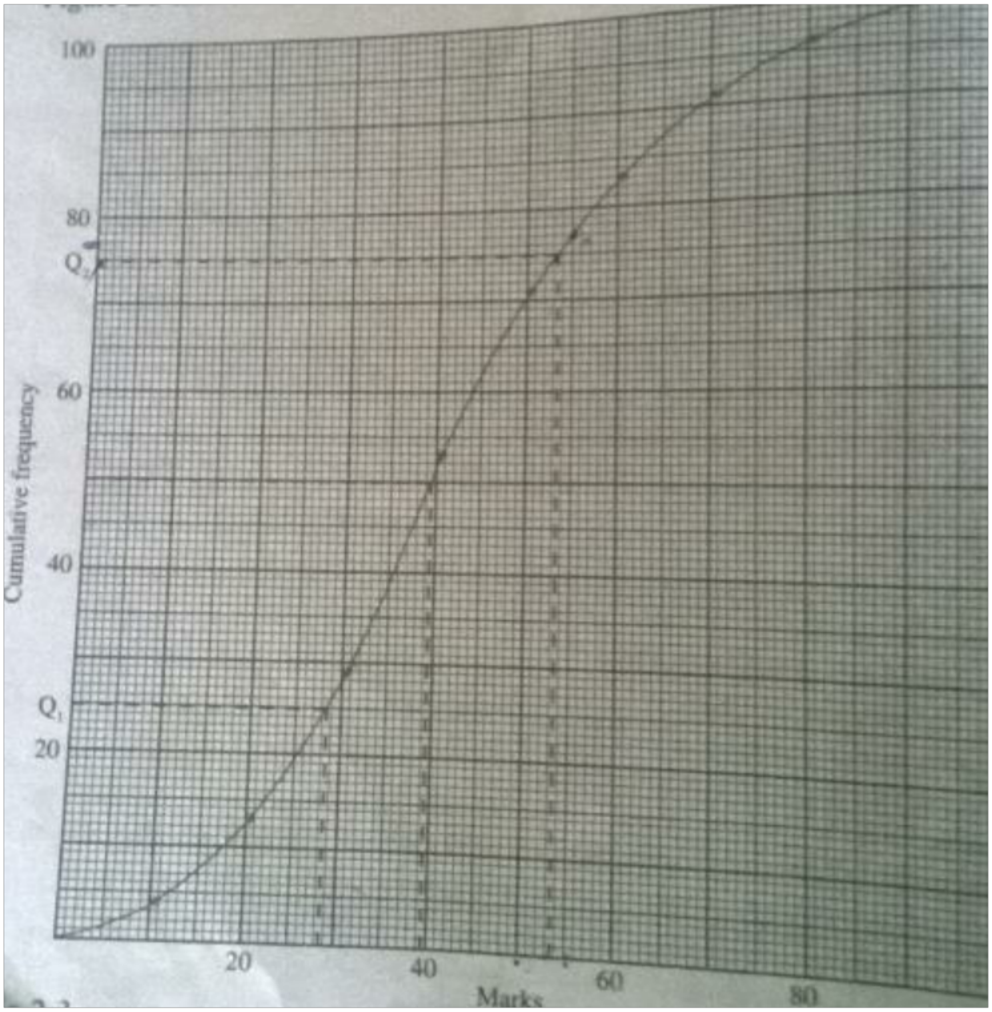
b.) 23 candidates scored 55 and over

c.) Pass mark is 31 if 70% of pupils are to pass

d.) (I) The middle 50% include the marks between the lower and the upper quartiles i.e. between 28.5 and 53.5 marks.

(II) The middle 80% include the marks between the first decile and the 9^{th} decile i.e between 18 and 69 marks

isabokemicah@gmail.com



Measure of Dispersion

Range

The difference between the highest value and the lowest value

Disadvantage

It depends only on the two extreme values

Interquartile range

The difference between the lower and upper quartiles. It includes the middle 50% of the values

Semi quartile range

The difference between the lower quartile and upper quartile divided by 2.It is also called the quartile deviation.

Mean Absolute Deviation

If we find the difference of each number from the mean and find their mean , we get the mean Absolute deviation

Variance

The mean of the square of the square of the deviations from the mean is called is called variance or mean deviation.

Example

Deviation from mean(d)	+1	-1	+6	-4	-2	-11	+1	10
f_i	1	1	36	16	4	121	1	100

Sum $d^2 = 1 + 1 + 36 + 16 + 4 + 121 + 1 + 100 = 280$

Variance = $\frac{\sum d^2}{N} = \frac{280}{8} = 35$

The square root of the variance is called the standard deviation.It is also called root mean square deviation. For the above example its standard deviation =$\sqrt{35} = 5.9$

Example

The following table shows the number of children per family in a housing estate

Number	***0***	1	2	3	4	5	6

of childred							
Number of families	*1*	5	11	27	10	4	2

Calculate

a.) The mean number of children per family

b.) The standard deviation

Solution

Number of children (x)	Number of Families (f)	fx	Deviations d= x -m	d^2	fd^2
o	1	0	− 3	9	9
1	5	5	− 2	4	20
2	11	22	-1	1	11
3	27	81	0	0	0
4	10	40	1	1	10
5	4	20	2	4	16
6	2	12	3	9	18
	$\Sigma f = 60$	$\sum fx = 180$			$\Sigma fd^2 = -40$

a.) Mean = $\frac{180}{60}$ = 3 childen

b.) Variance = $\frac{\sum fd^2}{\sum f}$

$= \frac{84}{60}$

$= 1.4$

$$\text{standard deviation} = \sqrt{1.4} = 1.183$$

Example

The table below shows the distribution of marks of 40 candidates in a test

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
frequency	2	2	3	9	12	5	2	3	1	1

Calculate the mean and standard deviation.

Marks	Midpoint (x)	Frequency (*f*)	fx	d= x - m	d^2	fd^2
1-10	5.5	2	11.0	- 39.5	1560.25	3120.5
11-20	15.5	2	31.0	-29.5	870.25	1740.5
21-30	25.5	3	76.5	-19.5	380.25	1140.75
31 -40	35.5	9	319.5	-9.5	90.25	812.25
41 -50	45.5	12	546.0	0.5	0.25	3.00
51-60	55.5	5	277.5	10.5	110.25	551.25
61- 70	65.5	2	131.0	20.5	420.25	840.5
71-80	75.5	3	226.5	30.5	930.25	2790.75
81 -90	85.5	1	85.5	40.5	1640.25	1640.25
91 -100	95.5	1	95.5	50.5	2550.25	2550.25
		$\Sigma f = 40$	Σf			$\Sigma fd^2 =$

			***x*=1800**			15190

Mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1800}{40} = 45$ marks

Variance $= \frac{\sum fd^2}{\sum f} = \frac{15190}{40} = 379.75$

$= 379.8$

Standard deviation = $\sqrt{379.8}$

$= 19.49$

Note;

Adding or subtracting a constant to or from each number in a set of data does not alter the value of the variance or standard deviation.

More formulas

The formula for getting the variance s^2 of a variance x is

$$d^2 = \frac{\sum (x-\bar{x})^2 f}{\sum f}$$

$$= \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

Example

The table below shows the length in centimeter of 80 plants of a particular species of tomato

length	152-156	157-161	162-166	167-171	172- 176	177-181
frequency	12	14	24	15	8	7

Calculate the mean and the standard deviation

Solution

Let A = 169

Length	Mid-point x	x-169	$t=\frac{x-169}{5}$	f	ft	ft^2
152 -156	154	-15	-3	12	-36	108
157 -161	159	-10	-2	14	-28	56
162 -166	164	-5	-1	24	-24	24
167 -171	169	0	0	15	0	0
172-176	174	5	1	8	8	8
177-181	179	10	2	7	14	28
				$\sum f = 80$	$\sum ft = -66$	$\sum ft^2 = 224$

$\bar{t} = \frac{-66}{80} = -0.825$

Therefore $\bar{x} = -0.825 \times 5 + 169$

= -4.125 + 169

= 164.875 (to 4 s.f)

Variance of t = $\frac{\sum ft^2}{\sum f} - \bar{t}^2$

$= \frac{224}{80} - (-0.825)^2$

= 2.8 – 0.6806

= 2.119

Therefore , variance of x = 2.119 x 5^2

= 52.975

= 52.98 (4 s.f)

Standard deviation of x = $\sqrt{52.98}$

= 7.279

= 7.28 (to 2 d.p)

End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic

1. Every week the number of absentees in a school was recorded. This was done for 39 weeks these observations were tabulated as shown below

Number of absentees	0.3	4 -7	8 -11	12 - 15	16 - 19	20 - 23
(Number of weeks)	6	9	8	11	3	2

Estimate the median absentee rate per week in the school

2. The table below shows high altitude wind speeds recorded at a weather station in a period of 100 days.

Wind speed (knots)	0 - 19	20 - 39	40 - 59	60-79	80- 99	100-119	120-139	140-159	160-179
Frequency (days)	9	19	22	18	13	11	5	2	1

(a) On the grid provided draw a cumulative frequency graph for the data

(b) Use the graph to estimate

(i) The interquartile range

(ii) The number of days when the wind speed exceeded 125 knots

3. Five pupils A, B, C, D and E obtained the marks 53, 41, 60, 80 and 56 respectively. The table below shows part of the work to find the standard deviation.

Pupil	Mark x	x - a	$(x-a)^2$
A	53	-5	
B	41	-17	
C	60	2	
D	80	22	
E	56	-2	

(a) Complete the table

(b) Find the standard deviation

4. In an agricultural research centre, the length of a sample of 50 maize cobs were measured and recorded as shown in the frequency distribution table below.

Length in cm	Number of cobs
8 – 10	4
11 – 13	7
14 – 16	11
17 – 19	15
20 – 22	8
23 - 25	5

Calculate

(a) The mean
(b) (i) The variance
(ii) The standard deviation

5. The table below shows the frequency distribution of masses of 50 new- born calves in a ranch

Mass (kg)	Frequency
15 – 18	2
19- 22	3
23 – 26	10
27 – 30	14
31 – 34	13
35 – 38	6
39 – 42	2

(a) On the grid provided draw a cumulative frequency graph for the data

(b) Use the graph to estimate

(i) The median mass

(ii) The probability that a calf picked at random has a mass lying between 25 kg and 28 kg.

4. The table below shows the weight and price of three commodities in a given period

Commodity Weight Price Relatives

X	3	125
Y	4	164
Z	2	140

Calculate the retail index for the group of commodities.

7. The number of people who attended an agricultural show in one day was 510 men, 1080 women and some children. When the information was represented on a pie chart, the combined angle for the men and women was 216^0. Find the angle representing the children.

8. The mass of 40 babies in a certain clinic were recorded as follows:

Mass in Kg	No. of babies.
1.0 – 1.9	6
2.0 – 2.9	14
3.0 -3.9	10
4.0 – 4.9	7
5.0 – 5.9	2
6.0 – 6.9	1

Calculate

(a) The inter – quartile range of the data.

(b) The standard deviation of the data using 3.45 as the assumed mean.

9. The data below shows the masses in grams of 50 potatoes

Mass (g)	25- 34	35-44	45 - 54	55- 64	65 - 74	75-84	85-94
No of potatoes	3	6	16	12	8	4	1

(a) On the grid provide, draw a cumulative frequency curve for the data

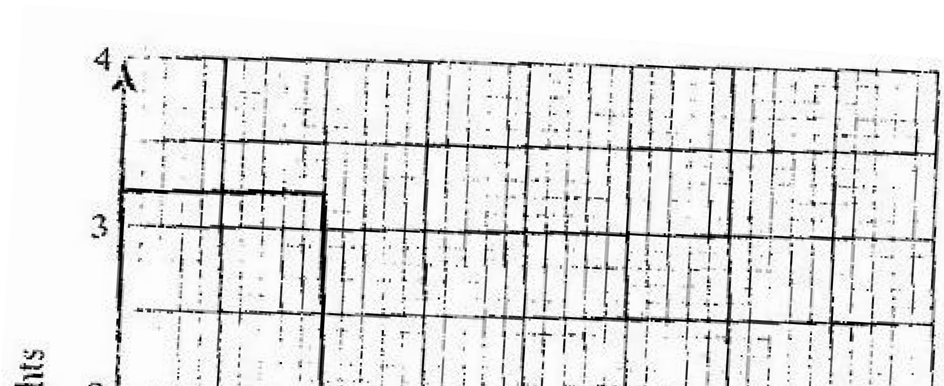
(b) Use the graph in (a) above to determine

(i) The 60^{th} percentile mass

(ii) The percentage of potatoes whose masses lie in the range 53g to 68g

10. The histogram below represents the distribution of marks obtained in a test.

The bar marked A has a height of 3.2 units and a width of 5 units. The bar marked B has a height of 1.2 units and a width of 10 units



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If the frequency of the class represented by bar B is 6, determine the frequency of the class represented by bar A.

11. A frequency distribution of marks obtained by 120 candidates is to be represented in a histogram. The table below shows the grouped marks. Frequencies for all the groups and also the area and height of the rectangle for the group 30 – 60 marks.

Marks	0-10	10-30	30-60	60-70	70-100
Frequency	12	40	36	8	24
Area of rectangle			180		
Height of rectangle			6		

(a) (i) Complete the table

(ii) On the grid provided below, draw the histogram

(b) (i) State the group in which the median mark lies

(ii) A vertical line drawn through the median mark divides the total area of the histogram into two equal parts

Using this information or otherwise, estimate the median mark

11. In an agriculture research centre, the lengths of a sample of 50 maize cobs were measured and recorded as shown in the frequency distribution table below

Length in cm	Number of cobs
8 – 10	4
11- 13	7
14 – 16	11

17- 19	15
20 – 22	8
23- 25	5

Calculate

(a) The mean

(b) (i) The variance

(ii) The standard deviation

12. The table below shows the frequency distribution of masses of 50 newborn calves in a ranch.

Mass (kg)	Frequency
15 – 18	2
19- 22	3
23 – 26	10
27 – 30	14
31- 34	13
35 – 38	6
39 - 42	2

(a) On the grid provided draw a cumulative frequency graph for the data

(b) Use the graph to estimate

(i) The median mass

(ii) The probability that a calf picked at random has a mass lying between 25 kg and 28 kg

14. The table shows the number of bags of sugar per week and their moving averages

Number of bags per week	340	330	x	343	350	345
Moving averages		331	332	y	346	

(a) Find the order of the moving average

(b) Find the value of X and Y axis

CHAPTER FIFTY EIGHT

THREE DIMENSIONAL GEOMETRY

Specific Objectives

By the end of the topic the learner should be able to:

(a) State the geometric properties of common solids;

(b) Identify projection of a line onto a plane;

(c) Identify skew lines;

(d) Calculate the length between two points in three dimensional geometry;

(e) Identify and calculate the angle between

(i) Two lines;

(ii) A line and a plane;

(ii) Two planes.

Content

(a) Geometrical properties of common solids

(b) Skew lines and projection of a line onto a plane

(c) Length of a line in 3-dimensional geometry

(d) The angle between

i) A line and a line

ii) A line a plane

iii) A plane and a plane

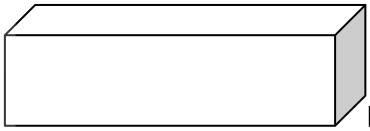
iv) Angles between skewlines.

Introduction

Geometrical properties of common solids

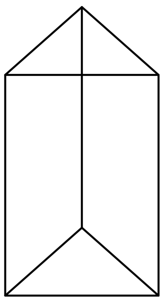
- A geometrical figure having length only is in one dimension
- A figure having area but not volume is in two dimension
- A figure having vertices (points),edges(lines) and faces (plans) is in three dimension

Examples of three dimensional figures



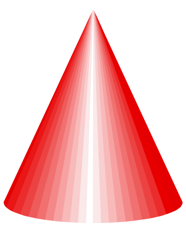
Rectangular Prism

A three-dimensional figure having 6 faces, 8 vertices, and 12 edges



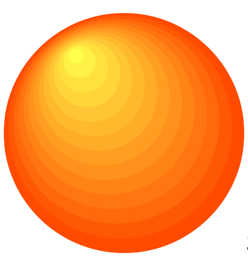
Triangular Prism

A three-dimensional figure having 5 faces, 6 vertices, and 9 edges.



Cone

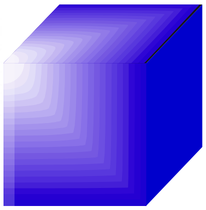
A three- dimensional figure having one face.



Sphere

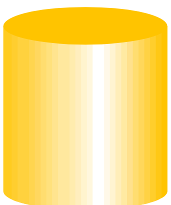
A three- dimensional figure with no straight lines or line segments

isabokemicah@gmail.com



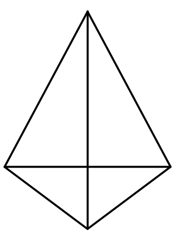
Cube

A three- dimensional figure that is measured by its length, height, and width.
It has 6 faces, 8 vertices, and 12 edges



Cylinder

A three- dimensional figure having 2 circular faces



Rectangular Pyramid

A three-dimensional figure having 5 faces, 5 vertices, and 8 edges

Angle between a line and a plane

The angle between a line and a plane is the angle between the line and its projection on the plane

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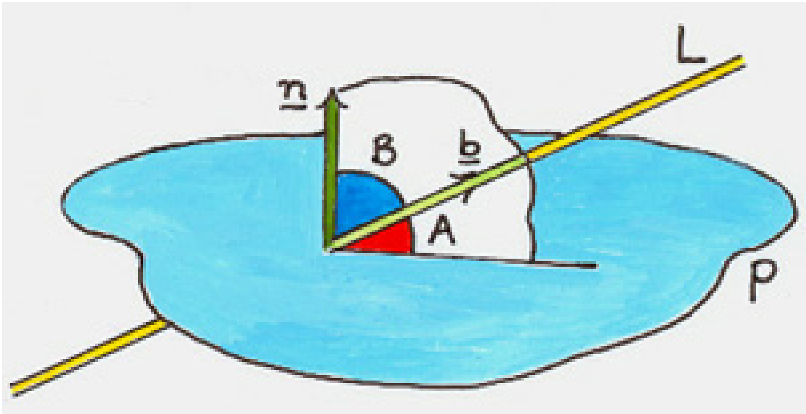
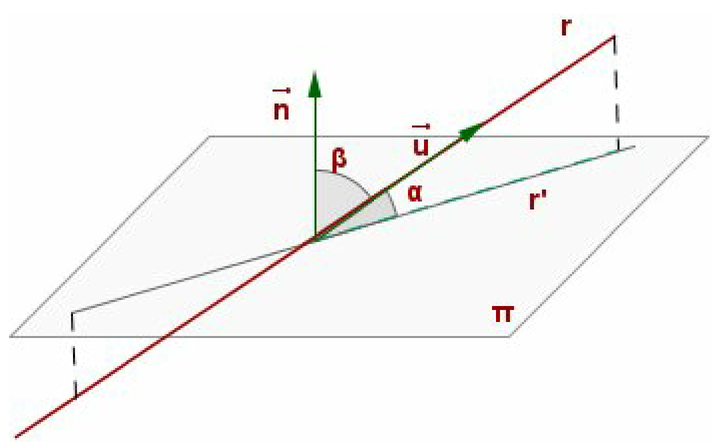


The angle between the line L and its projection or shadow makes angle A with the plan. Hence the angle between a line and a plane is A.

