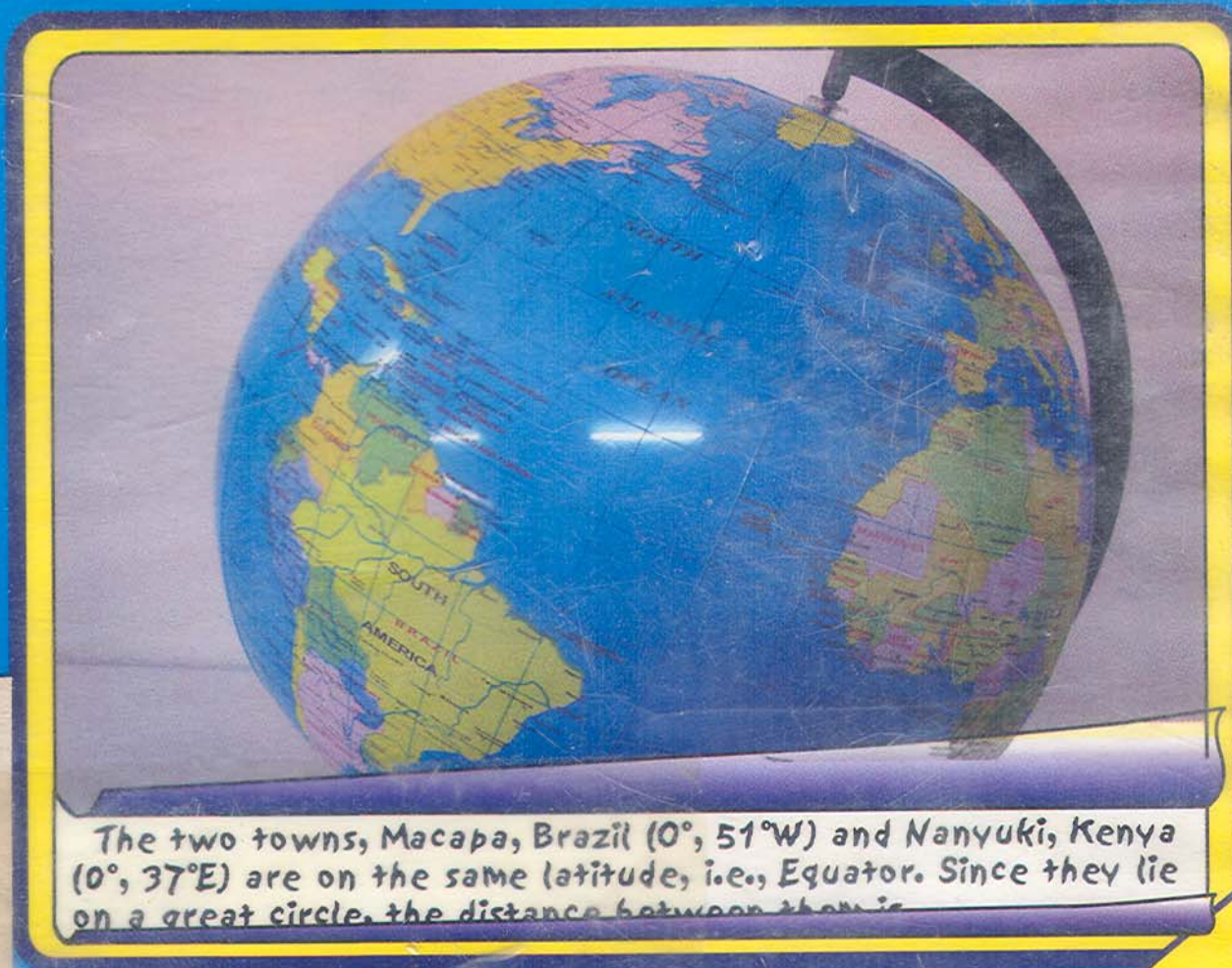


Secondary

MATHEMATICS

Students' Book Four

Third Edition



The two towns, Macapa, Brazil ($0^\circ, 51^\circ\text{W}$) and Nanyuki, Kenya ($0^\circ, 37^\circ\text{E}$) are on the same latitude, i.e., Equator. Since they lie on a great circle, the distance between them is



KENYA LITERATURE BUREAU

New Syllabus

CONTENTS

	<i>Page</i>
<i>Foreword</i>	<i>iii</i>
<i>Acknowledgement</i>	<i>ix</i>
<i>Introduction</i>	<i>xi</i>
1. MATRICES AND TRANSFORMATION	1
Matrices of Transformation	1
Finding the Matrix of a Transformation	6
Successive Transformation	16
Single Matrix of Transformation for Successive Transformations ..	21
Inverse of a Transformation	24
Area Scale Factor and Determinant of Matrix	26
Shear and Stretch	28
Isometric and Non-Isometric Transformations	35
2. STATISTICS II	38
Measures of Central Tendency	38
Mean Using Assumed (working) Mean	38
Quartiles, Deciles and Percentiles	46
Median from Cumulative Frequency Curve	48
3. LOCI	66
Introduction	66
Common Types of Loci	68
Intersecting Loci	75
Loci of Inequalities	81
Locus involving Chords	84
Mixed Exercise 1	85
4. TRIGONOMETRY III	90
Some Trigonometric Ratios	90
Waves	93
Trigonometric Equations	100
5. THREE DIMENSIONAL GEOMETRY	104
Geometrical Properties of Common Solids	104
Angle between a Line and a plane	106

	Angle between Two Planes	113
	Skew Lines	118
6.	LONGITUDES AND LATITUDES	125
	Great and Small Circles	125
	Position of a Place on the Earth's Surface	128
	Distances on the Surface of the Earth	130
	Longitude and Time	141
	Mixed Exercise 2	145
7.	LINEAR PROGRAMMING	150
	Forming Linear inequalities	150
	Solutions of Linear Inequalities	151
	Solution by Graphing	152
	Optimisation	157
8.	DIFFERENTIATION	162
	Average and Instantaneous rates of change	162
	Gradient of a Curve at a Point	163
	Gradient of $y = x^n$	164
	Delta Notation	167
	The Derivative of a Polynomial	170
	Equations of Tangents and Normals to a Curve	173
	Stationery Points	174
	Curve Sketching	180
	Application of Differentiation in Calculation of Velocity and Acceleration	182
	Maxima and Minima	186
9.	AREA APPROXIMATION	190
	Introduction	190
	Using Counting Technique to Approximate Area	190
	Approximation of Area by Trapezium Method	195
	The Mid-ordinate Rule	202
10.	INTEGRATION	207
	Reverse Differentiation	207
	Definite and Indefinite Integrals	212

Area Under a Curve	217
Application in Kinematics	223
Mixed Exercise 3	227
Revision Exercise 1	232
Revision Exercise 2	233
Revision Exercise 3	235
Revision Exercise 4	237
Revision Exercise 5	239
Revision Exercise 6	242
Revision Exercise 7	244
Revision Exercise 8	247
Revision Exercise 9	250
Revision Exercise 10	251
Sample Test Paper 1	255
Sample Test Paper 2	260
Sample Test Paper 3	265
Sample Test Paper 4	270
Sample Test Paper 5	274
Sample Test Paper 6	279

MATRICES AND TRANSFORMATION

1.1: Matrices of Transformation

A transformation changes the position, size or shapes of an object. Examples of transformations include reflection, rotation, enlargement and translation, all of which are covered in Book Two.

Consider a triangle PQR with vertices P(1, 2), Q(3, 2) and R(3, 5). The position vectors of the points P, Q and R are:

$$\mathbf{OP} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbf{OQ} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{OR} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

These position vectors are 2×1 matrices and can be premultiplied by any 2×2 matrix. Consider the effect of pre-multiplying them by the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{array}{cc} P & P' \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} Q & Q' \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} & = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} R & R' \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} & = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \end{array}$$

Pre-multiplying by the matrix A has mapped triangle PQR onto triangle P'Q'R', whose vertices are of the co-ordinates P'(-1, 2), Q'(-3, 2) and R'(-3, 5). The object and the image figures are shown in figure 1.1.

Notice that the image P'Q'R' can also be obtained if ΔPQR is reflected in the line $x = 0$, i.e., y-axis. Therefore, the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a matrix of reflection in the line $x = 0$ (y-axis).

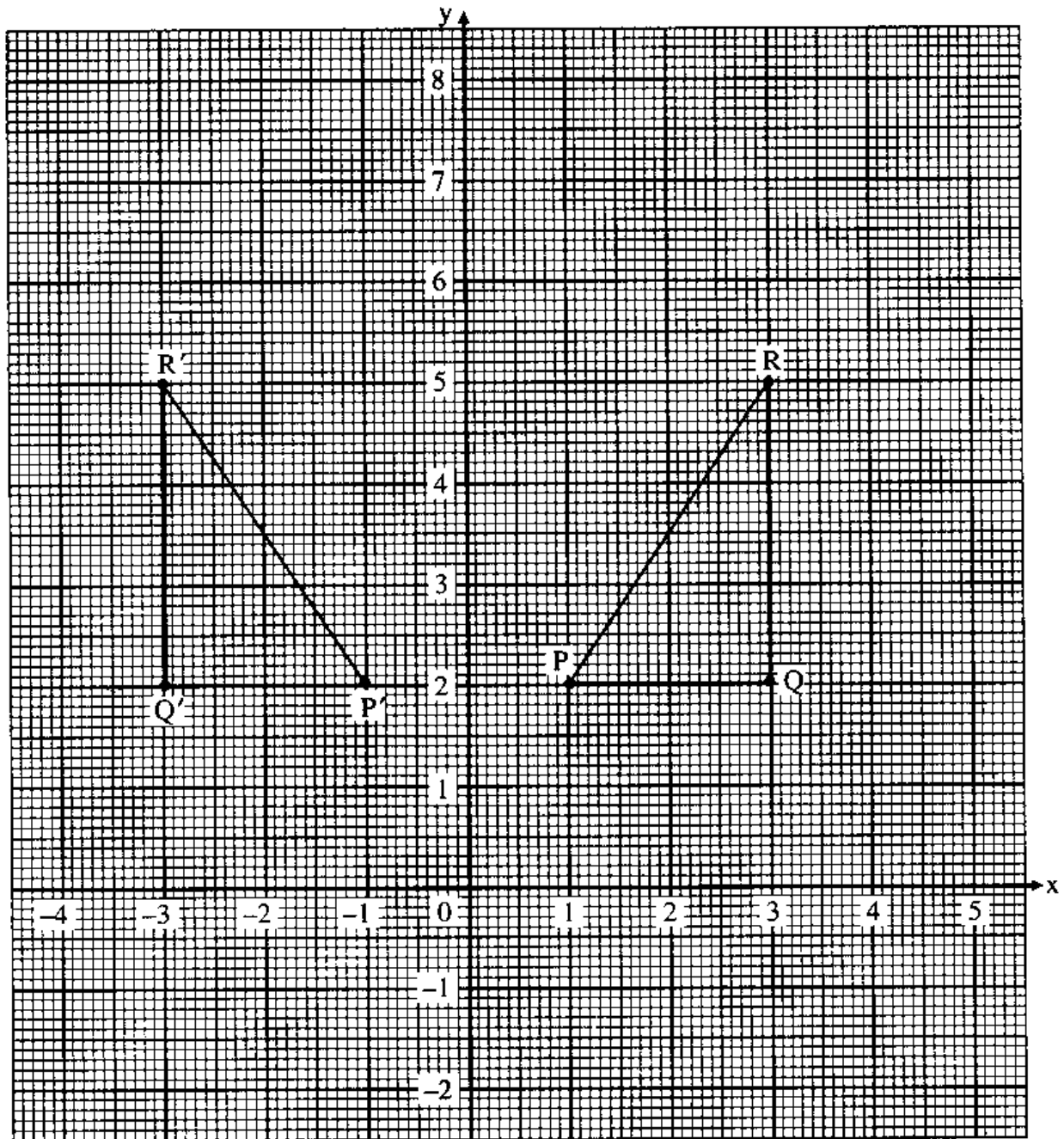


Fig. 1.1

Example 1

Identify the transformation whose matrix is:

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

We need to take an object figure and premultiply it with the transformation represented by each of the matrices. Let us take triangle ABC with vertices at A(1, 1), B(3, 1) and C(3, 4).

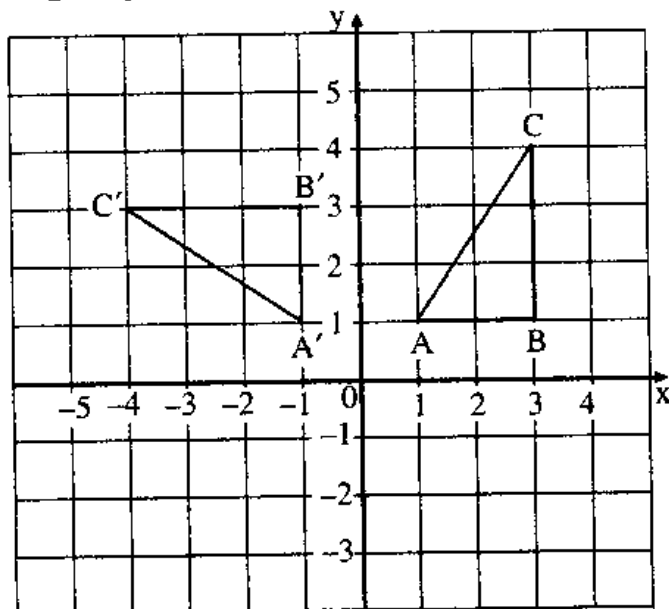
$$(a) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ -1 & -1 & -4 \\ 1 & 3 & 3 \end{pmatrix}$$

$$(b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{pmatrix}$$

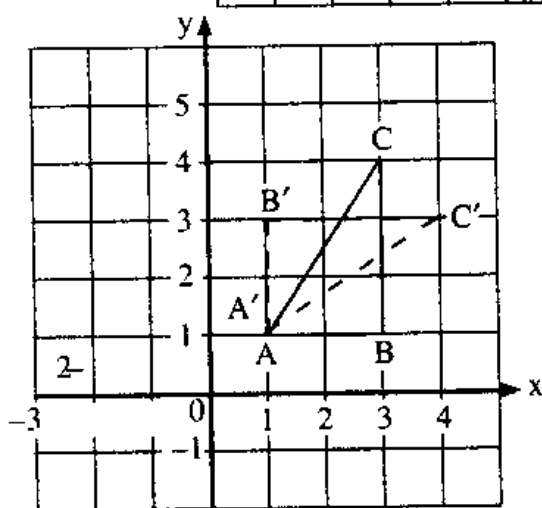
$$(c) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ 1 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix}$$

The object and image figures in each case are shown in figure 1.2:

(a)



(b)



(c)

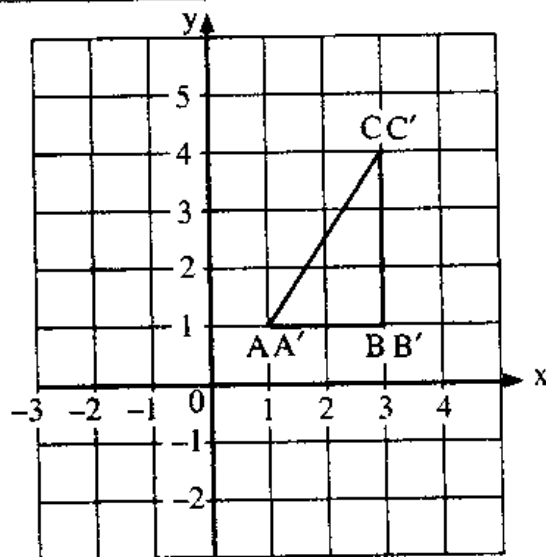


Fig. 1.2

From figure 1.2, we can see that the matrix:

- (i) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a positive quarter turn about the origin, i.e., a rotation of $+90^\circ$ or -270° about the origin.
- (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.
- (iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix. It takes the object back to the original position. It is called the **identity transformation**.

Example 2

Describe the transformation represented by the matrix;

$$\begin{pmatrix} -\cos 30^\circ & -\cos 60^\circ \\ \sin 30^\circ & -\sin 60^\circ \end{pmatrix}$$

Let us consider the effect of the transformation on square OABC with vertices at O(0, 0) A(10, 0) B(10, 10) and C(0, 10).

$$\begin{aligned} \begin{pmatrix} -\cos 30^\circ & -\cos 60^\circ \\ \sin 30^\circ & -\sin 60^\circ \end{pmatrix} \begin{pmatrix} O & A & B & C \\ 0 & 10 & 10 & 0 \\ 0 & 0 & 10 & 10 \end{pmatrix} &= \begin{pmatrix} -0.87 & -0.5 \\ +0.5 & -0.87 \end{pmatrix} \begin{pmatrix} 0 & 10 & 10 & 0 \\ 0 & 0 & 10 & 10 \end{pmatrix} \\ &= \begin{pmatrix} O' & A' & B' & C' \\ 0 & -8.7 & -13.7 & -5 \\ 0 & 5 & -3.7 & -8.7 \end{pmatrix} \end{aligned}$$

The object and image figures are shown in figure 1.3.

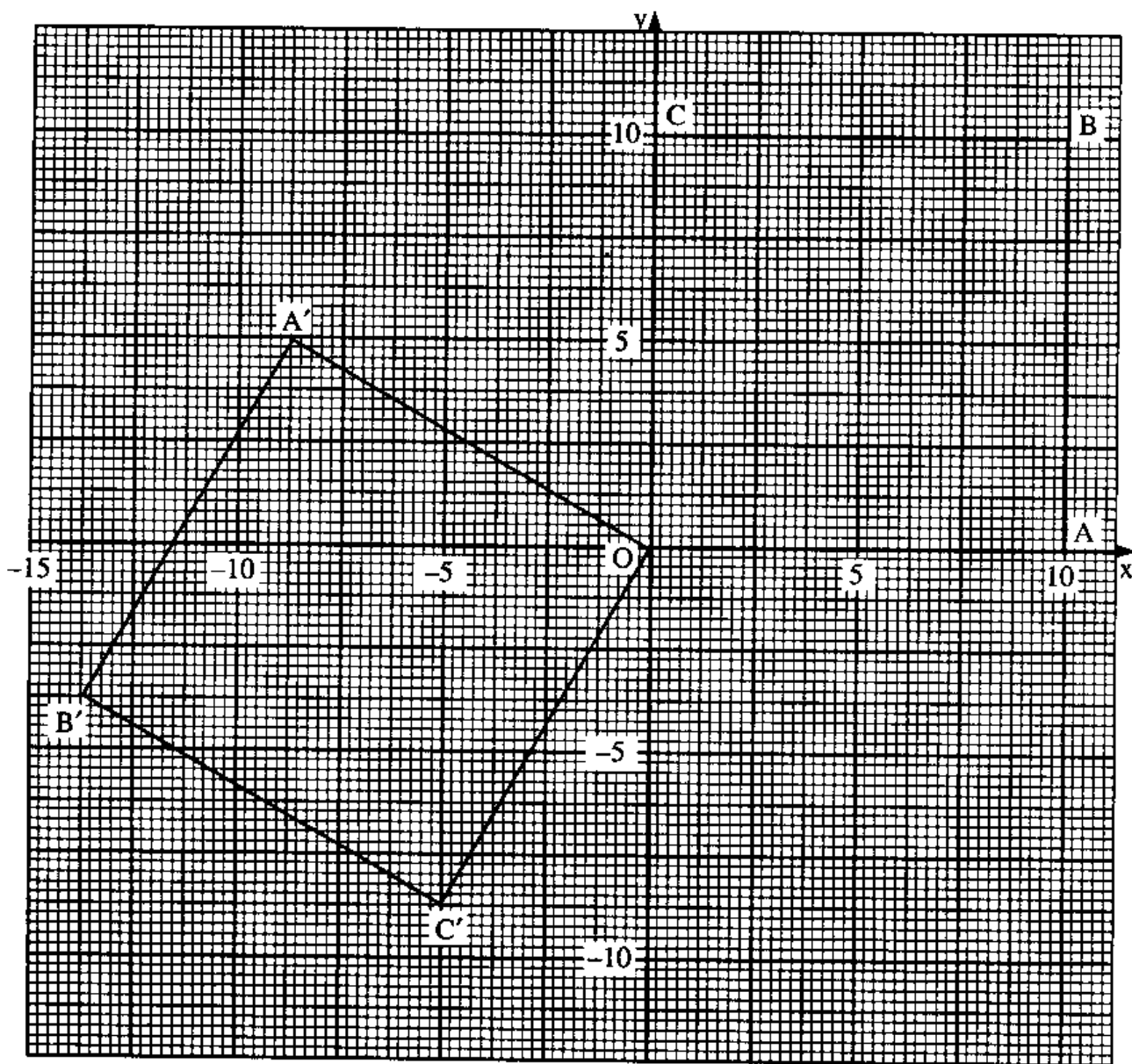


Fig. 1.3

The transformation is a rotation of $+150^\circ$ about the origin.

Exercise 1.1

1. Draw the rectangle ABCD with vertices at A(2, 3), B(2, 5), C(6, 5) and D(6, 3), and its image under the transformations represented by each of the given matrices. In each case, describe the transformation fully.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

2. Choose a simple object figure and use it to identify the transformation represented by each of the following matrices:

(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2.5 & 0 \\ 0 & 2.5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. Describe the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

4. What is the image of the origin under any transformation represented by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

Can a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, in which a, b, c and d are real numbers, represent a translation?

5. Use the triangle ABC with vertices at A(5, 0), B(9, 2) and C(4, 7) to find the transformation represented by the matrix $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$

6. Triangle PQR has vertices at P(2, 4), Q(5, 5) and R(9, 3). Draw on a squared paper the triangle PQR and its image under the transformation whose matrix is $\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$. Describe the transformation fully.

7. Use the triangle whose vertices are O(0, 0), A(5, 3), B(9, 2) to identify the

transformation represented by the matrix $\begin{pmatrix} \frac{15}{17} & \frac{-8}{17} \\ \frac{-8}{17} & \frac{15}{17} \end{pmatrix}$

8. Use a suitable object to identify the transformation whose matrix is:

(a) $\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$ (b) $\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{pmatrix}$

$$(c) \begin{pmatrix} -\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & -\cos 45^\circ \end{pmatrix} \quad (d) \begin{pmatrix} \cos 90^\circ & \cos 0^\circ \\ -\sin 90^\circ & \sin 0^\circ \end{pmatrix}$$

9. (a) Identify clearly the transformation whose matrix is:

$$(a) \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{pmatrix} \quad (b) \begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{pmatrix}$$

10. (a) Describe the transformation represented by the matrix $\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$

(b) What transformation does the matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ represent?

1.2: Finding the Matrix of a Transformation

We have seen that a matrix may represent a transformation. We will now look at how to obtain the matrix of a given transformation.

Suppose we want to find the matrix of a reflection in the x-axis. We need to take a simple object figure and its image under reflection in the x-axis. For example, let us take a triangle PQR with vertices P(1, 3), Q(3, 3) and R(2, 5). The vertices of the image of the figure are P'(1, -3), Q'(3, -3) and R'(2, -5).

Let the matrix of the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P & Q & R \\ 1 & 3 & 2 \\ 3 & 3 & 5 \end{pmatrix} = \begin{pmatrix} P' & Q' & R' \\ 1 & 3 & 2 \\ -3 & -3 & -5 \end{pmatrix}$$

$$\begin{pmatrix} a + 3b & 3a + 3b & 2a + 5b \\ c + 3d & 3c + 3d & 2c + 5d \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ -3 & -3 & -5 \end{pmatrix}$$

Equating the corresponding elements and solving simultaneously;

$$\begin{array}{rcl} a + 3b = 1 & c + 3d = -3 \\ \underline{3a + 3b = 3} & \underline{3c + 3d = -3} \end{array}$$

$$\begin{array}{rcl} 2a = 2 & 2c = 0 \\ a = 1 & c = 0 \\ \therefore b = 0 & \therefore d = -1 \end{array}$$

Therefore, the matrix of reflection in x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example 3

A trapezium with vertices A(1, 4), B(3, 1), C(5, 1) and D(7, 4) is mapped onto a trapezium whose vertices are A'(-4, 1), B'(-1, 3), C'(-1, 5) and D'(-4, 7). Describe the transformation and find its matrix.

Solution

The object and image figures are shown in figure 1.4.

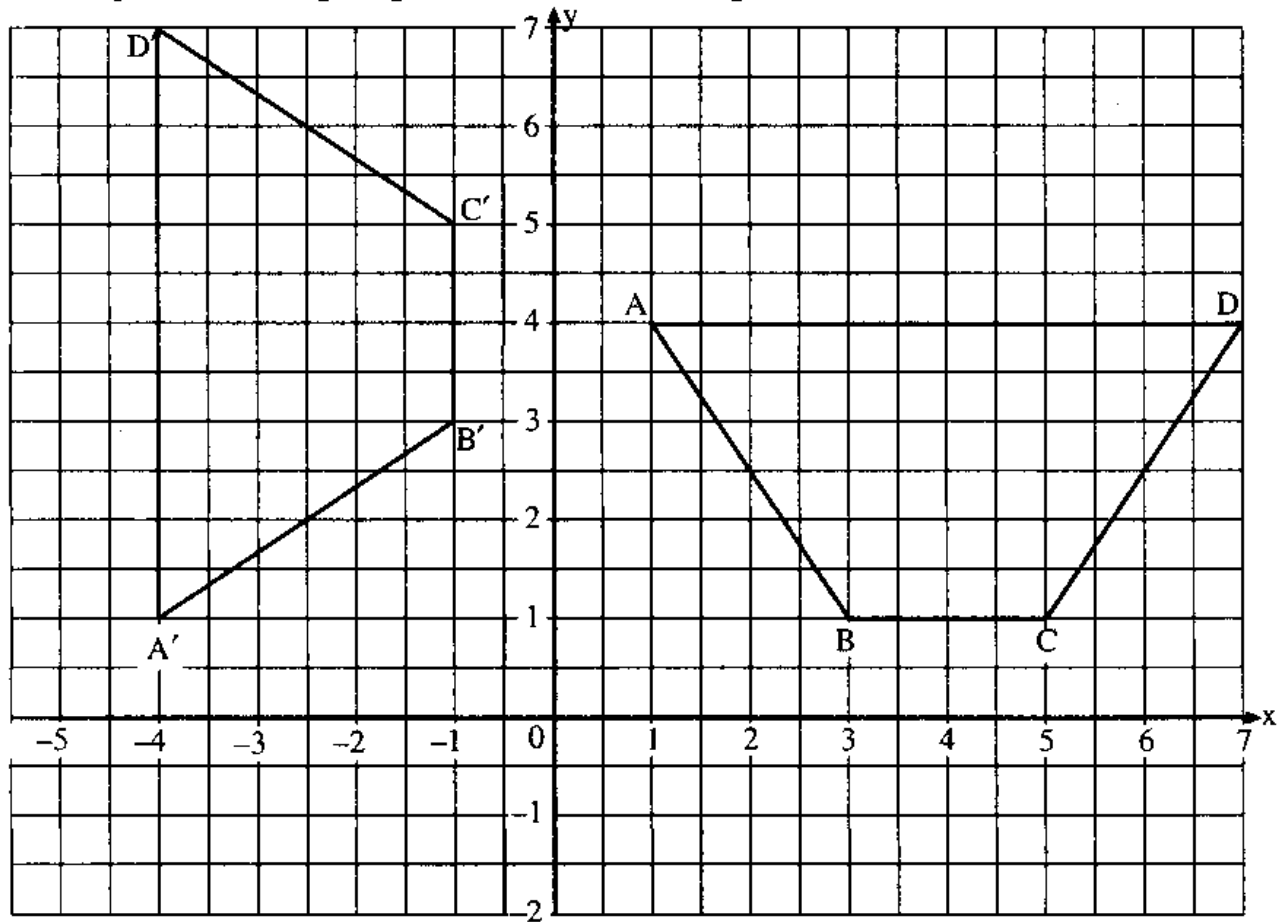


Fig. 1.4

The transformation is a rotation of $+90^\circ$ about the origin.

Let the matrix of transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B & C & D \\ 1 & 3 & 5 & 7 \\ 4 & 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ -4 & -1 & -1 & -4 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

$$\begin{pmatrix} a + 4b & 3a + b & 5a + b & 7a + 4b \\ c + 4d & 3c + d & 5c + d & 7c + 4d \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ -4 & -1 & -1 & -4 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

Equating the corresponding elements, we get;

$$a + 4b = -4 \quad c + 4d = 1$$

$$3a + b = -1 \quad 3c + d = 3$$

Solving the equations simultaneously;

$$3a + 12b = -12 \quad 3c + 12d = 3$$

$$3a + b = -1 \quad 3c + d = 3$$

$$\hline 11b = -11 \quad 11d = 0$$

$$b = -1 \quad d = 0$$

$$a = 0 \quad c = 1$$

The matrix of the transformation is therefore $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note:

Under any transformation represented by a 2×2 matrix, the origin is invariant, i.e., does not change its position. Therefore, if the transformation is a rotation, it must be about the origin. Similarly, if it is a reflection it must be on a mirror line which passes through the origin.