Example

The angle between a line, r, and a plane, π, is the angle between r and its projection onto π, r'.



height is 4 m

Example

Suppose r' is 10 cm find the angle α

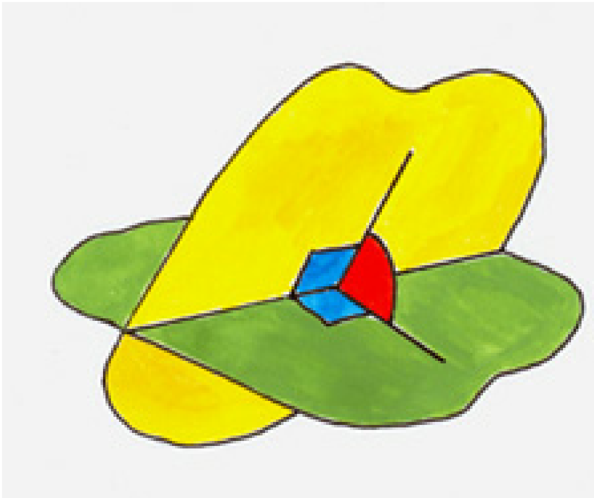
Solution

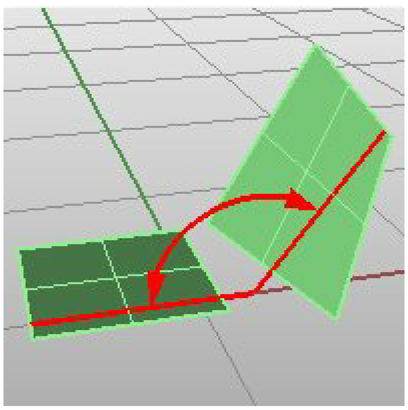
To find the angle we use tanθ = $\frac{\text{opposite}}{\text{adjacent}} = \frac{4}{10} = 0.4$

$$\tan^{-1}(0.4) = 21.8^0$$

Angle Between two planes

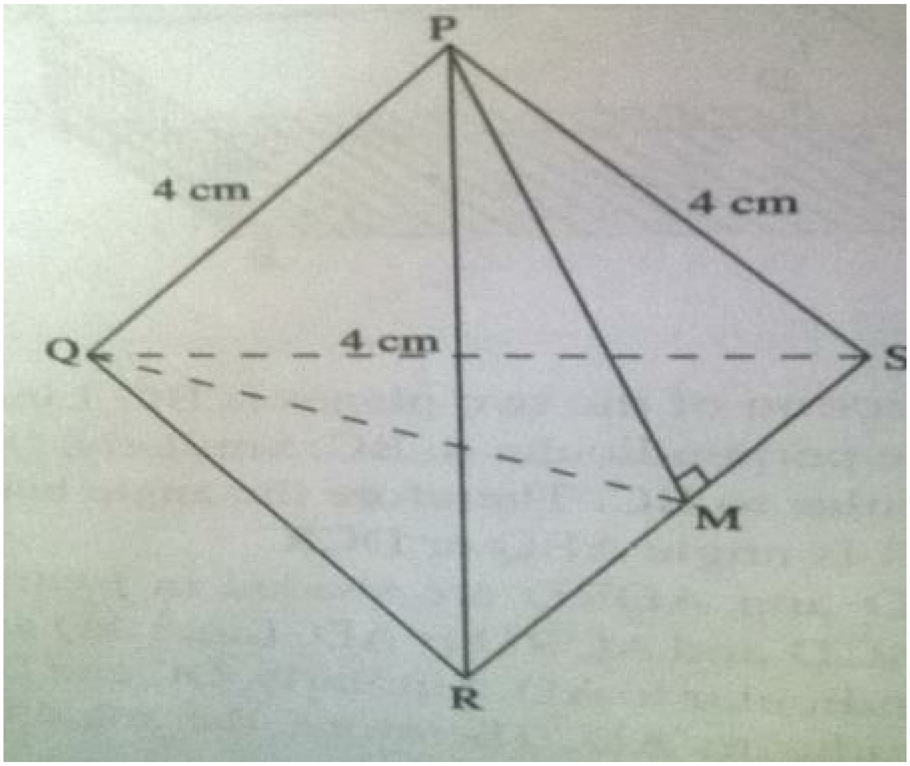
Any two planes are either parallel or intersect in a straight line. The angle between two planes is the angle between two lines, one on each plane, which is perpendicular to the line of intersection at the point





Example

The figure below PQRS is a regular tetrahedron of side 4 cm and M is the mid point of RS;



a.) Show that PM is $2\sqrt{3}$ cm long, and that triangle PMQ is isosceles

b.) Calculate the angle between planes PSR and QRS

c.) Calculate the angle between line PQ and plane QRS

Solution

a.) Triangle PRS is equilateral. Since M,is the midpoint of RS , PM is perpendicular bisector

$$PM^2 = 4^2 - 2^2$$

$$= 12$$

$PM = \sqrt{12}$ cm

$= \sqrt{4 \times 3} = 2\sqrt{3}$ cm

Similar triangle MQR is right angled at M

$$QM^2 = 4^2 - 2^2$$

$$= 12$$

QM = $\sqrt{12}$ cm

= $\sqrt{4 \times 3}$ = $2\sqrt{3}$ cm

Since PM = QM = $2\sqrt{12}$ cm Triangle PMQ is isosceleles

b.) The required angle is triangle PMQ .Using cosine rule

$$4^2 = (2\sqrt{3})^2 + (2\sqrt{3})^2 - 2(2\sqrt{3})2(2\sqrt{3})\cos m$$

$$16 = 12 + 12 - 2 \times 12\cos m$$

$$= 24 - 24\cos m$$

$$\cos m = \frac{24-16}{24} = 0.3333$$

$$\text{Therefore, } m = 70.53^0$$

c.) The required angle is triangle PQM

Since triangle PMQ is isosceles with triangle PMQ = 70.54^0;

<PQM = $\frac{1}{2}(180 - 70.54)$

= $\frac{1}{2}(109.46)$

$$= 54.73^0$$

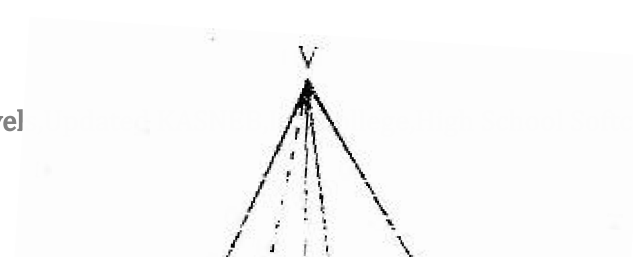
End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. The diagram below shows a right pyramid VABCD with V as the vertex. The base of the pyramid is rectangle ABCD, WITH ab = 4 cm and BC= 3 cm. The height of the pyramid is 6 cm.



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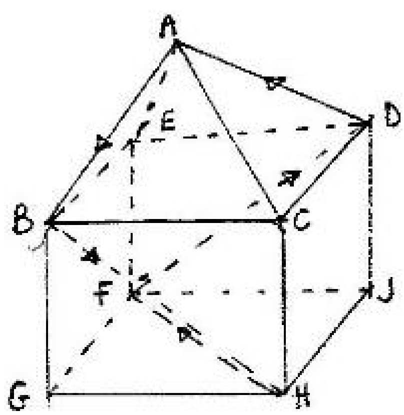
(a) Calculate the

(i) Length of the projection of VA on the base
(ii) Angle between the face VAB and the base

(b) P is the mid- point of VC and Q is the mid – point of VD.

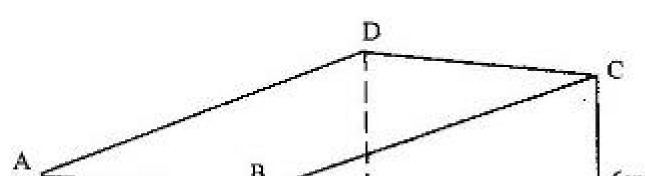
Find the angle between the planes VAB and the plane ABPQ

2. The figure below represents a square based solid with a path marked on it.



Sketch and label the net of the solid.

3. The diagram below represents a cuboid ABCDEFGH in which FG= 4.5 cm, GH = 8 cm and HC = 6 cm



Calculate:

(a) The length of FC

(b) (i) The size of the angle between the lines FC and FH

(ii) The size of the angle between the lines AB and FH

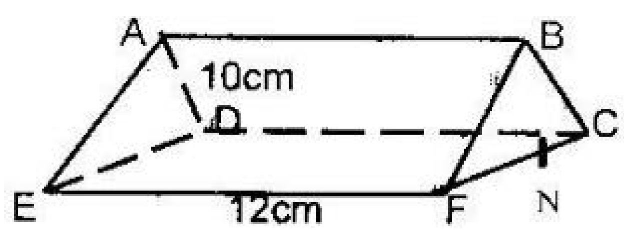
(c) The size of the angle between the planes ABHE and the plane FGHE

4. The base of a right pyramid is a square ABCD of side 2a cm. The slant edges VA, VB, VC and VD are each of length 3a cm.

(a) Sketch and label the pyramid

(b) Find the angle between a slanting edge and the base

5. The triangular prism shown below has the sides AB = DC = EF = 12 cm. the ends are equilateral triangles of sides 10cm. The point N is the mid point of FC.



Find the length of:

(a) (i) BN

(ii) EN

(b) Find the angle between the line EB and the plane CDEF

CHAPTER FIFTY NINE

TRIGONOMETRY III

Specific Objectives

By the end of the topic the learner should be able to:

(a) Recall and define trigonometric ratios;

(b) Derive trigonometric identity sin2x+cos2x = 1;

(c) Draw graphs of trigonometric functions;

(d) Solve simple trigonometric equations analytically and graphically;

(e) Deduce from the graph amplitude, period, wavelength and phase angles.

Content

(a) Trigonometric ratios

(b) Deriving the relation sin2x+cos2x =1

(c) Graphs of trigonometric functions of the form

y = sin x y = cos x, y = tan x

y = a sin x, y = a cos x,

y = a tan x y = a sin bx,

y = a cos bx y = a tan bx

y = a sin(bx ± 9)

y = a cos(bx ± 9)

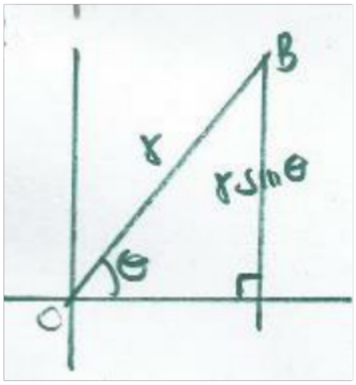
y = a tan(bx ± 9)

(d) Simple trigonometric equation

(e) Amplitude, period, wavelength and phase angle of trigonometric functions.

Introduction

Consider the right – angled triangle OAB



$$\sin\theta = \frac{AB}{r}$$

$AB = r\sin\theta$

$OA = r\cos\theta$

Since triangle OAB is right- angled

$$OA^2 + AB^2 = OB^2 (\text{pythagoras theorm})$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

Divide both sides by r^2 gives

$$\cos^2\theta + \sin^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example

If $\tan\theta = a$ show that;

$$\frac{\cos\theta\sin^2\theta + \cos^3\theta}{\sin\theta} = \frac{1}{a}$$

Solution

Factorize the numerator gives and since $\sin^2\theta + \cos^2 = 1$

$$\frac{\cos\theta(\sin^2\theta + \cos^2\theta)}{\sin\theta} = \frac{\cos\theta(1)}{\sin\theta}$$

But $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Therefore, $= \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{1}{a}$

Example

Show that

$$\frac{(1-\cos\theta)(1+\cos\theta)}{(1-\sin\theta)(1+\sin\theta)} = \tan^2\theta$$

Removing the brackets from the expression gives

$$\frac{1-\cos^2\theta}{1-\sin^2\theta} \qquad \text{reason}\left[(A-B)(A+B)=(A^2-B^2)\right]$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + 1 = \cos^2\theta$$

Also

$$1-\cos^2\theta = \sin^2\theta$$

Therefore

$$\frac{1-\cos^2\theta}{1-\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

Example

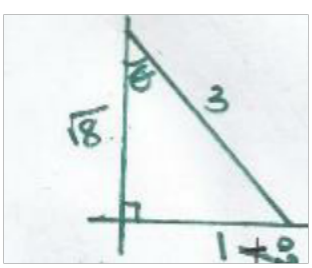
Given that $\sin\theta = \frac{1}{3}$

a.) $\cos^2\theta$

b.) $\tan^2\theta$

c.) $\tan^2\theta + \cos^2\theta$

Solution using the right angle triangle below.



a.) $\cos\theta = \frac{\sqrt{8}}{3}$

therefore $\cos^2\theta = (\frac{\sqrt{8}}{3})^2 = \frac{8}{9}$

b.) $\tan^2\theta = \left(\frac{1}{\sqrt{8}}\right)^2 = \frac{1}{8}$

c.) $\tan^2\theta + \cos^2\theta = \frac{1}{8} + \frac{8}{9} = 1\frac{1}{72}$

Waves

Amplitude

This is the maximum displacement of the wave above or below the x axis.

Period

The interval after which the wave repeats itself

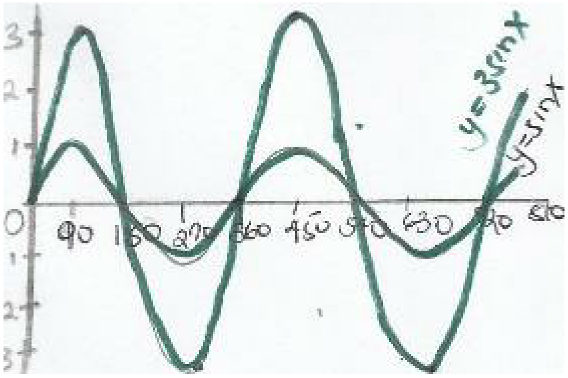
Transformations of waves

The graphs of y = sin x and y = 3 sin x can be drawn on the same axis. The table below gives the corresponding values of sin x and 3 sin x for $0^0 \leq x \geq 720^0$

x^2	0	30	60	90	120	150	180	210	240	270	300	330	360
Sin x	0	0.50	0.87	1.00	0.87	0.50	0	-0.50	-0.87	-0.50	-0.87	-0.50	0
3 sin x	0	1.50	2.61	3.00	2.61	1.50	0	-1.50	-2.61	-1.50	-2.61	-1.50	0

390	420	450	480	510	540	570	600	630	660	690	720
0.5	0.87	1.00	0.87	0.50	0	-0.50	-0.87	-1.00	-0.87	-0.50	0
1.50	2.61	3.00	2.61	1.50	0	-1.50	-2.61	-3.00	-2.61	-2.61	0

The wave of y = 3 sin x can be obtained directly from the graph of y = sin x by applying a stretch scale factor 3 , x axis invariant .

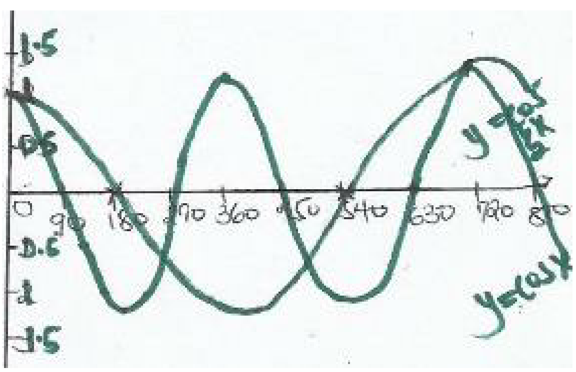


Note;

- The amplitude of y= 3sinx is y =3 which is three times that of y = sin x which is y =1.
- The period of the both the graphs is the same that is 360^0 or 2π

Example

Draw the waves y = cos x and y = cos $\frac{1}{2}x$. We obtain y = cos $\frac{1}{2}x$ from the graph y = cos x by applying a stretch of factor 2 with y axis invariant.



Note;

- The amplitude of the two waves are the same.
- The period of y = cos $\frac{1}{2}x$ is 4π that is, twice the period of y = cos x

Trigonometric Equations

In trigonometric equations, there are an infinite number of roots. We therefore specify the range of values for which the roots of a trigonometric equation are required.

Example

Solve the following trigonometric equations:

a.) Sin 2x = cos x, for $0 \leq x \leq 360^0$

b.) Tan 3x = 2, for$0 \leq x \leq 360^0$

c.) $2\sin\left(x-\frac{\pi}{6}\right)$

Solution

a.) Sin 2 x = cos x

Sin 2x = sin (90 – x)

Therefore 2 x = 90 – x

X = 30^0

For the given range, x =30^0 and 150^0.

b.) Tan 3x = 2

From calculator

3x =63.43^0 ,243.43^0 ,423.43^0 ,$603.43^0 783.43^0$ and 321.14^0.

In the given range;

$$x = 21.14^0 ,81.14^0 ,141.14^0 ,201.14^0 ,261.14^0 \text{ and } 321.14^0$$

c.) $2\sin\left(x-\frac{\pi}{6}\right) = -\sqrt{3}$

Sin$\left(x-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ sin $\left(x-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$$x - \frac{\pi}{6} = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$x - \frac{\pi}{6} = \frac{4}{3}\pi, \frac{5}{3}\pi$$

$$x = \frac{3}{2}\pi^c, \frac{11}{6}\pi^c$$

End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic

1. (a) Complete the table for the function y = 2 sin x

x	0^0	10^0	20^0	30^0	40^0	50^0	60^0	70^0	80^0	90^0	100^0	110^0	120^0
Sin 3x	0	0.5000							-08660				
y	0	1.00							-1.73				

(b) (i) Using the values in the completed table, draw the graph of $y = 2 \sin 3x$ for $0^0 \leq x \leq 120^0$ on the grid provided

(ii) Hence solve the equation 2 sin 3x = -1.5

2. Complete the table below by filling in the blank spaces

X^0	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
Cos x^0	1.00		0.50			-0.87		-0.87					
2 cos ½ x^0	2.00	1.93				0.52			-1.00				-2.00

Using the scale 1 cm to represent 30^0 on the horizontal axis and 4 cm to represent 1 unit on the vertical axis draw, on the grid provided, the graphs of $y = \cos x^0$ and $y = 2 \cos ½ x^0$ on the same axis.

(a) Find the period and the amplitude of $y = 2 \cos ½ x^0$

(b) Describe the transformation that maps the graph of $y = \cos x^0$ on the graph of $y = 2 \cos {}^1/_2\, x^0$

1. (a) Complete the table below for the value of y = 2 sin x + cos x.

X	0^0	30^0	45^0	60^0	90^0	120^0	135^0	150^0	180^0	225^0	270^0	315^0	360^0
2 sin x	0		1.4	1.7	2	1.7	1.4	1	0		-2	-1.4	0
Cos x	1		0.7	0.5	0	-0.5	-0.7	-0.9	-1		0	0.7	1
Y	1		2.1	2.2	2	1.2	0.7	0.1	-1		-2	-0.7	1

(b) Using the grid provided draw the graph of y=2sin x + cos x for 0^0. Take 1cm represent 30^0 on the x- axis and 2 cm to represent 1 unit on the axis.

(c) Use the graph to find the range of x that satisfy the inequalities
2 sin x cos x > 0.5

4. (a) Complete the table below, giving your values correct to 2 decimal places.

x	0	10	20	30	40	50	60	70
Tan x	0							
2 x + 300	30	50	70	90	110	130	150	170
Sin (2x + 30^0)	0.50			1				

b) On the grid provided, draw the graphs of y = tan x and y = sin (2x + 30^0) for 0^0 ≤ x 70^0
Take scale: 2 cm for 100 on the x- axis
4 cm for unit on the y- axis
Use your graph to solve the equation tan x- sin (2x + 30^0) = 0.

5. (a) Complete the table below, giving your values correct to 2 decimal places

X^0	0	30	60	90	120	150	180
2 sin x^0	0	1		2		1	
1 – cos x^0			0.5	1			

(b) On the grid provided, using the same scale and axes, draw the graphs of
y = sin x^0 and y = 1 – cos x^0 ≤ x ≤ 180^0
Take the scale: 2 cm for 30^0 on the x- axis
2 cm for I unit on the y- axis

(c) Use the graph in (b) above to
(i) Solve equation
2 sin x^o + cos x^0 = 1
(iii) Determine the range of values x for which 2 sin x^o> 1 – cos x^0

6. (a) Given that y = 8 sin 2x – 6 cos x, complete the table below for the missing values of y, correct to 1 decimal place.

X	0^0	15^0	30^0	45^0	60^0	75^0	90^0	105^0	120^0
Y = 8 sin 2x – 6 cos x	-6	-1.8		3.8	3.9	2.4	0		-3.9

(b) On the grid provided, below, draw the graph of y = 8 sin 2x – 6 cos for
0^0 ≤ x ≤ 120^0
Take the scale 2 cm for 15^0 on the x- axis
2 cm for 2 units on the y – axis

(c) Use the graph to estimate
(i) The maximum value of y
(ii) The value of x for which 4 sin 2x – 3 cos x =1

7. Solve the equation 4 sin (x + 30^0) = 2 for 0 ≤ x ≤ 360^0

8. Find all the positive angles not greater than 180^0 which satisfy the equation
$\frac{\text{Sin}^2 x}{\text{Cos } x} - 2 \tan x = 0$

9. Solve for values of x in the range 0^0 ≤ x ≤ 360^0 if 3 cos^2 x – 7 cos x = 6

10. Simplify $\underline{9 - y^2}$ where y = 3 cos θ

y

11. Find all the values of Ø between 0^0 and 360^0 satisfying the equation 5 sin Θ = -4

12. Given that sin (90 – x) = 0.8. Where x is an acute angle, find without using mathematical tables the value of tan x^0

13. Complete the table given below for the functions
y= -3 cos $2x^0$ and y = 2 sin ($^{3x}/_2{}^0$ + 30) for 0 ≤ x ≤ 180^0

X^0	0^0	20^0	40^0	60^0	80^0	100^0	120^0	140^0	160^0	180^0
-3cos $2x^0$	-3.00	-2.30	-0.52	1.50	2.82	2.82	1.50	-0.52	-2.30	-3.00
2 sin (3 x^0 + 30^0)	1.00	1.73	2.00	1.73	1.00	0.00	-1.00	-1.73	-2.00	-1.73

Using the graph paper draw the graphs of y = -3 cos $2x^0$ and y = 2 sin ($^{3x}/_2{}^0$ + 30^0)

(a) On the same axis. Take 2 cm to represent 20^0 on the x- axis and 2 cm to represent one unit on the y – axis

(b) From your graphs. Find the roots of 3 cos 2 x^0 + 2 sin ($^{3x}/_2{}^0$ + 30^0) = 0

14. Solve the values of x in the range 0^0 ≤ x ≤ 360^0 if 3 $\cos^2$x – 7cos x = 6

15. Complete the table below by filling in the blank spaces

x^0	0^0	30^0	60^0	90	1^0	150^0	180	210	240	270	300	330	360
Cosx^0	1.00		0.50			-0.87		-0.87					
2cos ½ x^0	2.00	1.93					0.5						

Using the scale 1 cm to represent 30^0 on the horizontal axis and 4 cm to represent 1 unit on the vertical axis draw on the grid provided, the graphs of y – cos x^0 and y = 2 cos ½ x^0 on the same axis

(a) Find the period and the amplitude of y =2 cos ½ x^0

Ans. Period = 720^0. Amplitude = 2

(c) Describe the transformation that maps the graph of y = cos x^0 on the graph of y = 2 cos ½ x^0

CHAPTER SIXTY

LONGITUDES AND LATITUDES

Specific Objectives

By the end of the topic the learner should be able to:

(a) Define the great and small circles in relation to a sphere (including the Earth);

(b) Establish the relationship between the radii of small and great circles;

(c) Locate a place on the earth's surface in terms of latitude and longitude;

(d) Calculate the distance between two points along the great circles and small circles (longitude and latitude) in nautical miles (nm) an kilometers (km);

(e) Calculate time in relation to longitudes;

(f) Calculate speed in knots and kilometers per hour.

Content

(a) Latitude and longitude (great and small circles)

(b) The Equator and Greenwich Meridian

(c) Radii of small and great circles

(d) Position of a place on the surface of the earth

(e) Distance between two points along the small and great circles in nautical miles and kilometers

(f) Distance in nautical miles and kilometres along a circle of latitude

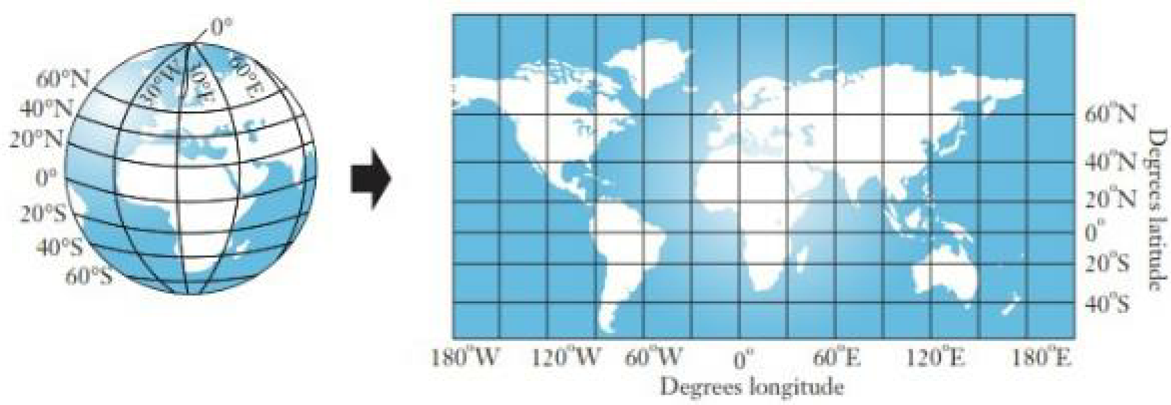
(g) Time and longitude

(h) Speed in knots and Kilometres per hour.

Introduction

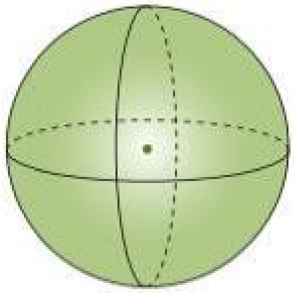
Just as we use a coordinate system to locate points on a number plane so we use latitude and longitude to locate points on the earth's surface.

Because the Earth is a sphere, we use a special grid of lines that run across and down a sphere. The diagrams below show this grid on a world globe and a flat world map.

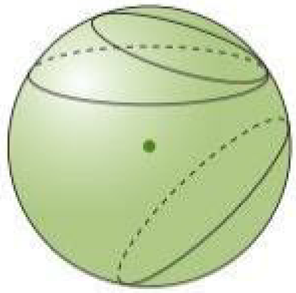


Great and Small Circles

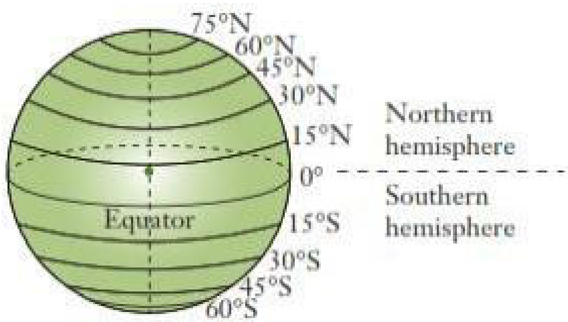
If you cut a 'slice' through a sphere, its shape is a circle. A slice through the **centre** of a sphere is called a **great circle**, and its radius is the same as that of the sphere. Any other slice is called a **small circle**, because its radius is smaller than that of a great circle.Hence great circles divides the sphere into two equal parts

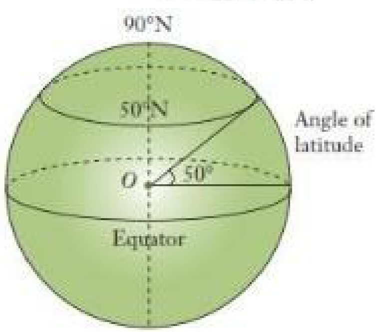


Great circles



Small circles





Latitude

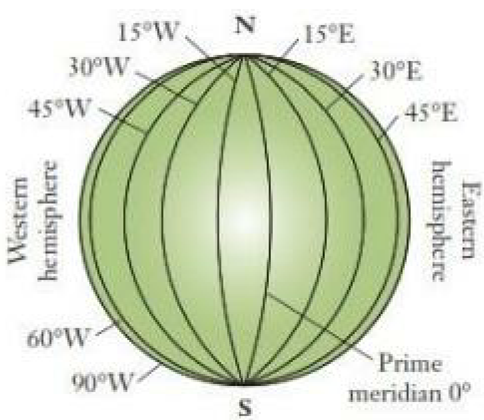
Latitudes are imaginary lines that run around the earth and their planes are perpendicular to the axis of the earth .The equator is the latitude tha divides the earth into two equal parts.Its the only great circles amoung the latitudes. The equator is , 0°.

The **angle of latitude** is the angle the latitude makes with the Equator at the centre, *O*, of the Earth. The diagram shows the 50°N parallel of latitude. Parallels of latitude range from 90°N (North Pole) to 90°S (South Pole).

The angle 50^0 subtended at the centre of the earth is the is the is the latitude of the circle passing through 50^0 north of equator.The maximum angle of latitude is 90^0 north or south of equator.

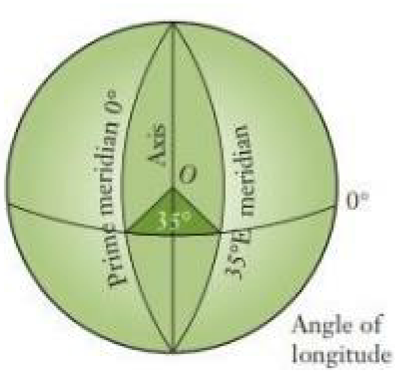
Longitudes /meridians

They are circles passing through the north and south poles



They can also be said that they are imaginary semicircles that run down the Earth. They are 'half' great circles that meet at the North and South Poles. The main meridian of longitude is the **prime meridian**, 0°. It is also called the **Greenwich meridian** since it runs through the Royal Observatory at Greenwich in London, England. The other meridians are measured in degrees east or west of the prime meridian.

The **angle of longitude** is the angle the meridian makes with the prime meridian at the centre, *O*, of the Earth. The diagram shows the 35°E meridian of longitude.



Meridians of longitude range from 180°E to 180°W. 180°E and 180°W are actually the same meridian, on the opposite side of the Earth to the prime meridian. It runs through the Pacific Ocean, east of Fiji.

Note

- If P is θ north of the equator and Q is α south of the quator , then the difference in latitude between them is given by $(\theta + \alpha)$

- If P and Q are on the same side of the equator , then the difference in latitude is ($\theta - \alpha$)

Position Coordinates

Locations on the Earth are described using latitude (°N or °S) and longitude (°E or °W) in that order. For example, Nairobi has coordinates (1°S, 37°E), meaning it is position is 1° south of the Equator and 37° east of the prime meridian.

EG

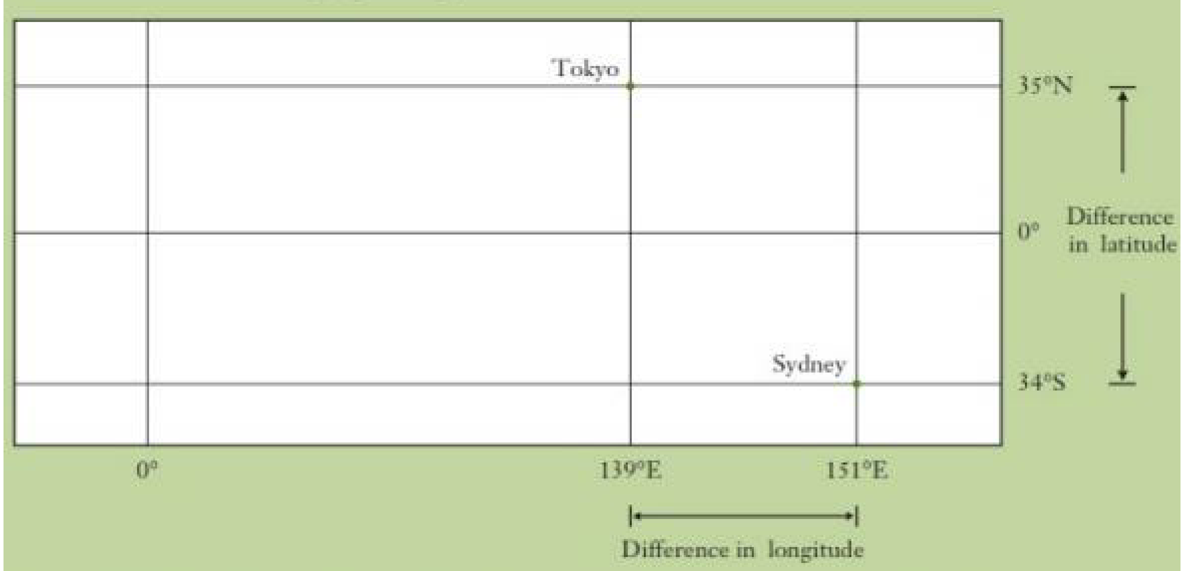
Sydney's coordinates are (34°S, 151°E) while Tokyo's are (35°N, 139°E).

a Find their difference in latitude.

b Find their difference in longitude.

c Which city is further west?

It is useful to draw a rough grid to position the cities.



a Difference in latitude = 35° + 34° = 69° Difference between 35°N and 34°S.

b Difference in longitude = 151° – 139° = 12° Difference between 151°E and 139°E.

c Sydney is further east, so Tokyo is further west.

Great Circle Distances

Remember the arc length of a circle is $l = \frac{\theta}{360} \times 2\pi r$ where θ is the degrees of the central angle, and the radius of the earth is 6370 km approx.

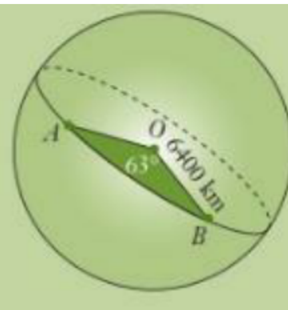
On a flat surface, the shortest distance between two points is a straight line. Since the Earth's surface is curved, the shortest distance between *A* and *B* is the arc length *AB* of the great circle that passes through *A* and *B*. This is called the **great circle distance** and the size of angle $\angle AOB$ where *O* is the centre of the Earth is called the **angular distance**.

Note

- The length of an arc of a great circle subtending an angle of 1^{i} (one minute) at the centre of the earth is 1 nautical mile nm.
- A nautical mile is the standard international unit from measuring distances travelled by ships and aeroplanes 1 nautical mile (nm) = 1.853 km

If an arc of a great circle subtends an angle θ at the centre of the earth,the arcs length is ($60 \times \theta$) nautical miles.

A and *B* on the Earth's surface have an angular distance of 63°. Calculate the great circle distance between *A* and *B*, correct to the nearest kilometre. The radius of the Earth is 6400 km.



Solution

$AB = \frac{\theta}{360} \times 2\pi r$

$= \frac{63}{360} \times 2 \times \pi \times 6400$

$= 7037.1675\ldots$

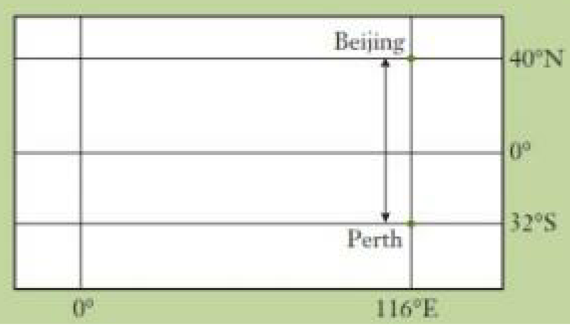
≈ 7037 km

Beijing, China and Perth, Australia have coordinates (40°N, 116°E) and (32°S, 116°E) respectively.

a What great circle joins Beijing and Perth?

b What is the angular distance between these two cities?

c Hence, calculate the shortest distance between Beijing and Perth, to the nearest kilometre, given that the Earth's radius is 6400 km.



Solution

a The 116°E meridian of longitude.

b Angular distance = 40° + 32° = 72°

c Distance $= \frac{72}{360} \times 2\pi \times 6400$

$= 8042.4772\ldots$

≈ 8042 km

Example

Find the distance between points P(40^{0} N ,50^{0}E) and Q ($20^{0}30^{1}$S, 50^{0}E) and express it in;

a.) Nm

b.) Km (Take radius of the earth to be 6370 km)

Solution

a.) Angle subtended at the centre is $40^0 + 20.5^0 = 60.5^0$

1^0 Is subtended by 60 nm

60.5^0 Is subtended by; 60 x 60.5 = 3630 nm

b.) The radius of the earth is 6370 km

Therefore, the circumference of the earth along a great circle is;

$$2\pi r = 6370 \text{ X } 2 \text{ X } \frac{22}{7}$$

Angle between the points is 60.5^0.Therefore, we find the length of an arch of a circle which subtends an angle of 60.5^0 at the centre is 360^0 is subtended by arc whose length is $6370 \text{ X } 2 \text{ X } \frac{22}{7}$.

Therefore, 60.5^0 Is subtended by ;$\frac{60.5}{360}$ x $6370 \text{ X } 2 \text{ X } \frac{22}{7} = 6729$ km

Example

Find the distance between points A (0^0,30^0E) and (0^0 ,50^0E) and express it in ;

a.) Nm.

b.) Km (Take the radius of the earth to be 6370 km)

Solution

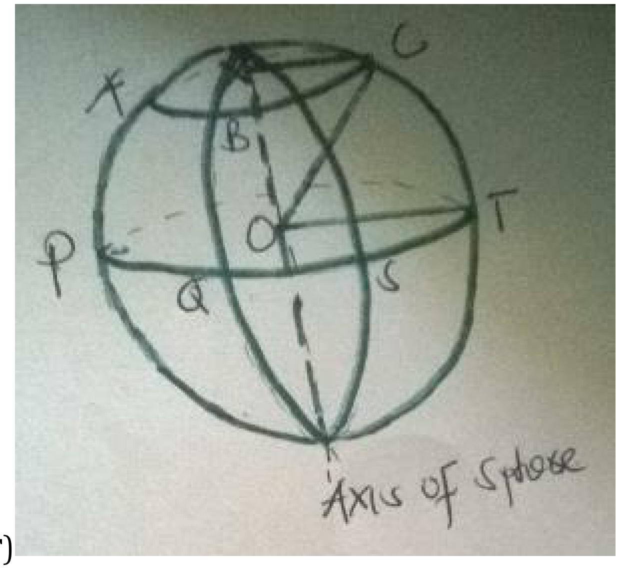
a.) The two points lie on the equator, which is great circle. Therefore ,we are calculating distance along a great circle.

Angle between points A and B is $(50^0 - 30^0) = 20^0$

b.) Distance in km $= \frac{20}{360}$ x $6370 \text{ X } 2 \text{ X } \frac{22}{7} = 2224$ km

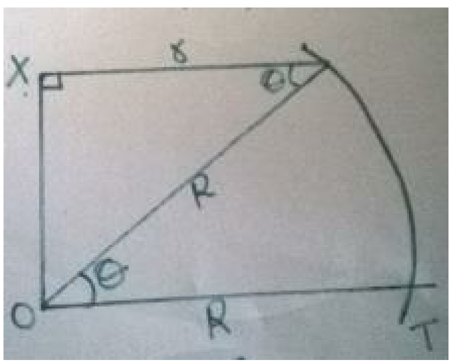
Distance along a small Circle (circle of latitude)

The figure below ABC is a small circle, centre X and radius r cm.PQST is a great circle ,centre O,radius R



cm.The angle θ is between the two radii.(OC and OT)

From the figure, XC is parallel to OT. Therefore, angle COT = angle XCO=θ(alternate angles).Angle CXO =90^0 (Radius XC is perpendicular to the axis of sphere).



Thus, from the right- angled triangle OXC

$$\text{Cos } \theta = \frac{r}{R}$$

Therefore, r = R cosθ

This expression can be used to calculate the distance between any two points along the small circle ABC, centre X and radius r.

Example

Find the distance in kilometers and nautical miles between two points (30^0N 45^0E) and Q (30^0N 60^0W).

Solution

Figure a shows the position of P and Q on the surface of the earth while figure b shows their relative positions on the small circle is the centre of the circle of latitude 30^0N with radius r.

The angle subtended by the arc PQ centre C is $45^0 + 60^0 = 105^0$.So, the length of PQ

$$= \frac{105}{360} \text{ X } 2\pi r$$

$$= \frac{105}{360} \text{ X } 2\pi R\cos 30^0 \text{ km}$$

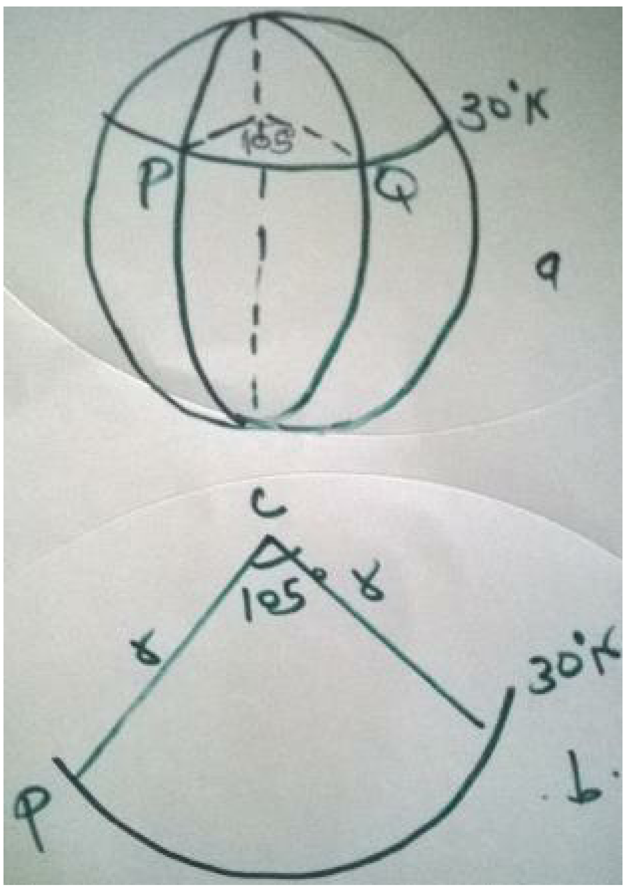
$$= \frac{105}{360} \text{ X } 2 \text{ X} \frac{22}{7} \text{ X } 6370\text{COS}\theta \text{ km}$$

$$= 10113 \text{ km}$$

The length of PQ in nautical miles

$$= 60 \text{ x } 105 \cos 30^0 \text{nm}$$

$= 60 \text{ x } 105 \text{ x } 0.8660 \text{ nm} = 5456 \text{ nm} =$



In general, if the angle at the centre of a circle of latitude θ is α,then the length of its arc is 60 $\alpha\cos\theta$ nm,where α is the angle between the longitudes along the same latitude.

Shortest distance between the two points on the earths surface

The shortest distance between two points on the earths surface is that along a great circle.

Example

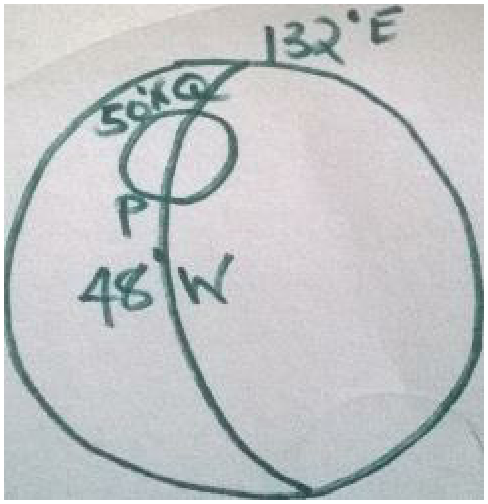
P and Q are two points on latitude 50^0N. They lie on longitude 40^0 W and 132 E^0 respectively. Find the distance from P to Q :

a.) Along a parallel of latitude

b.) Along a great circle

Solution

The positions of P and Q on earths surface are as shown below

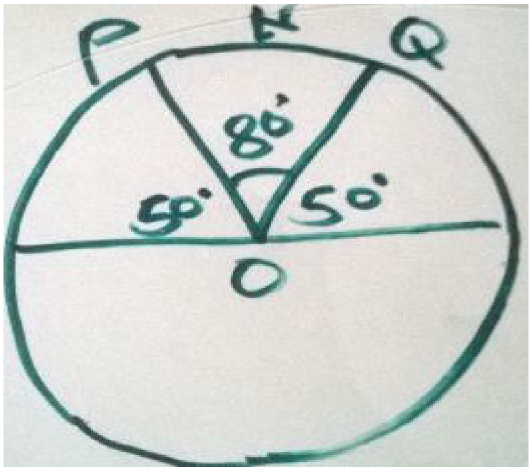


a.) The length of the circle parallel of latitude 50^0N is 2 πr km, which is $2\pi R\cos 50^0$km.The difference in longitude between P and Q is $132^{\,0} + 48^0 = 180^0$

PQ= $\frac{180}{360}$ X 2 X$\frac{22}{7}$ X 6370 $\cos 50^0$ = 12869 km

b.) The required great circle passes via the North Pole. Therefore, the angle subtended at the centre by the arc PNQ is;

= $180^0 - 2 \times 50^0 = 80^0$



Therefore the arc PNQ

$$= \frac{80}{360} \text{ X } 2\,\pi R$$

$$= \frac{80}{360} \text{ X } 2 \text{ X} \frac{22}{7} \text{ X } 6370 = 8898 \text{ km}$$

Note;

Notice that the distance between two points on the earth's surface along a great circle is shorter than the distance between them along a small circle

Longitude and Time

The earth rotates through 360^0 about its axis every 24 hours in west – east direction. Therefore for every 1^0 change in longitude there is a corresponding change in time of 4 minutes, or there is a difference of 1 hour between two meridians 15^0 apart.

All places in the same meridian have the same local time. Local time at Greenwich is called Greenwich Mean Time .GMT.

All meridians to the west of Greenwich Meridian have sunrise after the meridian and their local times are behind GMT.

All meridian to the east of Greenwich Meridian have sunrise before the meridian and their local times are ahead of GMT. Since the earth rotates from west to east, any point P is ahead in time of another point Q if P is east of Q on the earth's surface.

Example

Find the local time in Nairobi (1^0 S ,37^0 E), when the local time of Mandera (Nairobi (4^0 N ,42^0 E) is 3.00 pm

Solution

The difference in longitude between Mandera and Nairobi is (42^0 - 37^0) = 5^0, that is Mandera is 5^0 east of Nairobi.Therefore their local time differ by; 4 x 5 = 20 min.