Exercise 1.2

- The vertices of a triangle are $A(2, 5)$, $B(2, 2)$ and $C(4, 2)$. Find the matrix of the transformation which maps the triangle onto a triangle whose vertices are:
 - $A'(-5, -2)$, $B'(-2, -2)$ and $C'(-2, -4)$
 - $A'(-5, 2)$, $B'(-2, 2)$ and $C'(-2, 4)$
 - $A'(-2, -5)$, $B'(-2, -2)$ and $C'(-4, -2)$
 - $A'(1, 2.5)$, $B'(1, 1)$ and $C'(2, 1)$
 - $A'(2, 5)$, $B'(2, 2)$ and $C'(4, 2)$
 Describe fully the transformation in each case.
- The vertices of rectangle PQRS are $P(-6, 2)$, $Q(-2, 2)$, $R(-2, 4)$ and $S(-6, 4)$. Find the matrix of the transformation that maps it onto rectangle $P'Q'R'S'$ with the following vertices:
 - $P'(6, -2)$, $Q'(2, -2)$, $R'(2, -4)$ and $S'(6, -4)$
 - $P'(-6, -4)$, $Q'(-2, -4)$, $R'(-2, -2)$ and $S'(-6, -2)$
 - $P'(-6, 2)$, $Q'(-2, 2)$, $R'(-2, 4)$ and $S'(-6, 4)$
 - $P'(2, 2)$, $Q'(6, 2)$, $R'(6, 4)$ and $S'(2, 4)$
 - $P'(-4, 12)$, $Q'(-4, 4)$, $R'(-8, 4)$ and $S'(-8, 12)$
- Find the matrix representing:
 - reflection in the line $4y - 3x = 0$.
 - reflection in the line $5y + 4x = 0$.
- Find the matrix of:
 - enlargement, centre origin and scale factor 3.
 - enlargement, centre origin and scale factor -4 .
 - rotation through $+45^\circ$ about the origin.
 - rotation through -45° about the origin.
- The vertices of triangle LMN are $L(3, 1)$, $M(4, 3)$ and $N(-2, 1)$. A transformation maps the triangle onto one with vertices at $L'(-1, 3)$, $M'(0, 5)$ and $N'(2, -1)$. Describe the transformation and find its matrix.

6. A rectangle PQRS with vertices at P(-4, -1), Q(-4, -5), R(-3, -5) and S(-3, -1) is mapped onto a rectangle with vertices at P'(5, -4), Q'(9, -4), R'(8, -3) and S'(4, -3). Find the matrix of the transformation.
7. A triangle with vertices at A(2, 5), B(2, 2) and C(4, 2) is mapped onto a triangle with vertices at A'(-4, -10), B'(-4, -4) and C'(-8, -4). Describe the transformation and find its matrix.
8. The vertices of a T-shaped figure are A(1, -1), B(1, -2), C(3, -2), D(3, -6), E(4, -6), F(4, -2), G(6, -2) and H(6, -1). The vertices of its image after a transformation are A'(-1, -1), B'(-1, -2), C'(-3, -2), D'(-3, -6), E'(-4, -6), F'(-4, -2), G'(-6, -2) and H'(-6, -1). Describe the transformation and find its matrix.

The Unit Square

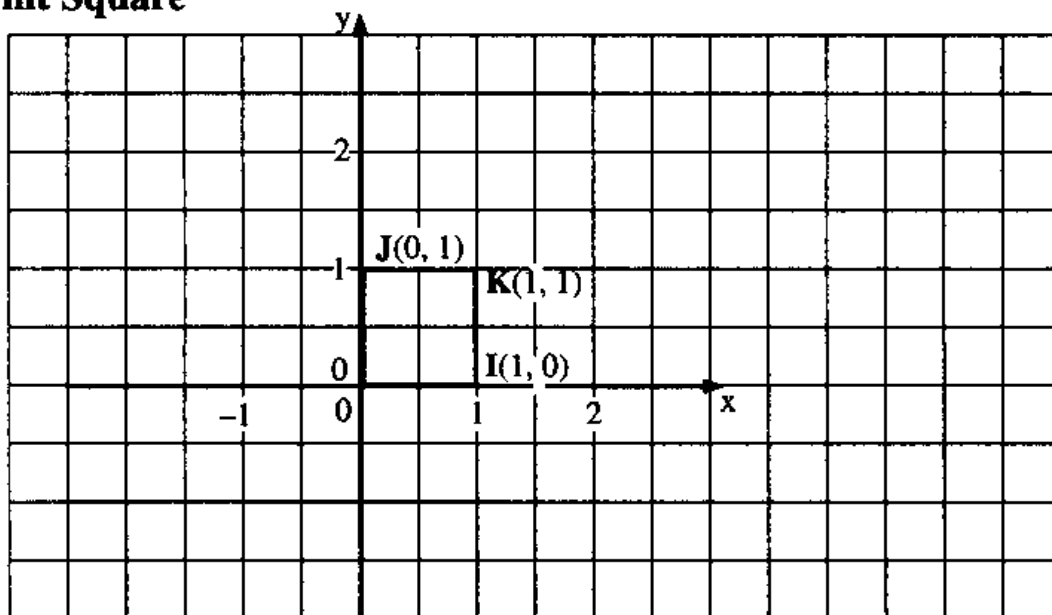


Fig. 1.5

The unit square OIKJ with vertices at O(0, 0), I(1, 0), K(1, 1) and J(0, 1) helps us to quickly identify what transformation a given matrix represents. It also helps us to get the transformation of a given matrix easily.

Example 4

Find the images of I and J under the transformation whose matrix is:

- (a) $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 6 \\ 4 & 5 \end{pmatrix}$

Solution

(a) $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} I & J \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I' & J' \\ 2 & 3 \\ 5 & 4 \end{pmatrix}$

$$(b) \quad \begin{pmatrix} -1 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} I & J \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I' & J' \\ -1 & 6 \\ 4 & 5 \end{pmatrix}$$

Notice that when I and J are used as objects points, then it is the same as post-multiplying a transformation matrix by the identity matrix of order 2. We can, therefore, read out the images directly.

In (a) above the first column $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ gives the image I' of I and the second column $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ the image J' of J. Thus, the co-ordinates of I' and J' are (2, 5) and (3, 4) respectively. Similarly, in (b), the co-ordinates of I' are (-1, 4) and J' (6, 5).

In general, the images of I and J under a transformation represented by any 2×2 matrix i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $I'(a, c)$ and $J'(b, d)$.

Example 5

Identify the transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution

Using the unit square, the co-ordinates of the image I' of I is (0, 1) while the co-ordinates of the image J' of J is (-1, 0), see figure 1.6.

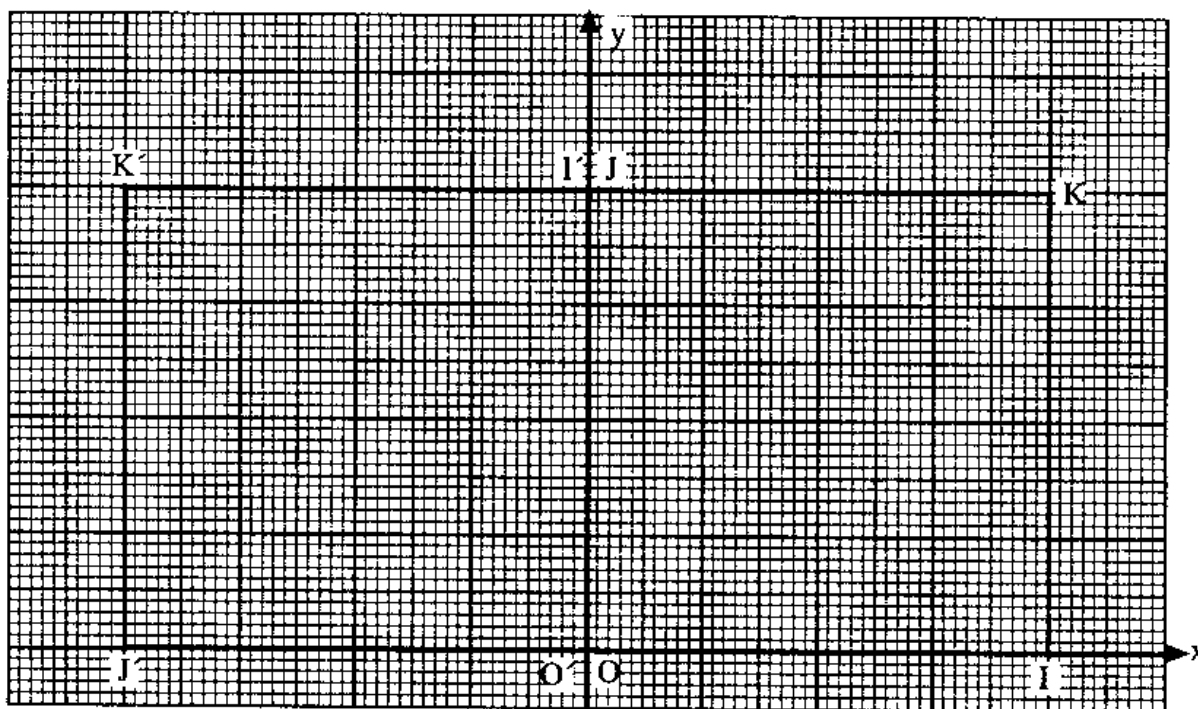


Fig 1.6

The transformation is a positive quarter turn about the origin. The image K' of K is used to check the answer.

In this case, K is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Example 6

Identify the transformation represented by the matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

Solution

Using the unit square;

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \text{ see figure 1.7.}$$

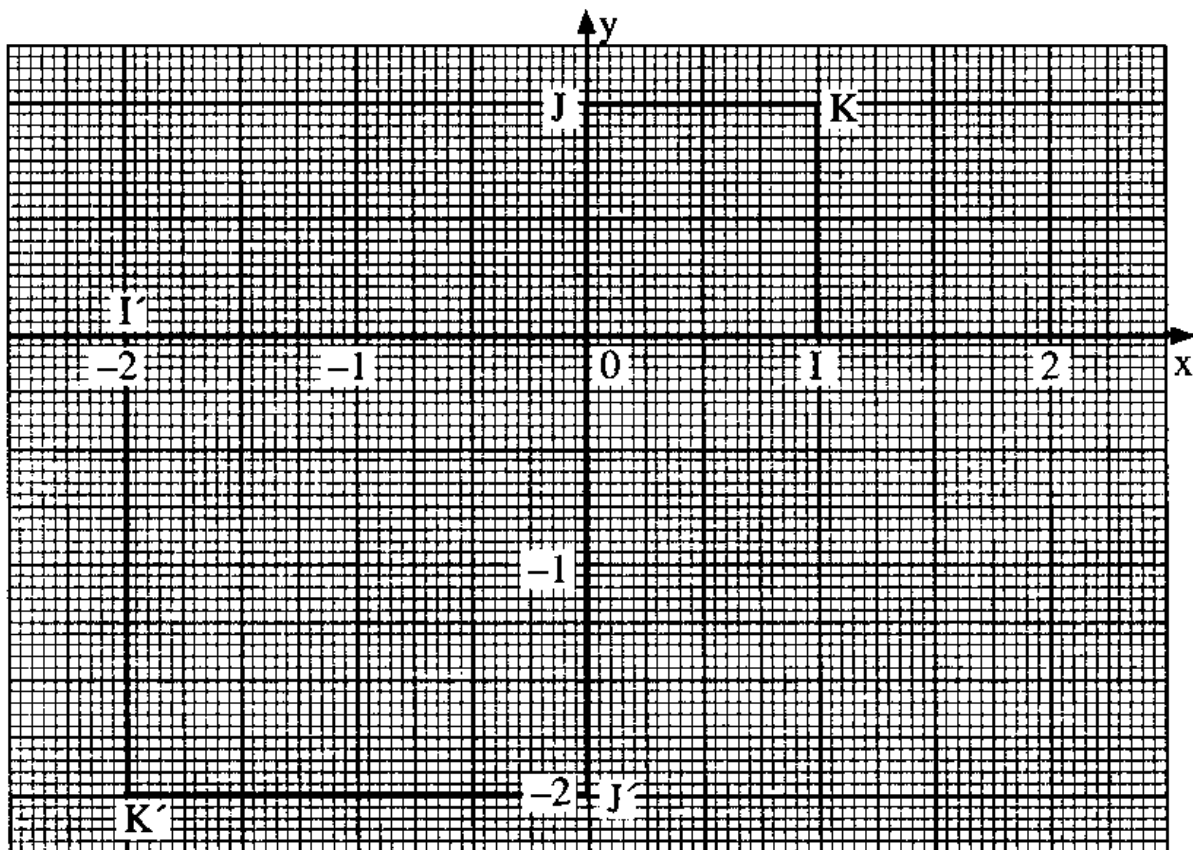


Fig 1.7

The transformation is an enlargement, centre origin and scale factor -2 .

Example 7

Find the matrix of reflection in the line $y = x$.

Solution

Using a unit square (see figure 1.8), the image of I is (0, 1) and the image of J is (1, 0).

Therefore, the matrix of the transformation is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

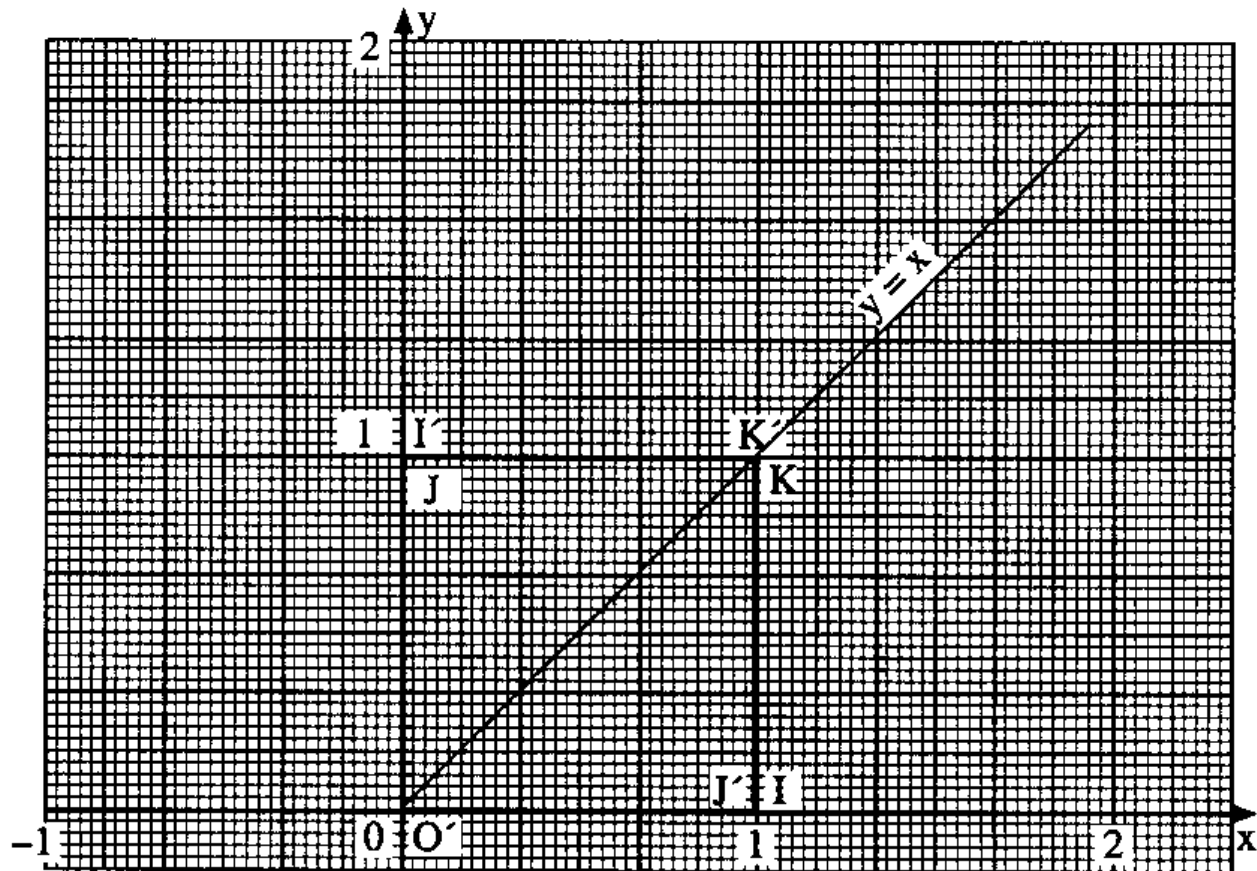


Fig. 1.8

Example 8

Show on a diagram the unit square and its image under the transformation represented by the matrix $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

Solution

Using a unit square (see figure 1.9), the image of I is (1, 0), the image of J is (4, 1),

the image of O is (0, 0) and that of K is; $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Therefore, K' , the image of K is (5, 1)

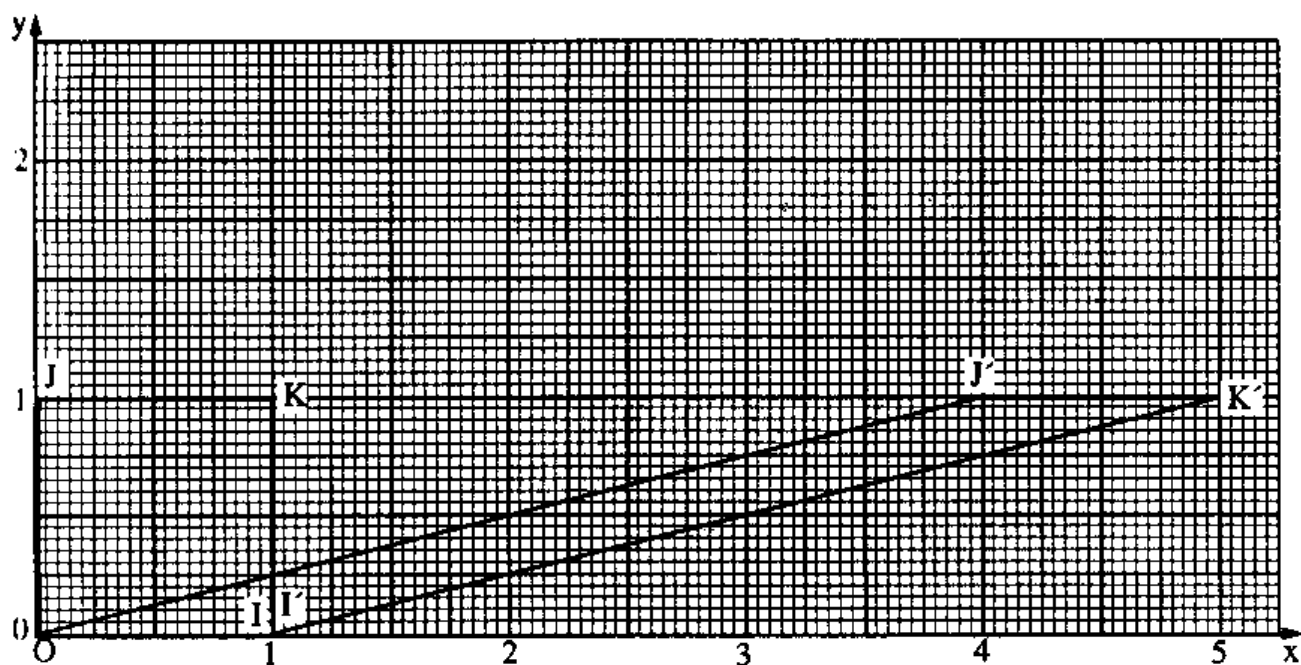


Fig. 1.9

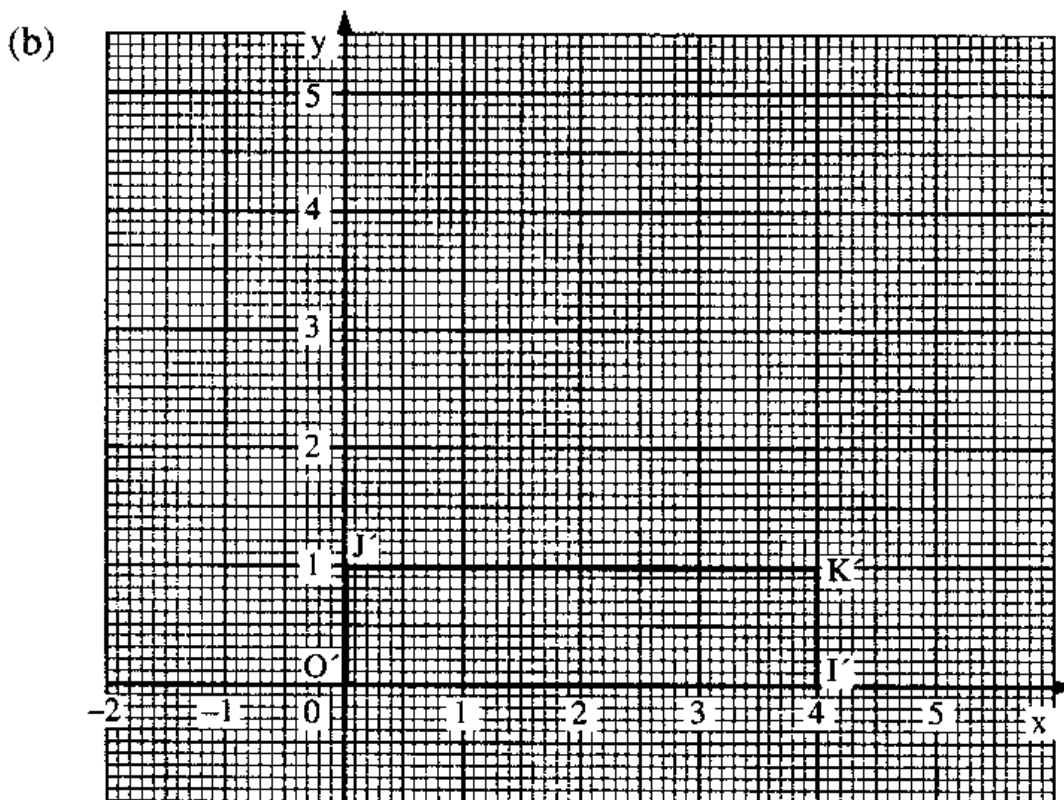
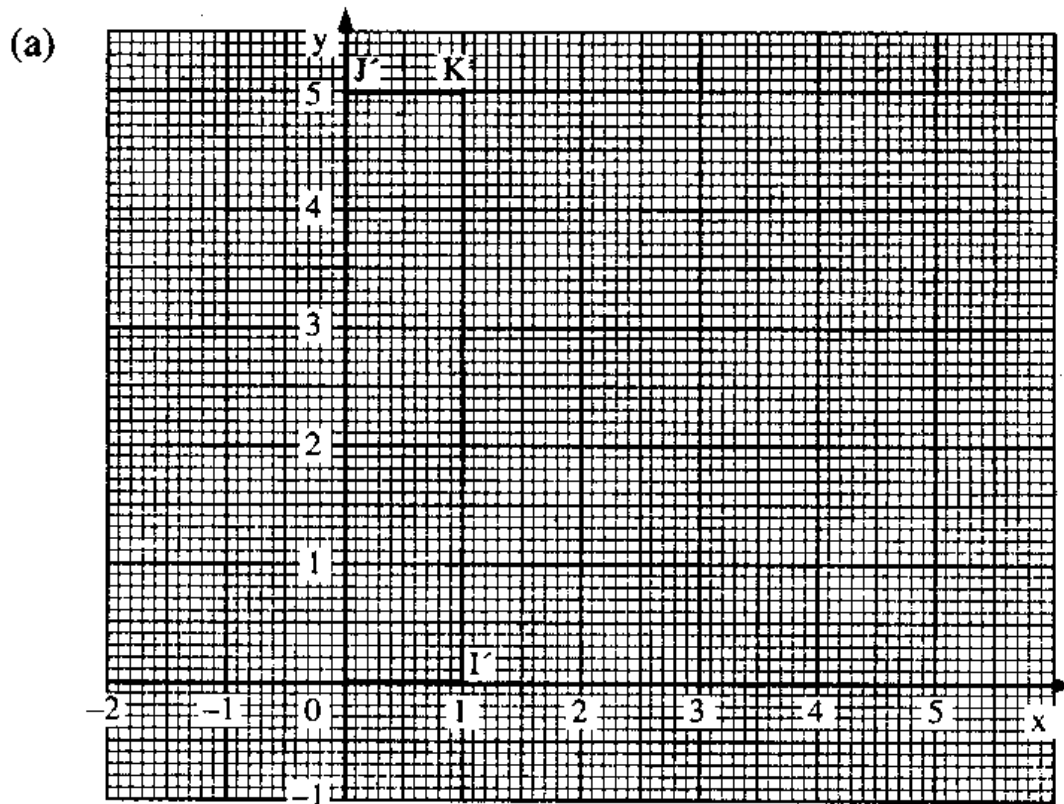
Exercise 1.3

1. Use the unit square to obtain the matrix of the transformation in each of the following:
 - (a) A positive quarter turn about $(0, 0)$.
 - (b) A half turn about $(0, 0)$.
 - (c) A negative quarter turn about $(0, 0)$.
 - (d) A reflection in the line $x = 0$.
 - (e) A reflection in the line $y + x = 0$.
 - (f) A reflection in the line $y = 0$.
 - (g) A rotation of 360° about $(0, 0)$.
 - (h) An enlargement, centre origin and scale factor 1.5.
 - (i) An enlargement, centre origin and scale factor -2.5 .