Since Nairobi is in the west of Mandera, we subtract 20 minutes from 3.00 p.m. This gives local time for Nairobi as 2.40 p.m.

Example

If the local time of London (52^0 N,0^0), IS 12.00 noon, find the local time of Nairobi (1^0 S,37^0 E),

Solution

Difference in longitude is ($37^0 + 0^0$) = 37^0

So the difference in time is 4 x 37 min = 148 min

= 2 hrs. 28 min

Therefore , local time of Nairobi is 2 hours 28 minutes ahead that of London that is,2.28 p.m

Example

If the local time of point A (0^0 N,170^0 E) is 12.30 a.m, on Monday,Find the local time of a point B (0^0 N,170^0 W).

Solution

Difference in longitude between A and B is $170^0 + 170^0 = 340^0$

In time is 4 x 340 = 1360 min

= 22 hrs. 40 min.

Therefore local time in point B is 22 hours 40 minutes behind Monday 12:30 p.m. That is, Sunday 1.50 a.m.

Speed

A speed of 1 nautical mile per hour is called a knot. This unit of speed is used by airmen and sailors.

Example

A ship leaves Mombasa (4^0S,39^0E)and sails due east for 98 hours to appoint K Mombasa (4^0S,80^0E)in the indian ocean.Calculate its average speed in;

a.) Km/h

b.) Knots

Solution

a.) The length x of the arc from Mombasa to the point K in the ocean

$$\frac{41}{360} \text{ x } 2\pi r$$

$= \frac{41}{360}$ X 2 $\pi R\cos 4^0$ km

$= \frac{41}{360}$ X 2 X$\frac{22}{7}$ X 6370cos 4^0 km = 4549 km

Therefore speed is = $\frac{4549}{98}$ = 46.41 km/h

b.) The length x of the arc from Mombasa to the point K in the ocean in nautical miles

$$x = 60 \text{ x } 41 \text{ x } \cos 4^0 \text{ nm}$$

$$= 60 \text{ x } 41 \text{ x } 0.9976 \text{ nm} = 2454 \text{ nm}$$

Therefore , speed =$\frac{2454}{98}$

= 25.04 knots

End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. An aeroplane flies from point A (1^0 15'S, 37^0 E) to a point B directly North of A. the arc AB subtends an angle of 45^0 at the center of the earth. From B, aeroplanes flies due west two a point C on longitude 23^0 W.)

(Take the value of π $^{22}/_{7}$ as and radius of the earth as 6370km)

(a) (i) Find the latitude of B

(ii) Find the distance traveled by the aeroplane between B and C

(b) The aeroplane left at 1.00 a.m. local time. When the aeroplane was leaving B, what was the local time at C?

2. The position of two towns X and Y are given to the nearest degree as X (45^0 N, 10^0W) and Y (45^0 N, 70^0W)

Find

(a) The distance between the two towns in

(i) Kilometers (take the radius of the earth as 6371)
(ii) Nautical miles (take 1 nautical mile to be 1.85 km)

(b) The local time at X when the local time at Y is 2.00 pm.

3. A plane leaves an airport A (38.5^0N, 37.05^0W) and flies dues North to a point B on latitude 52^0N.

(a) Find the distance covered by the plane

(b) The plane then flies due east to a point C, 2400 km from B. Determine the position of C

Take the value π of as $^{22}/_7$ and radius of the earth as 6370 km

4. A plane flying at 200 knots left an airport A (30^0S, 31^0E) and flew due North to an airport B (30^0 N, 31^0E)

(a) Calculate the distance covered by the plane, in nautical miles

(b) After a 15 minutes stop over at B, the plane flew west to an airport C (30^0 N, 13^0E) at the same speed.

Calculate the total time to complete the journey from airport C, though airport B.

5. Two towns A and B lie on the same latitude in the northern hemisphere.

When its 8 am at A, the time at B is 11.00 am.

a) Given that the longitude of A is 15^0 E find the longitude of B.

b) A plane leaves A for B and takes $3^1/_2$ hours to arrive at B traveling along a parallel of latitude at 850 km/h. Find:

(i) The radius of the circle of latitude on which towns A and B lie.

(ii) The latitude of the two towns (take radius of the earth to be 6371 km)

6. Two places A and B are on the same circle of latitude north of the equator. The longitude of A is 118^0W and the longitude of B is 133^0 E. The shorter distance between A and B measured along the circle of latitude is 5422 nautical miles.

Find, to the nearest degree, the latitude on which A and B lie

7. (a) A plane flies by the short estimate route from P (10^0S, 60^0 W) to Q (70^0 N, 120^0 E) Find the distance flown in km and the time taken if the aver age speed is 800 km/h.

(b) Calculate the distance in km between two towns on latitude 50^0S with long longitudes and 20^0 W. (take the radius of the earth to be 6370 km)

8. Calculate the distance between M (30^0N, 36^0E) and N (30^0 N, 144^0 W) in nautical miles.

(i) Over the North Pole

(ii) Along the parallel of latitude 30^0 N

9. (a) A ship sailed due south along a meridian from 12^0 N to $10^0 30'$S. Taking the earth to be a sphere with a circumference of 4×10^4 km, calculate in km the distance traveled by the ship.

(b) If a ship sails due west from San Francisco (37^0 47'N, 122^0 26'W) for distance of 1320 km. Calculate the longitude of its new position (take the radius of the earth to be 6370 km and π = 22/7).

CHAPTER SIXTY ONE

LINEAR PROGRAMMING

Specific Objectives

By the end of the topic the learner should be able to:

(a) Form linear inequalities based on real life situations;

(b) Represent the linear inequalities on a graph;

(c) Solve and interpret the optimum solution of the linear inequalities,

(d) Apply linear programming to real life situations.

Content

(a) Formation of linear inequalities

(b) Analytical solutions of linear inequalities

(c) Solutions of linear inequalities by graphs

(d) Optimisation (include objective function)

(e) Application of quadratic equations to real life situations.

Forming linear inequalities

In linear programing we are going to form inequalities representing given conditions involving real life situation.

Example

Esha is five years younger than his sister. The sum of their age is less than 36 years. If Esha's age is x years, form all the inequalities in x for this situation.

Solution

The age of Esha's sister is x +5 years.

Therefore, the sum of their age is;

X + (x +5) years

Thus;

2x +5 < 36

2x < 31

X > 15.5

X > 0 (age is always positive)

Linear programming

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the "best" value obtainable under those conditions. A typical example would be taking the limitations of materials and labor, and then determining the "best" production levels for maximal profits under those conditions.

In "real life", linear programming is part of a very important area of mathematics called "optimization techniques". This field of study are used every day in the organization and allocation of resources. These "real life" systems can have dozens or hundreds of variables, or more. In algebra, though, you'll only work with the simple (and graph able) two-variable linear case.

The general process for solving linear-programming exercises is to graph the inequalities (called the "constraints") to form a walled-off area on the x,y-plane (called the "feasibility region"). Then you figure out the coordinates of the corners of this feasibility region (that is, you find the intersection points of the various pairs of lines), and test these corner points in the formula (called the "optimization equation") for which you're trying to find the highest or lowest value.

Example

Suppose a factory want to produce two types of hand calculators, type A and type B. The cost, the labor time and the profit for every calculator is summarized in the following table:

Type	Cost	Labor Time	Profit
A	Sh 30	1 (hour)	Sh 10
B	Sh 20	4 (hour)	Sh 8

Suppose the available money and labors are ksh 18000 and 1600 hours. What should the production schedule be to ensure maximum profit?

Solution

Suppose x_1 is the number of type A hand calculators and x_2 is the number of type B hand calculators and y to be the cost. Then, we want to maximize $p = 10x_1 + 8x_2$ subject to

$$30x + 20y \leq 18000$$

$$x + 4y \leq 1600$$

$$x \geq 0, \quad y \geq 0$$

where P is the total profit.

Solution by graphing

Solutions to inequalities formed to represent given conditions can be determined by graphing the inequalities and then reading off the appropriate values (possible values)

Example

A student wishes to purchase not less than 10 items comprising books and pens only. A book costs sh.20 and a pen sh.10.if the student has sh.220 to spend, form all possible inequalities from the given conditions and graph them clearly, indicating the possible solutions.

Solution

Let the number of books be x and the number of pens then, the inequalities are;

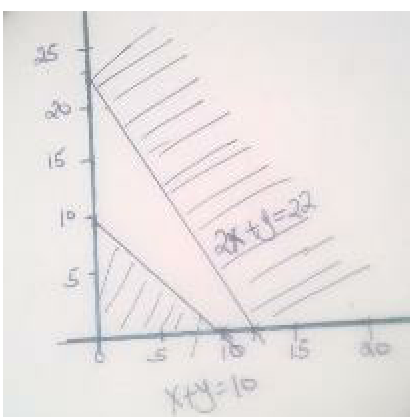
i.) $x + y \geq 10$ (the items bought to be atlest ten)

ii.) $x + y \leq 220$ (only sh.220 is available)

This simplifies to $2x + y \leq 22$

iii.) $x >$ and $y > 0$(number of items bought cannot be negative).

The graph below represents the inequalities



All the points in the unshaded region represent possible solutions. A point with co-ordinates (x ,y) represents x books and y pens. For example, the point (3, 10) means 3 books and 10 pens could be bought by the students.

Optimization

The determination of the minimum or the maximum value of the objective function ax + by is known as optimization. Objective function is an equation to be minimized or maximized .

Example

A contractor intends to transport 1000 bags of cement using a lorry and a pick up. The lorry can carry a maximum of 80 bags while a pick up can carry a maximum of 20 bags. The pick up must make more than twice the number of trips the lorry makes and the total number of trip to be less than 30.The cost per trip for the lorry is ksh 2000, per bag and ksh 900 for the pick up.Find the minimum expenditure.

Solution

If we let x and y be the number of trips made by the lorry and the pick up respectively. Then the conditions are given by the following inequalities;

i.) $x + y < 30$

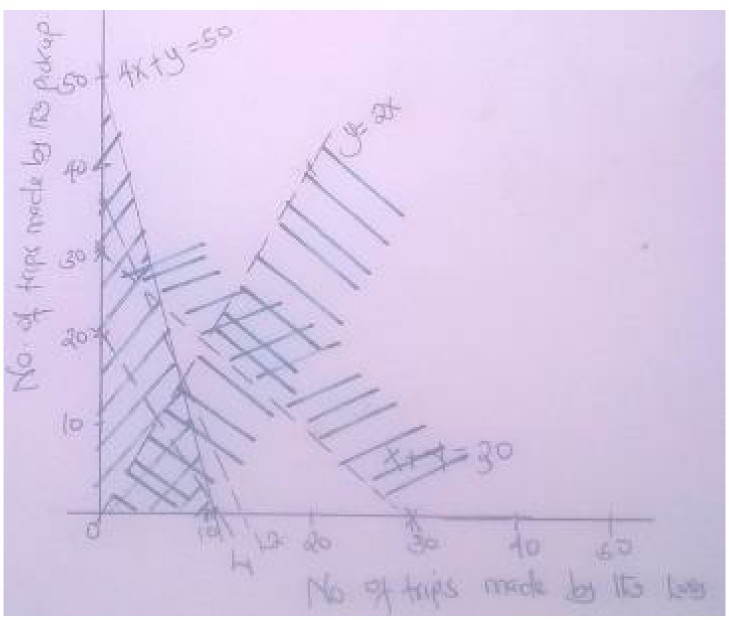
ii.) $80x + 20y \geq 1000$, which simplifies as $4x + y \geq 50$

iii.) $y > 2x$

iv.) $x < 0$

The total cost of transporting the cement is given by sh 2000x + 900y.This is called the objective function.

The graph below shows the inequalities.



From the graph we can identify 7 possibilities

(7,22),(8,18),(8,19),(8,20),(8,21),(9,19),(9,20)

Note;

Co-ordinates stands for the number of trips. For example (7, 22) means 7 trips by the lorry and 22 trips by the pickup. Therefore the possible amount of money in shillings to be spent by the contractor can be calculated as follows.

i.) $(2000 \times 7) + (900 \times 22) = 33800$

ii.) $(2000 \times 8) + (900 \times 18) = 32200$

iii.) $(2000 \times 8) + (900 \times 19) = 33100$

iv.) $(2000 \times 8) + (900 \times 20) = 34000$

v.) $(2000 \times 8) + (900 \times 21) = 34900$

vi.) $(2000 \times 9) + (900 \times 19) = 35100$

vii.) $(2000 \times 9) + (900 \times 20) = 36000$

We note that from the calculation that the least amount the contractor would spend is sh.32200.This is when the lorry makes 8 trips and the pick- up 18 trips. When possibilities are many the method of determining the solution by calculation becomestedious. The alternative method involves drawing the graph of the function we wish to maximize or minimize, the objective function. This function is usually of the form ax +by , where a and b ar constants.

For this ,we use the graph above which is a convenient point (x , y) to give the value of x preferably close to the region of the possibilities. For example the point (5, 10) was chosen to give an initial value of thus ,2000x + 900y = 19000.we now draw the line 2000x + 900y=19000.such a line is referred to us a search line.

Using a ruler and a set square, slide the set square keeping one edge parallel to l_1 until the edge strikes the feasible point nearest l_1 (see the dotted line l_2) From the graph this point is (8,18),which gives the minimum expenditure as we have seen earlier.The feasible point furthest from the line l_1 gives the maximum value of the objective function.

The determination of the minimum or the maximum value of the objective function ax + by is known as optimization.

Note;

The process of solving linear equations are as follows

i.) Forming the inequalities satisfying given conditions

ii.) Formulating the objective function .

iii.) Graphing the inequalities

iv.) Optimizing the objective function

This whole process is called linear programming .

Example

A company produces gadgets which come in two colors: red and blue. The red gadgets are made of steel and sell for ksh 30 each. The blue gadgets are made of wood and sell for ksh 50 each. A unit of the red gadget requires 1 kilogram of steel, and 3 hours of labor to process. A unit of the blue gadget, on the other hand, requires 2 board meters of wood and 2 hours of labor to manufacture. There are 180 hours of labor, 120 board meters of wood, and 50 kilograms of steel available. How many units of the red and blue gadgets must the company produce (and sell) if it wants to maximize revenue?

Solution

The Graphical Approach

Step 1. *Define all decision variables.*

Let: x_1 = number of red gadgets to produce (and sell)
x_2 = number of blue gadgets to produce (and sell)

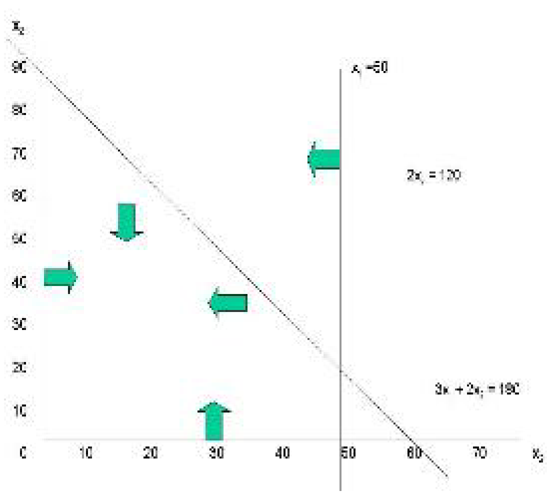
Step 2. *Define the objective function.*

Maximize $R = 30x_1 + 50x_2$ (total revenue in ksh)

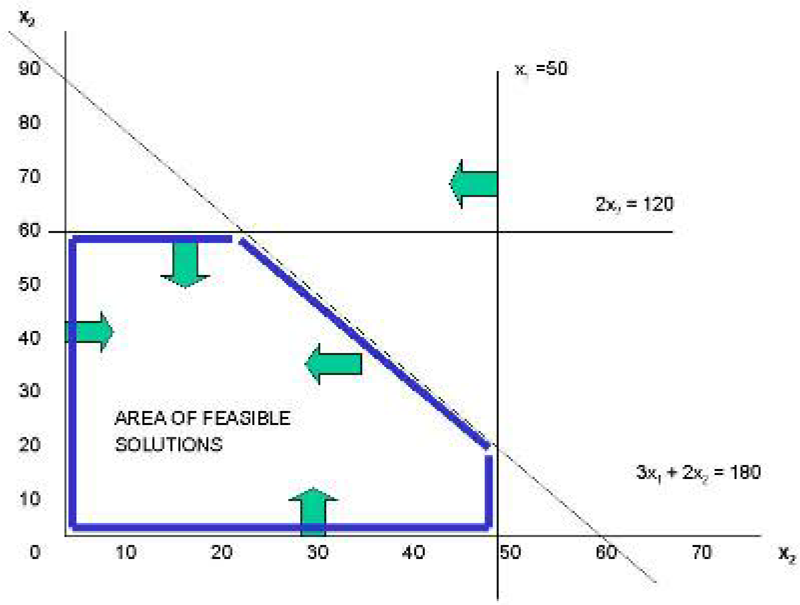
Step 3. *Define all constraints.*

(1) $x_1 \leq 50$ (steel supply constraint in kilograms)
(2) $2x_2 \leq 120$ (wood supply constraint in board meters)
(3) $3x_1 + 2x_2 \leq 180$ (labor supply constraint in man hours)
$x_1, x_2 \geq 0$ (non-negativity requirement)

Step 4. *Graph all constraints.*



Then determine area of feasible study



Note;

- The area under the line marked blue is the needed area or area of feasible solutions.
- We therefore shade the unwanted region out the trapezium marked blue

Optimization

List all corners (identify the corresponding coordinates), and pick the best in terms of the resulting value of the objective function.

(1) $x_1 = 0$ $x_2 = 0$ $R = 30\ (0) + 50\ (0) = 0$

(2) $x_1 = 50$ $x_2 = 0$ $R = 30\ (50) + 50\ (0) = 1500$

(3) $x_1 = 0$ $x_2 = 60$ $R = 30\ (0) + 50\ (60) = 3000$

(4) $x_1 = 20$ $x_2 = 60$ $R = 30\ (20) + 50\ (60) = 3600$ (the optimal solution)

(5) $x_1 = 50$ $x_2 = 15$ $R = 30\ (50) + 50\ (15) = 2250$

End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. A school has to take 384 people for a tour. There are two types of buses available, type X and type Y. Type X can carry 64 passengers and type Y can carry 48 passengers. They have to use at least 7 buses.

   (a) Form all the linear inequalities which will represent the above information.

   (b) On the grid [provide, draw the inequalities and shade the unwanted region.

   (c) The charges for hiring the buses are

   Type X: Ksh 25,000

   Type Y Ksh 20,000

   Use your graph to determine the number of buses of each type that should be hired to minimize the cost.

2. An institute offers two types of courses technical and business courses. The institute has a capacity of 500 students. There must be more business students than technical students but at least 200 students must take technical courses. Let x represent the number of technical students and y the number of business students.

   (a) Write down three inequalities that describe the given conditions

   (b) On the grid provided, draw the three inequalities

   (c) If the institute makes a profit of Kshs 2, 500 to train one technical students and Kshs 1,000 to train one business student, determine

      (i) The number of students that must be enrolled in each course to maximize the profit

      (ii) The maximum profit.

3. A draper is required to supply two types of shirts A and type B.

   The total number of shirts must not be more than 400. He has to supply more type A than of type B however the number of types A shirts must be more than 300 and the number of type B shirts not be less than 80.

   Let x be the number of type A shirts and y be the number of types B shirts.

   (a) Write down in terms of x and y all the linear inequalities representing the information above.
   (b) On the grid provided, draw the inequalities and shade the unwanted regions
   (c) The profits were as follows
   Type A: Kshs 600 per shirt

   Type B: Kshs 400 per shirt

(i) Use the graph to determine the number of shirts of each type that should be made to maximize the profit.

(ii) Calculate the maximum possible profit.

4. A diet expert makes up a food production for sale by mixing two ingredients N and S. One kilogram of N contains 25 units of protein and 30 units of vitamins. One kilogram of S contains 50 units of protein and 45 units of vitamins. The food is sold in small bags each containing at least 175 units of protein and at least 180 units of vitamins. The mass of the food product in each bag must not exceed 6kg.

If one bag of the mixture contains x kg of N and y kg of S

(a) Write down all the inequalities, in terms of x and representing the information above (2 marks)

(b) On the grid provided draw the inequalities by shading the unwanted regions (2 marks)

(c) If one kilogram of N costs Kshs 20 and one kilogram of S costs Kshs 50, use the graph to determine the lowest cost of one bag of the mixture.

5. Esha flying company operates a flying service. It has two types of aeroplanes. The smaller one uses 180 litres of fuel per hour while the bigger one uses 300 litres per hour.

The fuel available per week is 18,000 litres. The company is allowed 80 flying hours per week.

(a) Write down all the inequalities representing the above information

(b) On the grid provided on page 21, draw all the inequalities in (a) above by

shading the unwanted regions

(c) The profits on the smaller aeroplane is Kshs 4000 per hour while that on the

bigger one is Kshs. 6000 per hour. Use your graph to determine the maximum profit that the company made per week.

6. A company is considering installing two types of machines. A and B. The information about each type of machine is given in the table below.

Machine	Number of operators	Floor space	Daily profit
A	2	$5m^2$	Kshs 1,500
B	5	$8m^2$	Kshs 2,500

The company decided to install x machines of types A and y machines of type B

(a) Write down the inequalities that express the following conditions

i. The number of operators available is 40
ii. The floor space available is $80m^2$
iii. The company is to install not less than 3 type of A machine
iv. The number of type B machines must be more than one third the number of type A machines

(b) On the grid provided, draw the inequalities in part (a) above and shade the unwanted region.

(c) Draw a search line and use it to determine the number of machines of each type that should be installed to maximize the daily profit.

CHAPTER SIXTY TWO

LOCUS

Specific Objectives

By the end of the topic the learner should be able to:

(a) Define Locus;

(b) Describe common types of Loci;

(c) Construct;

i) Loci involving inequalities;

ii) Loci involving chords;

iii) Loci involving points under given conditions;

iv) Intersecting loci.

Content

(a) Common types of Loci

(b) Perpendicular bisector loci

(c) Locus of a point at a given distance from a fixed point

(d) Angle bisector loci

(e) Other loci under given condition including intersecting loci

(f) Loci involving inequalities

(g) Loci involving chords (constant angle loci).

Introduction

Locus is defined as the path, area or volume traced out by a point, line or region as it moves according to some given laws

In construction the opening between the pencil and the point of the compass is a fixed distance, the length of the radius of a circle. The point on the compass determines a fixed point. If the length of the radius remains the same or unchanged, all of the point in the plane that can be drawn by the compass from a circle and any points that cannot be drawn by the compass do not lie on the circle. Thus the circle is the set of all points at a fixed distance from a fixed point. This set is called a locus.

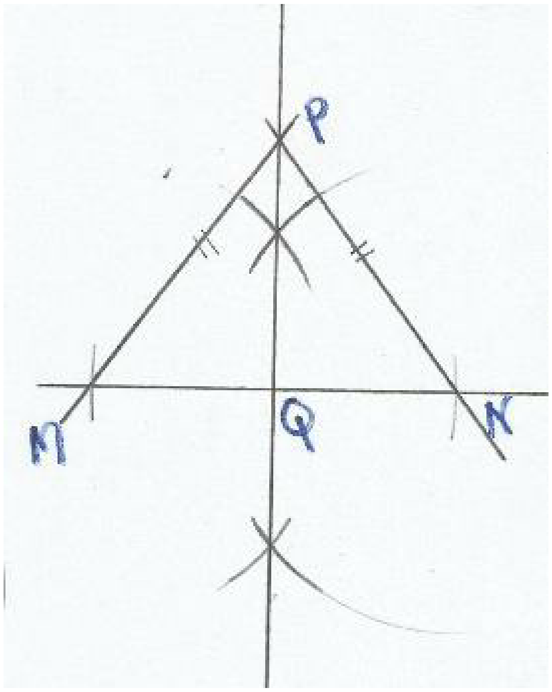
Common types of Loci

Perpendicular bisector locus

The locus of a point which are equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points. This locus is called the perpendicular bisector locus.

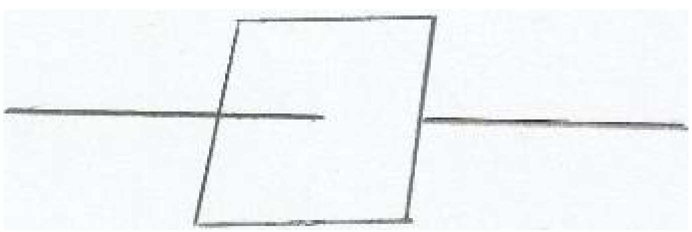
So to find the point equidistant from two fixed points you simply find the perpendicular bisector of the two points as shown below.

Q is the mid-point of M and N.



In three Dimensions

In three dimensions, the perpendicular bisector locus is a plane at right angles to the line and bisecting the line into two equal parts. The point P can lie anywhere in the line provided its in the middle.

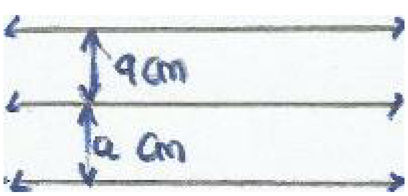


The Locus of points at a Given Distance from a given straight line.

In two Dimensions

In the figure below each of the lines from the middle line is marked a centimeters on either side of the given line MN.

The 'a' centimeters on either sides from the middle line implies the perpendicular distance.



The two parallel lines describe the locus of points at a fixed distance from a given straight line.

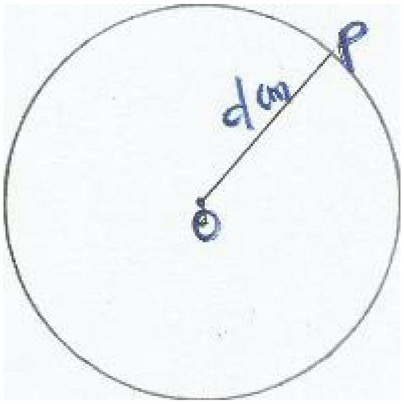
In three Dimensions

In three dimensions the locus of point 'a' centimeters from a line MN is a cylindrical shell of radius 'a' c, with MN as the axis of rotation.

Locus of points at a Given Distance from a fixed point.

In two Dimension

If O is a fixed point and P a variable point'd' cm from O,the locus of p is the circle O radius 'd' cm as shown below.

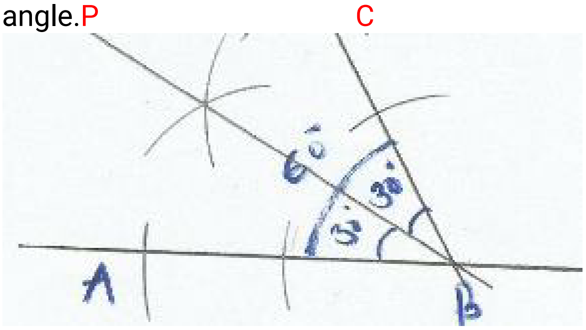


All points on a circle describe a locus of a point at constant distance from a fixed point. In three dimesion the locus of a point 'd' centimetres from a point is a spherical shell centre O and radius d cm.

Angle Bisector Locus

The locus of points which are equidistant from two given intersecting straight lines is the pair of perpendicular lines which bisect the angles between the given lines.

Conversely ,a point which lies on a bisector of given angle is equidistant from the lines including that angle.P C



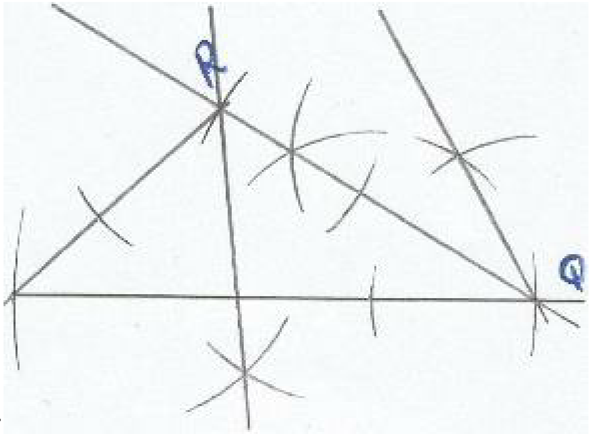
Line PB bisect angle ABC into two equal parts.

Example

Construct triangle PQR such that PQ= 7 cm, QR = 5 cm and angle PQR = 30^0.Construct the locus L of points equidistant from RP and RQ.

Solution

L is the bisector of Angle PRQ.



P

L

Constant angle loci

A line PQ is 5 cm long, Construct the locus of points at which PQ subtends an angle of 70^0.

Solution

i.) Draw PQ = 5 cm

ii.) Construct TP at P such that angle QPT = 70^0

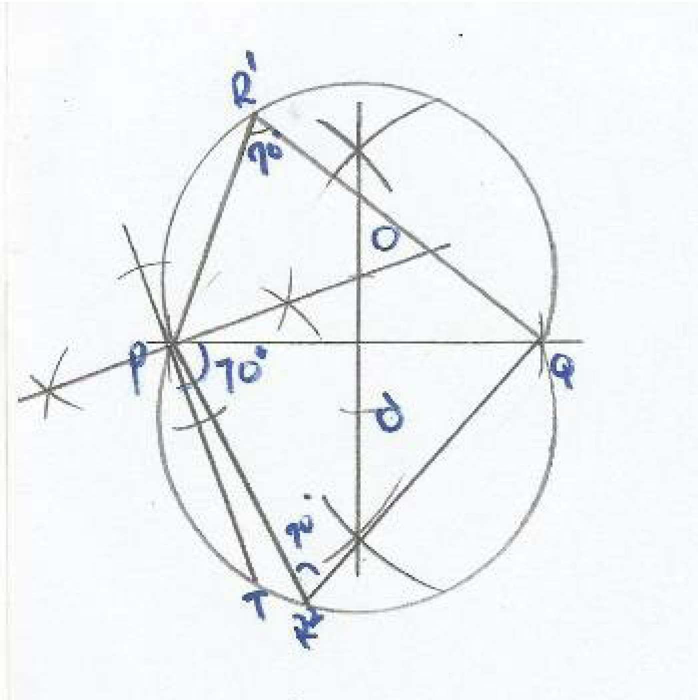
iii.) Draw a perpendicular to TP at P(radius is perpendicular to tangent)

iv.) Construct the perpendicular bisector of PQ to meet the perpendicular in (iii) at O

v.) Using O as the centre and either OP or OQ as radius, draw the locus

vi.) Transfer the centre on the side of PQ and complete the locus.

vii.) Transfer the centre on the opposite sides of PQ and complete the locus as shown below.



- To O^1 are of the same radius,
- Angle subtended by the same chord on the circumference are equal ,
- This is called the constant angle locus.

Intersecting Loci

a.) Construct triangle PQR such that PQ =7 cm, OR = 5 cm and angle PQR = 30^0

b.) Construct the locus L_1 of points equidistant from P and Q to meet the locus L_2 of points equidistant from Q and R at M .Measure PM

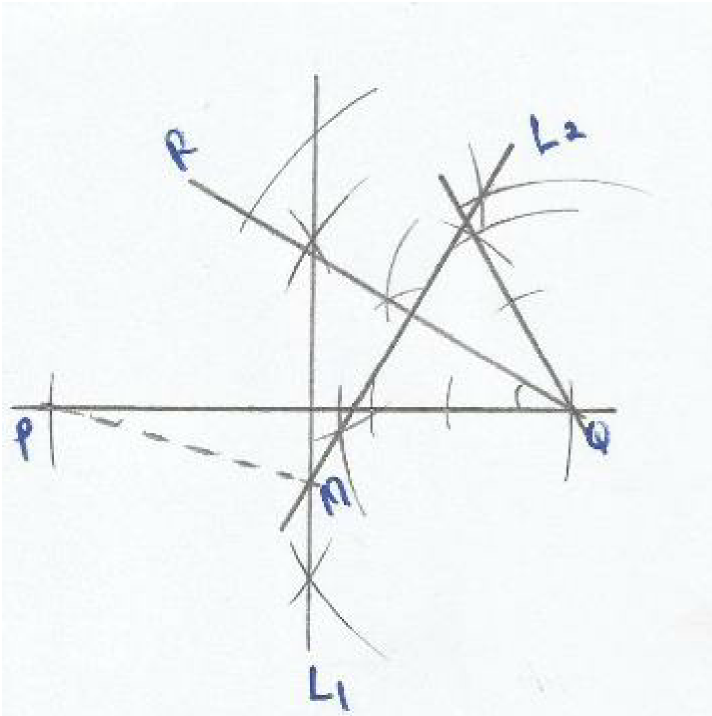
Solution

In the figure below

i.) L_1 is the perpendicular bisector of PQ

ii.) L_2 is the perpendicular bisector of PQ

iii.) By measurement, PM is equal to 3.7 cm



Loci of inequalities

An inequality is represented graphically by showing all the points that satisfy it.The intersection of two or more regions of inequalities gives the intersection of their loci.

Remember we shed the unwanted region

Example

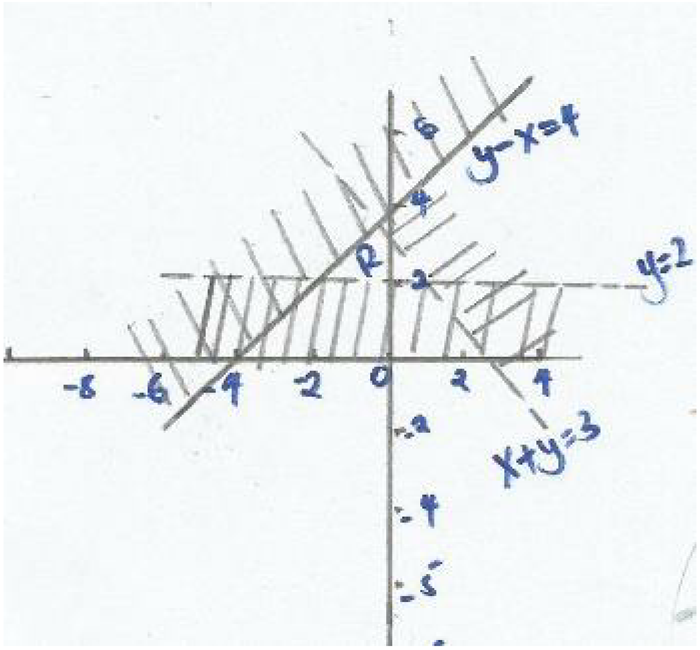
Draw the locus of point (x, y) such that $x + y < 3$, $y - x \leq 4$ and $y > 2$.

Solution

Draw the graphs of $x + y = 3$,$y - x = 4$ and $y = 2$ as shown below.

The unwanted regions are usually shaded. The unshaded region marked R is the locus of points (x ,y), such that $x + y < 3$, $y - X \leq 4$ and $y > 2$.

The lines of greater or equal to ad less or equal to (≤ , ≥) are always solid while the lines of greater or less (<>) are always broken.

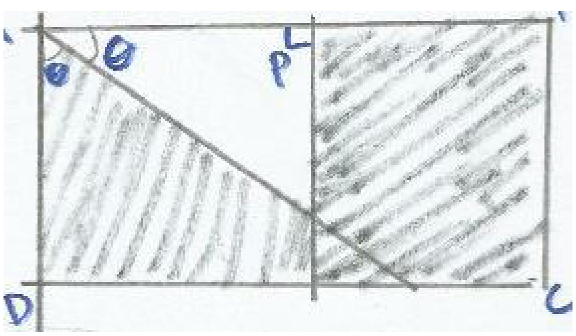


Example

P is a point inside rectangle ABCD such that AP≤PB and Angle DAP ≥ Angle BAP. Show the region on which P lies.

Solution

A B



Draw a perpendicular bisector of AP=PB and shade the unwanted region. Bisect <DAB (< DAP = < BAP) and shade the unwanted region lies in the unshaded region.

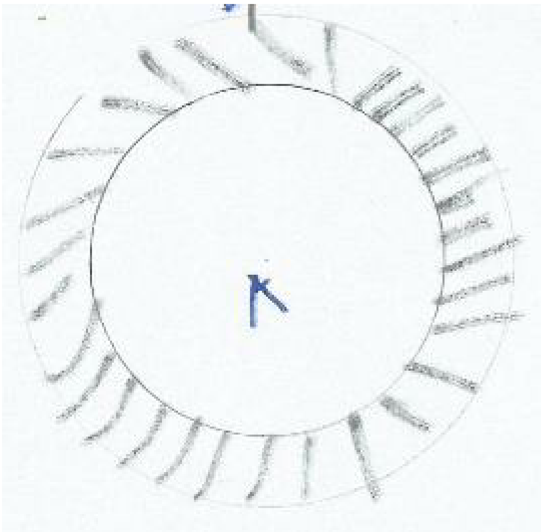
Example

Draw the locus of a point P which moves that AP ≤ 3 cm.

Solution

i.) Draw a circle, centre A and radius 3 cm

ii.) Shade the unwanted region.



Locus involving chords

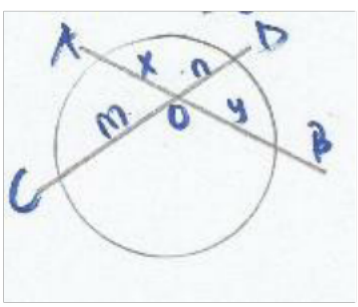
The following properties of chords of a circle are used in construction of loci

(I)Perpendicular bisector of any chord passes through the centre of the circle.

(ii) The perpendicular drawn from a centre of a circle bisects the chord.

(III) If chords of a circle are equal, they are equidistant from the centre of the circle and vice -versa

(IV) In the figure below, if chord AB intersects chord CD at O, AO = x ,BO = y, CO = m and DO =n then $m \times n = x \times y$



End of topic

Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. Using a ruler and a pair of compasses only,

a. Construct a triangle ABC such that angle ABC = 135°C, AB = 8.2cm and BC = 9.6cm
b. Given that D is a position equidistant from both AB and BC and also from B and C
i. Locate D
ii. Find the area of triangle DBC.

2. (a) Using a ruler, a pair of compasses only construct triangle XYZ such that XY = 6cm,

YZ = 8cm and ∠XYZ = 75°

(b) Measure line XZ and ∠XZY

(c) Draw a circle that passes through X, Y and Z

(d) A point M moves such that it is always equidistant from Y and Z. construct the locus of M and define the locus

3. (a) (i) Construct a triangle ABC in which AB=6cm, BC = 7cm and angle ABC = 75°

Measure:-

(i) Length of AC

(ii) Angle ACB

(b) Locus of P is such that BP = PC. Construct P

(c) Construct the locus of Q such that Q is on one side of BC, opposite A and angle BQC = 30°

(d) (i) Locus of P and locus of Q meet at X. Mark ***x***

(ii) Construct locus R in which angle BRC 120°

(iii) Show the locus S inside triangle ABC such that XS ≥ SR

4. *Use a ruler and compasses only for all constructions in this question.*

a) i) Construct a triangle ABC in which AB=8cm, and BC=7.5cm and ∠ABC=112½°

ii) Measure the length of AC

b) By shading the unwanted regions show the locus of P within the triangle ABC such that

i) AP ≤ BP

ii) AP >3cm

Mark the required region as **P**

c) Construct a normal from C to meet AB produced at D

d) Locate the locus of **R** in the same diagram such that the area of triangle ARB is ¾ the area of the triangle ABC.

5. On a line AB which is 10 cm long and on the same side of the line, use a ruler and a pair of compasses only to construct the following.

a) Triangle ABC whose area is 20 cm^2 and angle ACB = 90^o

b) (i) The locus of a point P such that angle APB = 45^o.

(ii) Locate the position of P such that triangle APB has a maximum area and calculate this area.

6. A garden in the shape of a polygon with vertices A, B, C, D and E. AB = 2.5m, AE = 10m,

ED = 5.2M and DC=6.9m. The bearing of **B** from **A** is 030º and **A** is due to east of **E**

while **D** is due north of E, angle EDC = 110º,

a) Using a scale of 1cm to represent 1m construct an accurate plan of the garden

b) A foundation is to be placed near to CD than CB and no more than 6m from A,

i) Construct the locus of points equidistant from CB and CD.

ii) Construct the locus of points 6m from **A**

c) i) shade and label **R** ,the region within which the foundation could be placed in the garden

ii) Construct the locus of points in the garden 3.4m from AE.

iii) Is it possible for the foundation to be 3.4m from AE and in the region?

7. a) Using a ruler and compasses **only** construct triangle PQR in which QR= 5cm, PR = 7cm and angle PRQ = 135°

b) Determine < PQR

c) At P drop a perpendicular to meet QR produced at T d) Measure PT

e) Locate a point **A** on **TP** produced such that the area of triangle AQR is equal to one- and – a -half times the area of triangle PQR

f) Complete triangle AQR and measure angle AQR

8. Use ruler and a pair of compasses only in this question.

(a) Construct triangle ABC in which AB = 7 cm, BC = 8 cm and ∠ABC = 60^0.

(b) Measure (i) side AC (ii) ∠ ACB

(c) Construct a circle passing through the three points A, B and C. Measure the radius of the circle.

(d) Construct Δ PBC such that P is on the same side of BC as point A and ∠ PCB = ½ ∠ ACB,∠ BPC = ∠ BAC measure ∠ PBC.

9. Without using a set square or a protractor:-

(a) Construct triangle **ABC** in which **BC** is 6.7cm, angle **ABC** is 60^o and ∠**BAC** is 90^o.

(b) Mark point **D** on line **BA** produced such that line **AD** =3.5cm

(c) Construct:-

(i) A circle that touches lines **AC** and **AD**

(ii) A tangent to this circle parallel to line **AD**

Use a pair of compasses and ruler only in this question;

(a) Draw acute angled triangle **ABC** in which angle **CAB** = 37½ °, **AB** = 8cm and **CB** = 5.4cm. Measure the length of side **AC** (**hint 37½ ° = ½ x 75°**)

(b) On the triangle **ABC** below:

(i) On the same side of **AC** as **B,** draw the locus of a point **X** so that angle **Ax C** = 52½ °

(ii) Also draw the locus of another point **Y**, which is 6.8cm away from **AC** and on the same side as **X**

(c) Show by shading the region **P** outside the triangle such that angle **APC ≥ 52 ½°** and

P is not less than 6.8cm away from **AC**

CHAPTER SIXTY THREE

DIFFERENTIATION

Specific Objectives

By the end of the topic the learner should be able to:

(a) Find average rates of change and instantaneous rates of change;

(b) Find the gradient of a curve at a point using tangent;

(c) Relate the delta notation to rates of change;

(d) Find the gradient function of a function of the form y = x n (n is a positive

Integer);

(e) Define derivative of a function, derived function of a polynomial anddifferentiation;

(f) Determine the derivative of a polynomial;

(g) Find equations of tangents and normal to the curves;

(h) Sketch a curve;

(i) Apply differentiation in calculating distance, velocity and acceleration;

(j) Apply differentiation in finding maxima and minima of a function.

Content

(a) Average and instantaneous rates of change

(b) Gradient of a curve at a point

(c) Gradient of y= x n (where n is a positive integer)

(d) Delta notation (A) or 5

(e) Derivative of a polynomial

(f) Equations of tangents and normals to the curve

(g) Stationery points

(h) Curve sketching

(i) Application of differentiation in calculation of distance, velocity andacceleration

(j) Maxima and minima

Introduction

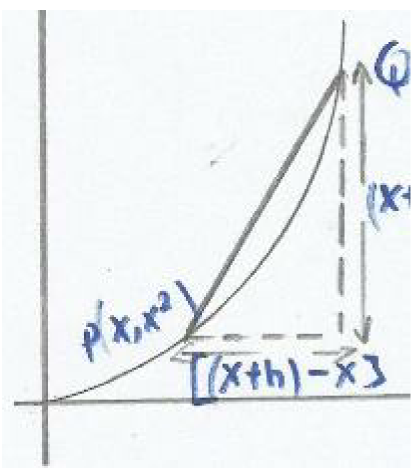
Differentiation is generally about rate of change

Example

If we want to get the gradient of the curve $y = x^2$ at a general point (x ,y).We note that a general point on the curve $y = x^2$ will have coordinates of the form (x ,x^2)The gradient of the curve $y= x^2$at a general point (x, y) can be established as below.