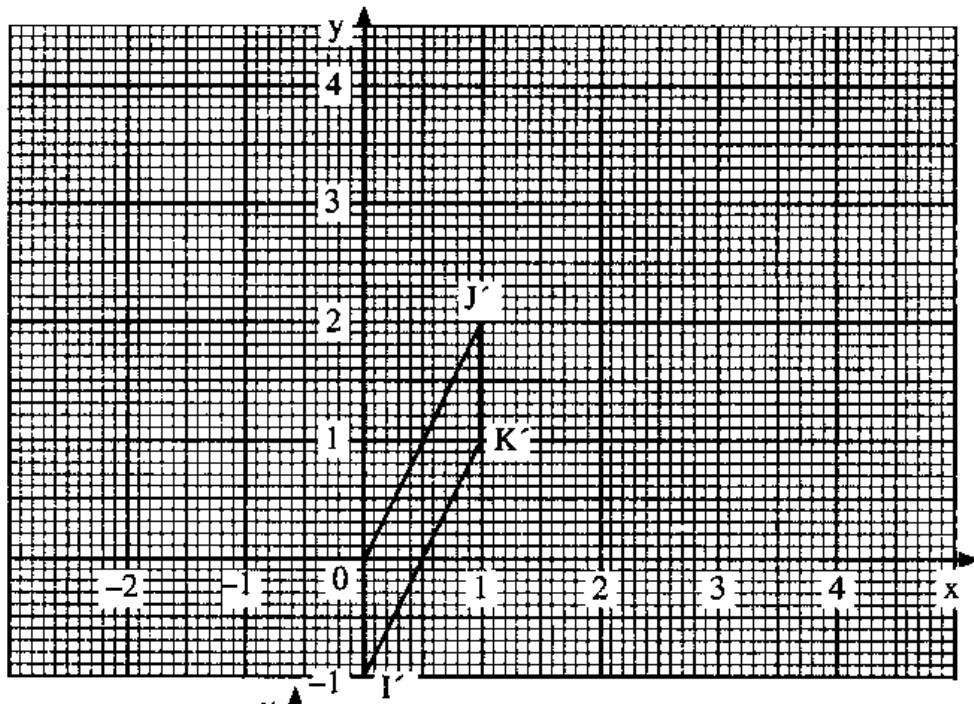
2. Draw the image of the unit square under the transformation represented by:

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	(b) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	(c) $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$	(d) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$
(e) $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$	(f) $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$	(g) $\begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$	(h) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
(i) $\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$	(j) $\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$		

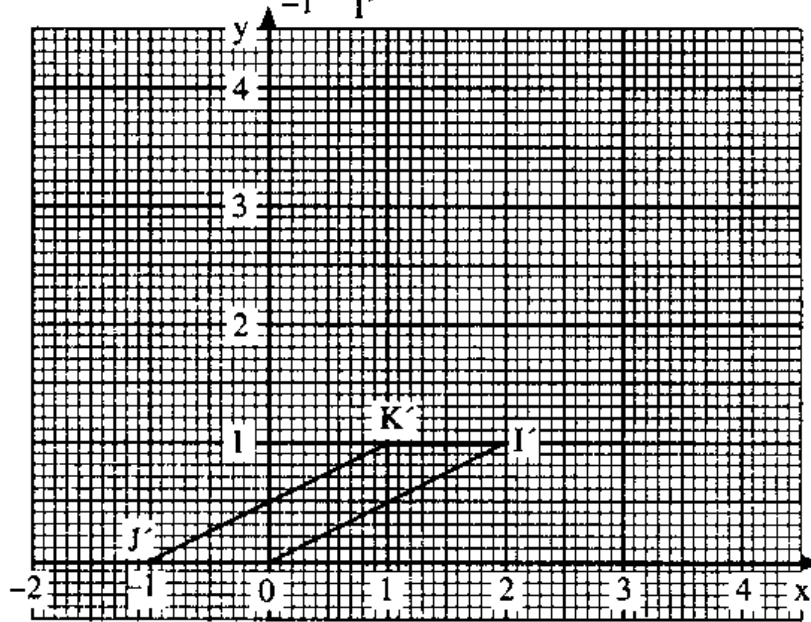
3. The figures below show the image of the unit square under different transformations. Find the matrix of the transformation in each case:



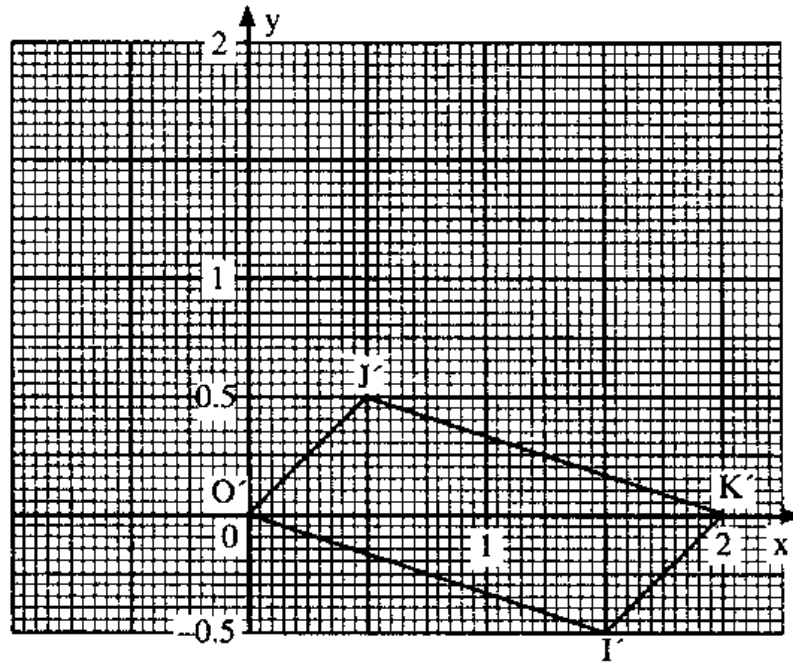
(c)



(d)



(e)



(f)

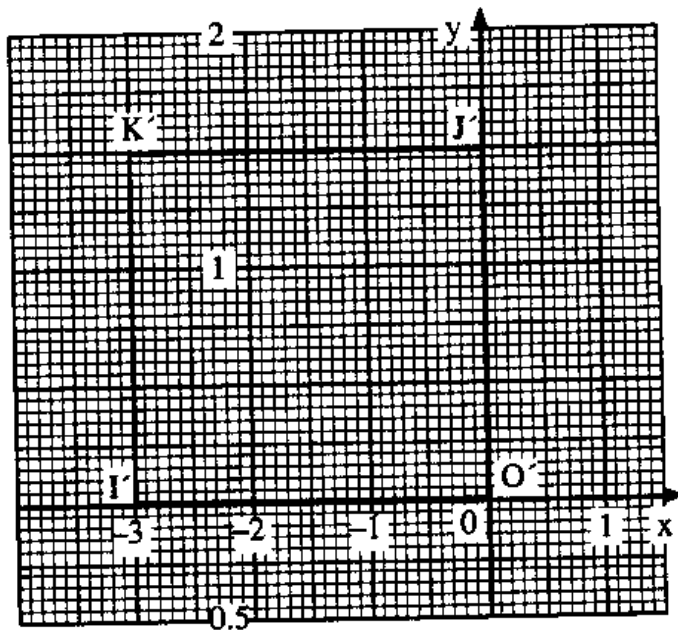


Fig. 1.10

4. Draw the image of the unit square under the transformation given by the matrix;
- $$\begin{pmatrix} 2 \cos 60^\circ & -2 \sin 60^\circ \\ 2 \sin 60^\circ & 2 \cos 60^\circ \end{pmatrix}$$
5. Draw the unit square and its image under the transformation given by the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

1.3: Successive Transformations

Suppose \mathbf{H} is a transformation whose matrix is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ and \mathbf{Y} a transformation whose matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

The image $O'I'K'J'$ of the unit square under \mathbf{H} is shown in figure 1.11. $O'I''K''J''$, the image of $O'I'K'J'$ under \mathbf{Y} is also shown on the same figure.

$$\text{Thus, } \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} O & I & K & J \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} O' & I' & K' & J' \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O' & I' & K' & J' \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix} = \begin{pmatrix} O'' & I'' & K'' & J'' \\ 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

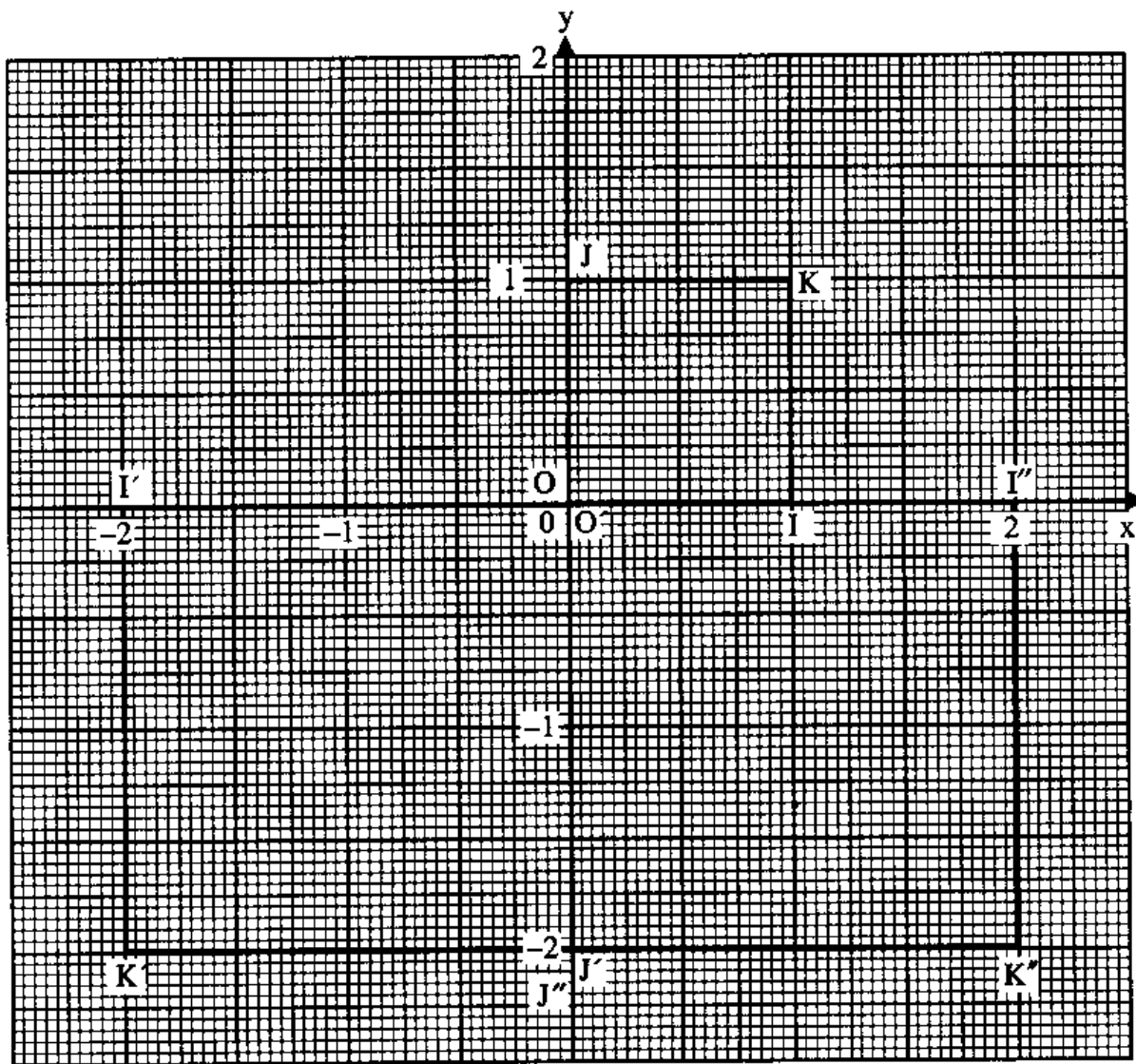


Fig. 1.11

The vertices of the final image are $O'(0, 0)$, $I'(2, 0)$, $K'(2, -2)$ and $J'(0, -2)$. This process of performing transformation **H** followed by transformation **Y** is written as **YH** and it is called **successive transformation**.

If **X**, **Y** and **T** are transformations, then **XYT** means perform **T** first, then **Y** and finally **X**, in that order.

Example 9

A quadrilateral has vertices at $A(1, 1)$, $B(4, 1)$, $C(2, 3)$ and $D(2, 2)$. Show on a diagram the image of the quadrilateral under the combined transformation:

- (a) **UTEQ**,
- (b) **TQ²**,

given that **Q**, **E**, **T** and **U** are transformations whose matrices are:

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively.

Solution

Figure 1.12 shows the object and its image under the combined transformation $UTEQ$:

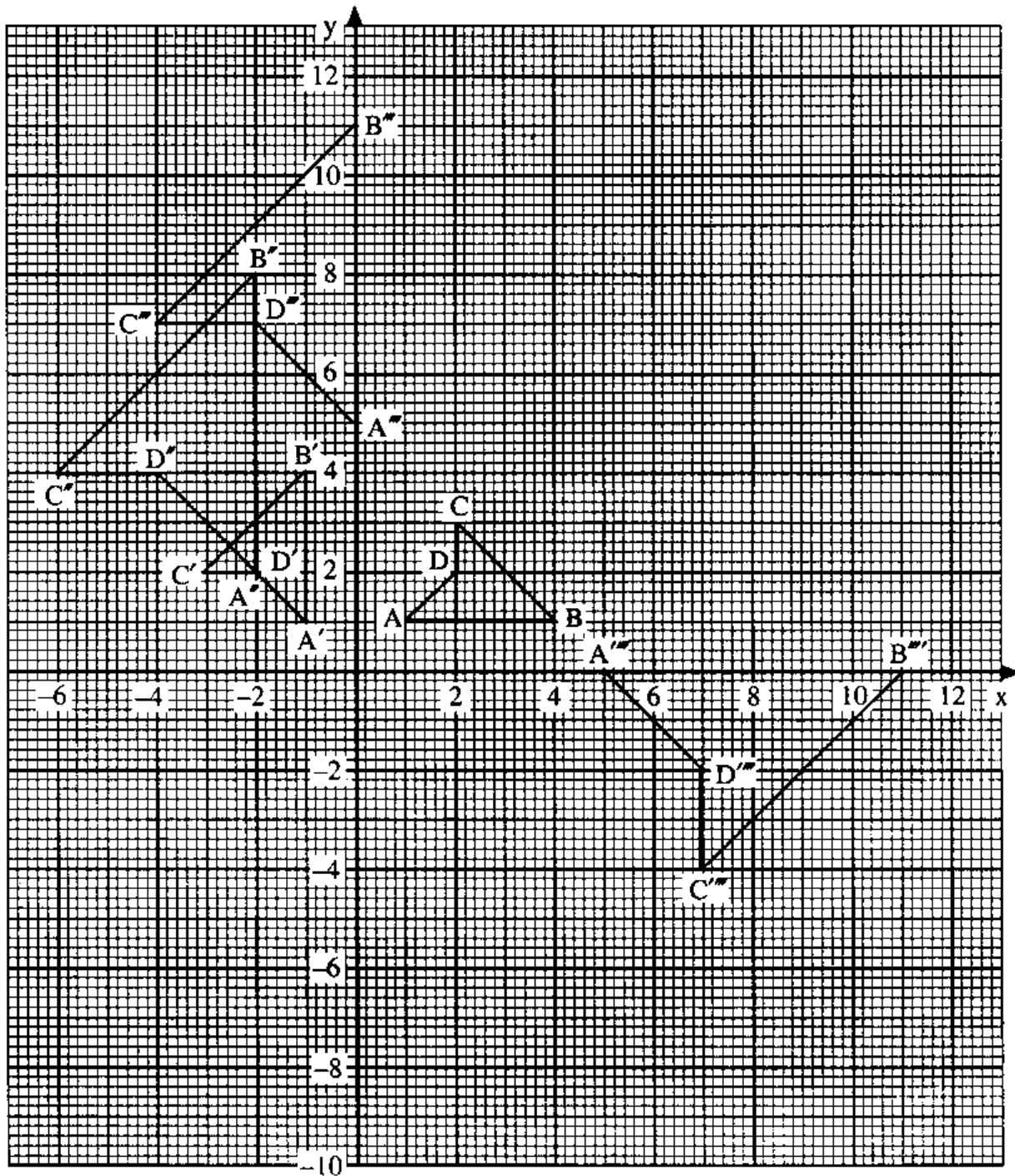


Fig. 1.12

Figure 1.13 shows the object and its image under the combined transformation TQ^2 :

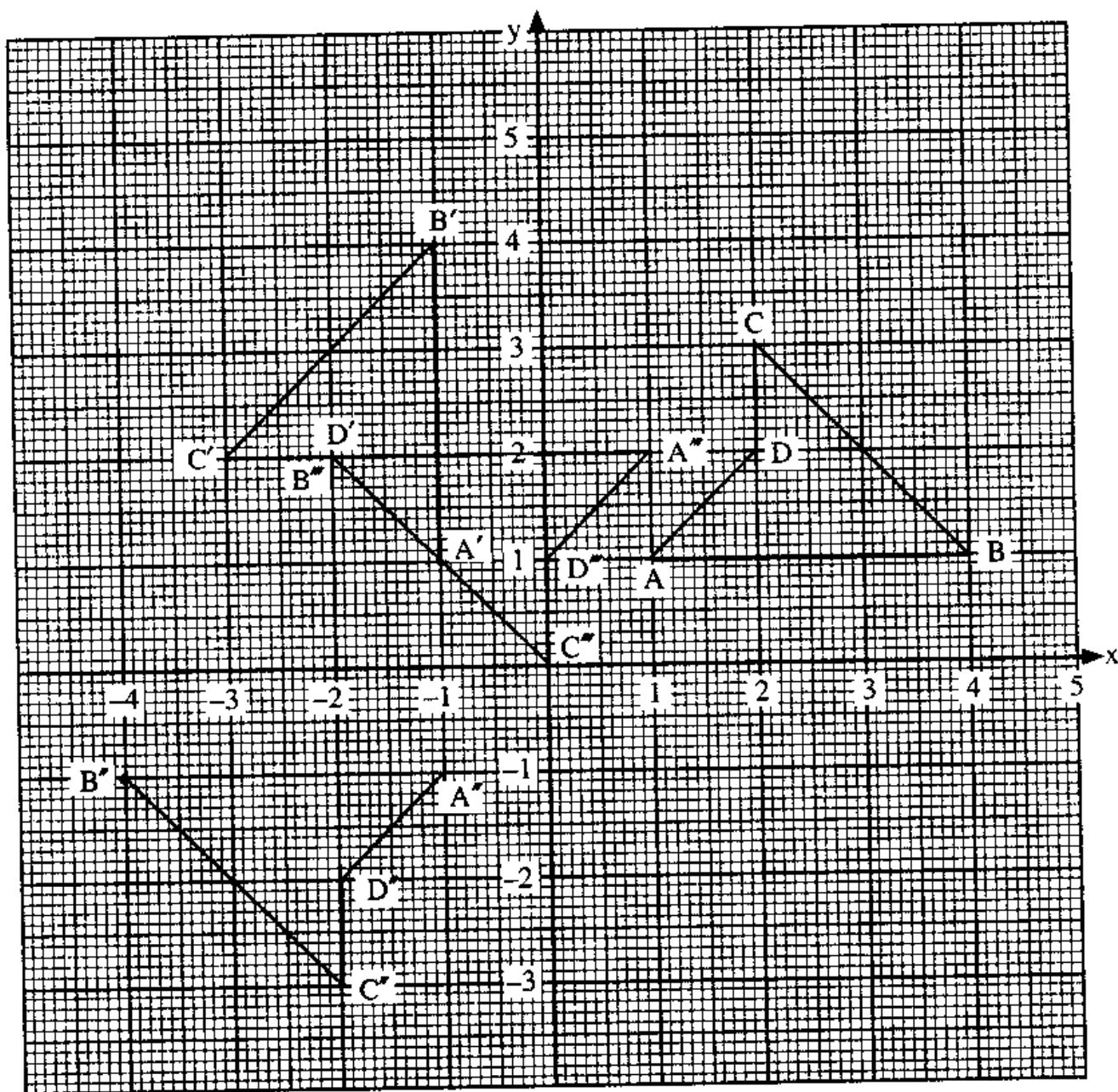


Fig. 1.13

Example 10

Two transformations are defined by $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show on a diagram the images of triangle ABC with vertices A(-3, 2), B(-3, -1) and C(-1, -1) under the combined transformations PQ and QP.

Solution

Figure 1.14(a) shows the image of triangle ABC under PQ while 1.14(b) shows the image of triangle ABC under QP:

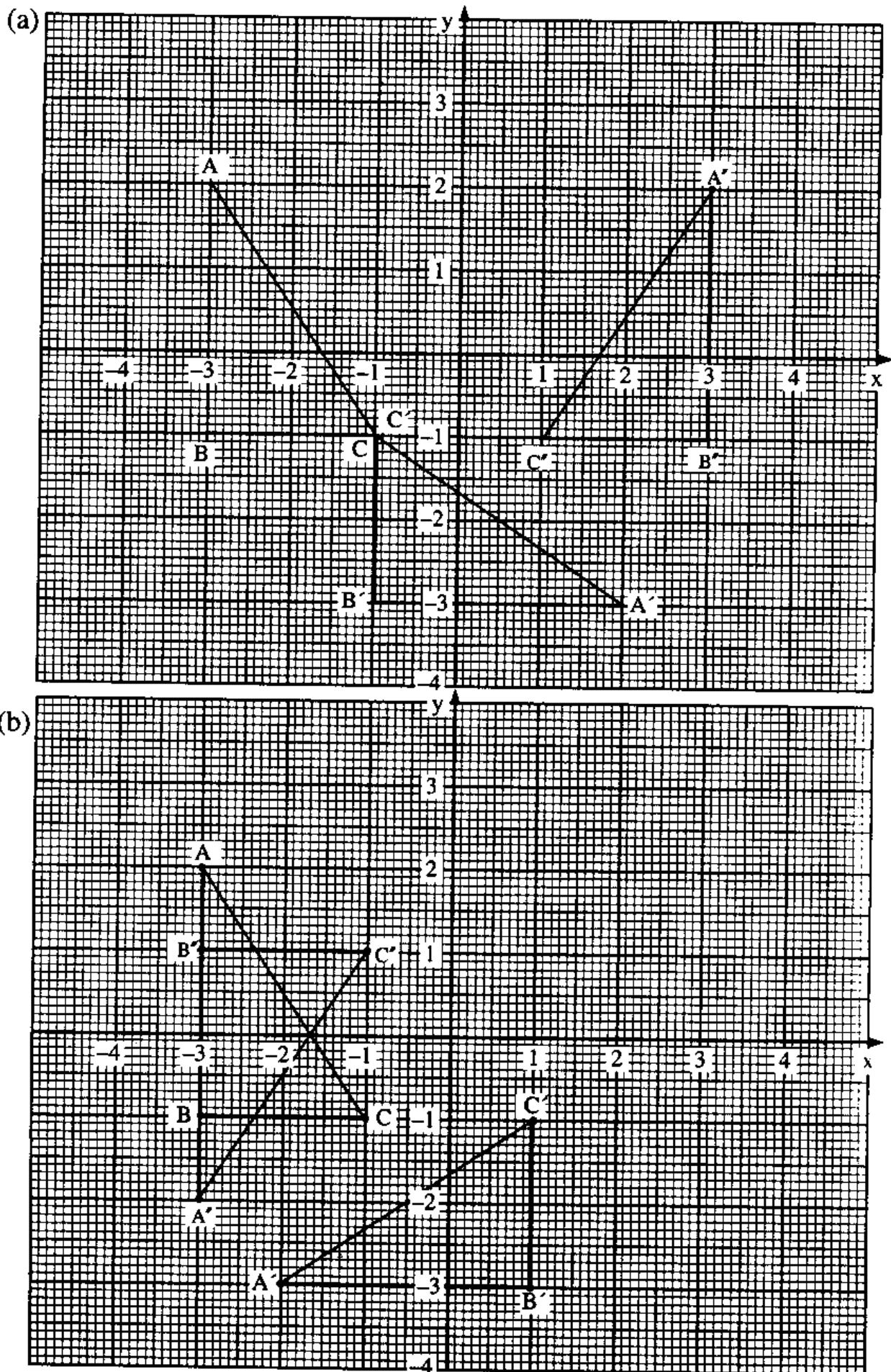


Fig. 1.14

This example illustrates that if \mathbf{P} and \mathbf{Q} are transformations, then in general, $\mathbf{PQ} \neq \mathbf{QP}$.

1.4: Single Matrix of Transformation for Successive Transformations

Example 11

Triangle ABC has vertices at A(1, 4), B(1, 1) and C(3, 1).

- Show on a diagram the image $A''B''C''$ of triangle ABC under the transformation \mathbf{VR} , where \mathbf{V} is a reflection on a line $y = x$ and \mathbf{R} is a negative quarter-turn about the origin.
- Describe a single transformation that would map triangle ABC onto triangle $A''B''C''$ and find its matrix.
- Find the relationship between the matrices of \mathbf{V} , \mathbf{R} and that of the transformation in (b).

Solution

- Figure 1.15 shows ΔABC and its image $\Delta A''B''C''$ under \mathbf{VR} .

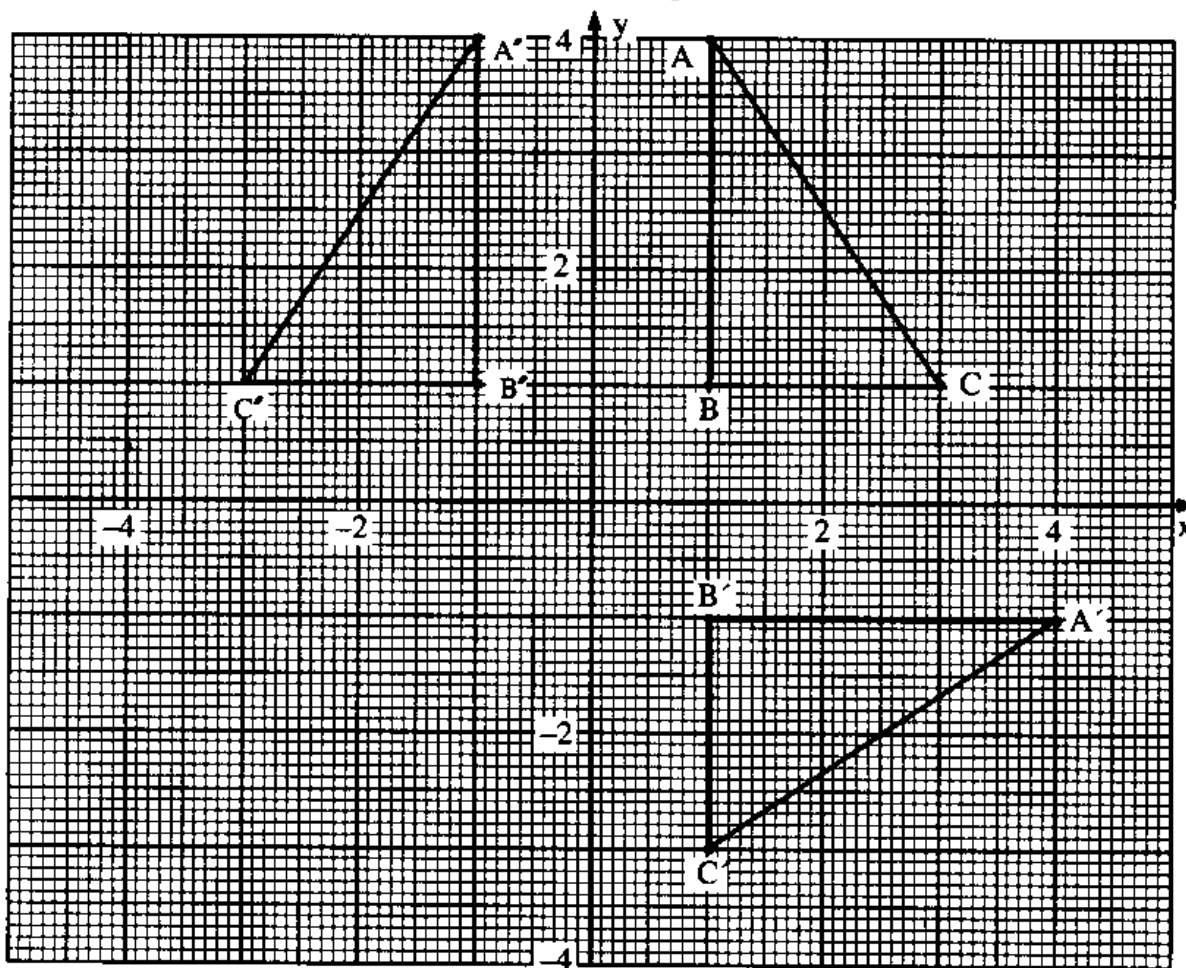


Fig. 1.15

- Reflection in the line $x = 0$ maps ΔABC onto $\Delta A'B'C'$.

Let the matrix of reflection be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & 1 & 3 \\ 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ -1 & -1 & -3 \\ 4 & 1 & 1 \end{pmatrix}$, from which;

$a = -1$, $b = 0$, $c = 0$ and $d = 1$

The matrix of reflection is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

- (c) The matrix for **V** is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that for **R** is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The product of the two matrices in the order in which the transformation is written is;

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix of the single transformation which is equivalent to two transformations combined is the same as the product of the matrices of the individual transformations (written in the order in which the transformations are performed).

Example 12

A trapezium has vertices $A(4, 1)$, $B(6, 1)$, $C(8, 3)$ and $D(3, 3)$. The image of $ABCD$ under a transformation **M** is $A'(2, 1)$, $B'(4, 1)$, $C'(2, 3)$ and $D'(-3, -3)$.

- (a) Find the matrix **M**.
 (b) $A^*B^*C^*D^*$ is the image of $A'B'C'D'$ under a reflection on the line $y = x$. Find a single matrix that maps $ABCD$ onto $A^*B^*C^*D^*$.

Solution

- (a) Let matrix **M** be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Therefore; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B & C & D \\ 4 & 6 & 8 & 3 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ 2 & 4 & 2 & -3 \\ 1 & 1 & 3 & -3 \end{pmatrix}$, from which;

$a = -\frac{1}{2}$, $b = 4$, $c = 0$ and $d = 1$.

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

- (b) Using a unit square, the matrix of reflection on the line $y = x$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
 Therefore, a single matrix of transformation that maps $ABCD$ onto $A^*B^*C^*D^*$ is;

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

Exercise 1.4

In this exercise, letters will be used to represent transformations as shown below.

<i>Letter</i>	<i>Matrix</i>	<i>Transformation</i>
I	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Identity
X	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Reflection on the x-axis
Y	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Reflection on the y-axis
U	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Reflection on the line $y = x$
V	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	Reflection on the line $y + x = 0$
Q	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	Positive quarter turn about the origin
H	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	Half turn about the origin
R	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Positive three quarter turn about the origin
E	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	An enlargement, centre origin, scale factor k .

Take k in the transformation matrix wherever applicable to be equal to 2.

- Triangle ABC has vertices at A(1, 2), B(3, 0) and C(3, 2). Show on a diagram the triangle ABC and its final image in each case under the combined transformation:

(a) UV	(b) HV	(c) XQ	(d) QR
(e) HY	(f) YU	(g) HR	(h) EI

(Take k in the transformation matrix wherever applicable to be equal to 2.)
- A quadrilateral ABCD has vertices at A(1, 1), B(4, 2), C(1, 3) and D(2, 2). Show on a diagram the image of the quadrilateral under the combined transformation:

(a) IXY	(b) VQH	(c) XYU	(d) QHR
(e) YUV	(f) HRE	(g) UVQ	(h) EVX
- A triangle LMN has vertices at L(3, 1), M(5, 2) and N(2, 3). Show on a diagram the triangle LMN and its image under the composite transformations:

(a) RH	(b) EQ	(c) RV	(d) HU
(e) RX	(f) EX	(g) Q ² V	(h) R ³ E
- A trapezium has vertices at A(2, 4), B(1, 1), C(6, 1) and D(5, 4). Show on a diagram the trapezium ABCD and its image under the composite transformation Q²P if $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Describe fully a single transformation that would map the object onto the final image.