If we take a small change in x , say h. This gives us a new point on the curve with co-ordinates

$[(x +h), (x + h)^2]$. So point Q is $[(x +h), (x + h)^2]$ while point P is (x ,x^2).



To find the gradient of PQ $= \dfrac{\text{change in Y}}{\text{Change in X}}$

Change in y = $(x + h)^2 - x^2$

Change in x = (x + h) - x

Gradient $= \dfrac{(x + h)^2 - x^2}{(x + h) - x}$

$= \dfrac{x^2+2xh+ h^2-x^2}{x +h-x}$

$= \frac{2xh + h^2}{h}$

$= 2x + h$

By moving Q as close to p as possible, h becomes sufficiently small to be ignored. Thus, 2x +h becomes2x.Therefore, at general point (x,y)on the curve $y = x^2$,the gradient is 2x.

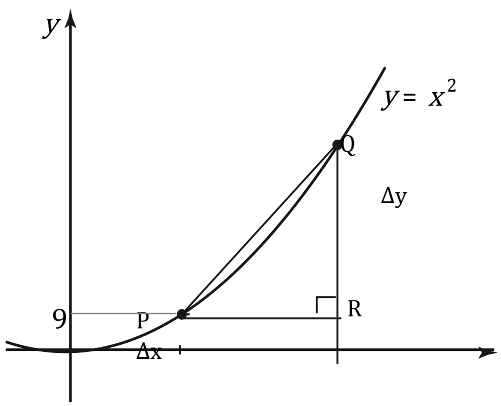
2x is called the gradient function of the curve $y = x^2$.We can use the gradient function to determine the gradient of the curve at any point on the curve.

In general, the gradient function of $y = x^n$ is given by nx^{n-1},where n is a positive integer. The gradient function is called the derivative or derived function and the process of obtaining it is called differentiation.

The function $y=x^5$ becames $5x^{5-1} = 5x^4$when we differentiate it

Delta Notation

A small increase in x is usually denoted byΔxsimilarly a small increase in y is denoted by Δy.Let us consider the points P (x ,y) and Q [$(x + \Delta x),(y + \Delta y)$ on the curve $y = x^2$



Note;

X is a single quantity and not a product of Δ and x .similarly Δy is a single quantity.

The gradient of PQ, $\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{(x+\Delta x)-x}$

$= \frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{x+\Delta x-x}$

$= 2x + \Delta x$

As Δx tends to zero;

i.) Δx can be ignored

ii.) $\frac{\Delta y}{\Delta x}$ gives the derivative which is denoted by $\frac{dy}{dx}$

iii.) thus $\frac{dy}{dx} = 2x$

When we find $\frac{dy}{dx}$, we say we are differentiating with respect to x, For example given y =x^4; then $\frac{dy}{dx} = 4x^3$

In general the derivatives of y =ax^n is nax^{n-1} e.g. $y = 5x^2 = 10x$, $y = 6x^3 = (6x\ 3)x^{3-1} = 18x^2$

Derivative of a polynomial.

A polynomial in x is an expression of the form $a_0x^n + a_1x^{n-1} + ...a_n$; where $a_0, a_1 .... a_n$ are constants

To differentiate a polynomial function, all you have to do is multiply the coefficients of each variable by their corresponding exponents/powers, subtract each exponent/powers by one , and remove any constants.

Steps involved in solving polynomial areas follows

Identify the variable terms and constant terms in the equation.

A variable term is any term that includes a variable and a constant term is any term that has only a number without a variable. Find the variable and constant terms in this polynomial function: y = $5x^3 + 9x^2 + 7x + 3$

- The variable terms are $5x^3$, $9x^2$, and 7x
- The constant term is 3

Multiply the coefficients of each variable term by their respective powers.

Their products will form the new coefficients of the differentiated equation. Once you find their products, place the results in front of their respective variables. For example:

- 5x3 = 5 x 3 = 15
- 9x2 = 9 x 2 = 18
- 7x = 7 x 1 = 7

Lower each exponent by one.

To do this, simply subtract 1 from each exponent in each variable term. Here's how you do it:

- $5x^3 = 15x^2$
- $9x^2 = 18x$
- $7x^{1-1} = 7$

Replace the old coefficients and old exponents/powers with their new counterparts.

To finish differentiating the polynomial equation, simply replace the old coefficients with their new coefficients and replace the old powers with their values lowered by one. The derivative of constants is zero so you can omit 3, the constant term, from the final result.

The derivative of the polynomial y = $15x^2 + 18x + 7$

In general, the derivative of the sum of a number of terms is obtained by differentiating each term in turn.

Examples

Find the derived function of each of the following

a.) $S=2t^3 - 3t^2 + 4t$ b.) $A = V^2 - 2V + 10$

Solution

a.) $\frac{dS}{dt} = 6t^2 - 6t + 4$

b.) $\frac{dA}{dv} = 2v - 2$

Equations of tangents and Normal to a curve.

The gradient of a curve is the same as the gradient of the tangent to the curve at that point. We use this principle to find the equation of the tangent to a curve at a given point.

Find the equation of the tangent to the curve;

$y = x^3 + 2x + 1$ at $(1,4)$

Solution

$$\frac{dy}{dx} = 3x^2 + 2$$

At the point $(1,4)$, the gradient is $3 \times 1^2 + 2 = 5$(we have substituted the value of x with 1)

We want the equation of straight line through (1, 4) whose gradient is 5.

Thus $\frac{y-4}{x-1} = 5$

$$y - 4 = 5x - 5$$

$$y = 5x - 1 \text{ (this is the equation of the tangent)}$$

A normal to a curve at appoint is the line perpendicular to the tangent to the curve at the given point.

In the example above the gradient of the tangent of the tangent to the curve at (1, 4) is 5. Thus the gradient of the normal to the curve at this point is$-\frac{1}{5}$.

Therefore, equation of the normal is:

$$\frac{y-4}{x-1} = -\frac{1}{5}$$

$5(y - 4) = -1(x - 1)$

$$y = \frac{-x+21}{5}$$

Example

Find the equation of the normal to the curve $y = x^3 - 2x - 1$ at $(1, -2)$

Solution

$$y = x^3 - 2x - 1$$

$$\frac{dy}{dx} = 3x^2 - 2$$

At the point (1,-2) gradient of the tangent line is 1.Therefore the gradient of the normal is -1.the required equation is

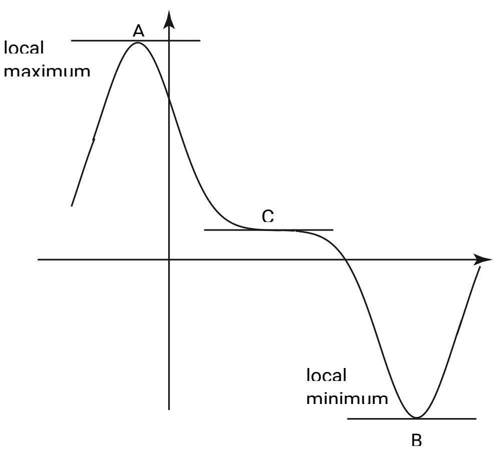
$$\frac{y-(-2)}{x-1} = -1$$

$$\frac{y+2}{x-1} = -1$$

$$y + 2 = -x + 1$$

The equation of the normal is y = -x -1

Stationary points



Note;

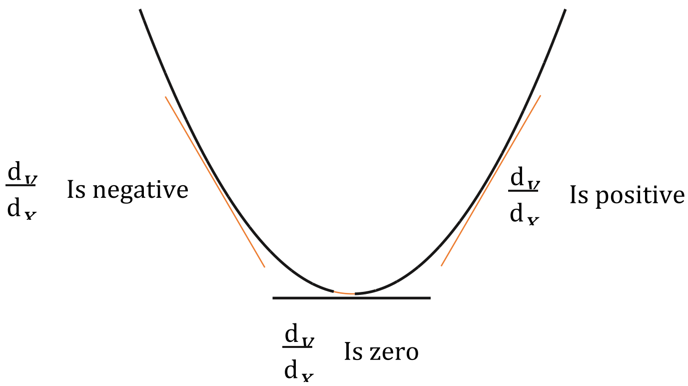
- In each of the points A ,B and C the tangent is horizontal meaning at these points the gradient is zero.so $\frac{dy}{dx} = 0$ at points A,B, C.
- Any point at which the tangent to the graph is horizontal is called a stationary point. We can locate stationary points by looking for points at which $\frac{dy}{dx} = 0$.

Turning points

The point at which the gradient changes from negative through zero to positive is called minimum point while the point which the gradient changes from positive through zero to negative is called maximum point .In the figure above A is the maximum while B is the minimum.

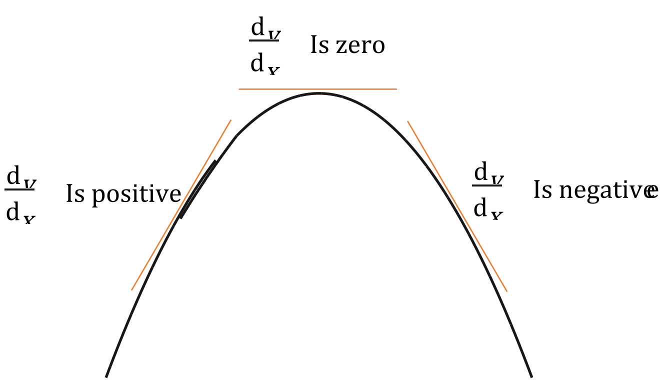
Minimum point .

Gradient moves from negative through zero to positive.



Maximum point

Gradient moves from positive through zero to negative.



The maximum and minimum points are called turning points.

A point at which the gradient changes from positive through zero to positive or from negative zero to negative is called point of inflection.

Example

Identify the stationary points on the curve $y = x^2 - 3x + 2$.for each point, determine whether it is a maximum, minimum or a point of inflection.

Solution

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

At stationary point, $\frac{dy}{dx} = 0$

Thus $3x^2 - 3 = 0$

$3(x^2-1) = 0$

$3(x+1)(x-1) = 0$

$$x = -1 \text{ or } x = 1$$

when $= -1$, $y = 4$

when $x = 1$, $y = 0$

Therefore, stationary points are (-1 , 4) and (1 ,0).

Consider the sign of the gradient to the left and right of x = 1

x	0	1	2
$\frac{dy}{dx}$	-3	0	9
Diagrammatic representation	\	___	/

Therefore (1 , 0) is a minimum point.

Similarly, sign of gradient to the left and right of x = -1 gives

x	-2	-1	0
$\frac{dy}{dx}$	9	0	-3
Diagrammatic representation	/	___	\

Therefore (-1 , 4) is a maximum point.

Example

Identify the stationary points on the curve $y = 1 + 4x^3 - x^4$. Determine the nature of each stationary point.

Solution

$y = 1 + 4x^3 - x^4$

$$\frac{dy}{dx} = 12x^2 - 4x^3$$

At stationary points, $\frac{dy}{dx} = 0$

$$12x^2 - 4x^3 = 0$$

$$4x^2(3-x) = 0$$

$$x = 0 \text{ or } x = 3$$

Stationary points are (0, 1) and (3, 28)

Therefore (0, 1) is a point of inflection while (3, 28) is a maximum point.

Application of Differentiation in calculation of velocity and acceleration.

Velocity

If the displacement, S is expressed in terms of time t, then the velocity is $v = \frac{dS}{dt}$

Example

The displacement, S metres, covered by a moving particle after time, t seconds, is given by

$S = 2t^3 + 4t^2 - 8t + 3$.Find:

a.) Velocity at :

   i.) $t = 2$

   ii.) $t = 3$

b.) Instant at which the particle is at rest.

Solution

$S = 2t^3 + 4t^2 - 8t + 3$

The gradient function is given by;

$V = \frac{dS}{dt}$

$= 6t^2 + 8t - 8$

a.) velocity

   i.) at t = 2 is ;

   $v = 6 \times 2^2 + 8 \times 2 - 8$

   = 24 + 16 – 8

   =32m/s

   ii.) at t = 3 is ;

   $v = 6 \times 3^2 + 8 \times 3 - 8$

   = 54 + 24 – 8

   =70m/s

b.) the particle is at rest when v is zero

$$6t^2 + 8t - 8 = 0$$

$$2(3t^2+4t-4) = 0$$

$$2(3t-2)(t+2) = 0$$

$$t = \frac{2}{3} \text{ or } t = -2$$

It is not possible to have t = -2

The particle is therefore at rest at $t = \frac{2}{3}$ seconds

Acceleration

Acceleration is found by differentiating an equation related to velocity. If velocity v , is expressed in terms of time, t , then the acceleration, a, is given by a $=\frac{dv}{dt}$

Example

A particle moves in a straight line such that is its velocity v ms^{-1} after t seconds is given by

$v = 3 + 10t - t^2$.

Find

a.) the acceleration at :

i.) t =1 sec

ii.) t =3 sec

b.) the instant at which acceleration is zero

Solution

a.) $v = 3 + 10t + t^2$

$$a = \frac{dv}{dt} = 10 - 2t$$

i.) At t = 1 sec a = 10 – 2 x 1

$$= 8 \text{ ms}^{-2}$$

ii.) At t = 3 sec a = 10 – 2 x 3

$$= 4 \text{ ms}^{-2}$$

b.) Acceleration is zero when $\frac{dy}{dt} = 0$

Therefore, 10 – 2t = 0 hence t = 5 seconds

Example

A closed cylindrical tin is to have a capacity of 250π ml. if the area of the metal used is to be minimum, what should the radius of the tin be?

Solution

Let the total surface area of the cylinder be A cm^2,radius r cm and height h cm.

Then, $A = 2\pi r^2 + 2\pi rh$

Volume = $2\pi r^2 h = 250\pi\ cm^2$

$$\pi r^2 h = 250\pi$$

Making h the subject, $h = \frac{250\pi}{\pi r^2}$

$$= \frac{250}{r^2}$$

Put $h = \frac{250}{r^2}$ in the expression for surface area to get;

$$A = 2\pi r^2 + 2\pi r.\frac{250}{r^2}$$

$$= 2\pi r^2 + 500\pi r^{-1}$$

$$\frac{dA}{dr} = 4\pi r - 500\pi r^{-2}$$

For minimum surface area, $\frac{dA}{dr} = 0$

$$4\pi r - \frac{500\pi}{r^2} = 0$$

$$4\pi r^3 - 500\pi = 0$$

$$4r^3 = 500$$

$$r^3 = \frac{500}{4} = 125$$

$$r = \sqrt[3]{125}$$

= 5

Therefore the minimum area when r = 5 cm

Example

A farmer has 100 metres of wire mesh to fence a rectangular enclosure. What is the greatest area he can enclose with the wire mesh?

Solution

Let the length of the enclosure be x m. Then the width is $\frac{100-2x}{2}$ m = $(50-x)$m

Then the area A of the rectangle is given by;

A = x (50 –x)m^2

= 50x - x^2m^2

For maximum or minimum area,

$$\frac{dA}{dx} = 0$$

Thus, 50 – 2x = 0

$$x = 25$$

The area is maximum when x = 25 m

That is A = 50 X 25 – $(25)^2$

= 625 m^2.

CHAPTER SIXTY FOUR

INTERGRATION

Specific Objectives

By the end of the topic the learner should be able to:

(a) Carry out the process of differentiation;

(b) Interpret integration as a reverse process of differentiation;

(c) Relate integration notation to sum of areas of trapezia under a curve;

(d) Integrate a polynomial;

(e) Apply integration in finding the area under a curve,

(f) Apply integration in kinematics.

Content

(a) Differentiation

(b) Reverse differentiation

(c) Integration notation and sum of areas of trapezia

(d) Indefinite and definite integrals

(e) Area under a curve by integration

(f) Application in kinematics.

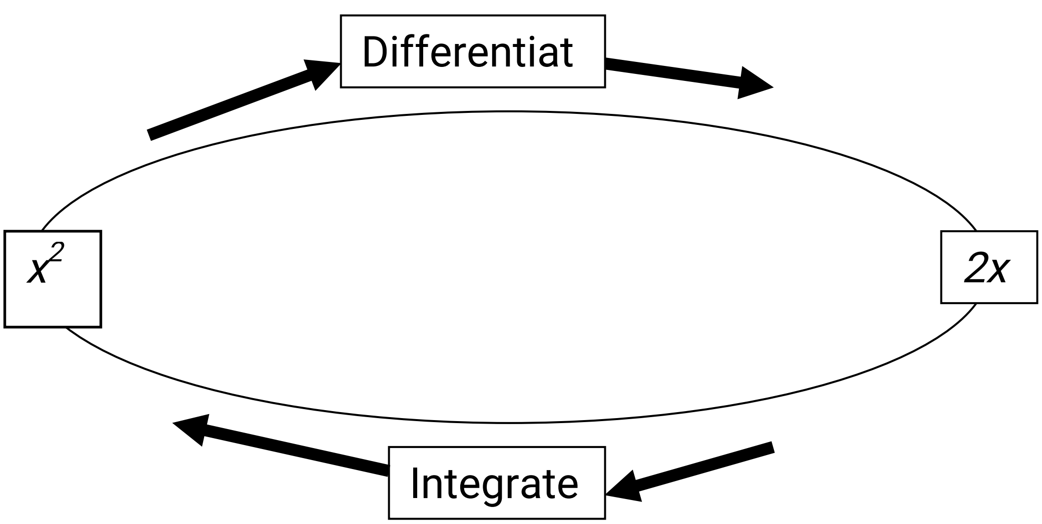
Introduction

The process of finding functions from their gradient (derived) function is called integration

Suppose we differentiate the function $y=x^2$. We obtain

$$\frac{dy}{dx} = 2x$$

Integration reverses this process and we say that the integral of 2x is x^2.



From differentiation we know that the gradient is not always a constant. For example, if $\frac{dy}{dx}$ = 2x, then this comes from the function of the form $y=x^2 + c$, Where c is a constant.

Example

Find y if $\frac{dy}{dx}$is:

a.) $3x^2$

b.) $4x^3$

Solution

a.) $\frac{dy}{dx} = 3x^2$

Then, y =$x^3 + c$

b.) $\frac{dy}{dx} = 4x^3$

Then, y =$x^4 + c$

Note;

To integrate we reverse the rule for differentiation. In differentiation we multiply by the power of x and reduce the power by 1.In integration we increase the power of x by one and divide by the new power.

If $\frac{dy}{dx} = x^n$,then $y = \frac{x^{n+1}}{n+1} + c$, where c is a constant and n≠ -1.since c can take any value we call it an arbitrary constant.

Example

Integrate the following expression

a.) $2x^5$

b.) x^{-1}

c.) $5x^3$- 2x +4

Solution

A.) $\frac{dy}{dx} = 2x^5$

Then, y = $\frac{2x^{5+1}}{5+1} + c$

= $\frac{2x^6}{6} + c$

$$= \frac{x^6}{3} + c$$

B.) $\frac{dy}{dx} = x^{-2}$

Then, $y = \frac{x^{-2+1}}{-2+1} + c$

$= \frac{x^{-1}}{-1} + c$

$= -x^{-1} + C$

C.) $\frac{dy}{dx} = 5x^3 - 2x + 4$

Then, $y = \frac{5x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} - \frac{4x^{0+1}}{0+1} + C$

$= \frac{5}{4}x^4 - \frac{2}{2}x^2 + 4x + C$

$= \frac{5}{4}x^4 - x^2 + 4x + C$

Example

Find the equation of a line whose gradient function is $\frac{dy}{dx} = 2x + 3$ and passes through (0,1)

Solution

Since $\frac{dy}{dx} = 2x + 3$,the general equation is y $= x^2 + 3x + c$.The curve passes through (0,1).Substituting these values in the general equation ,we get 1 = 0 + 0 + c

1 = c

Hence, the particular equation is y $= x^2 + 3x + 1$

Example

Find v in terms of h if $\frac{dV}{dh} = 3h^2 + 4$ and V =9 when h=1

Solution

The general solution is

$V = \frac{3h^3}{3} + 4h + c$

$= h^3 + 4h + c$

V= 9 when h= 1.Therefore

$$9=1^3 + 4 + c$$

$9 = 5 + c$

$4 = C$

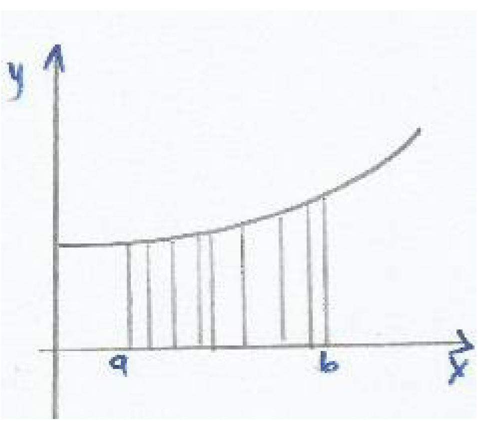
Hence the particular solution is ;

$V = h^3 + 4h + 1$

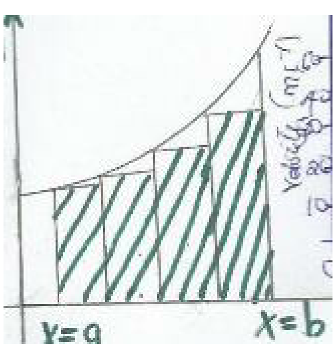
Definite and indefinite integrals

It deals with finding exact area.

Estimate the area shaded beneath the curve shown below



The area is divided into rectangular strips as follows.



The shaded area in the figure above shows an underestimated and an overestimated area under the curve. The actual area lies between the underestimated and overestimated area. The accuracy of the area can be improved by increasing the number of rectangular strips between x = a and x = b.

The exact area beneath the curve between x = a and b is given by

$$\int_a^b y\delta x$$

The symbol $\int$is an instruction to intergrate.