5. Copy and complete the following table of combination of transformations. Note that the first matrix means written down first, not performed first, e.g., $UQ = X$

		<i>First transformation</i>							
		I	X	Y	V	U	Q	H	R
Second transformation	I								
	X								
	Y								U
	U								
	V								
	Q					X			
	H								
	R								

1.5: Inverse of a Transformation

Suppose that A is a transformation which maps an object T onto an image T' . Then, a transformation which can map T' back to T is called the inverse of the transformation A , written as A^{-1} . For example, if Q is a positive quarter turn about the origin, then Q^{-1} is a negative quarter turn about the origin. The matrix for Q is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the matrix for Q^{-1} is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Note that $Q^{-1}Q = QQ^{-1} = I$.

Example 13

T is a triangle with vertices $A(2, 4)$, $B(1, 2)$ and $C(4, 2)$. S is a transformation represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$.

- Draw T and its image T' under the transformation S .
- Find the matrix of the inverse of the transformation S .

Solution

- Using transformation matrix $S = \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$;

$$\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} A & B & C \\ 2 & 1 & 4 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ 6 & 3 & 6 \\ 2 & 1 & 1 \end{pmatrix}$$

Figure 1.16 shows T and T'.

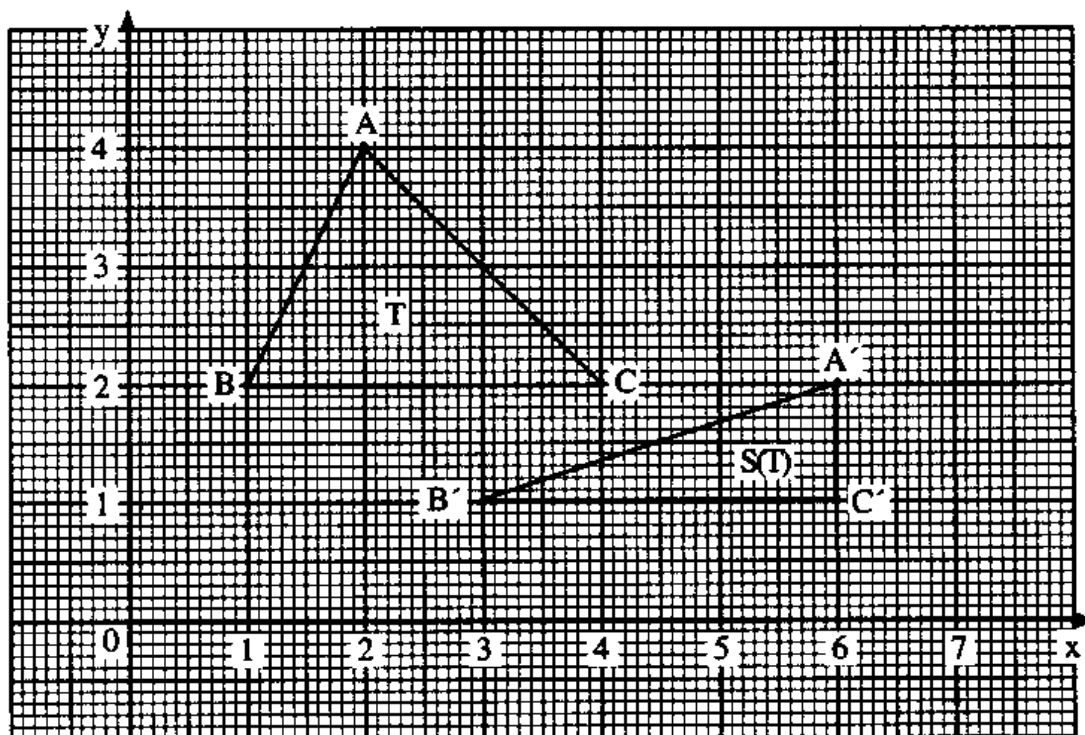


Fig. 1.16

(b) Let the inverse of the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This can be arrived at in three ways;

(i) $S^{-1}S = I$

$$\text{Therefore, } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating corresponding elements and solving simultaneously;

$$a = 1, b = -2, c = 0 \text{ and } d = 2$$

$$\text{Therefore, } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

(ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A' & B' & C' \\ 6 & 3 & 6 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & B & C \\ 2 & 1 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

$$a = 1, b = -2, c = 0 \text{ and } d = 2$$

$$S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

(iii) S^{-1} is the inverse of S.

$$\text{Therefore, } S^{-1} = \text{inverse of } \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned}
 &= \frac{1}{\frac{1}{2}} \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix} \\
 &= 2 \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}
 \end{aligned}$$

State the inverse of each of the following:

- (i) Reflection in a given mirror line M.
- (ii) A rotation of $+60^\circ$ about $(2, 1)$.
- (iii) A translation described by $T = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.
- (iv) An enlargement with centre $(1, 1)$ and scale factor 10.

1.6: Area Scale Factor and Determinant of a Matrix

Under any transformation, the ratio of area of image to area object is known as the area scale factor (A.S.F.).

$$\text{Area scale factor} = \frac{\text{area of image}}{\text{area of object}}$$

A triangle ABC has vertices $A(3, 5)$, $B(2, 1)$ and $C(4, 1)$. The table below is about the image of triangle ABC under transformations whose matrices are given. Copy and complete the table.

<i>Matrix</i>	$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$
<i>Area of object</i>	4					
<i>Area of image</i>	36					
<i>Area scale factor (A.S.F.)</i>	9					4
<i>Determinant of matrix</i>		1				

In each case, compare area scale factor and the determinant of the matrix.

Note:

Area scale factor is numerically equal to the determinant. In cases where the determinant is negative, the area scale factor is equal to the determinant, with the sign ignored.

Exercise 1.5

- Describe the inverse of each of the following operations:
 - 'Add 5'
 - 'Subtract 4'
 - 'Multiply by 10'
 - 'Divide by 3'
 - 'Multiply by 10, then subtract 4'
 - 'Put on socks'
 - 'Put on socks, then put on shoes'
 - 'Go into a room, then close the door'
- State the inverse of each of the following transformations:
 - Positive rotation about $(3, 5)$ through an angle of 80° .
 - A reflection on the line $y + x = 0$.
 - A translation with displacement vector $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$
 - An enlargement about $(10, 10)$ and scale factor of 0.5.
- If the area of an object is 10 square units, state the area of the image after a transformation whose matrix is:
 - $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$
 - $\begin{pmatrix} -2 & 3 \\ 5 & 1 \end{pmatrix}$
 - $\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}$
 - $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$
- A translation \mathbf{X} maps the point $P(7, 2, 0)$ onto $P'(11, 2, 14)$. Find the matrix of the inverse of the translation.
- Draw the image of a trapezium PQRS with vertices at $P(-8, 4)$, $Q(-6, 2)$, $R(-2, 2)$ and $S(0, 4)$ under a transformation whose matrix is $\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$. Find the area of the image.
- Under a transformation, a quadrilateral ABCD with vertices at $A(-5, 2)$, $B(-2, 4)$, $C(-6, 7)$ and $D(-8, 5)$ is mapped onto a quadrilateral $A'B'C'D'$ with vertices at $A'(-12, 6)$, $B'(-8, 12)$, $C'(-19, 21)$ and $D'(-21, 15)$. Find:
 - the matrix of transformation.
 - the area of the image.
- Draw the image of the triangle RST whose vertices are $R(4, 3)$, $S(2, 1)$ and $T(4, 1)$ under the transformation whose matrix is $\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$. Find the area of the image and the inverse matrix of the transformation.
- The vertices of a triangle XYZ are $X(1, 2)$, $Y(3, 2)$ and $Z(2, 5)$. \mathbf{Q} is a transformation represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If \mathbf{Q}^2 maps triangle XYZ onto $X'Y'Z'$, find the inverse of \mathbf{Q}^2 and the area of triangle $X'Y'Z'$.

9. A parallelogram ABCD has vertices at A(-4, -1), B(-5, -3), C(-2, -3) and D(-1, -1). Under a transformation, the parallelogram is mapped onto another with vertices at A'(4, -2), B'(5, -6), C'(2, -6) and D'(1, -2). Find the matrix of the inverse transformation.
10. A rectangle has vertices at A(1, 1), B(3, 1), C(3, 3) and D(1, 3). Find the area of the image of the rectangle under the transformation represented by:
- (a) $\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -5 & 1 \\ -7 & 7 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}$
11. **T** represents a translation defined by $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, **E** an enlargement with centre (2, 3) and scale factor 3, and **Q** a positive quarter turn about the point (5, 1). Use the symbols T^{-1} , E^{-1} and Q^{-1} to describe the inverse of the following combined transformations:
- (a) **QT** (b) **TQ** (c) **QE** (d) **TEQ** (e) **QTE**
12. A kite has vertices at P(4, 1), Q(6, 5), R(4, 7) and S(x, y). Find the co-ordinates of S. Find also the area of the image under the transformations whose matrices are:
- (a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 1 \\ 4 & 5 \end{pmatrix}$

1.7: Shear and Stretch

Shear

Figure 1.17 shows a rectangle ABCD and its image A'B'C'D' under a transformation. Points Q and M are on line AD and BC respectively, and their images are also shown.

The transformation which maps ABCD onto A'B'C'D' is called a **shear**.

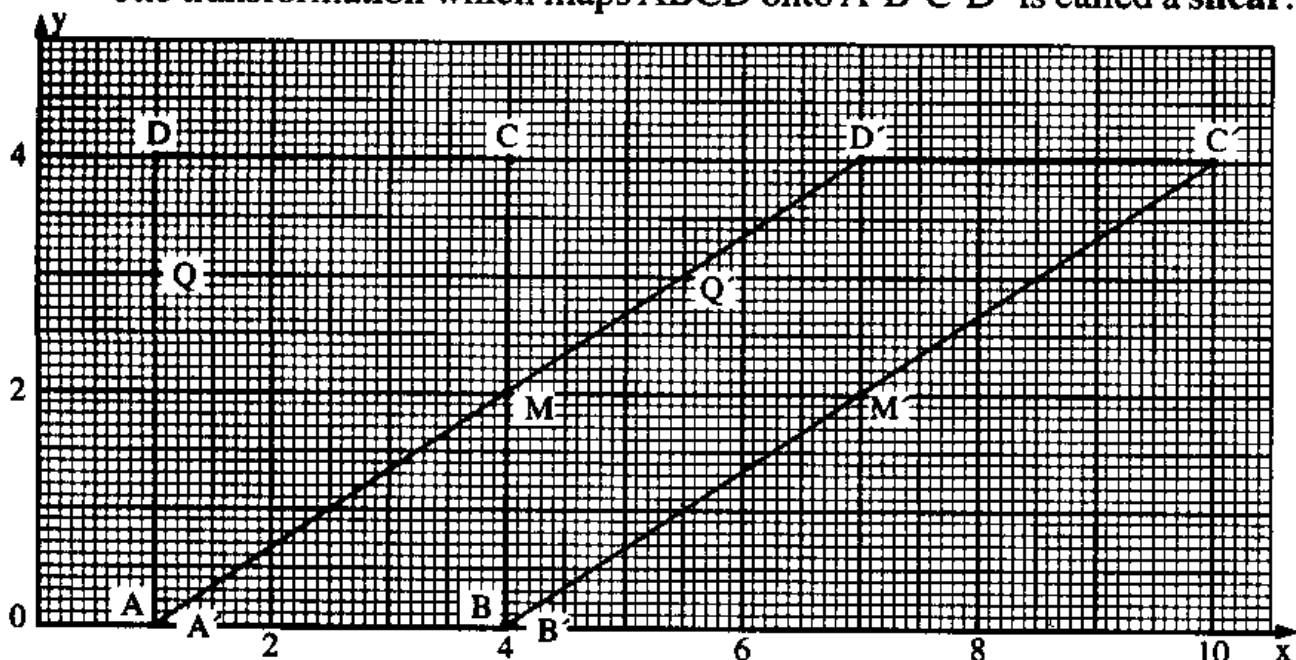


Fig. 1.17

The parallelograms $ABCD$ and $A'B'C'D'$ have same base and equal heights. Therefore, their areas are equal. Under any shear, area is always invariant. Each line not parallel to the invariant line intersects with its image in the invariant line. For example, AD and $A'D'$ intersect on the invariant line at A' .

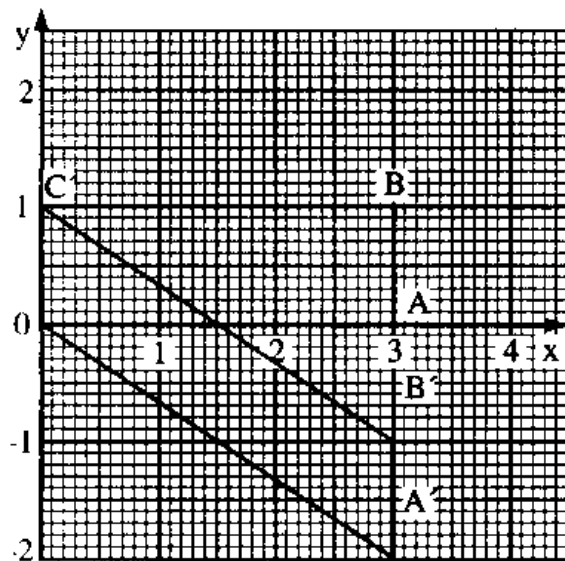
A shear is fully described by giving:

- (i) the invariant line.
- (ii) a point not on the invariant line, and its image.

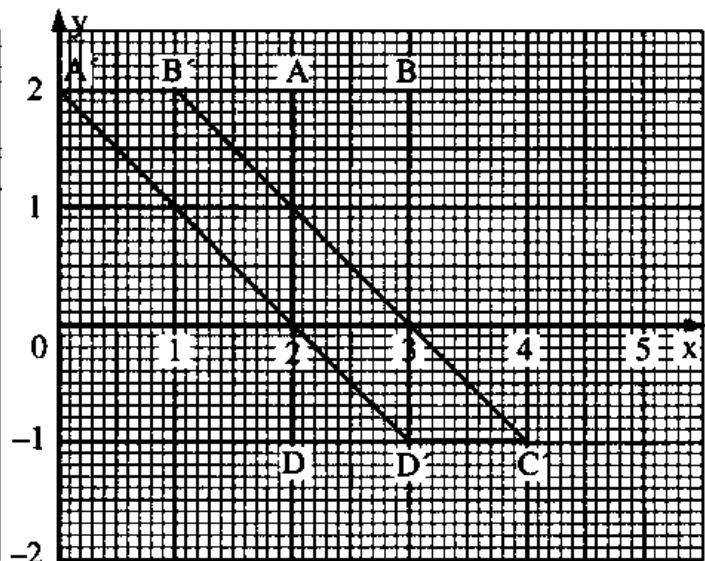
Example 14

Figure 1.18 shows objects and their images. Describe the transformation in each case:

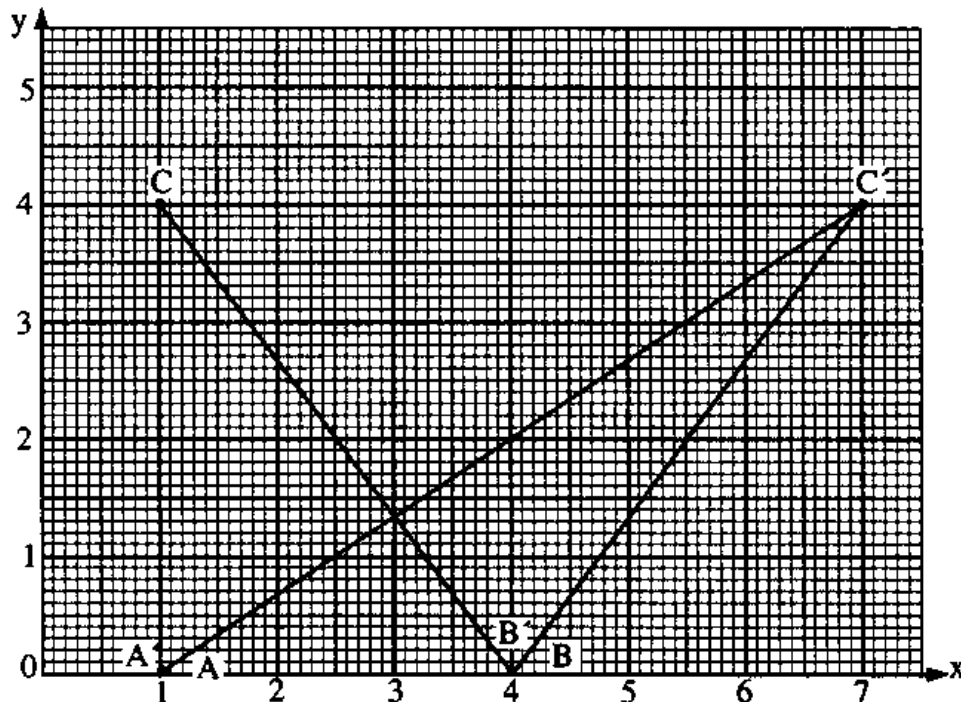
(a)



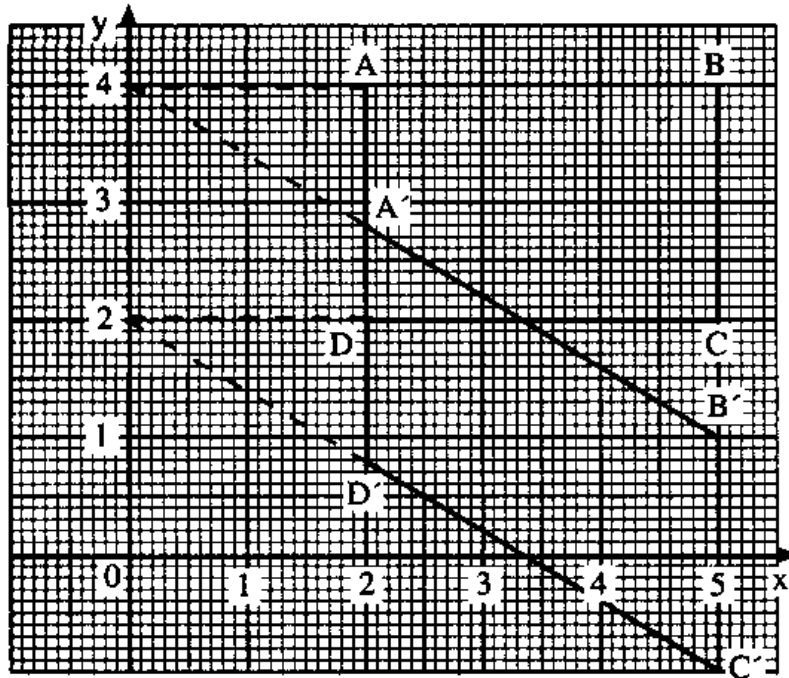
(b)



(c)



(d)

*Fig. 1.18***Solution**

The transformation in:

- (a) is a shear, y-axis invariant and $A(3, 0)$ is mapped onto $A'(3, -2)$.
- (b) is a shear, x-axis invariant and $B(3, 2)$ is mapped onto $B'(1, 2)$.
- (c) is a shear, x-axis invariant and $C(1, 4)$ is mapped onto $C'(7, 4)$.
- (d) is a shear, y-axis invariant and $B(5, 4)$ is mapped onto $B'(5, 1)$.

Example 15

Triangle AOB has vertices at $O(0, 0)$, $A(3, 0)$ and $B(3, 4)$. Show on a diagram its image under a shear whose matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Solution

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O & A & B \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} O' & A' & B' \\ 0 & 3 & 11 \\ 0 & 0 & 4 \end{pmatrix}$$

The object and its image are shown in figure 1.19.

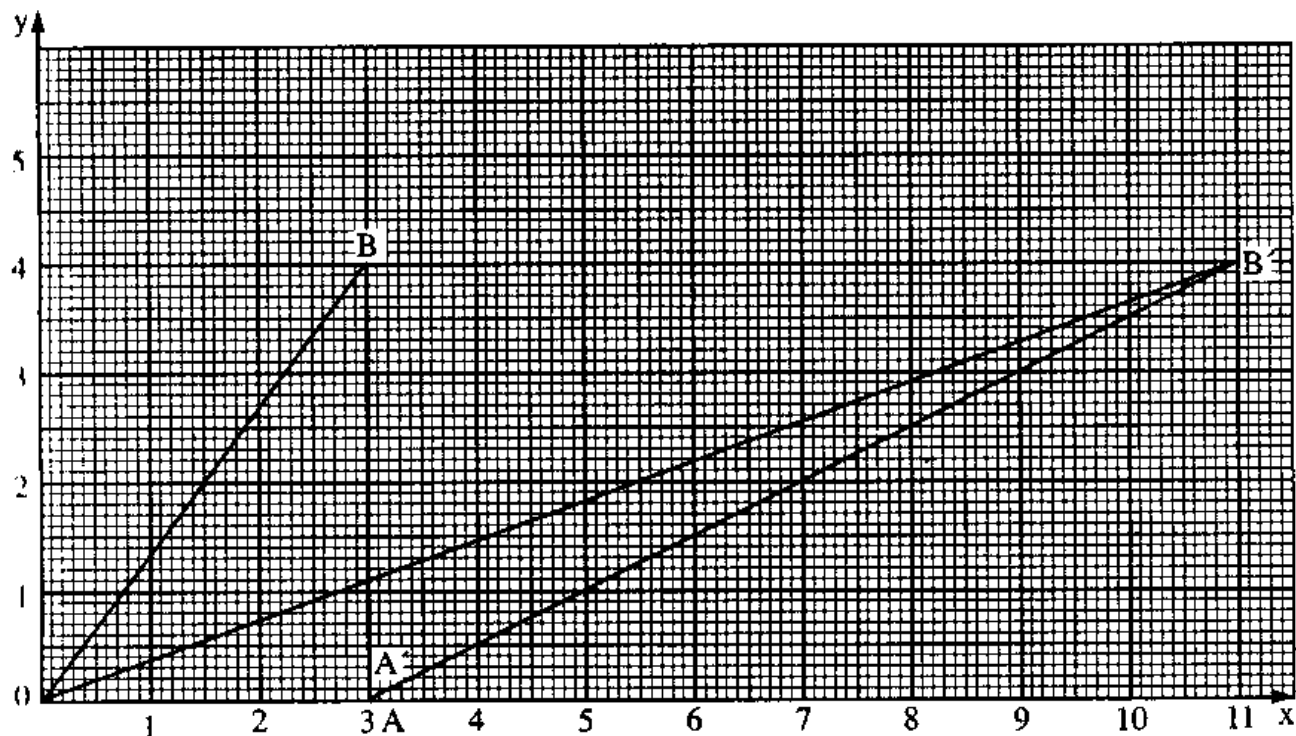


Fig. 1.19

Note that if a matrix is used to describe a shear, then the invariant line of the shear must pass through the origin.

In general, a shear with x-axis invariant is represented by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Under this transformation, $J(0, 1)$ is mapped onto $J'(k, 1)$.

Likewise, a shear with y-axis invariant is represented by a matrix of the form $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$. Under this transformation, $I(1, 0)$ is mapped onto $I'(1, k)$.

Example 16

Triangle ABC has vertices $A(-1, 2)$, $B(-1, -1)$ and $C(1, -1)$. Find the matrix of the transformation which maps ΔABC onto $\Delta A'B'C'$ whose vertices are $A'(-3, 2)$, $B'(0, -1)$ and $C'(2, -1)$. Describe the transformation fully.

Solution

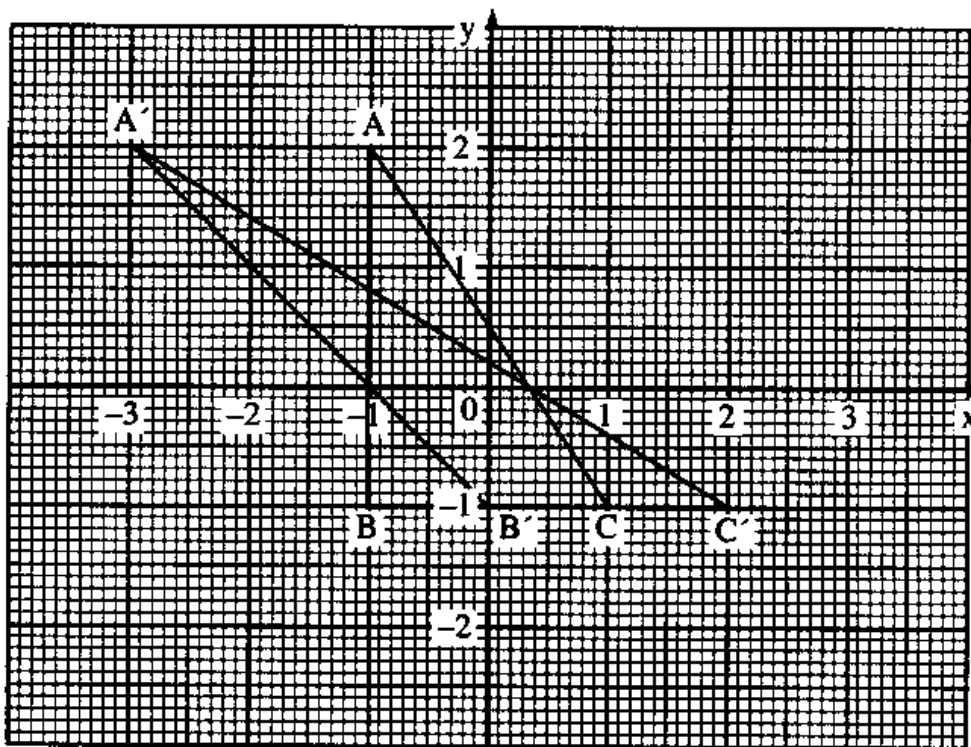


Fig. 1.20

Let the matrix of the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\text{Then; } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B & C \\ -1 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ -3 & 0 & 2 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\text{This gives; } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The transformation is a shear with the x-axis invariant and $A(-1, 2) \rightarrow A'(-3, 2)$.

Stretch

Figure 1.21 shows a rectangle ABCD and its image under a transformation:

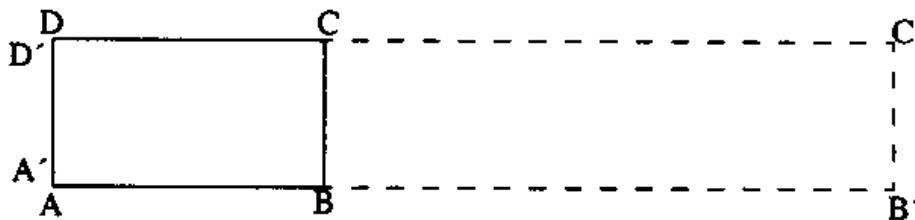


Fig. 1.21

Note:

- (i) The line AD is invariant.
- (ii) All other points have moved in a direction perpendicular to AD. We call such a transformation a one-way **stretch**.

The ratio $\frac{D'C'}{DC} = \frac{A'B'}{AB}$ is called the scale factor of the stretch.

Figure 1.22 shows a rectangle PQRS and its image under a stretch with PQ invariant and scale factor 3.

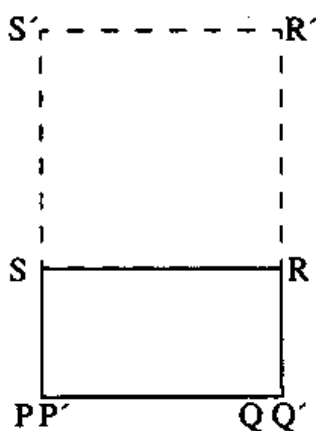


Fig. 1.22

A one-way stretch is completely defined by giving:

- (i) the invariant line.
- (ii) the scale factor.

Example 17

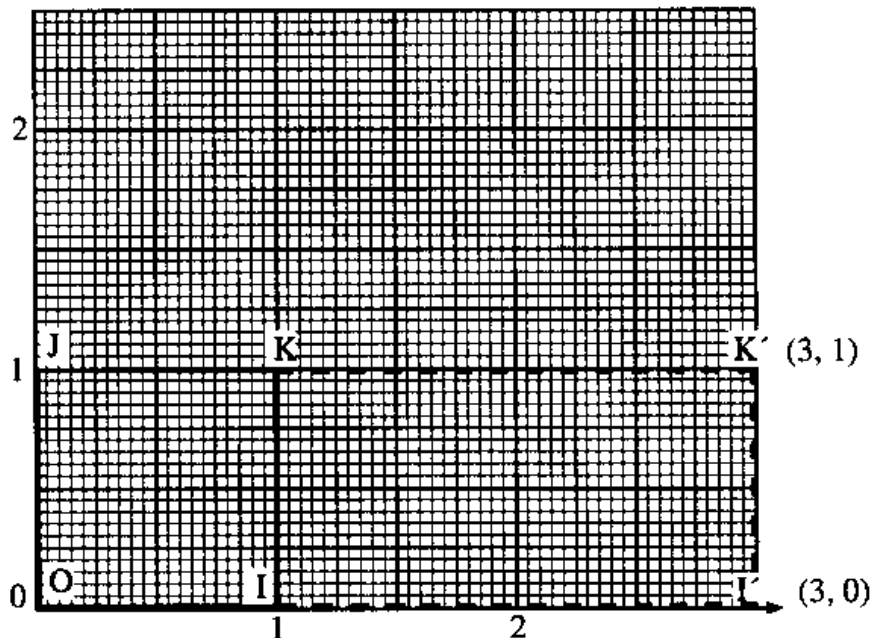
Using the unit square, find the matrix of a stretch with:

- (a) y-axis invariant and scale factor 3.
- (b) x-axis invariant and scale factor 5.

Solution

(See figure 1.23 (a) and (b).)

(a)



(b)

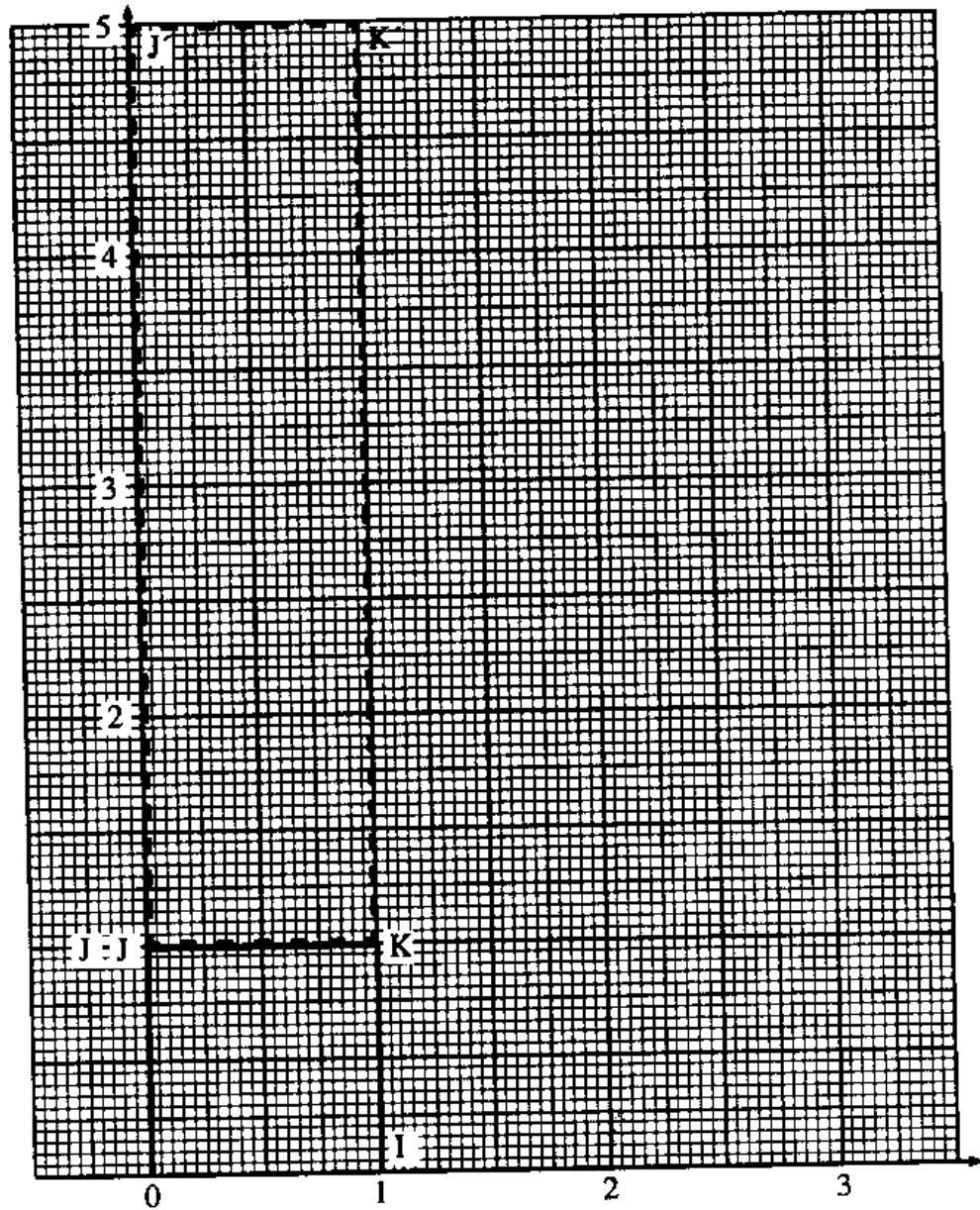


Fig. 1.23

- (a) The image of I is $I'(3, 0)$ and the image of J is $(0, 1)$, as shown in figure 1.23 (a). Therefore, the matrix of the stretch is $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.
- (b) The image of I is $I'(1, 0)$ and the image of J is $J'(0, 5)$, as shown in figure 1.23 (b). Therefore, the matrix of stretch is $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$.

In general, the matrix of a stretch with the y-axis invariant and a scale factor k is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, the matrix of a stretch with x-axis invariant and scale factor k is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

Find the matrix of a stretch with the x-axis invariant and scale factor:

- (i) 4
 (ii) -2

1.8: Isometric and Non-isometric Transformations

Isometric transformations are those in which the object and image have the same shape and size (congruent), e.g., rotation, reflection and translation.

Non-isometric transformations are those in which the object and the image are not congruent, e.g., shear, stretch and enlargement.

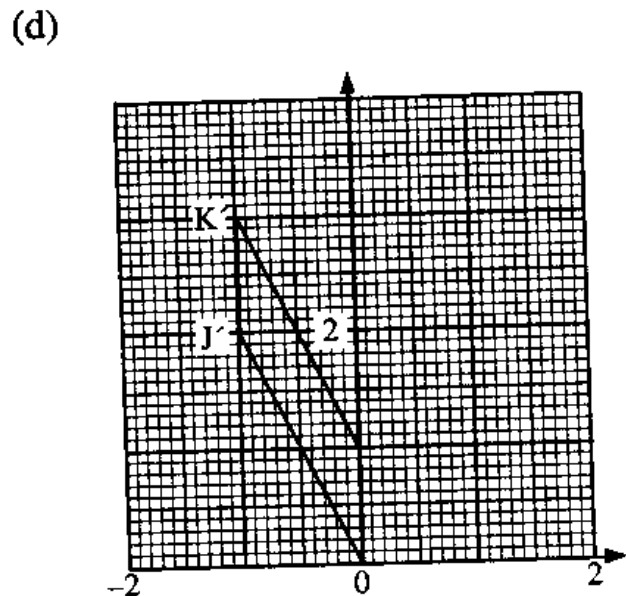
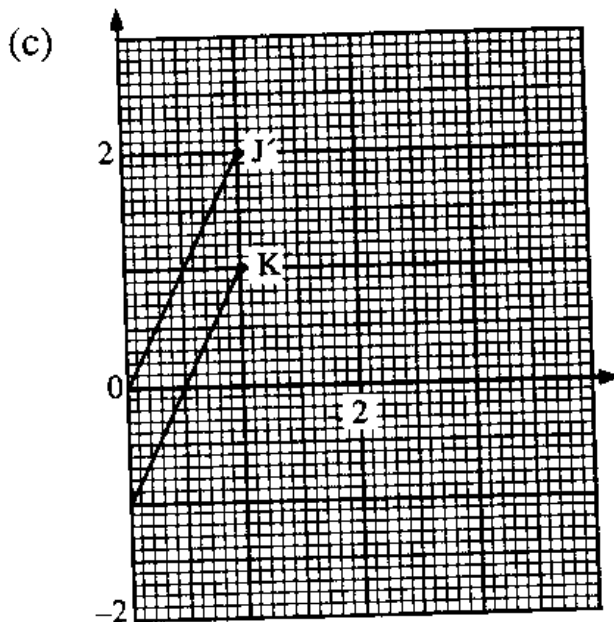
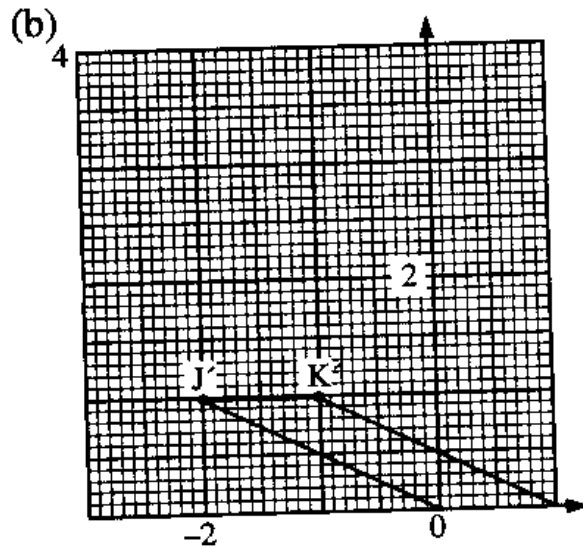
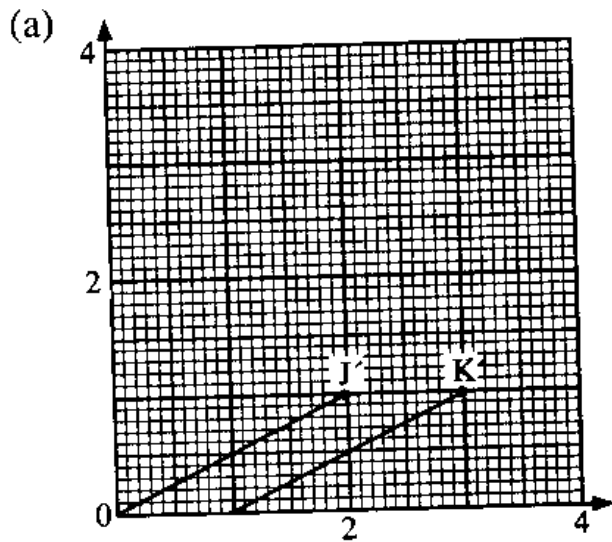
Identify the transformation represented by each of the following matrices:

- (i) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ (v) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

For each, state whether it is isometric or non-isometric.

Exercise 1.6

1. The following figures show the image of the unit square under various transformations. Identify each transformation and find its matrix.



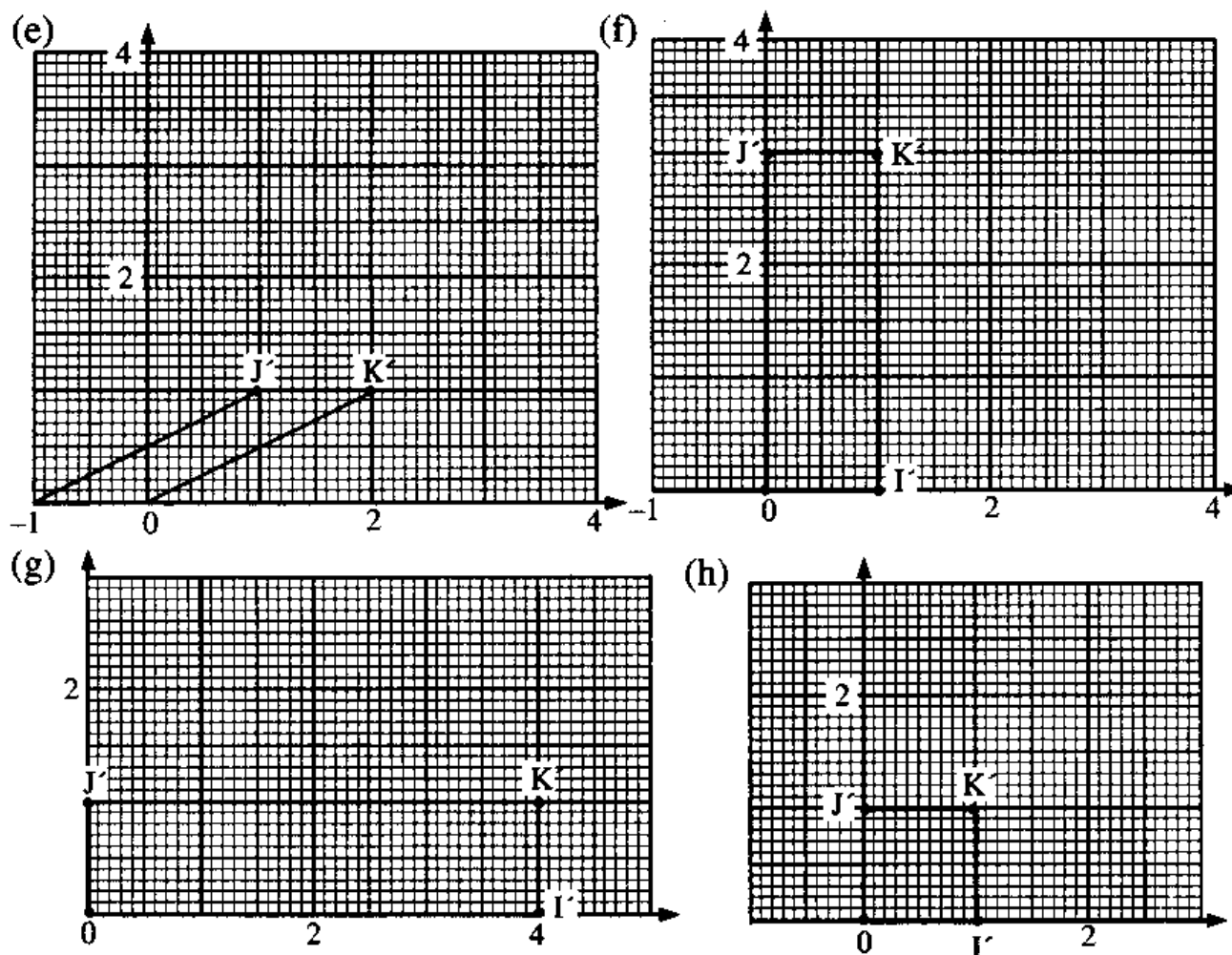


Fig. 1.24

2. Copy figure 1.25 and draw its image under a shear with:

- x-axis invariant and $B \rightarrow B'(4, 2)$
- y-axis invariant and $B \rightarrow B'(2, 3)$
- The line $y = x$ invariant and $C \rightarrow C'(0, 2)$

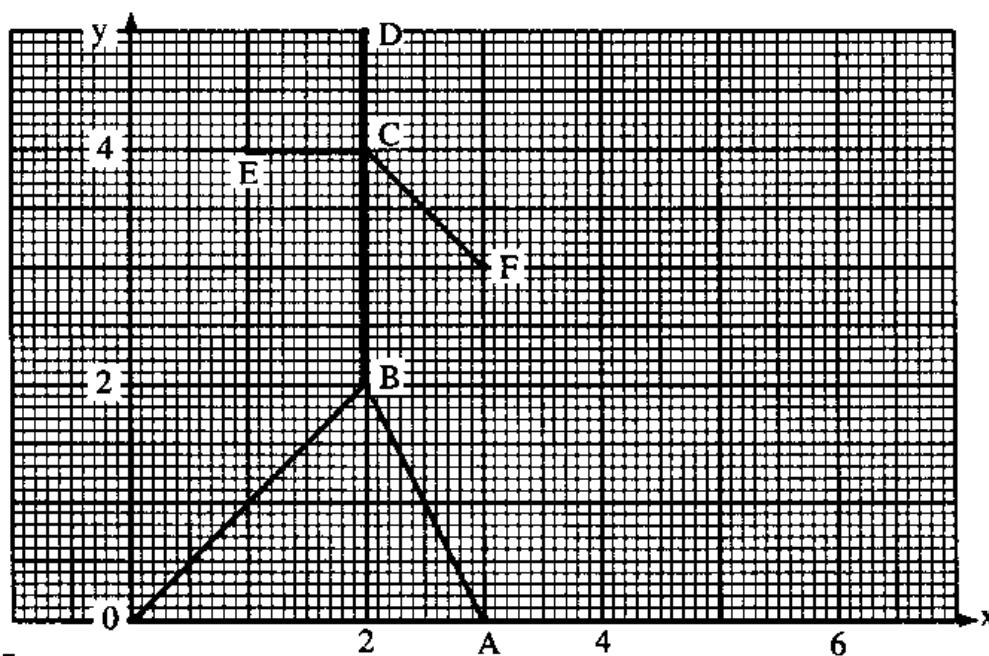


Fig 1.25

3. For each of the matrices below, find out whether it is isometric or non-isometric:

(a) $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

4. An enlargement matrix of $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is not an isometry, but there are two values of k for which the matrix is isometrical. What are these values?

5. Copy the figure below and draw its image under a shear with:

- (a) x -axis invariant and $E \rightarrow E'(3, 3)$;
 (b) y -axis invariant and $E \rightarrow E'(5, 1)$;
 (c) $y = 1$ invariant and $A \rightarrow A'(0, -1)$;
 (d) $x = 3$ invariant and $F \rightarrow F'(0, 1)$.

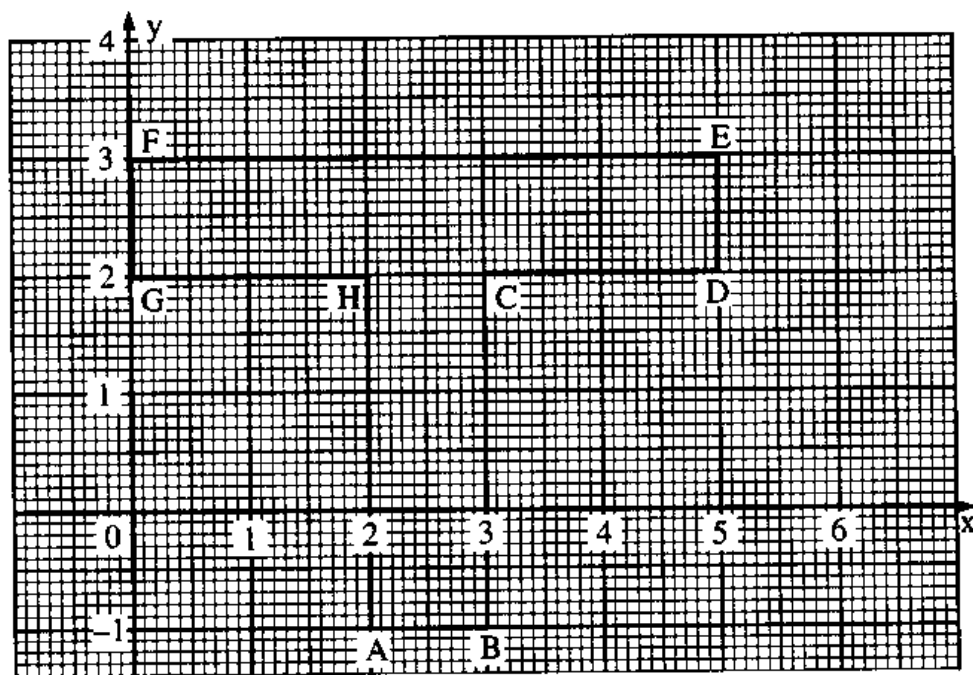


Fig 1.26

6. A rectangle OABC has vertices at $O(0, 0)$, $A(0, -1)$, $B(4, -1)$ and $C(4, 0)$. Find the co-ordinates of the vertices of the image under a stretch scale factor 4 (y -axis invariant).
7. A triangle OPQ with vertices at $O(0, 0)$, $P(3, 0)$ and $Q(3, 4)$ is mapped onto $\Delta O'P'Q'$ with vertices at $O'(0, 0)$, $P'(3, 0)$ and $Q'(3, 8)$. Describe the transformation and find the matrix of the transformation.
8. A triangle ABC with vertices $A(-5, 2)$, $B(-3, 2)$ and $C(-3, 5)$ is mapped onto $\Delta A'B'C'$ with vertices at $A'(-5, 2)$, $B'(-3, 2)$ and $C'(0, 5)$. Describe the transformation.

Chapter Two

STATISTICS II

2.1: Measures of Central Tendency

In Book Two, three statistical measures were discussed. These are mean, median and mode. These statistical measures are called measures of central tendency.

2.2: Mean using Working (Assumed) Mean

Consider the following sets of data:

A: 35, 43, 45, 48, 48, 49, 52, 54, 62, 64

$$\text{Mean} = \frac{500}{10} = 50$$

B: 47, 55, 57, 60, 60, 61, 64, 66, 74, 76

$$\text{Mean} = \frac{620}{10} = 62$$

C: 27, 35, 37, 40, 40, 41, 44, 46, 54, 56

$$\text{Mean} = \frac{420}{10} = 42$$

Data B is obtained from Data A by adding 12 to each of the numbers. Similarly, Data C is obtained from Data A by subtracting 8 from each of the numbers.

Can you see the connection between the means?

If 12 is added to the mean of Data A, we obtain the mean of Data B. Similarly, if we subtract 8 from the mean of Data A, we obtain the mean of Data C.

In general, if a constant k is added to or subtracted from each figure in a set of data, the mean of the new data can be obtained from the mean of the old data by adding or subtracting the same constant.

Multiply each of the numbers in Data A by 4 and find the mean of the new data. What connection do you notice between the means? Repeat the same procedure but this time divide each number by 10.

In general, if each number in a given data is multiplied or divided by a constant, then the mean of the new set of data can be obtained by multiplying or dividing the mean of the first set of data by the same constant.

Example 1

The data below represents the quantities to the nearest millilitre of 18 packets of milk:

495, 496, 497, 498, 499, 499, 500, 500, 501, 501, 502, 503, 503, 504, 504, 505, 506, 507.

Calculate the mean.

Solution

We can start by subtracting a constant from each of the numbers. Let the constant be 500.

x	$t = x - 500$
495	-5
496	-4
497	-3
498	-2
499	-1
499	-1
500	0
500	0
501	1
501	1
502	2
503	3
503	3
504	4
504	4
505	5
506	6
507	7
	$\Sigma t = 20$

$$\begin{aligned} \text{Mean of } t &= \frac{\Sigma t}{18} \\ &= \frac{20}{18} \\ &= 1.11 \end{aligned}$$

$$\begin{aligned} \text{Mean of } x &= 500 + \text{mean of } t \\ &= 500 + 1.11 \\ &= 501.11 \end{aligned}$$

The number 500 is referred to as **working (assumed) mean**. The assumed mean is used to make calculations easier.

Example 2

The masses to the nearest kilogram of 40 students in a form 4 class were measured and recorded in the table below. Calculate the mean mass.

Mass (kg)	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Frequency	2	0	1	2	3	2	5	6	7	5	3	2	1	1

Solution

Let us use a working mean of 53.

Mass x (kg)	$t = x - 53$	f	ft
47	-6	2	-12
48	-5	0	0
49	-4	1	-4
50	-3	2	-6
51	-2	3	-6
52	-1	2	-2
53	0	5	0
54	1	6	6
55	2	7	14
56	3	5	15
57	4	3	12
58	5	2	10
59	6	1	6
60	7	1	7
		$\Sigma f = 40$	$\Sigma ft = 40$

$$\begin{aligned} \text{Mean of } t &= \frac{\Sigma ft}{\Sigma f} \\ &= \frac{40}{40} = 1 \end{aligned}$$

$$\begin{aligned} \text{Mean of } x (\bar{x}) &= 53 + \text{mean of } t \\ &= 53 + 1 \\ &= 54 \end{aligned}$$

Example 3

The masses to the nearest gram of 100 eggs were as follows:

Mass (g)	100-103	104-107	108-111	112-115	116-119	120-123
Frequency	1	15	42	31	8	3

Find the mean mass.

Solution

Let us use a working mean of 109.5.

<i>Class</i>	<i>midpoint x</i>	$t = x - 109.5$	<i>f</i>	<i>ft</i>
100-103	101.5	-8	1	-8
104-107	105.5	-4	15	-60
108-111	109.5	0	42	0
112-115	113.5	4	31	124
116-119	117.5	8	8	64
120-123	121.5	12	3	36
			$\Sigma f = 100$	$\Sigma ft = 156$

Mean of $t = \bar{t}$

$$= \frac{156}{100}$$

$$= 1.56$$

Therefore, mean of x (\bar{x}) = 109.5 + mean of t

$$= 109.5 + \bar{t}$$

$$= 109.5 + 1.56$$

$$= 111.06 \text{ g}$$

With grouped data, we can obtain easier values to work with by dividing each figure by the class width (after subtracting the assumed mean). In order to obtain the mean of the original data from the mean of the new set of data, we will have to reverse the steps. The correct order of reversing the steps is to multiply by the class width and then add the working mean. Let us repeat example 3 using this approach.

<i>Class</i>	<i>midpoint x</i>	$t = \frac{x - 109.5}{4}$	<i>f</i>	<i>ft</i>
100-103	101.5	-2	1	-2
104-107	105.5	-1	15	-15
108-111	109.5	0	42	0
112-115	113.5	1	31	31
116-119	117.5	2	8	16
120-123	121.5	3	3	9
			$\Sigma f = 100$	$\Sigma ft = 39$

$$\bar{t} = \frac{\Sigma ft}{\Sigma f}$$

$$= \frac{39}{100}$$

$$= 0.39$$

Therefore, $\bar{x} = 0.39 \times 4 + 109.5$

$$= 1.56 + 109.5$$

$$= 111.06 \text{ g}$$

If we had chosen to divide by the class width first and then subtract a working mean, the correct procedure to follow in reversing these steps is to first add the working mean and then multiply by the class width. Using this approach, the solution to example 3 can be set out as follows:

<i>Class</i>	<i>midpoint x</i>	$\frac{x}{4}$	$t = \frac{x}{4} - 27.375$	<i>f</i>	<i>ft</i>
100-103	101.5	25.375	-2	1	-2
104-107	105.5	26.375	-1	15	-15
108-111	109.5	27.375	0	42	0
112-115	113.5	28.375	1	31	31
116-119	117.5	29.375	2	8	16
120-123	121.5	30.375	3	3	9
				$\Sigma f = 100$	$\Sigma ft = 39$

$$\begin{aligned}\bar{t} &= \frac{39}{100} \\ &= 0.39\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \bar{x} &= (0.39 + 27.375)4 \\ &= 27.765 \times 4 \\ &= 111.06 \text{ g}\end{aligned}$$

Example 4

The table below shows the distribution of marks scored by 40 form 4 students in a Mathematics test:

<i>Marks</i>	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
<i>Frequency</i>	3	4	2	5	8	11	2	1	3	1

Calculate the mean mark.

Solution

Method 1

We subtract an assumed mean of 45.5 and then divide by the class width.

<i>Class</i>	<i>midpoint x</i>	$x - 45.5$	$t = \frac{x - 45.5}{10}$	<i>f</i>	<i>ft</i>
1-10	5.5	-40	-4	3	-12
11-20	15.5	-30	-3	4	-12
21-30	25.5	-20	-2	2	-4
31-40	35.5	-10	-1	5	-5
41-50	45.5	0	0	8	0
51-60	55.5	10	1	11	11
61-70	65.5	20	2	2	4
71-80	75.5	30	3	1	3
81-90	85.5	40	4	3	12
91-100	95.5	50	5	1	5
				$\Sigma f = 40$	$\Sigma ft = 2$

$$\begin{aligned}\bar{t} &= \frac{2}{40} \\ &= 0.05\end{aligned}$$

$$\begin{aligned}\bar{x} &= 0.05 \times 10 + 45.5 \\ &= 0.5 + 45.5 \\ &= 46\end{aligned}$$

Method 2

We first divide by the class width and then subtract 4.55.

x	$\frac{x}{10}$	$t = \frac{x}{10} - 4.55$	f	ft
5.5	0.55	-4	3	-12
15.5	1.55	-3	4	-12
25.5	2.55	-2	2	-4
35.5	3.55	-1	5	-5
45.5	4.55	0	8	0
55.5	5.55	1	11	11
65.5	6.55	2	2	4
75.5	7.55	3	1	3
85.5	8.55	4	3	12
95.5	9.55	5	1	5
			$\Sigma f = 40$	$\Sigma ft = 2$

$$\begin{aligned}\bar{t} &= \frac{2}{40} \\ &= 0.05\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \bar{x} &= (0.05 + 4.55)10 \\ &= 4.60 \times 10 \\ &= 46\end{aligned}$$

Exercise 2.1

1. Using 48 as an assumed mean, find the mean of each of the following sets of data:

(a) 60 45 38 55 51 58 59 49 46 39

(b) 30 35 48 39 41 45 42 43 49 47

2. Calculate the mean of each of the following sets of data:

(a) 0.655 0.685 0.705 0.725 0.745 0.765
 0.665 0.695 0.715 0.735 0.755 0.775

(b) 0.021 0.023 0.027 0.30 0.041
 0.022 0.023 0.028 0.032 0.043
 0.024 0.025 0.028 0.033 0.045

(c) 225 300 400 525 600
 225 325 400 525 625
 250 350 475 550 650
 275 375 475 475 675

3. The data below represents the number of tourists visiting a certain hotel in one year:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
No. of tourists	1 500	1 430	790	500	600	750	900	1 700	1 800	2 010	1 602	1 540

Find the mean number of tourists visiting the hotel per month.