Thus $\int ydx$ means integrate the expression for y with respect to x.

The expression $\int_a^b y\delta x$,where a and b are limits , is called a definite integral. 'a' is called the lower limit while b is the upper limit. Without limits, the expression is called an indefinite integral.

Example

$$\int_2^6 (2x^2+3)\, dx$$

The following steps helps us to solve it

i.) Integrate $2x^2 + 3$ with respect to x , giving $\frac{2}{3}x^3 + 3x + c$.

ii.) Place the integral in square brackets and insert the limits, thus

$$\left[\frac{2}{3}x^3+3\,x+c\right]_2^6$$

iii.) Substitute the limits ;

X = 6 gives $\frac{2 \text{ x } 6^3}{3} + 3 \text{ x } 6 + c = 162 + c$

x = 6 gives$\frac{2 \text{ x } 2^3}{3} + 3 \text{ x } 2 + c = \frac{34}{3} + c$

iv.) Subtract the results of the lower limit from that of upper limit, that is;

$(162 + c) - (\frac{34}{3} + c) = 150\frac{2}{3}$

We can summarize the steps in short form as follows:

$$\int_2^6 (2x^2+3)\,dx = \left[\frac{2}{3}x^3+3x+c\right]_2^6$$

$$=\left[\frac{2\times 6^3}{3}+(3\times 6)\right]-\left[\frac{2\times 2^3}{3}+3\times 2\right]$$

$$=150\,\frac{2}{3}$$

Example

a.) Find the indefinite integral

i.) $\int (x^2+1)\,dx$

ii.) $\int (x^2+4x)\,dx$

b.) Evaluate

i.) $\int_0^1 (x^4-5)\,dx$

i.) $\int_{-1}^2 (-x^3+5x-2)\,dx$

ii.) $\int_2^3 (3x^2-4x+5)\,dx$

Solution

a.) i.) $\int (x^2+1)\,dx = \frac{x^3}{3}+x+c$

ii.) $\int (x^2+4x)\,dx = \frac{x^4}{4}+2x^2+c$

Evaluate

i.) $\int_0^1 (x^4-5)\,dx\left[\frac{x^5}{5}-5x\right]_0^1$

$$\left(\frac{1}{5}-5\right)-\left(\frac{0}{5}-0\right)$$

$$=-4\frac{4}{5}$$

ii.) $\int_{-1}^2 (-x^3+5x-2)\,dx=\left[\frac{-x^4}{4}+\frac{5x^2}{2}-2x\right]_{-1}^2$

$$(-4+10-4) - \left(-\frac{1}{4} + \frac{5}{2} + 2\right)$$

$$= 2 - 4\frac{1}{4}$$

$$= -2\frac{1}{4}$$

iii.) $\int_2^3 (3x^2 - 4x + 5)\,dx = \left[\frac{3x^2}{3} - \frac{4x^2}{2} + 5x\right]_2^3$

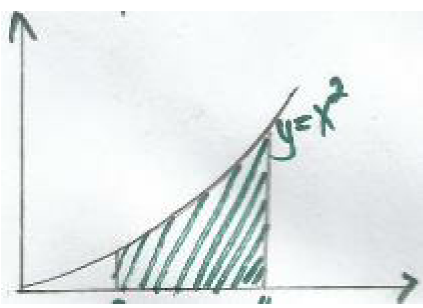
$$= (27 - 18 + 15) - (8 - 8 + 10)$$

$$= 14$$

Area under the curve

Find the exact area enclosed by the curve $y = x^2$, the axis, the lines $x = 2$ and $x = 4$

Solution



2 4

The area is given by;

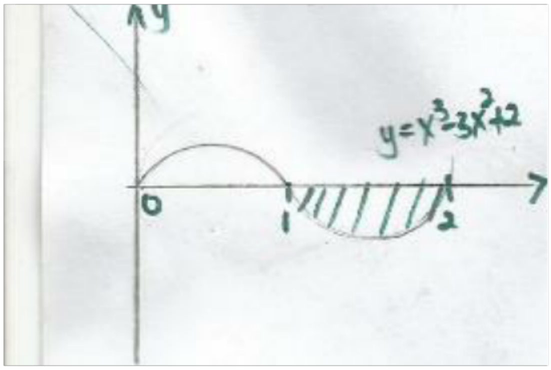
$$\int_3^4 x^2\,dx = \left[\frac{1}{3}x^3\right]_2^4$$

$$\frac{64}{3} - \frac{8}{3} = 18\frac{2}{3} \text{ square units}$$

Example

Find the area of the region bounded by the curve = $x^3 - 3x^2 + 2x$, the x axis x = 1 and x = 2

Solution



The area is given by;

$$\int_1^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$$

$$= (4 - 8 + 4) - (\frac{1}{4} - 1 + 1)$$

$$= 0 - \frac{1}{4} \quad = -\frac{1}{4}$$

Note;

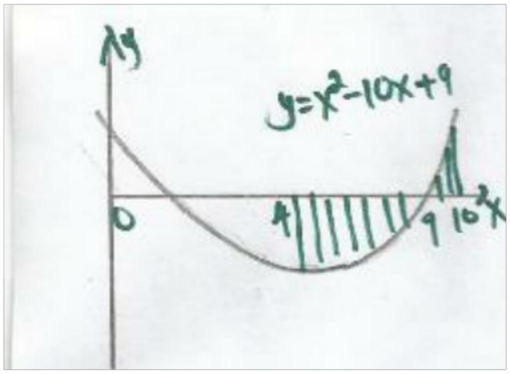
The negative sign shows that the area is below the x – axis. We disregard the negative sign and give it as positive as positive .The answer is $\frac{1}{4}$ square units.

Example

Find the area enclosed by the curve $x^2 - 10x + 9$,the x – axis and the lines x = 4 and x =10.

Solution

The required area is shaded below.



Area = $\int_4^9 (x^2-10x+9)dx + \int_9^{10}(x^2-10x+9)dx$

$$= \left[\frac{x^3}{3}-5x^2+9x\right]_4^9 + \left[\frac{x^3}{3}-5x^2+9x\right]_9^{10}$$

$$= \left[(243-405+81)-\left(\frac{64}{3}-80+36\right)\right] + \left[\left(\frac{1000}{3}-500+90\right)-(243-405+81)\right]$$

$$= -58\frac{1}{3} + 4\frac{1}{3} \text{ (Drop negative sign for area under x - axis)}$$

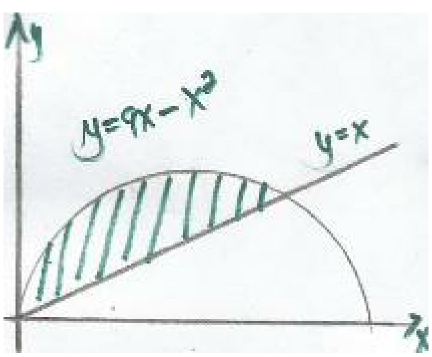
$$= 62\frac{2}{3} \text{ square units}$$

Example

Find the area enclosed by the curve $y = 9x - x^2$, and the line y =x.

Solution

The required area is



To find the limits of integration, we must find the x co-ordinates of the points of intersection when;

$$x = 9x - x^2$$

$$\rightarrow 0 = 8x - x^2$$

$$0 = x\,(8 - x)$$

$$x = 0 \text{ or } x = 8$$

The required area is found by subtracting area under y = x from area under y = = $9x - x^2$

The required area $= \int_0^8 (9x - x^2)\,dx - \int_0^8 x\,dx$

$$\left[\frac{9x^2}{2} - \frac{x^3}{3}\right]_0^8 - \left[\frac{x^3}{2}\right]_0^8$$

$$= 117\frac{1}{3} - 32$$

$$= 85\frac{1}{3} \text{ square units}$$

Application in kinematics

The derivative of displacement S with respect to time t gives velocity v, while the derivative of velocity with respect to time gives acceleration, a

Differentiation.		Integration
Displacement.		displacement
	↓↑	
Velocity.		Velocity
	↓↑	
Acceleration.		Acceleration

Note;

Integration is the reverse of differentiation. If we integrate velocity with respect to time we get displacement while if velocity with respect to time we get acceleration.

Example

A particle moves in a straight line through a fixed point O with velocity (4 – 1)m/s.Find an expression for its displacement S from this point, given that S = when t = 0.

Solution

Since $\frac{dS}{dt} = 4 - t$

$S = 4t - \frac{t^2}{2} + c$

Substituting S = 4, t = 0 to get C;

$$4 = 4 \text{ x } 0 - \frac{0^2}{2} + c$$

$$4 = C$$

Therefore $S = 4t - \frac{t^2}{2} + 4.$

Example

A ball is thrown upwards with a velocity of 40 m s

a.) Determine an expression in terms of t for

i.) Its velocity

ii.) Its height above the point of projection

b.) Find the velocity and height after:

i.) 2 seconds

ii.) 5 seconds

iii.) 8 seconds

c.) Find the maximum height attained by the ball. (Take acceleration due to gravity to be 10 m/s^2.

Solution

a.) $\frac{dv}{dt}$ = -10 (since the ball is projected upwards)

Therefore, v =-10 t + c

When t = 0, v = 40 m/s

Therefore, 40 = 0 + c

40 = c

i.) The expression for velocity is v = 40 – 10t

ii.) Since $\frac{dS}{dt} = v = 40 - 10t;$

$$S = 40t - 5t^3 + c$$

When t = 0 , S = 0

C = 0

The expression for displacement is ;

$$S = 40t - 5t^2$$

b.) Since v = 40 – 10t

i.) When t = 2

$v = 40 - 10\ (2)$

$= 40 - 20$

$= 20$ m/s

$S = 40t - 5t^2$

$= 40\ (2) - 5\ (2)^2$

$= 80 - 20$

$= 60$ m

ii.) When t = 5

$V = 40 - 10\ (5)$

$= -10$ m/s

$S = 40\ (5) - 4\ (5)^2$

$= 200 - 125$

$= 75$ m

iii.) When t = 8

$V = 40 - 10(8)$

$= -40$ m/s

$S = 40(8) - 5\ (8)^2$

$= 320 - 320$

$= 0$

c.) Maximum height is attained when v = 0.

Thus , $40 - 10t = 0$

$t = 4$

Maximum height $S = 160 - 80$

$= 80$ m

Example

The velocity v of a particle is 4 m/s. Given that S = 5 when t =2 seconds:

a.) Find the expression of the displacement in terms of time.

b.) Find the :

i.) Distance moved by the particle during the fifth second.

ii.) Distance moved by the particle between t =1 and t =3.

Solution

a.) $\frac{dS}{dt} = 4t + c$

S=4t + c

Since S = 5 m when t =2;

5 = 4 (2) + C

5 – 8 = C

-3 = C

Thus, S =4t – 3

b.) I.) $[4t\text{-}3]_4^5 = [(20\text{-}3) - (16\text{-}3)]$

$= 17 - 13$

$= 4 \text{ m}$

II.) $[4t\text{-}3]_1^3 = [(12\text{-}3) - (4\text{-}3)]$

$= 9 - 1 = 8$

CHAPTER SIXTY FIVE

AREA APPROXIMATION

Specific Objectives

By the end of the topic the learner should be able to:

(a) Approximate the area of irregular shapes by counting techniques;

(b) Derive the trapezium rule;

(c) Apply trapezium rule to approximate areas of irregular shapes;

(d) Apply trapezium rule to estimate areas under curves;

(e) Derive the mid-ordinate rule;

(f) Apply mid-ordinate rule to approximate area under curves.

Content

(a) Area by counting techniques

(b) Trapezium rule

(c) Area using trapezium rule

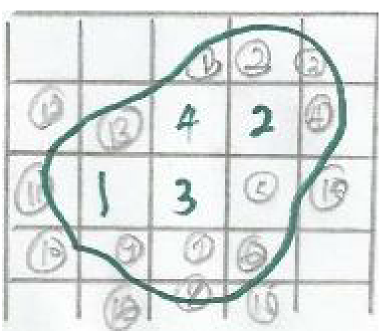
(d) Mid-ordinate

(e) Area by the mid-ordinate rule

Introduction

Estimation of areas of irregular shapes such as lakes, oceans etc. using counting method. The following steps are followed

- Copy the outline of the region to be measured on a tracing paper
- Put the tracing on a one centimeter square grid shown below



- Count all the whole squares fully enclosed within the region
- Count all the partially enclosed squares and take them as half square centimeter each
- Divide the number of half squares by two and add it to the number of full squares.

Number of compete squares = 4

Number of half squares = 16/ 2 = 8

Therefore the total number of squares = 25 + 8

= 33

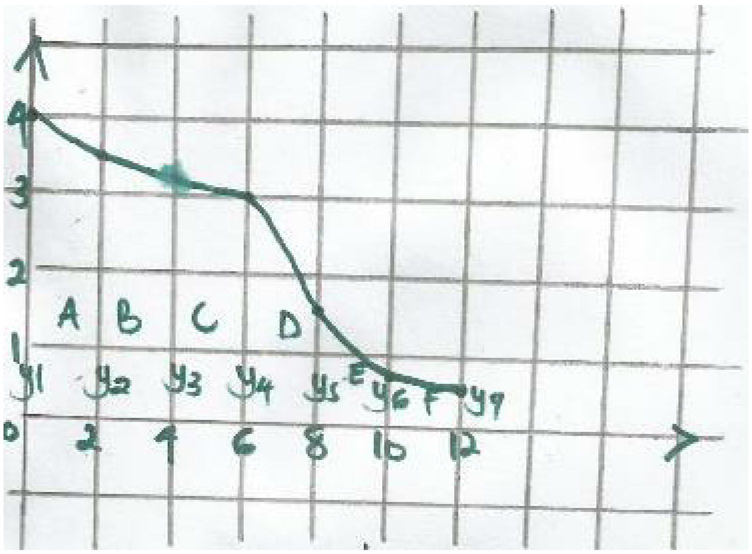
The area of the land mass on the paper is therefore 33cm^2

Note;

The smaller the subdivisions, the greater the accuracy in approximating area.

Approximating Area by Trapezium Method.

Find the area of the region shown, the region may be divided into six trapezia of uniform as shown



The area of the region is approximately equal to the sum of the areas of the six trapezia.

Note;

The width of each trapezium is 2 cm, and 4 and 3.5 are the lengths of the parallel sides of the first trapezium.

The area of the trapezium A = $\frac{1}{2}$ x 2 (4+3.5) = 7.5 cm^2

Area of the trapezium B = $\frac{1}{2}$ x 2 (3.5+3.5) = 6.7 cm^2

Area of the trapezium C = $\frac{1}{2}$ x 2 (3+1.5) = 6.2cm^2

Area of the trapezium D = $\frac{1}{2}$ x 2 (3+1.5) = 4.5 cm^2

Area of the trapezium E = $\frac{1}{2}$ x 2 (1.5+0.8) = 2.3 cm^2

Area of the trapezium F = $\frac{1}{2}$ x 2 (0.8+3.5) = 1.3 cm^2

Therefore, the total area of the region is

$$(7.5 + 6.7 + 6.2 + 4.5 + 2.3 + 1.3)\text{cm}^2 = 28.5 \text{ cm}^2$$

If the lengths of the parallel sides of the trapezia (ordinates) are $y_1,\ y_2, y_3, y_4, y_5, y_6 y_7$

Note;

In trapezium rule, except for the first and last lengths, each of the other lengths is counted twice. Therefore, the expression for the area can be simplified to:

$$\frac{1}{2} \text{ x } 2\{(y_1 + y_7) + 2\ (y_2 + y_3 + y_4 + y_5 + y_6\ )\}$$

In general, the approximate area of a region using trapezium method is given by:

$A = \frac{1}{2} h\ \{(y_0 + y_n) + 2\ (y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ y_{n-1}\ )\};$

Where h is the uniform width of each trapezium, y_0 - y_n are the first and last length respectively. This method of approximating areas of irregular shape is called trapezium rule.

Example

A car start from rest and its velocity is measured every second from 6 seconds.

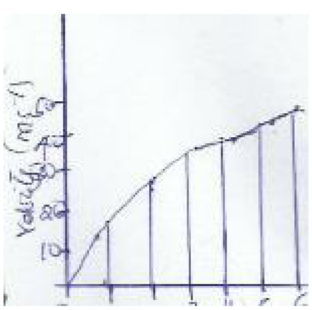
Time (t)	0	1	2	3	4	5	6
Velocity v (m/s	0	12	24	35	41	45	47

Use the trapezium rule to calculate distance travelled between t = 1 and t = 6

Note;

The area under velocity – time graph represents the distance covered between the given times.

To find the required displacement, we find the area of the region bounded by graph, t =1 and t =6



0 1 2 3 4 5 6

Solution

Divide the required area into five trapezia, each of with 1 unit. Using the trapezium rule;

$A = \frac{1}{2}h\{(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)\}$;

The required displacement = $\frac{1}{2} \times 1\{(12+47) + 2(24+35+41+45)\}$

$\frac{1}{2}(59 + 2 \times 145)$

= 174.5 m

Example

Estimate the area bounded by the curve $y = \frac{1}{2}x^2 + 5$, the x – axis, the line x =1 and x = 5 using the trapezium rule.

Solution

To plot the graph $y = \frac{1}{2}x^2 + 5$, make a table of values of x and the corresponding values of y as follows:

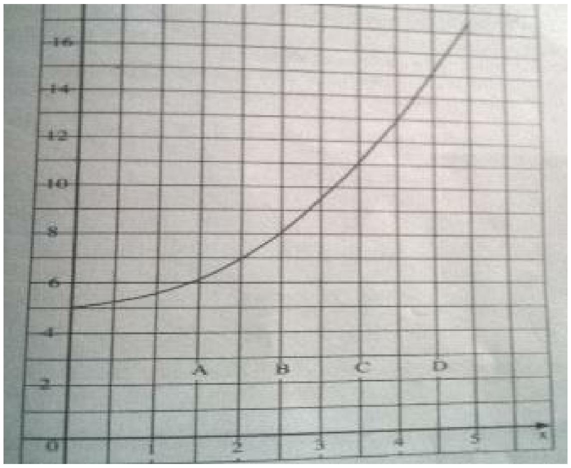
x	0	1	2	3	4	5
$Y = \frac{1}{2}x^2 + 5$	5	5.5	7	9.5	13	17.5

By taking the width of each trapezium to be 1 unit, we get 4 trapezium .A, B , C and D .The area under curve is approximately;

$A = \frac{1}{2}h\{(y_1 + y_5) + 2(y_2 + y_3 + y_4)\} = \frac{1}{2}(5.5 + 17.5) + 2(7 + 9.5 + 13)$ sq.units

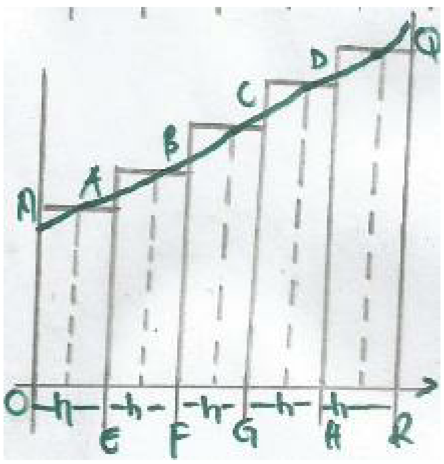
$= \frac{1}{2}(23.0 + 59)$ sq.units

= 41 sq.units



The Mid- ordinate Rule

The area OPQR is estimated:



y_1

The area of OPQR is estimated as follows

- Divide the base OR into a number of strips, each of their width should be the same .In the example we have 5 strips where h $=\frac{\text{length of the base OR}}{\text{number of strips}}$
- From the midpoints of OE ,EF ,FG ,GH and HR , draw vertical lines (mid- ordinates) to meet the curve PQ as shown above
- Label the mid-ordinates $y_1 y_2 y_3 y_4$ and y_5
- We take the area of each trapezium to be equal to area of a rectangle whose width is the length of interval (h) and the length is the value of mid –ordinates. Therefore, the area of the region OPQR is given by;

$$A = (y_1 \times h) + (y_2 \times h) + (y_3 \times h) + (y_4 \times h) + (y_5 \times h)$$

$$= h\,(y_1 + y_2 + y_3 + y_4 + y_5)$$

This the mid –ordinate rule $h\,(y_1 + y_2 + y_3 + y_4 + y_5)$.

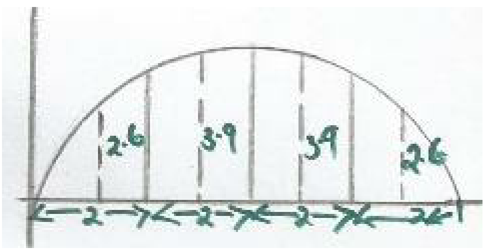
Note:

The mid-ordinate rule for approximating areas of irregular shapes is given by ;

Area = (width of interval) x (sum of mid – ordinates)

Example

Estimate the area of a semi-circle of radius 4 cm using the mid – ordinate rule with four equal strips, each of width 2 cm.



Solution

The above shows a semicircle of radius 4 cm divided into 4 equal strips, each of width 2 cm. The dotted lines are the mid-ordinates whose length are measured.

By mid- ordinate rule;

$$= h\,(y_1 + y_2 + y_3 + y_4 + y_5)$$

= 2 (2.6 + 3.9 + 3.9 + 2.6)

= 2 x 13

= 26 cm^2

The actual area is $\pi r^2 = \dfrac{3.142 \times 4^2}{2}$

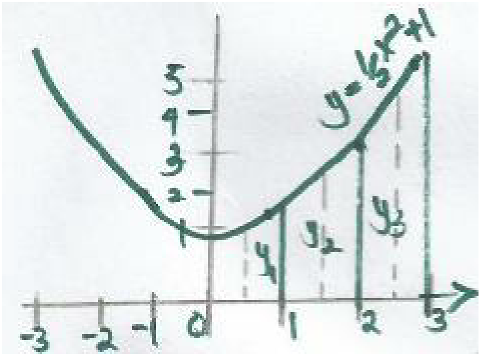
= 25.14 cm^2 to 4 s.f

Example

Estimate the area enclosed by the curve y $= \frac{1}{2}x^2 + 1$, $x = 0$, $x = 3$ and the x – axis using the mid-ordinate rule.