4. Below are the masses, to the nearest kilogram, of forty adults. Using classes 45-54, 55-64, etc, group the data and hence find the mean mass:

45 65 66 67 72 74 79 69 57 58
 65 58 65 67 66 70 69 49 50 52
 51 70 79 80 84 49 52 55 63 64
 85 90 87 69 81 82 68 69 58 60

5. The following table gives the breaking stress, in Newtons per mm^2 , of 20 cubes of concrete:

Breaking stress	19.0	20.0	21.0	22.0	23.0	24.0	25.0	26.0
No. of cubes	2	3	3	5	2	1	2	2

Find the mode, mean and median stress.

6. In an athletics competition, each of the 40 entrants for the javelin event were allowed three attempts and the best throw recorded for each. The table below shows the results.

Length (m)	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
Frequency	3	1	4	2	1	3	5	4	2	3	4	0	2	3	1	2

Find the mode, median and mean.

7. The distribution of heights, in metres, of men in a military recruitment exercise was as follows:

<i>Height (m)</i>	1.50-1.54	1.55-1.59	1.60-1.64	1.65-1.69	1.70-1.74	1.75-1.79
<i>Frequency</i>	6	48	63	87	114	83

1.80-1.84	1.85-1.89	1.90-1.94	1.95-1.99	2.00-2.04	2.05-2.09	2.10-2.14
32	25	18	12	3	7	2

Calculate the mean height.

8. The heights of 100 pea seedlings after two weeks' growth were recorded as below:

<i>Height in cm.</i>	0.5-1.0	1.1-1.6	1.7-2.2	2.3-2.8	2.9-3.4	3.5-4.0	4.1-4.6	4.7-5.2
<i>No. of seedlings</i>	2	10	15	19	24	14	11	5

Using an assumed mean, find the mean height of the seedlings.

9. The percentage content of iron in 100 tablets of a certain drug was determined and the results entered in a table as shown below:

<i>Percentage content of iron</i>	0.001-0.002	0.003-0.004	0.005-0.006	0.007-0.008	0.009-0.010
<i>No. of tablets</i>	5	7	42	38	8

Copy and complete the following table. Hence, find the mean percentage content.

<i>Class</i>	<i>Midpoint x</i>	<i>$1000x$</i>	<i>$t = 1000x - 5.5$</i>	<i>f</i>	<i>ft</i>
0.001-0.002	0.0015	1.5	-4	5	-20
0.003-0.004	0.0035	3.5	-2	7	-14
0.005-0.006					
0.007-0.008					
0.009-0.010					
				$\Sigma f =$	$\Sigma ft =$

10. 150 samples of a particular cement mixture were allowed to stand for a certain time and then the moisture content of each (as a percentage) determined. The results were as shown in the table below:

<i>Percentage moisture content</i>	23	24	25	26	27	28	29	30
<i>Number of samples</i>	6	11	16	20	22	27	30	18

Determine the median and the mean.

11. The thickness of 120 slabs are shown in the table below:

<i>Thickness (mm)</i>	5.2	5.4	5.6	5.8	6.0	6.2	6.4	6.6	6.8	7.0
<i>Number of slabs</i>	1	4	4	18	28	34	26	2	2	1

Find the median and the mean thickness.

12. Calculate mean mass of the data given below:

(a) without grouping the data,

(b) by grouping the data in class intervals of 45-49, 50-54,, e.t.c.

54 66 63 47 58 67 59 50 63 45 56

57 64 62 48 55 60 52 68 52 61 46

58 49 55 51 58 53 66 45 52 63 62

55 61 68 52 63 46 65

2.3: Quartiles, Deciles and Percentiles

A median divides a set of data into two parts, in each of which there are an equal number of items.

Quartiles divide a set data into four equal parts. The lower quartile is the median of the bottom half. The upper quartile is the median of the top half and the middle coincides with the median of the whole set of data.

Deciles divide a set of data into ten equal parts. Percentiles divide a set of data into the hundred equal parts. For quartiles, deciles and percentiles, the data is arranged in order of size.

Example 5

The table below shows the distribution of height to the nearest cm of 40 students:

Height (cm)	145-149	150-154	155-159	160-164	165-169	170-174	175-179
Frequency	2	5	16	9	5	2	1

Calculate the: (a) median height.

(b) (i) lower quartile.

(ii) upper quartile.

(c) 80th percentile.

Solution

(a) There are 40 students. Therefore, the median height is the average of the heights of the 20th and 21st students.

Class	Frequency	Cumulative frequency
145-149	2	2
150-154	5	7
155-159	16	23
160-164	9	32
165-169	5	37
170-174	2	39
175-179	1	40

Both the 20th and 21st students fall in the 155-159 class. This class is called the median class.

$$\text{Using the formula } m = L + \frac{\left(\frac{n}{2} - C\right)i}{f}$$

where L is the lower class limit of the median class.

n is the total frequency.

C is the cumulative frequency above the median class.

i is the class interval, and,

f is the frequency of the median class.

Therefore;

$$\begin{aligned} \text{Height of the 20}^{\text{th}} \text{ student} &= 154.5 + \frac{13}{16} \times 5 \\ &= 154.5 + 4.0625 \\ &= 158.5625 \end{aligned}$$

$$\begin{aligned} \text{Height of the 21}^{\text{st}} \text{ student} &= 154.5 \times \frac{14}{16} \times 5 \\ &= 154.5 + 4.375 \\ &= 158.875 \end{aligned}$$

$$\begin{aligned} \therefore \text{Median height} &= \frac{158.5625 + 158.875}{2} \\ &= 158.7 \text{ cm} \end{aligned}$$

$$(b) \quad (i) \quad \text{Lower quartile } (Q_1) = L + \frac{\left(\frac{n}{4} - C\right)i}{f}$$

The 10th student falls in the 155 – 159 class.

$$\begin{aligned} \therefore Q_1 &= 154.5 + \frac{\left(\frac{40}{4} - 7\right)5}{16} \\ &= 154.5 + 0.9375 \\ &= 155.4375 \end{aligned}$$

$$(ii) \quad \text{Upper quartile } (Q_3) = L + \frac{\left(\frac{3n}{4} - C\right)i}{f}$$

$$\begin{aligned} \therefore Q_3 &= 159.5 + \frac{\left(\frac{3}{4} \times 40 - 23\right)5}{9} \\ &= 159.5 + 3.888 \\ &= 163.3889 \end{aligned}$$

Note:

The median corresponds to the middle quartile (Q_2), or the 50th percentile .

$$(c) \quad \frac{80}{100} \times 40 = 32$$

The 32nd student falls in the 160-164 class.

$$\begin{aligned} \text{The 80th percentile} &= L + \frac{\left(\frac{80}{100}n - C\right)i}{f} \\ &= 159.5 + \frac{(32 - 23)5}{9} \\ &= 159.5 + 5 \\ &= 164.5 \end{aligned}$$

Project

Obtain the masses of the members of your class. Group the data into suitable classes. Determine the:

- (i) modal class.
- (ii) median.
- (iii) lower and upper quartiles.
- (iv) 60th percentile.

Do the same for the heights.

Example 6

Determine the lower quartile and upper quartile for the following set of numbers:
5, 10, 6, 5, 8, 7, 3, 2, 7, 8, 9

Solution

Arranging in ascending order;

2, 3, 5, 5, 6, 7, 7, 8, 8, 9, 10

The median is 7.

The lower quartile is the median of the first half, which is 5.

The upper quartile is the median of the second half, which is 8.

2.4: Median from Cumulative Frequency Curve

The median of grouped data can also be estimated using a cumulative frequency curve. Consider the following data which shows marks scored by 120 students in a mathematics test:

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	2	6	15	20	24	32	12	4	3	2

From the table, we can see that 8 students scored 20 or less marks. Similarly, 23 students scored 30 or less marks, and so on.

The table is shown below with a third row which shows these cumulative totals.

<i>Marks</i>	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
<i>Frequency</i>	2	6	15	20	24	32	12	4	3	2
<i>Cumulative frequency</i>	2	8	23	43	67	99	111	115	118	120

Such a table is called a cumulative frequency table. We may draw a graph of the cumulative frequency against the marks, as shown in figure 2.1.

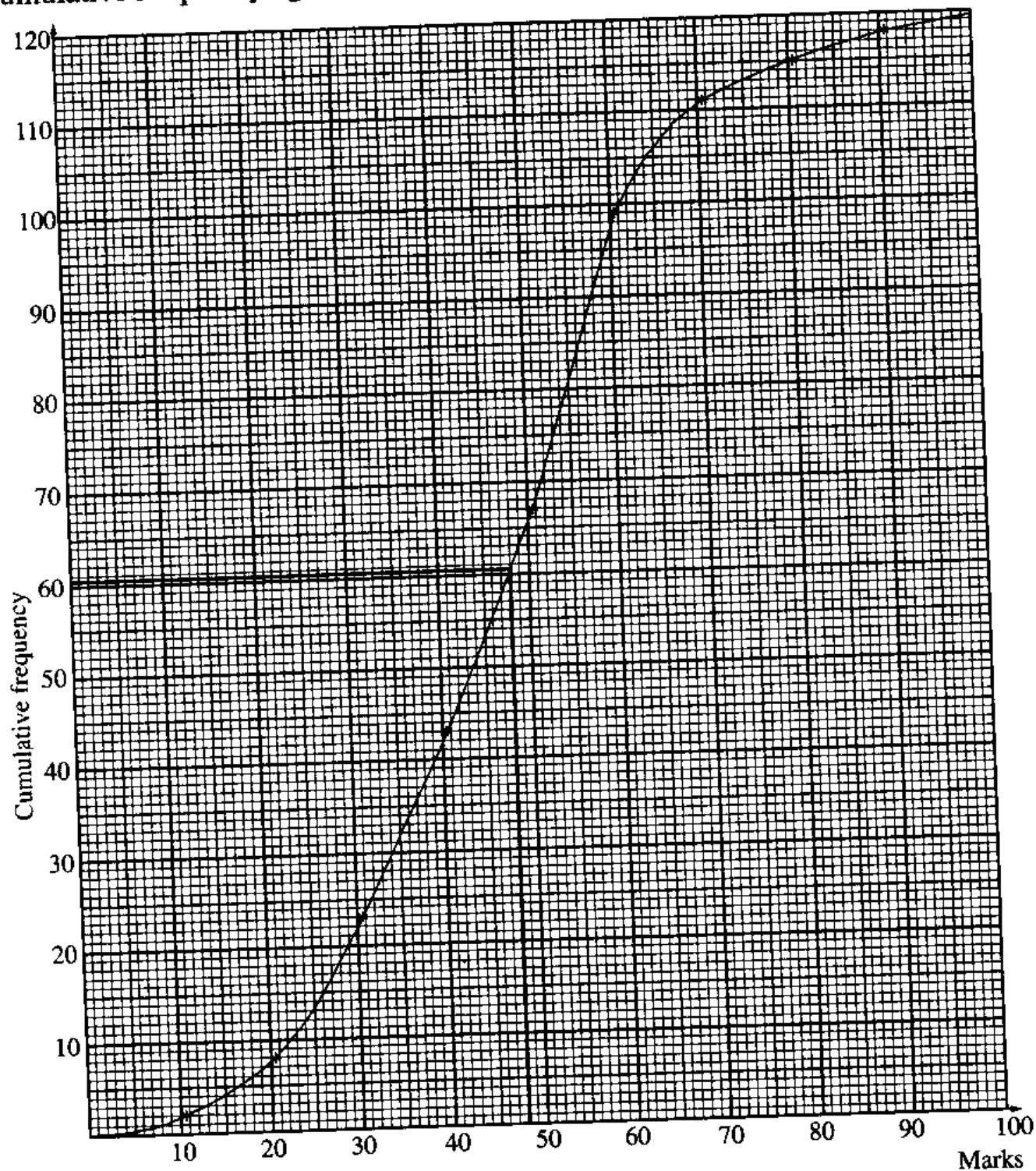


Fig 2.1

Note that in this graph the cumulative frequency is plotted against the upper class limit in each case. The points are joined with a smooth curve to give a cumulative frequency curve. A cumulative frequency curve has a typical extended S-shape.

A cumulative frequency curve is also called an **ogive**. We can estimate the median mark from the curve by reading the mark corresponding to the middle candidate.

Reading from our curve, the mark corresponding to a cumulative frequency of 60.5 is 48. Thus, the median is 48 marks.

Example 7

The table below shows the reading speeds in words per minute of a sample of 90 adults:

Speed (wpm)	121-140	141-160	161-180	181-200	201-220	221-240	241-260	261-280	281-300
Frequency	2	6	21	26	18	9	4	3	1

- Draw a cumulative frequency curve.
- From the graph, estimate the median and the quartiles.

Solution

- The cumulative frequency table is shown below:

Speed (wpm)	121-140	141-160	161-180	181-200	201-220	221-240	241-260	261-280	281-300
Frequency	2	6	21	26	18	9	4	3	1
Cumulative frequency	2	8	29	55	73	82	86	89	90

We plot the cumulative frequency against the upper class limits, as shown in figure 2.2. We read the values of the lower quartile, median and upper quartile as indicated on the graph.

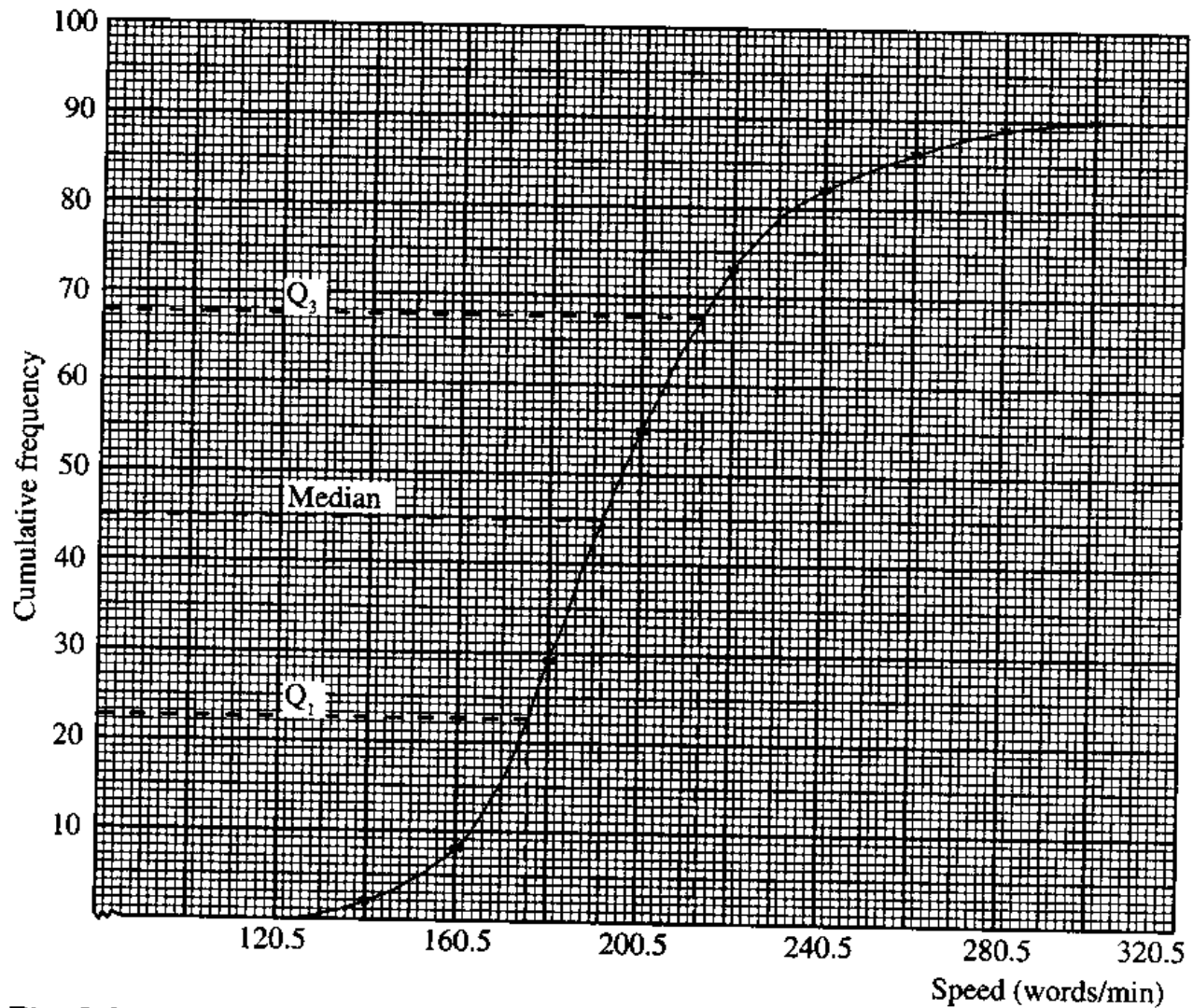


Fig. 2.2

From the graph:

Q_1 , the lower quartile, is 176.5

Q_2 , the median, is 192.5

Q_3 , the upper quartile, is 214.5

Example 8

The table below shows the marks of 100 candidates in an examination:

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	4	9	16	24	18	12	8	5	3	1

- Determine the median and the quartiles.
- If 55 marks is the pass mark, estimate how many students passed.
- Find the pass mark if 70% of the students are to pass.
- Determine the range of marks obtained by:
 - the middle 50% of the students.
 - the middle 80% of the students.

Solution

The cumulative frequency table is shown below:

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	4	9	16	24	18	12	8	5	3	1
Cumulative frequency	4	13	29	53	71	83	91	96	99	100

Figure 2.3 shows the cumulative frequency curve:

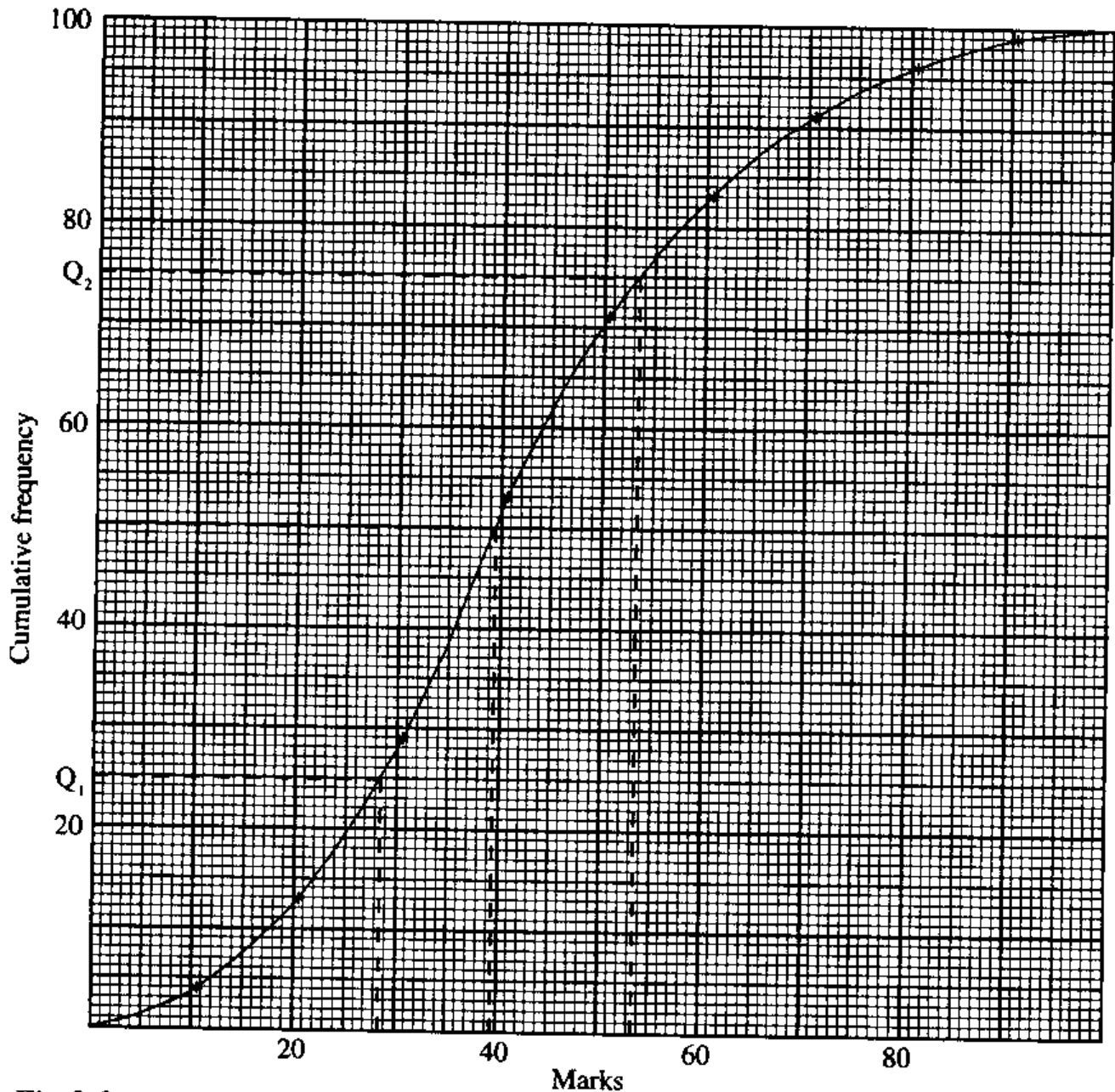


Fig 2.3

- (a) Reading from the graph:
- (i) The median = 39.5
 - (ii) The upper quartile $Q_1 = 28.5$
 - (iii) The upper quartile $Q_2 = 53.5$

- (b) 23 candidates scored 55 and over.
 (c) Pass mark is 31 if 70% of pupils are to pass.
 (d) (i) The middle 50% includes the marks between the lower and the upper quartiles, i.e., between 28.5 and 53.5 marks.
 (ii) The middle 80% includes the marks between the first decile and the 9th decile, i.e., between 18 and 69 marks.

Exercise 2.2

1. Find the mode, median and quartiles of the following:
 (a) 15, 3, 2, 7, 11, 4, 9, 13, 8, 6, 2, 7, 12
 (b) 1 143, 1 293, 989, 1 007, 1 261, 1 104, 1 357, 950, 982, 1 073, 1 357, 1 000, 1 197, 1 204, 1 333, 1 417.
2. 80 boxes of matches were selected at random and the number of sticks in each box counted. The table below shows the distribution of the number of sticks per box:

<i>Number of sticks</i>	32-33	34-35	36-37	38-39	40-41	42-43
<i>Number of boxes</i>	1	3	14	27	20	15

Calculate:

- (a) the mean number of matches per box.
 (b) the median and the quartiles.
3. The densities in kg/m^3 of 100 wooden cubes are shown in the table below:

<i>Density</i>	401-450	451-500	501-550	551-600	601-650	651-700	701-750	751-800	801-850	851-900
<i>Number of cubes</i>	3	7	13	18	32	16	4	4	2	1

- (a) Find the modal class.
 (b) Calculate the:
 (i) median.
 (ii) lower and upper quartiles.
4. The data below shows the times taken by 30 athletes who took part in 400 m heats during a school sports day:

<i>Time in seconds</i>	43-46	47-50	51-54	55-58	59-62	63-66
<i>Number of athletes</i>	0	3	12	10	4	1

Calculate the median and mean time.

5. The salaries in £ per annum of 60 employees of a company are as shown in the table below:

<i>Salary in £ per annum</i>	2201-2700	2701-3200	3201-3700	3701-4200
<i>No. of employees</i>	7	12	16	10
	4201-4700	4701-5200	5201-5700	5701-6200
	8	4	2	1

- (a) Determine the modal class.
- (b) Using a cumulative frequency curve, estimate the lower quartile, the median and the upper quartile.
- (c) Given the mean, mode, and the median, which measure do you think best represents the wage of a typical worker in this company? Give reasons for your answer.
6. The table below shows the times in seconds taken by 50 entrants in the heats of 100 metres freestyle swimming competition:

<i>Time in seconds</i>	41.0-41.8	41.9-42.7	42.8-43.6	43.7-44.5	44.6-45.4	45.5-46.3	46.4-47.2
<i>Frequency</i>	2	4	10	5	8	6	15

Draw a cumulative frequency curve and use it to find:

- (a) the median.
- (b) the second and the sixth deciles.
- (c) the number of swimmers who recorded 44.6 s and above.
- (d) the number of swimmers who recorded between 42.0 s and 46.0 s.
7. The set of data below is a record of the amount of maize meal in kilograms sold in a supermarket in one week:

<i>Amount in kilograms</i>	1-3	4-6	7-9	10-12	13-15	16-18	19-21	22-24	25-27	28-30
<i>Frequency</i>	15	6	1	10	8	4	3	5	1	2

Find:

- (a) the number of people who bought maize meal from the supermarket.
- (b) the range of masses of maize meal bought by the middle 80%.
- (c) the number of people who bought 23 kg or less.
- (d) the percentage of people who bought 20 kg or more.
8. The table below shows a frequency distribution for the final marks in a Mathematics examination:

<i>Marks</i>	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
<i>Number of pupils</i>	3	6	5	11	4	3	2	3	2	1

Calculate:

- the median and the quartiles.
- the 4th and the 6th decile.
- the 30th and the 70th percentile.

9. The data below shows the number of bags of maize taken to a co-operative society depot by 100 farmers in one week:

<i>Bags of maize</i>	5-14	15-24	25-34	35-44	45-54	55-64	65-74	75-84	85-94	95-104
<i>Number of farmers</i>	18	26	30	7	1	5	4	2	5	2

Use an ogive to estimate:

- the median.
- the quartiles.
- the 5th and the 7th deciles.

10. The table below shows the average earnings in shillings per hour of 200 employees of a company:

<i>Earnings in shillings per hour</i>	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-69
<i>Number of people</i>	21	59	52	22	20	20	4	2

- Calculate the mean.
- Draw a cumulative frequency curve and from it determine:
 - the median and the quartiles.
 - the percentage of people earning less than the mean wage.
 - the range of earnings of the middle 50% of the workers.
 - the 20th and 80th percentiles.

11. The table below shows the number of pairs of shoes of different sizes a dealer sold during one month:

<i>Shoes size</i>	12	13	1	2	3	4	5	6	7	8	9
<i>Number of pairs</i>	16	18	21	32	15	15	14	14	11	7	4

Find:

- the modal size.
- the median size.
- the range of sizes between which the middle 50% of the buyers lie.

2.5: Measures of Dispersion

The end-of-term marks for two pupils in eight subjects are shown below:

<i>Pupil A</i>	57	55	62	52	54	45	57	66
<i>Pupil B</i>	85	32	55	87	24	78	49	38

Both pupils have the same mean mark of 56.

Although the candidates have the same mean mark, most of A's marks are 5 marks away from (above or below) the mean while most of B's marks are about 8 marks off the mean.

We can see that the marks of pupil B are much more spread out than those of pupil A. We need a statistical measure to show this spread or dispersion. One measure of dispersion is range. The range is the difference between the highest and lowest values.

For pupil A, the range is $66 - 45 = 21$ marks.

For pupil B, the range is $87 - 24 = 63$ marks.

One disadvantage of the range as a measure of dispersion is that it depends only on the two extreme values. Another measure of dispersion is the interquartile range. This is the difference between the lower and the upper quartiles. It includes the middle 50% of the values.

Another measure used is the semi-interquartile range. It is also called the quartile deviation.

For pupil B, the lower quartile is $\frac{32 + 38}{2} = 35$

The upper quartile is $\frac{78 + 85}{2} = 81.5$

and the interquartile range is $81.5 - 35 = 46.5$

The quartile deviation = $\frac{46.5}{2} = 23.25$

Determine the interquartile range and the quartile deviation of the marks of pupil A.

Mean Absolute Deviation

The list below shows the deviation (difference) of each mark from the mean:

Pupil A: +1 -1 +6 -4 -2 -11 +1 +10

Pupil B: +29 -24 -1 +31 -32 +22 -7 -18

In each case, the sum of the deviations is zero. The sum of deviations from the mean is zero for any set of data. This follows from the definition of the mean. It does not tell us much about the dispersion of the data.

Since for dispersion we are interested in how far the values are below or above the mean, we may ignore the signs and take the absolute value of each deviation.

The absolute value of a number is:

- (i) the number itself, if it is positive or zero.
- (ii) the number with the negative sign ignored, if it is negative.

The symbol for absolute value of a number x is $|x|$.

For example $|5| = 5$, $|-5| = 5$, $|3 - 5| = |-2| = 2$.

The mean of the absolute deviations is called the **mean absolute deviation**.

Consider Pupil A:

<i>Deviation (d)</i>	+1	-1	+6	-4	-2	-11	+1	+10
<i>Absolute deviation d </i>	1	1	6	4	2	11	1	10

Sum of absolute deviations = 36

$$\begin{aligned} \text{Mean absolute deviation} &= \frac{36}{8} \\ &= 4.5 \end{aligned}$$

Pupil B:

<i>Deviation (d)</i>	+29	-24	-1	+31	-32	+22	-7	-18
<i>Absolute deviation d </i>	29	24	1	31	32	22	7	18

Sum of absolute deviation = 164

$$\begin{aligned} \text{Mean absolute deviation} &= \frac{164}{8} \\ &= 20.5 \end{aligned}$$

Calculate the mean absolute deviation of each of the following sets of data:

(i) 1 4 5 6 7 8 9

(ii) 32 33 34 37 44 45 49 50 50 52

Variance

Instead of ignoring the sign of each deviation, we can square each deviation, so that all become positive. The mean of the square of the deviations from the mean is called **variance** or **mean deviation**. The variance can be worked out as follows for pupil A:

<i>Deviation from the mean (d)</i>	+1	-1	+6	-4	-2	-11	+1	10
d^2	1	1	36	16	4	121	1	100

$$\begin{aligned} \text{Sum of } d^2 &= 1 + 1 + 36 + 16 + 4 + 121 + 1 + 100 \\ &= 280 \end{aligned}$$

$$\text{Variance} = \frac{\Sigma d^2}{N} = \frac{280}{8} = 35$$

The square root of the variance is called the **standard deviation**. It is also called **root mean square deviation**.