Solution

Take 3 strips. The dotted lines are the mid – ordinate and the width of each of the 3 strips is 1 unit.



By calculation, y_1, y_2 and y_3 are obtained from the equation;

$y = \frac{1}{2}x^2 + 1$

When x = 0.5, $y_1 = \frac{1}{2}x(0.5)^2 + 1$

$= 1.125$

When x = 1.5, $y_1 = \frac{1}{2}x(1.5)^2 + 1$

$= 2.125$

When x = 2.5, $y_1 = \frac{1}{2}x(0.5)^2 + 1$

$= 4.125$

Using the mid ordinate rule the area required is

A = 1 $(y_1 + y_2 + y_3)$

= 1 (1.125 + 2.125 + 4.125)

= 7.375 square units

End of topic

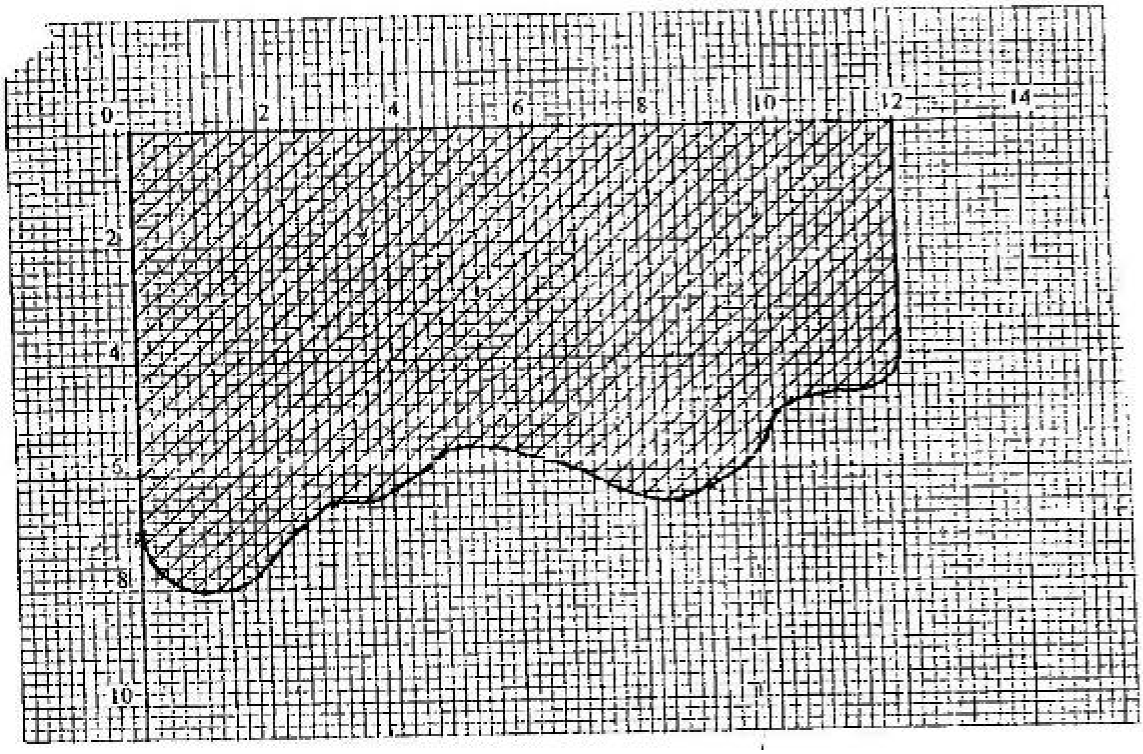
Did you understand everything?

If not ask a teacher, friends or anybody and make sure you understand before going to sleep!

Past KCSE Questions on the topic.

1. The shaded region below represents a forest. The region has been drawn to scale

where 1 cm represents 5 km. Use the mid – ordinate rule with six strips to estimate the area of forest in hectares. (4 marks)

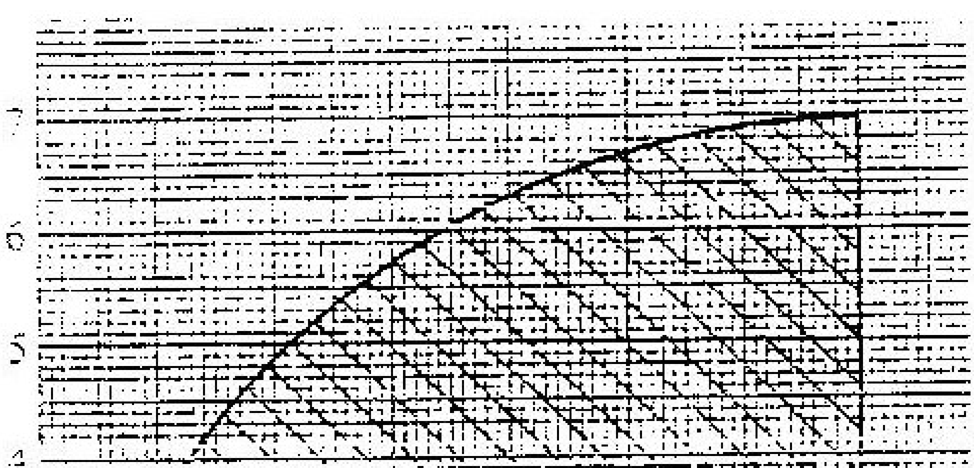


2. Find the area bounded by the curve $y=2x^3 - 5$, the x-axis and the lines x=2 and x=4.

3. Complete the table below for the function $y=3x^2 - 8x + 10$ (1 mk)

x	0	2	4	6	8	10
y	10	6		70		230

Using the values in the table and the trapezoidal rule, estimate the area bounded by the curve $y= 3x^2 - 8x + 10$ and the lines y=0, x=0 and x=104. Use the trapezoidal rule with intervals of 1 cm to estimate the area of the shaded region below

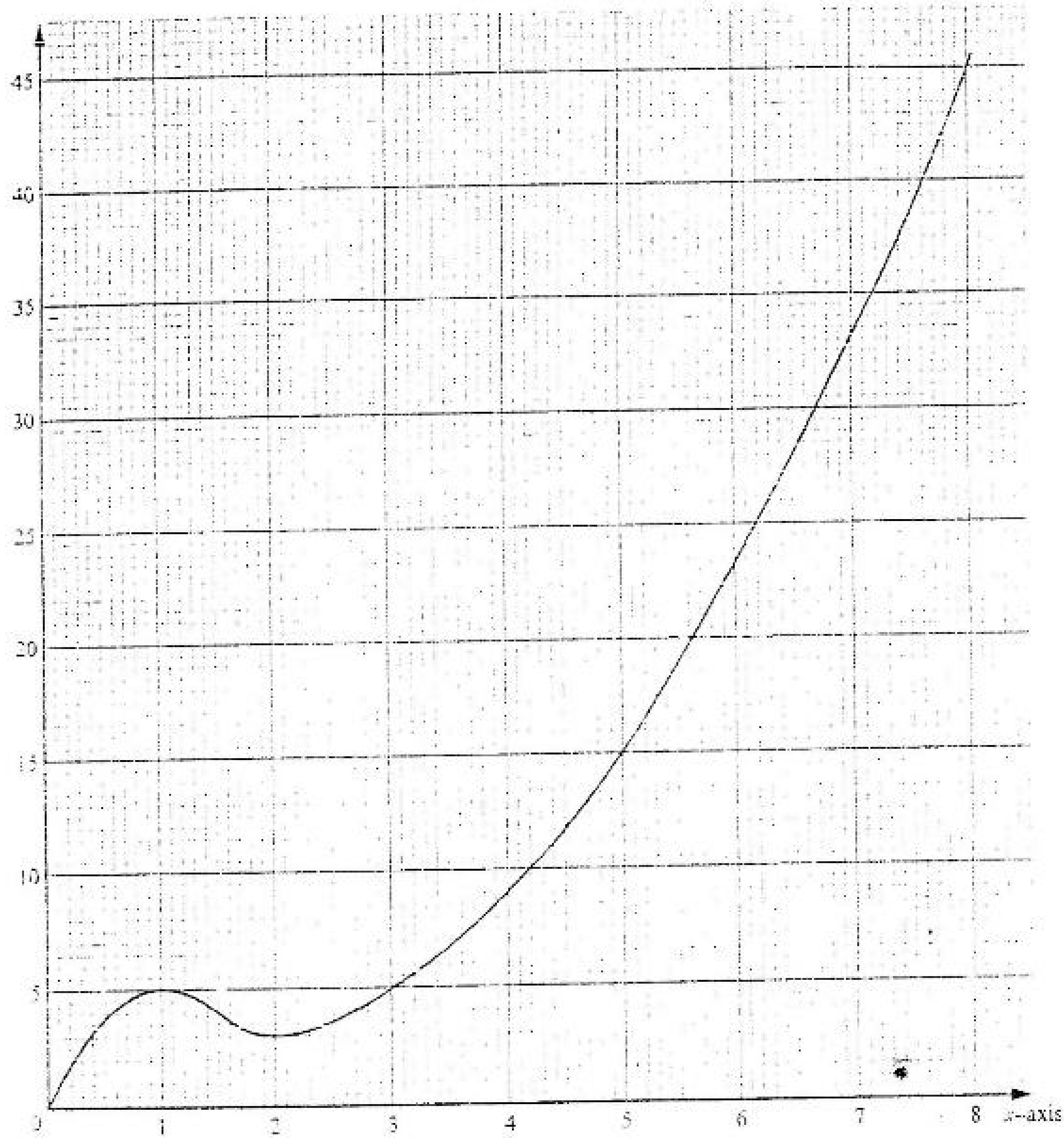


5. (a) Find the value of x at which the curve $y= x- 2x^2 - 3$ crosses the x- axis

(b) Find $\int(x^2 - 2x - 3)\, dx$

(c) Find the area bounded by the curve $y = x^2 - 2x - 3$, the axis and the lines $x= 2$ and $x = 4$.

6. The graph below consists of a non- quadratic part ($0 \leq x \leq 2$) and a quadrant part ($2 \leq x$ 8). The quadratic part is $y = x^2 - 3x + 5$, $2 \leq x \leq 8$



(a) Complete the table below

x	2	3	4	5	6	7	8

y	3						

(1mk)

(b) Use the trapezoidal rule with six strips to estimate the area enclosed by the curve, x = axis and the line x = 2 and x = 8 (3mks)

(c) Find the exact area of the region given in (b) (3mks)

(d) If the trapezoidal rule is used to estimate the area under the curve between x = 0 and x = 2, state whether it would give an under- estimate or an over- estimate. Give a reason for your answer.

7. Find the equation of the gradient to the curve Y= $(x^2 + 1)(x - 2)$ when x = 2

8. The distance from a fixed point of a particular in motion at any time t seconds is given by

$$S = \frac{t^3 - 5t^2}{2t^2} + 2t + 5$$

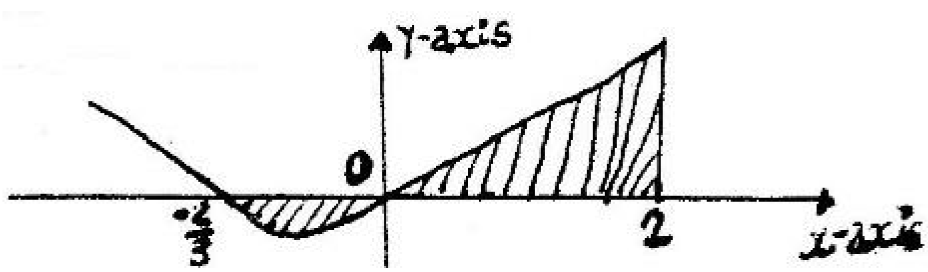
Find its:

(a) Acceleration after 1 second

(b) Velocity when acceleration is Zero

9. The curve of the equation $y = 2x + 3x^2$, has x = -2/3 and x = 0 and x intercepts.

The area bounded by the axis x = -2/3 and x = 2 is shown by the sketch below.



Find:

(a) $(2x + 3x^2)\,dx$

(b) The area bounded by the curve x – axis, x = - $^2/_3$ and x =2

10. A particle is projected from the origin. Its speed was recorded as shown in the table below

Time (sec)	0	5	10	15	20	25	39	35
Speed (m/s)	0	2.1	5.3	5.1	6.8	6.7	4.7	2.6

Use the trapezoidal rule to estimate the distance covered by the particle within the 35 seconds.

11. (a) The gradient function of a curve is given by $\frac{dy}{dx} = 2x^2 - 5$

Find the equation of the curve, given that y = 3, when x = 2

(b) The velocity, vm/s of a moving particle after seconds is given:

$v = 2t^3 + t^2 - 1$. Find the distance covered by the particle in the interval $1 \le t \le 3$

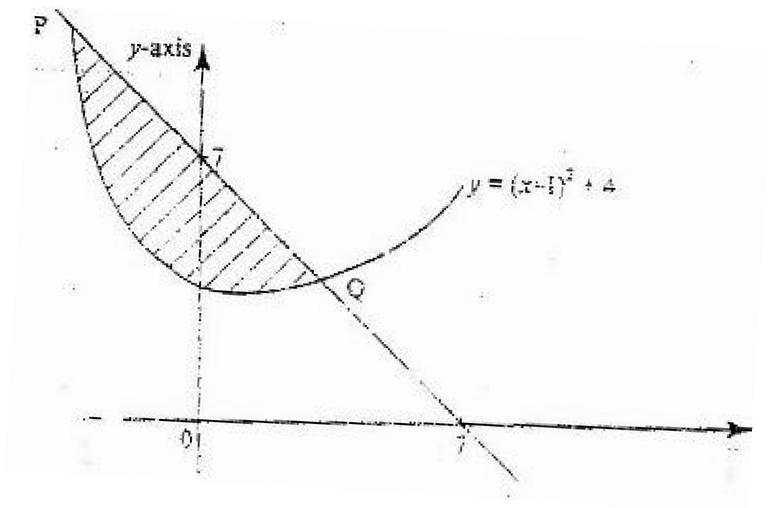
12. Given the curve $y = 2x^3 + {}^1/_2x^2 - 4x + 1$. Find the:

i) Gradient of curve at $\{1, -{}^1/_2\}$

ii) Equation of the tangent to the curve at $\{1, -{}^1/_2\}$

13. The diagram below shows a straight line intersecting the curve $y = (x-1)^2 + 4$

At the points P and Q. The line also cuts x-axis at (7, 0) and y axis at (0, 7)



a) Find the equation of the straight line in the form y = mx +c.

b) Find the coordinates of p and Q.

c) Calculate the area of the shaded region.

14. The acceleration, a ms^{-2}, of a particle is given by $a = 25 - 9t^2$, where t in seconds after the particle passes fixed point O.

If the particle passes O, with velocity of 4 ms^{-1}, find

(a) An expression of velocity V, in terms of t

(b) The velocity of the particle when t = 2 seconds

15. A curve is represented by the function $y = {}^1/_3\, x^3 + x^2 - 3x + 2$

(a) Find: $\dfrac{dy}{dx}$

(b) Determine the values of y at the turning points of the curve

$y = {}^1/_3 x^3 + x^2 - 3x + 2$

(c) In the space provided below, sketch the curve of $y = {}^1/_3\, x^3 + x^2 - 3x + 2$

16. A circle centre O, ha the equation $x^2 + y^2 = 4$. The area of the circle in the first quadrant is divided

into 5 vertical strips of width 0.4 cm

(a) Use the equation of the circle to complete the table below for values of y correct to 2 decimal places

X	0	0.4	0.8	1.2	1.6	2.0
Y	2.00			1.60		0

(b) Use the trapezium rule to estimate the area of the circle

17. A particle moves along straight line such that its displacement S metres from a given point is $S = t^3 - 5t^2 + 4$ where t is time in seconds

Find

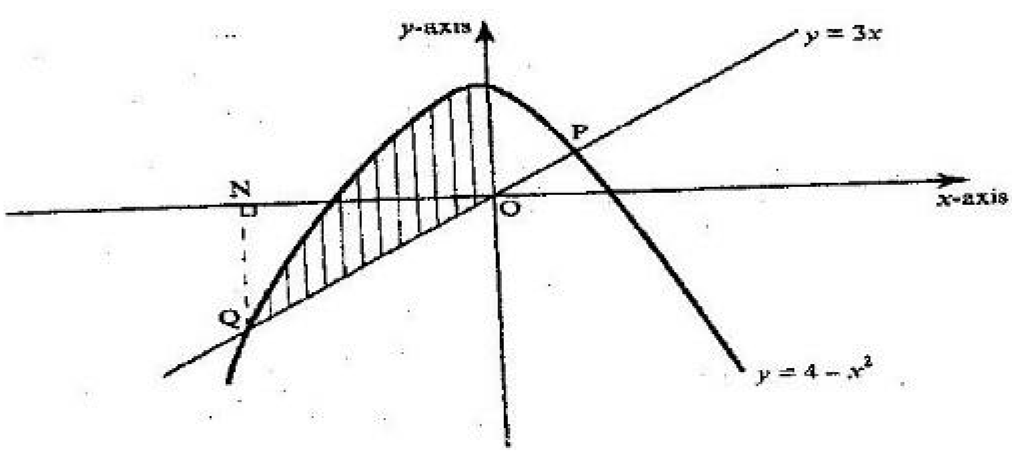
(a) The displacement of particle at t = 5

(b) The velocity of the particle when t = 5

(c) The values of t when the particle is momentarily at rest

(d) The acceleration of the particle when t = 2

18. The diagram below shows a sketch of the line y = 3x and the curve $y = 4 - x^2$ intersecting at points P and Q.



(a) Find the coordinates of P and Q

(b) Given that QN is perpendicular to the x- axis at N, calculate

(i) The area bounded by the curve y = 4 – x2, the x- axis and the line QN (2 marks)

(ii) The area of the shaded region that lies below the x- axis

(iii) The area of the region enclosed by the curve $y = 4-x^2$, the line y – 3x and the y-axis.

19. The gradient of the tangent to the curve $y = ax^3 + bx$ at the point (1, 1) is -5

Calculate the values of a and b.

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20. The diagram on the grid below represents as extract of a survey map showing two adjacent plots belonging to Kazungu and Ndoe.

The two dispute the common boundary with each claiming boundary along different smooth curves coordinates (x, y) and (x, y_2) in the table below, represents points on the boundaries as claimed by Kazungu Ndoe respectively.

X	0	1	2	3	4	5	6	7	8	9
Y_1	0	4	5.7	6.9	8	9	9.8	10.6	11.3	12
Y_2	0	0.2	0.6	1.3	2.4	3.7	5.3	7.3	9.5	12

(a) On the grid provided above draw and label the boundaries as claimed by Kazungu and Ndoe.

(b) (i) Use the trapezium rule with 9 strips to estimate the area of the section of the land in dispute

(ii) Express the area found in b (i) above, in hectares, given that 1 unit on each axis represents 20 metres

21. The gradient function of a curve is given by the expression 2x + 1. If the curve passes through the point (-4, 6);

(a) Find:

(i) The equation of the curve

(ii) The vales of x, at which the curve cuts the x- axis

(b) Determine the area enclosed by the curve and the x- axis

22. A particle moves in a straight line through a point P. Its velocity v m/s is given by v= 2 -t, where t is time in seconds, after passing P. The distance s of the particle from P when t = 2 is 5 metres. Find the expression for s in terms of t.

23. Find the area bonded by the curve y=2x – 5 the x-axis and the lines x=2 and x = 4.

23. Complete the table below for the function

$Y = 3x^2 - 8x + 10$

X	0	2	4	6	8	10
Y	10	6	-	70	-	230

Using the values in the table and the trapezoidal rule, estimate the area bounded by the curve $y = 3x^2 - 8x + 10$ and the lines y – 0, x = 0 and x = 10

24. (a) Find the values of x which the curve $y = x^2 - 2x - 3$ crosses the axis

(b) Find $(x^2 - 2x - 3)\,dx$

(c) Find the area bounded by the curve $Y = x^2 - 2x - 3$. The x – axis and the

lines x = 2 and x = 4

25. Find the equation of the tangent to the curve y = (x + 1) (x- 2) when x = 2

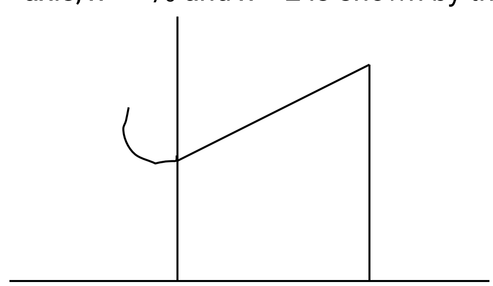
26. The distance from a fixed point of a particle in motion at any time t seconds is given by $s = t - {}^5/_2t^2 + 2t + s$ metres

Find its

(a) Acceleration after t seconds

(b) Velocity when acceleration is zero

27. The curve of the equation $y = 2x + 3x^2$, has $x = -{}^2/_3$ and x = 0, as x intercepts. The area bounded by the curve, x – axis, $x = -{}^2/_3$ and x = 2 is shown by the sketch below.



(a) Find $\int(2x + 3x^2)\,dx$

(b) The area bounded by the curve, x axis $x = -{}^2/_3$ and x = 2

28. A curve is given by the equation $y = 5x^3 - 7x^2 + 3x + 2$

Find the

(a) Gradient of the curve at x = 1

(b) Equation of the tangent to the curve at the point (1, 3)

29. The displacement x metres of a particle after t seconds is given by $x = t^2 - 2t + 6, t > 0$

(a) Calculate the velocity of the particle in m/s when t = 2s

(b) When the velocity of the particle is zero,

Calculate its

(i) Displacement

(ii) Acceleration

30. The displacement s metres of a particle moving along a straight line after t seconds is given by $s = 3t + {}^3/_2t^2 - 2t^3$

(a) Find its initial acceleration

(b) Calculate

(i) The time when the particle was momentarily at rest.

(ii) Its displacement by the time it comes to rest momentarily when

t = 1 second, s = 1 ½ metres when t = ½ seconds

(c) Calculate the maximum speed attained