For pupil A, the standard deviation = $\sqrt{35}$
= 5.9 marks

For pupil B, the standard deviation can be worked out as follows:

<i>d</i>	29	-24	-1	31	-32	22	-7	-18
<i>d</i> ²	841	576	1	961	1 024	484	49	324

Sum of *d*² = 841 + 576 + 1 + 961 + 1 024 + 484 + 49 + 324
= 4 260

Variance = $\frac{\Sigma d^2}{N} = \frac{4260}{8}$
= 532.5

Standard deviation = $\sqrt{532.5}$
= 23.08 marks

Calculate the standard deviation of each of the following sets of marks:

- (i) 35, 39, 40, 42, 46, 48, 51, 52, 55.
(ii) 26, 33, 36, 48, 52, 57, 63, 68, 71, 74, 83, 85, 89, 93, 97.

Example 9

The following table shows the number of children per family in a housing estate:

<i>Number of children</i>	0	1	2	3	4	5	6
<i>Number of families</i>	1	5	11	27	10	4	2

Calculate:

- (a) the mean number of children per family.
(b) the standard deviation.

<i>Number of children</i> (<i>x</i>)	<i>No. of families</i> (<i>f</i>)	<i>fx</i>	<i>d = x - m</i>	<i>d</i> ²	<i>fd</i> ²
0	1	0	-3	9	9
1	5	5	-2	4	20
2	11	22	-1	1	11
3	27	81	0	0	0
4	10	40	1	1	10
5	4	20	2	4	16
6	2	12	3	9	18
	$\Sigma f = 60$	$\Sigma fx = 180$			$\Sigma fd^2 = 84$

$$\begin{aligned} \text{(a) Mean} &= \frac{180}{60} \\ &= 3 \text{ children} \end{aligned}$$

$$\begin{aligned} \text{(b) Variance} &= \frac{\sum fd^2}{\sum f} \\ &= \frac{84}{60} \\ &= 1.4 \end{aligned}$$

$$\begin{aligned} \text{Standard deviation} &= \sqrt{1.4} \\ &= 1.183 \end{aligned}$$

Example 10

The table below shows the distribution of marks of 40 candidates in a test:

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	2	2	3	9	12	5	2	3	1	1

Calculate the mean and standard deviation.

Marks	Midpoint (x)	f	fx (Σ)	$d = x - m$	d^2	fd^2
1-10	5.5	2	11.0	-39.5	1560.25	3120.5
11-20	15.5	2	31.0	-29.5	870.25	1740.5
21-30	25.5	3	76.5	-19.5	380.25	1140.75
31-40	35.5	9	319.5	-9.5	90.25	812.25
41-50	45.5	12	546.0	0.5	0.25	3.00
51-60	55.5	5	277.5	10.5	110.25	551.25
61-70	65.5	2	131.0	20.5	420.25	840.5
71-80	75.5	3	226.5	30.5	930.25	2790.75
81-90	85.5	1	85.5	40.5	1640.25	1640.25
91-100	95.5	1	95.5	50.5	2550.25	2550.25
		$\Sigma f = 40$	$\Sigma fx = 1800$			$\Sigma fd^2 = 15190$

$$\text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{1800}{40} = 45 \text{ marks}$$

$$\begin{aligned} \text{Variance} &= \frac{\Sigma fd^2}{\Sigma f} = \frac{15190}{40} = 379.75 \\ &= 379.8 \text{ (to 4 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{Standard deviation} &= \sqrt{379.8} \\ &= 19.49 \end{aligned}$$

Exercise 2.3

1. Calculate the range and the quartile deviation for each of the following:

(a) 16, 13, 24, 40, 6, 20, 18, 17

(b) 53, 42, 36, 41, 43, 39, 25

Calculate the mean and the standard deviation in each of the following:

2. 9 2 3 4 5 5 7 8 1

3. 17 2 4 5 6 8 10 11 12 14 15 16 2 18

4.

x	0	1	2	3	4	5	6	7	8	9
f	2	3	2	0	1	5	3	2	6	1

5.

x	1	5	9	13	17	21	25	29	30	33	40	41
f	8	14	25	15	12	9	5	6	1	3	1	1

6.

<i>Class</i>	1-5	6-10	11-15	16-20	21-25	26-30
<i>Frequency</i>	3	3	4	5	3	2

7.

<i>Class</i>	10-12	13-15	16-18	19-21	22-23	24-26	27-29
<i>Frequency</i>	2	3	6	8	6	4	1

8. Find the standard deviation of the data in question 4, Exercise 2.1.

Other Formulae for Standard Deviation

If we let s stand for the standard deviation, the formula for the variance (s^2) from the explanation in the last section is;

$$s^2 = \frac{\Sigma(x - \bar{x})^2 f}{\Sigma f}, \text{ where } \bar{x} \text{ is the mean.}$$

Expansion and simplification of the formula shows that it is equivalent to;

$$s^2 = \frac{\Sigma fx^2}{\Sigma f} - \bar{x}^2$$

This means that we square x , multiply by the corresponding frequency and find the sum of fx^2 terms. Divide this sum by the number of items. To get the variance, we subtract \bar{x}^2 from the answer.

Use this formula to find the variance of the following numbers:
1, 4, 5, 6, 7, 8, 9

Let us use the above version of the formula to work out Example 9.

x	x^2	f	fx^2	fx
0	0	1	0	0
1	1	5	5	5
2	4	11	44	22
3	9	27	243	81
4	16	10	160	40
5	25	4	100	20
6	36	2	72	12
		$\Sigma f = 60$	$\Sigma fx^2 = 624$	$\Sigma fx = 180$

$$s^2 = \frac{\Sigma fx^2}{\Sigma f} - \bar{x}^2 = \frac{624}{60} - \left(\frac{180}{60}\right)^2$$

$$= 10.4 - 9$$

$$= 1.4$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{1.4}$$

$$= 1.18 \text{ (to 2 d.p.)}$$

Use the formula $s^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$ to work out the standard deviation in questions 5 and 7, Exercise 2.3.

We are going to see the effect on the standard deviation of changing the variable by adding/subtracting a constant from each of the values of x given. Let us work out Example 10 again. We will first reduce each value of x by 55.5.

Marks	Midpoint x	$t = x - 55.5$	f	ft	ft^2
1-10	5.5	-50	2	-100	5 000
11-20	15.5	-40	2	-80	3 200
21-30	25.5	-30	3	-90	2 700
31-40	35.5	-20	9	-180	3 600
41-50	45.5	-10	12	-120	1 200
51-60	55.5	0	5	0	0
61-70	65.5	10	2	20	200
71-80	75.5	20	3	60	1 200
81-90	85.5	30	1	30	900
91-100	95.5	40	1	40	1 600
			$\sum f = 40$	$\sum ft = -420$	$\sum ft^2 = 19 600$

$$\text{Mean of } t = \frac{-420}{40} = -10.5$$

$$\text{Variance of } t = \frac{\sum ft^2}{\sum f} - \bar{t}^2$$

$$= \frac{19600}{40} - (-10.5)^2$$

$$= 490 - 110.25$$

$$= 379.75$$

$$\text{Therefore, standard deviation of } t = \sqrt{379.75}$$

$$= 19.49 \text{ (4 s.f.)}$$

We see that we have obtained the same value for the standard deviation as before. Adding/subtracting a constant to/from each number in a set of data does not alter the value of the variance or standard deviation. Verify this fact by working through question 5 and 6 of Exercise 2.3 using a change of variable $t = x - A$, where A is a constant.

Finally, let us carry the change of variable further by dividing by a constant c , in this case 10.

Marks	Midpoint (x)	$x - 55.5$	$t = \frac{x - 55.5}{10}$	f	ft	ft^2
1-10	5.5	-50	-5	2	-10	50
11-20	15.5	-40	-4	2	-8	32
21-30	25.5	-30	-3	3	-9	27
31-40	35.5	-20	-2	9	-18	36
41-50	45.5	-10	-1	12	-12	12
51-60	55.5	0	0	5	0	0
61-70	65.5	10	1	2	2	2
71-80	75.5	20	2	3	6	12
81-90	85.5	30	3	1	3	9
91-100	95.5	40	4	1	4	16
				$\Sigma f = 40$	$\Sigma ft = -42$	196

$$\text{Mean of } t = \frac{-42}{40} = -1.05$$

$$\begin{aligned} \text{Variance of } t &= \frac{\Sigma ft^2}{\Sigma f} - \bar{t}^2 \\ &= \frac{196}{40} - (-1.05)^2 \\ &= 4.9 - 1.1025 \\ &= 3.7975 \end{aligned}$$

$$\text{Standard deviation of } t = \sqrt{3.7975} = 1.949$$

We note that in order to obtain the standard deviation of x from the standard deviation of t , we need to multiply the standard deviation of t by the constant c . Alternatively, to obtain the variance of x from the variance of t , we need to multiply the variance of t by c^2 .

$$\begin{aligned} \text{Thus, the variance of } x &= 3.7975 \times 100 \\ &= 379.75 \end{aligned}$$

$$\begin{aligned} \text{Standard deviation of } x &= \sqrt{379.75} \\ &= 19.49 \end{aligned}$$

In summary, the formula for getting the variance s^2 of a variable x is;

$$\begin{aligned} s^2 &= \frac{\Sigma(x - \bar{x})^2 f}{\Sigma f} \\ &= \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f} \right)^2 \end{aligned}$$

$$s^2 = c^2 \left[\frac{\sum ft^2}{\sum f} - \left(\frac{\sum ft}{\sum f} \right)^2 \right]$$

where $t = \frac{x-A}{c}$. A is the assumed mean and c the class size.

Therefore, the standard deviation is given by;

$$s = c \sqrt{\frac{\sum ft^2}{\sum f} - \left(\frac{\sum ft}{\sum f} \right)^2}$$

Example 11

The table below shows the length in centimetres of 80 plants of a particular species of tomato:

<i>Length</i>	152-156	157-161	162-166	167-171	172-176	177-181
<i>Frequency</i>	12	14	24	15	8	7

Calculate the mean and the standard deviation.

Solution

Let $A = 169$ and $c = 5$.

<i>Length</i>	<i>Midpoint x</i>	$x - 169$	$t = \frac{x-169}{5}$	<i>f</i>	<i>ft</i>	<i>ft</i> ²
152-156	154	-15	-3	12	-36	108
157-161	159	-10	-2	14	-28	56
162-166	164	-5	-1	24	-24	24
167-171	169	0	0	15	0	0
172-176	174	5	1	8	8	8
177-181	179	10	2	7	14	28
				$\sum f = 80$	$\sum ft = -66$	$\sum ft^2 = 224$

$$\bar{t} = \frac{-66}{80} = -0.825$$

$$\begin{aligned} \therefore \bar{x} &= -0.825 \times 5 + 169 \\ &= -4.125 + 169 \\ &= 164.875 \\ &= 164.9 \text{ (to 4 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{Variance of } t &= \frac{\sum ft^2}{\sum f} - \bar{t}^2 \\ &= \frac{224}{80} - (-0.825)^2 \\ &= 2.8 - 0.6806 \\ &= 2.119 \end{aligned}$$

$$\begin{aligned} \text{Therefore, variance of } x &= 2.119 \times 5^2 \\ &= 52.975 \\ &= 52.98 \text{ (4 s.f.)} \end{aligned}$$

$$\begin{aligned}\text{Standard deviation of } x &= \sqrt{52.98} \\ &= 7.279 \\ &= 7.28 \text{ (to 2 d.p.)}\end{aligned}$$

Work through questions 6 and 7 of Exercise 2.3 using the change of variable $t = \frac{x-A}{c}$, where A and c are constants.

Exercise 2.4

- Find the mean and standard deviation of the following sets of data:
 - 179 152 144 135 163 108 122 118 173 149 131 197.
 - 24 349 8 43 70 136 214 157 380 412 270 450
644 4 21 176 322 66 54.

- The marks scored by 30 pupils in a Geography examination were:

59 65 68 69 70 75 73 72 80 78
60 62 61 71 72 58 85 82 81 68
73 69 58 61 63 70 76 79 82 90

Find the standard deviation of the data.

- The following are the marks scored in Mathematics and Kiswahili by 20 students in a class:

Mathematics	45	48	52	64	70	44	56	38	42	70
	58	56	47	54	58	62	60	41	62	42
Kiswahili	62	56	46	34	48	82	56	72	73	42
	42	62	59	68	72	78	52	59	58	73

- Calculate the mean mark for each subject.
 - Find the interquartile range of the marks in each subject.
 - Which mark is better for this class, a score of 60 in Mathematics or 70 in Kiswahili?
- In an interview, 20 hawkers were asked to state their earnings in a certain week. The figures they stated were:

3000	5850	6250	3200	2850
3780	4000	4350	5160	6340
5500	3500	4200	5200	6570
1230	1450	1600	2150	2500

Calculate the mean and the standard deviation.

- The times to the nearest second taken by 30 athletes in 400 metres heats were recorded as below:

Time (sec.)	44	45	46	47	49	50	51	53	55	58	61
Frequency	1	1	4	4	6	3	3	2	2	3	1

Calculate the mean and standard deviation of the data.

6. Find the mean and standard deviation of the data given below:

<i>x</i>	21-25	26-30	31-35	36-40	41-45	46-50	51-55	56-60	61-65	66-70
<i>f</i>	7	13	9	18	12	21	11	9	7	3

7. The masses of 100 patients in a hospital were distributed as shown in the table below:

<i>Mass (kg)</i>	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
<i>Frequency</i>	3	7	8	9	12	18	25	10	6	2

Calculate the mean and the standard deviation of the data.

8. The table below shows the distribution of ages in months of 80 form four students in a school:

<i>Age in months</i>	210-212	213-215	216-218	219-221	222-224	225-227	228-230
<i>Frequency</i>	2	8	25	18	12	10	5

Calculate the mean and the standard deviation for the data.

9. The weekly earnings of 90 workers in a firm are distributed as shown in the table below:

<i>Earnings (sh) per week</i>	3001-3500	3501-4000	4001-4500	4501-5000	5001-5500	5501-6000
<i>Number of workers</i>	5	3	10	15	12	11

<i>Earnings (sh) per week</i>	6001-6500	6501-7000	7001-7500	7501-8000	8001-8500
<i>Number of workers</i>	10	10	8	4	2

Find:

- the mean.
 - the standard deviation.
 - the interquartile range for the data.
10. Use the formula $s = c \sqrt{\frac{\sum ft^2}{\sum f} - \left(\frac{\sum ft}{\sum f}\right)^2}$, where $t = \frac{x-A}{c}$, to calculate the standard deviation of the data given in Exercise 2.2, question 6.
11. Use the data given in Exercise 2.2, question 10 to calculate the standard deviation of the average earnings of the employees.
12. In an examination, the mean mark for Mathematics was 58 with a standard deviation of 6. The mean mark for Physics was 50 with a standard deviation of 8. Which mark was better, a mark of 60 in Physics or a mark of 60 in Mathematics?

Chapter Three

LOCI

3.1: Introduction

Figure 3.1 shows a door ABCD hinged at P and Q and open through an angle of 60° . The corner A of the door traces the path AE as the door closes. This path, which subtends an angle of 60° at D, is an arc and is called the **locus** of A as the door closes.

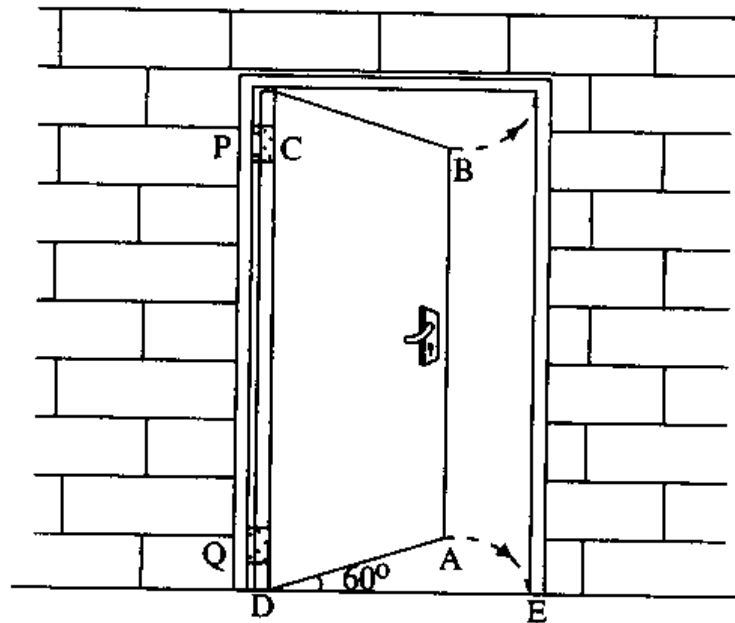


Fig. 3.1

The edge AD of the door moves to ED when the door is closed. What will be the region traced out by the edge AD after the door is closed?

You notice that the edge AD traces a sector ADE, see figure 3.2.

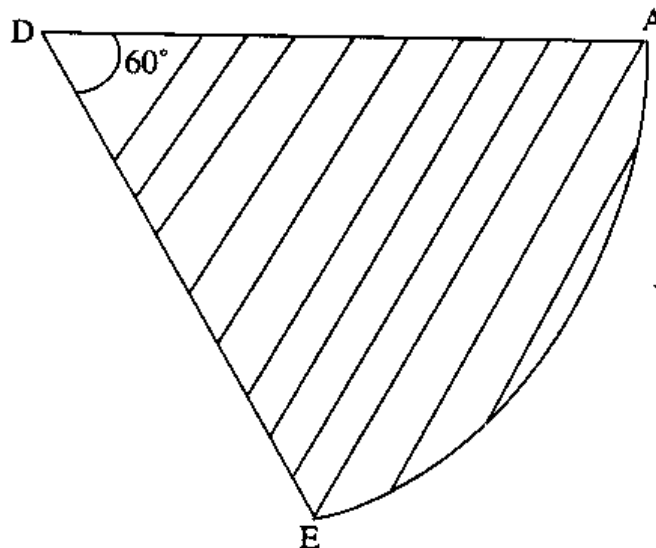


Fig. 3.2

If we consider the door ABCD, what will be its locus after it is closed?

You notice that in this case, the locus of the door is a solid which resembles a portion of a cylinder, as shown in figure 3.3.

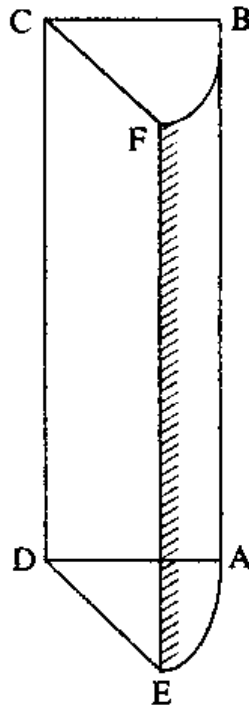


Fig. 3.3

From figure 3.1 to 3.3, you notice that the locus traced out by a point is a path, that traced by a line is a region and that traced by a region is a solid with some volume. **Locus (plural: loci) is therefore defined as the path, area or volume traced out by a point, line or region as it moves according to some given law(s).** In the example of the door closing, the laws we followed were:

- (i) the door was already open through an angle of 60° .
- (ii) the door was moved (rotated) about CD to close.

Activity

- (i) Pin the corners of a rectangular piece of paper on a softboard.
- (ii) Fix a pin at the centre of the paper.
- (iii) Tie the ends of a string, 15 cm long, together.
- (iv) Move the tip of the pen on the paper, keeping the string taut. What shape have you drawn?

Exercise 3.1

In questions 1 to 14, state and sketch the locus:

1. The tip of the minute hand of a clock during a 30-minute interval.
2. The minute hand of a clock during a 30-minute interval.
3. The axle of a bicycle as the bicycle moves along a level road.

4. The bob of a swinging pendulum.
5. The tip of one arm of a sea-saw as it swings.
6. The arm of a sea-saw as it swings.
7. The variable point T which is equidistant from two fixed points P and Q.
8. The point P which is always the same distance from a fixed point X.
9. A ball which rolls in such a way that it is always 4 metres from a straight wall.
10. A line AB rotated through 180° about A.
11. A protractor lies flat on a table. It is rotated about its straight edge until it is flat on the table again. State and sketch the locus of the protractor.
12. A rectangle ABCD is resting in a vertical plane. It is given a horizontal translation by a vector perpendicular to AB through a fixed distance. Describe the locus.
13. A solid set-square is rotated about one of its shorter sides through 360° . Describe the locus.
14. Sketch the locus of a cyclist who moves such that he is one metre from the edge of the road.

3.2: Common Types of Loci

Perpendicular Bisector Locus

In Two Dimensions

In figure 3.4, M and N are two fixed points. P is a point such that $PM = PN$. If Q is the midpoint of MN, then we need to show that PQ is perpendicular to MN.

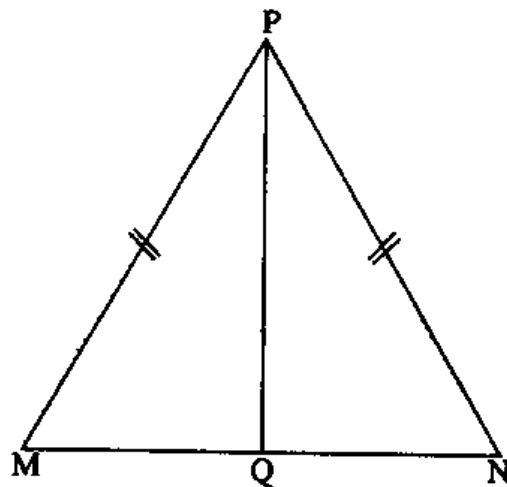


Fig. 3.4

Considering the triangles PMQ and PNQ;

$PM = PN$ (given)

$PQ = PQ$ (common)

$MQ = NQ$ (construction)

Therefore, $\Delta PMQ = \Delta PNQ$ (SSS).

This means that all corresponding sides and angles are equal. In particular;

$$\angle PQM = \angle PQN$$

But;

$$\angle PQM + \angle PQN = 180^\circ \text{ (adj } \angle\text{s on straight line)}$$

Therefore, $\angle PQM = \angle PQN = 90^\circ$.

Hence, PQ is a perpendicular bisector of MN.

In general, the locus of points which are equidistant from two fixed points is the perpendicular bisector of the straight line joining the fixed points. This locus is called the perpendicular bisector locus.

In Three Dimensions

In three dimensions, the perpendicular bisector locus is a plane at right angles to MN and bisecting MN, as shown in figure 3.5. Notice that the point P can lie anywhere in the plane provided $PN = PM$.

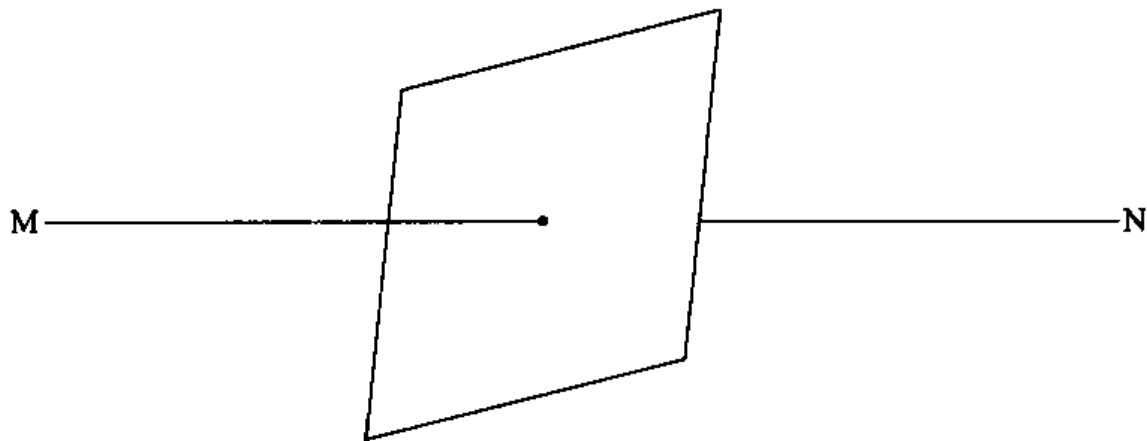


Fig. 3.5

The Locus of Points at a Given Distance from a Given Straight Line

In Two Dimensions

Figure 3.6 shows a given line MN. Each of the points is marked 'a' centimetres on either side of the given line MN ('a' centimetres from the given line implies the perpendicular distance).

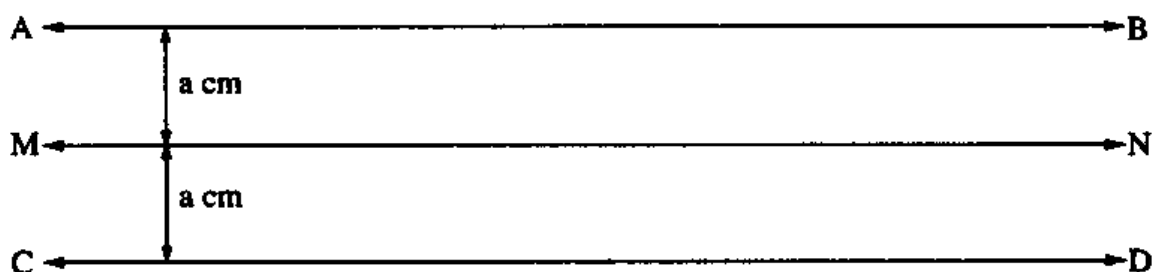


Fig 3.6

Notice that the locus of points form two parallel lines. The two parallel lines describe the locus of points at a fixed distance from a given straight line.

In Three Dimensions

In three dimensions, the locus of a point 'a' centimetres from a line MN is a cylindrical shell of radius 'a' cm, with MN as the axis of rotation, as shown in figure 3.7.

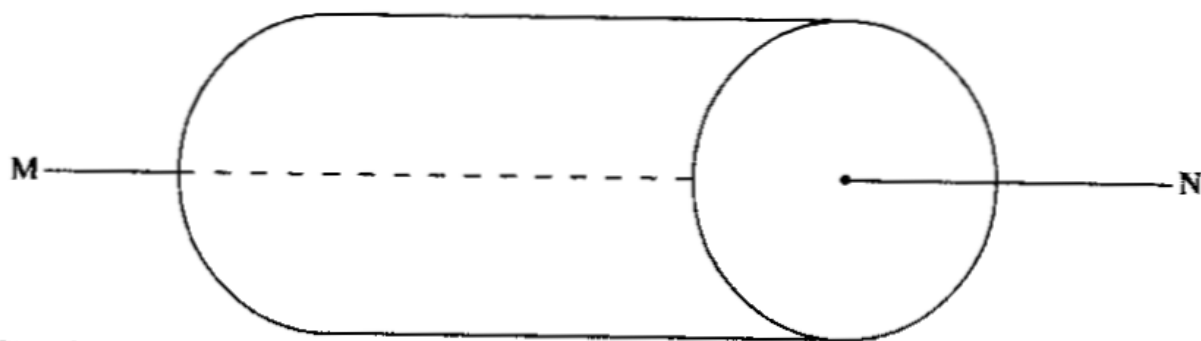


Fig. 3.7

Describe the locus in both two and three dimensions of a point T, which is always 3 cm from a line segment PQ, 10 cm long.

Locus of Points at a Given Distance from a Fixed Point

In Two Dimensions

If O is a fixed point and P a variable point 'd' cm from O, the locus of P is the circle O radius 'd' cm, as shown in figure 3.8.

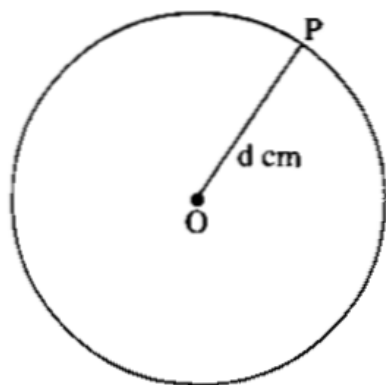


Fig. 3.8

We say that all points on a circle describe a locus of a point at constant distance from a fixed point.

In Three Dimensions

In three dimensions, the locus of a point 'd' centimetres from a point is a spherical shell centre O and radius d cm, as shown in figure 3.9.

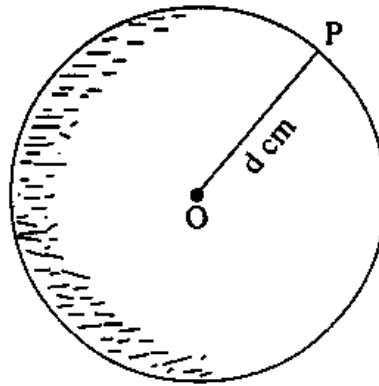


Fig. 3.9

Angle Bisector Locus

Figure 3.10 shows two straight lines PQ and RS, intersecting at M.

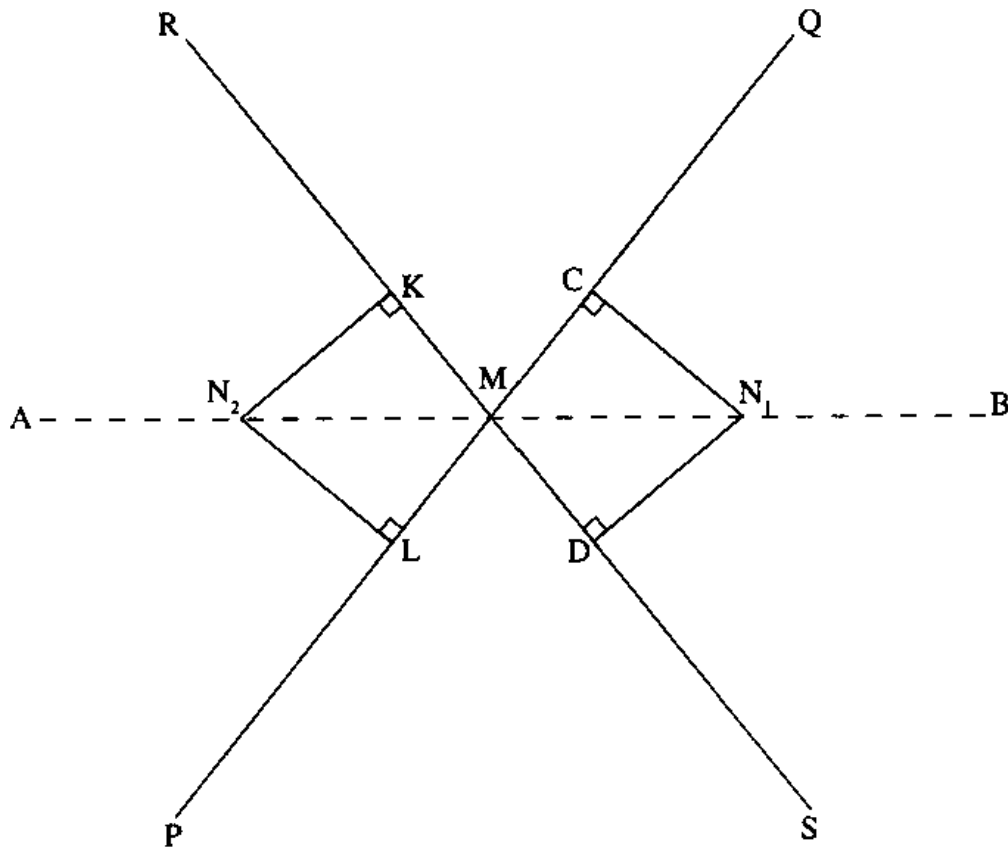


Fig. 3.10

AB is a line such that any point on it, say N_1 or N_2 , is equidistant from PQ and RS. The line AB bisects the angles RMP and QMS, because the triangle MCN_1 , is congruent to triangle MDN_1 (RHS).

Similarly, in figure 3.11, if XY is a set of points which are equidistant from the lines PQ and RS, then XY bisects the angles QMR and PMS.

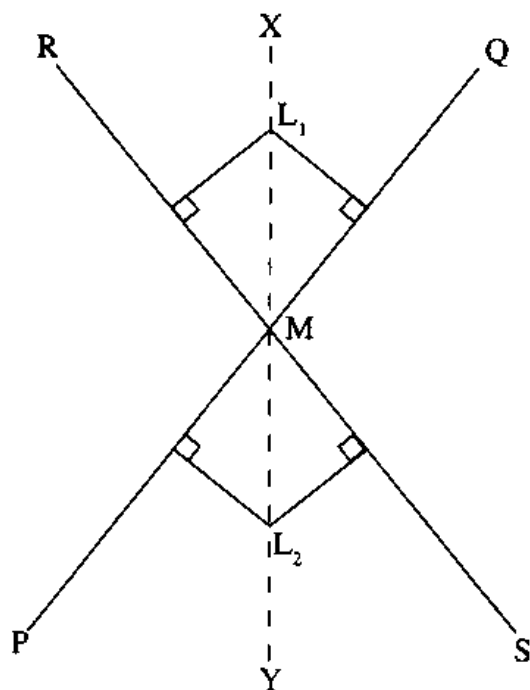


Fig. 3.11

From figures 3.10 and 3.11, we conclude that the locus of points which are equidistant from two intersecting lines (PQ and RS) bisect the angles between the two lines PQ and RS and are perpendicular to each other.

In general, the locus of points which are equidistant from two given intersecting straight lines is the pair of perpendicular lines which bisect the angles between the given lines.

Conversely, a point which lies on a bisector of a given angle is equidistant from the lines including that angle.

Example 1

Construct triangle PQR such that $PQ = 7$ cm, $QR = 5$ cm and $\angle PQR = 30^\circ$. Construct the locus L of points equidistant from RP and RQ.

Solution

L is the bisector of $\angle PRQ$.

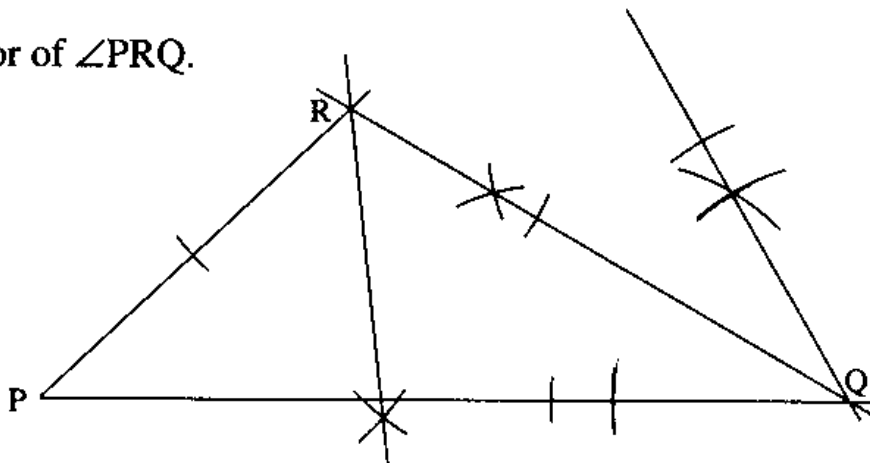


Fig. 3.12

Constant Angle Loci

In figure 3.13, two circles AQBC and APBD with centres O_1 and O_2 respectively intersect at points A and B.

Points P_1, P_2, P_3 and P_4 are points at the circumferences of the two equal circles, each with a radius of 3.1 cm. The angles AP_1B, AP_2B, AP_3B and AP_4B are subtended by the common chord AB .

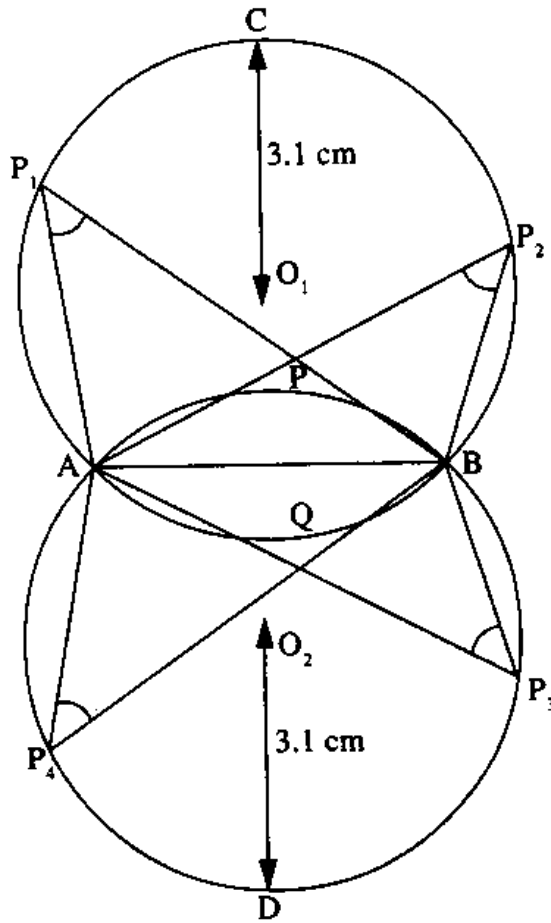


Fig. 3.13

Draw the figure and measure the sizes of all angles at the circumferences.

You notice that $\angle AP_1B = \angle AP_2B = \angle AP_3B = \angle AP_4B$. The locus of point P is the major arcs ACB and ADB . This is called the constant angle locus.

Example 2

A line PQ is 5 cm long. Construct the locus of points at which PQ subtends an angle of 70° .

Solution

Method 1

The angle subtended by the chord PQ on the circumference is 70° .

Let O be the centre of the circle. Construct an isosceles triangle OPQ such that $\angle OPQ = \angle OQP = 20^\circ$. Why? Using O as the centre and either OP or OQ as radius, draw the locus as shown in figure 3.14.

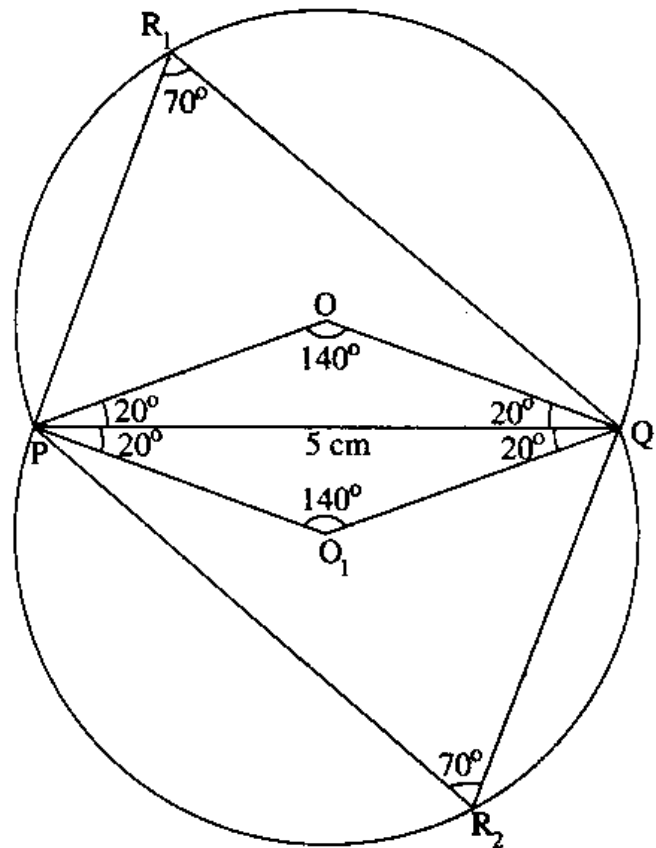


Fig. 3.14

Method 2

- (i) Draw $PQ = 5$ cm.
- (ii) Construct TP at P such that $\angle QPT = 70^\circ$.
- (iii) Draw a perpendicular to TP at P (radius is perpendicular to tangent).
- (iv) Construct the perpendicular bisector of PQ to meet the perpendicular in (iii) at O .
- (v) Using O as centre and either OP or OQ as radius, draw the locus.
- (vi) Transfer the centre on the opposite side of PQ and complete the locus as shown below.

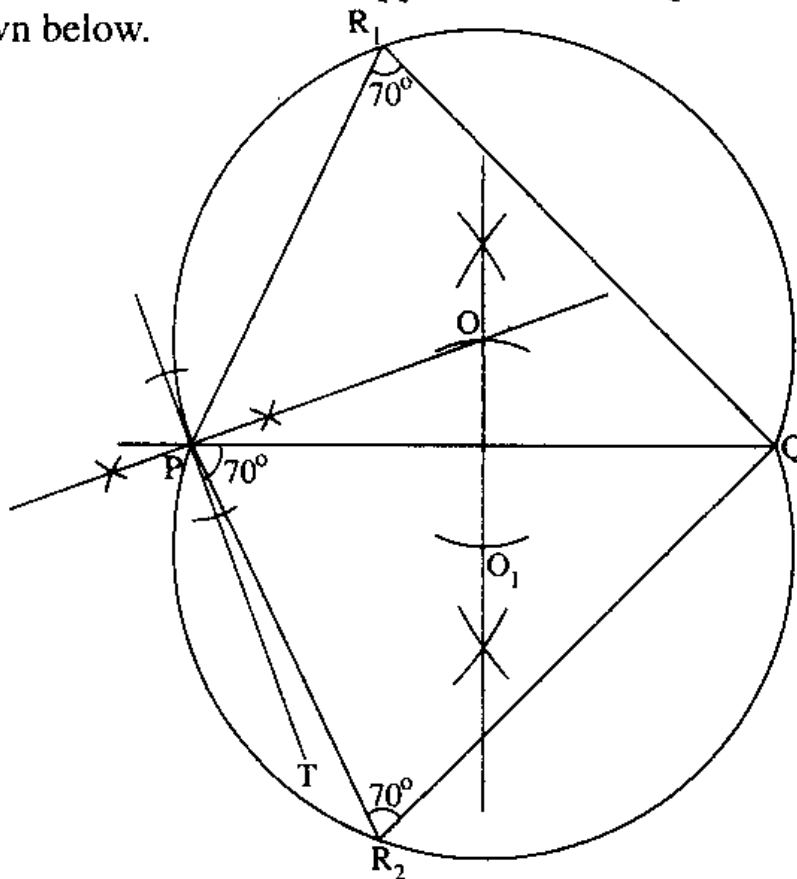


Fig. 3.15

Exercise 3.2

1. Draw a line AB, 5 cm long. Construct the locus of a point P which moves 3 cm from the line.
2. Draw a circle of radius 4 cm. Construct the locus of a point X which is 1 cm from the circumference of the circle.
3. Construct an equilateral triangle ABC of side 6 cm. On the diagram, construct the locus of:
 - (a) a point P which is equidistant from the sides AB and BC.
 - (b) a point Q equidistant from A and B.
4. Describe the locus of a point equidistant from the y and x-axis.
5. Draw a line XY, 4 cm long. Construct the locus of a point P such that $\angle XPY = 60^\circ$. Estimate the area enclosed by the locus of P.
6. Given that line XY = 8 cm, construct the locus of the point Z such that the area of triangle XZY is 20 cm^2 .
7. A variable point P moves such that it is 3.5 cm from a fixed line XY, 7 cm long. Describe the locus of P in three dimensions. Find the volume of the solid described by the locus.
8. On a grid, draw the locus of a point which moves 3 cm from the point (2, 5). Write the equation of the locus.
9. Sketch and describe the locus of the vertex V of a triangle VAB of area 10 cm^2 , given that AB is 4 cm.
10. Draw a line AB, 5 cm long. Construct the locus of all points P such that $\angle APB = 80^\circ$. Measure the maximum length of AP.
11. The line AB = 6 cm. Determine the locus of a point P which moves such that the area of triangle APB is 12 cm^2 .

3.3: Intersecting Loci

Consider the triangle PQR in figure 3.16.

We can construct L_1 , the locus of a point equidistant from QR and PR. On the same diagram, we can construct L_2 and L_3 , the locus of points 2 cm from PQ. The two loci intersect at the points T_1 and T_2 .

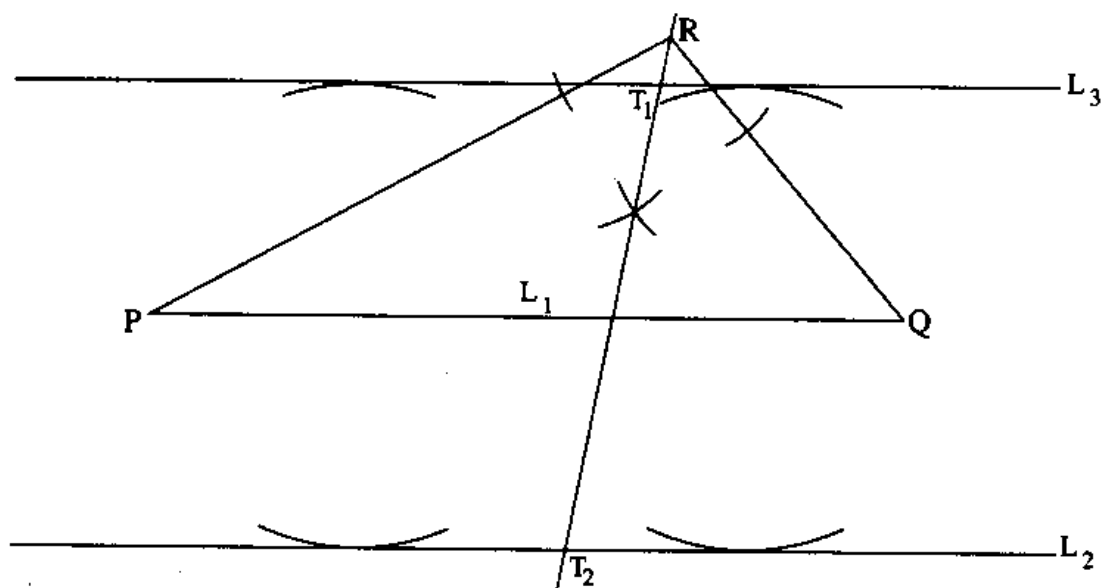


Fig. 3.16

Notice that the two points T_1 and T_2 satisfy the two sets of conditions.

Example 3

- Construct triangle PQR such that $PQ = 7$ cm, $QR = 5$ cm and $\angle PQR = 30^\circ$.
- Construct the locus L_1 of points equidistant from P and Q to meet the locus L_2 of points equidistant from Q and R at M. Measure PM.

Solution

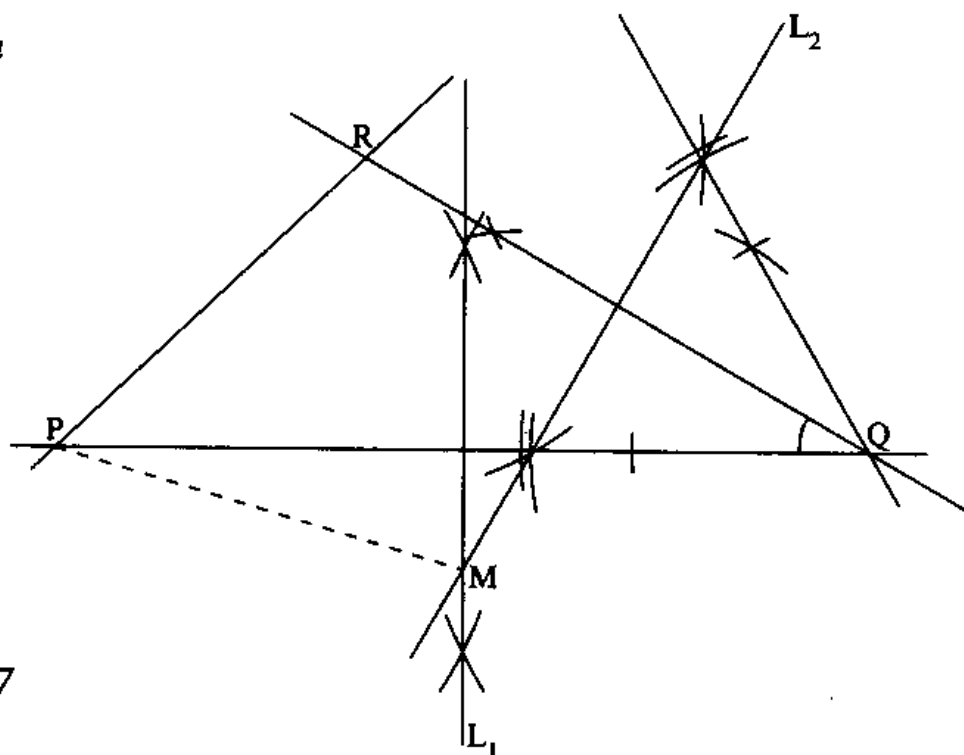


Fig. 3.17

In the figure:

- L_1 is perpendicular bisector of PQ.
- L_2 is the perpendicular bisector of QR.
- By measurement, $PM = 3.7$ cm.

Example 4

Using the triangle PQR in Example 3, construct the angle bisector locus L_3 of points equidistant from the lines PQ and RQ produced to X and Y respectively. If L_3 meets L_2 at S inside triangle PQR and at T outside triangle PQR, measure:

- (a) RS.
- (b) $\angle QTS$.
- (c) the perpendicular distance from S to RQ and S to PQ.
- (d) $\angle RQS$.

Solution

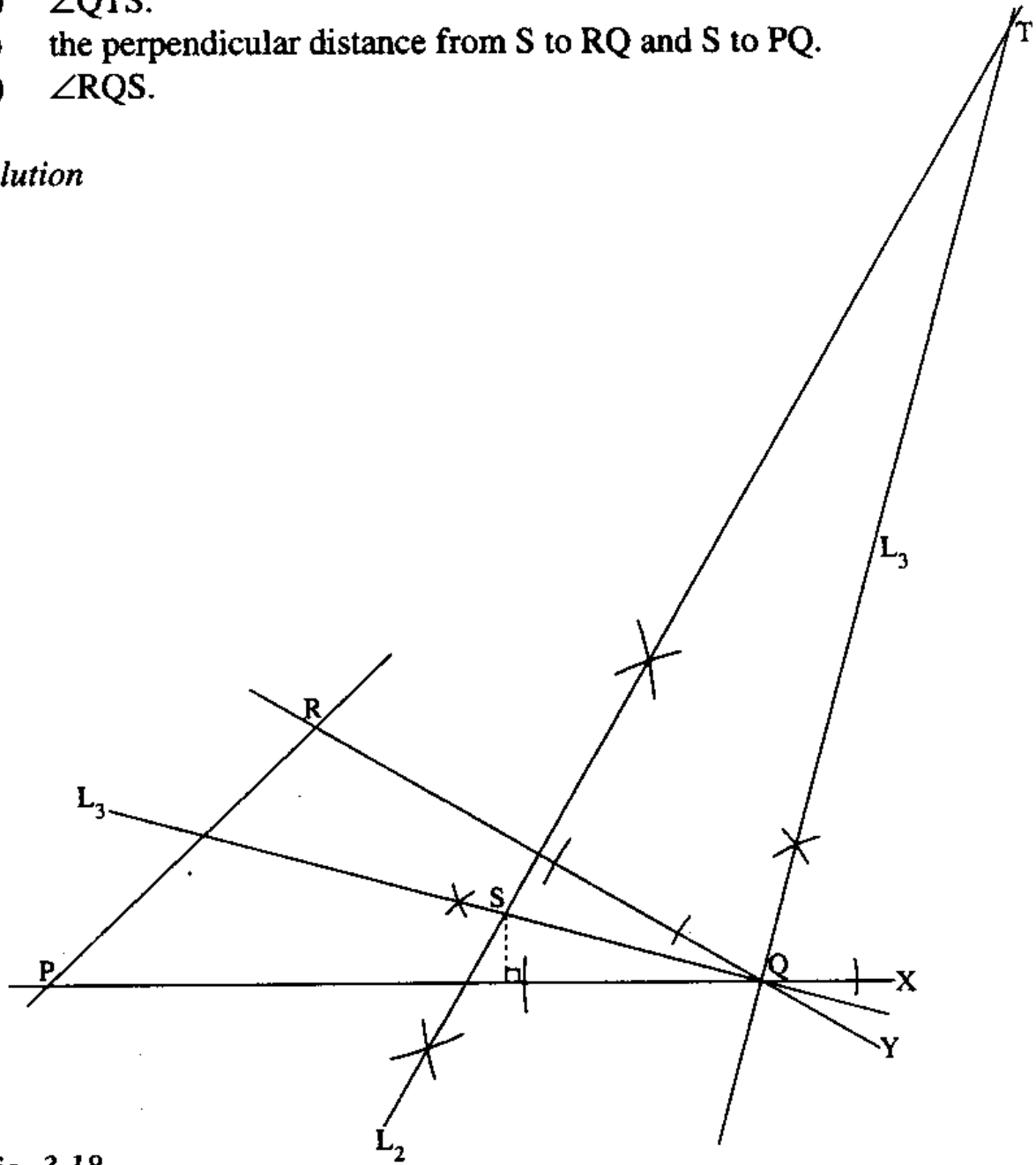


Fig. 3.18

In figure 3.18, L_3 is in two parts. SQ is the bisector of $\angle PQR$ and QT is perpendicular to SQ. By measurement:

- (a) $RS = 2.6$ cm
- (b) $\angle QTS = 15^\circ$
- (c) S to RQ is the same as S to PQ, which is 0.7 cm.
- (d) $\angle RQS = 15^\circ$

Intersecting Loci in a Triangle

- (i) In figure 3.19, L_1 is the locus of points equidistant from B and C (perpendicular bisector of BC) and L_2 is the locus of points equidistant from A and C (perpendicular bisector of AC). The two loci intersect at O. O is equidistant from the vertices of the triangle. Thus, if a circle is drawn with O as centre and OC, OB or OA as radius, it will pass through the vertices of triangle ABC. Such a circle is called a **circumcircle** and O the **circumcentre**.

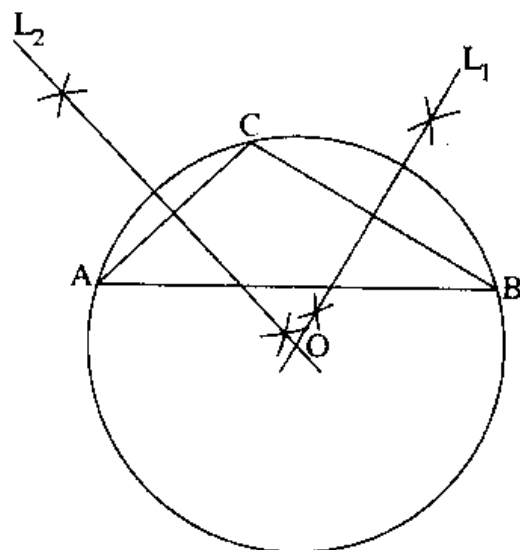


Fig. 3.19

- (ii) In figure 3.20, the locus L_1 of points equidistant from the side PQ and QR of triangle PQR, intersects with the locus L_2 of points equidistant from PQ and PR at O. Thus, if a circle is drawn with O as centre and OM as radius, it touches the three sides of the triangle. Such a circle is called an **inscribed circle (in-circle)**. The centre O (equidistant from the sides of the triangle) is called the **in-centre**.

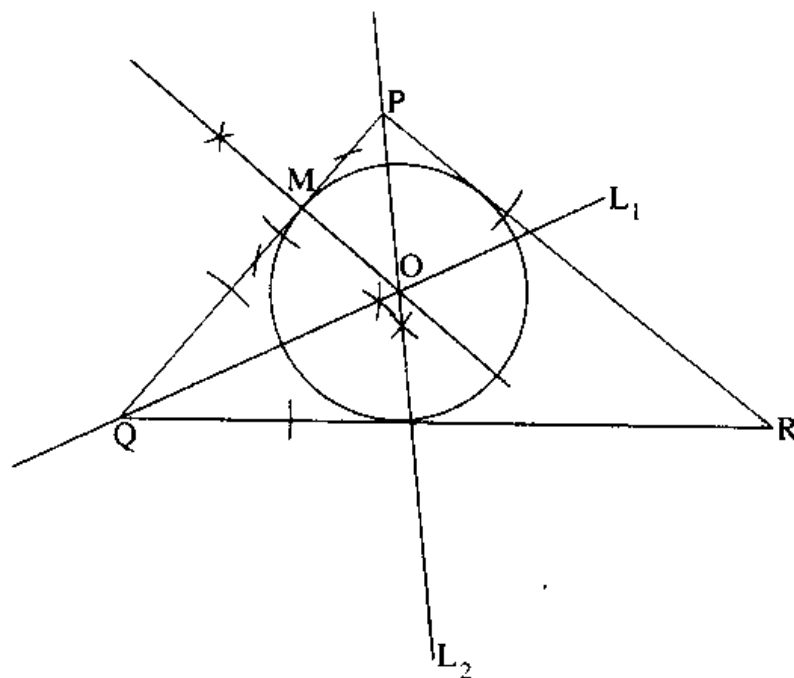


Fig. 3.20

- (iii) In figure 3.21, the locus L_1 of points equidistant from PQ produced and QR , meets L_2 , the locus of points equidistant from PR produced and QR at B (L_1 and L_2 are external angle bisectors). Point B is equidistant from PR produced, QR and PQ produced. Therefore, a circle drawn with B as centre and BN as radius will touch QR , PQ produced and PR produced. Such a circle is called an **escribed circle (ex-circle)** and B is the **ex-centre**. Locate the other ex-centres of the triangle.

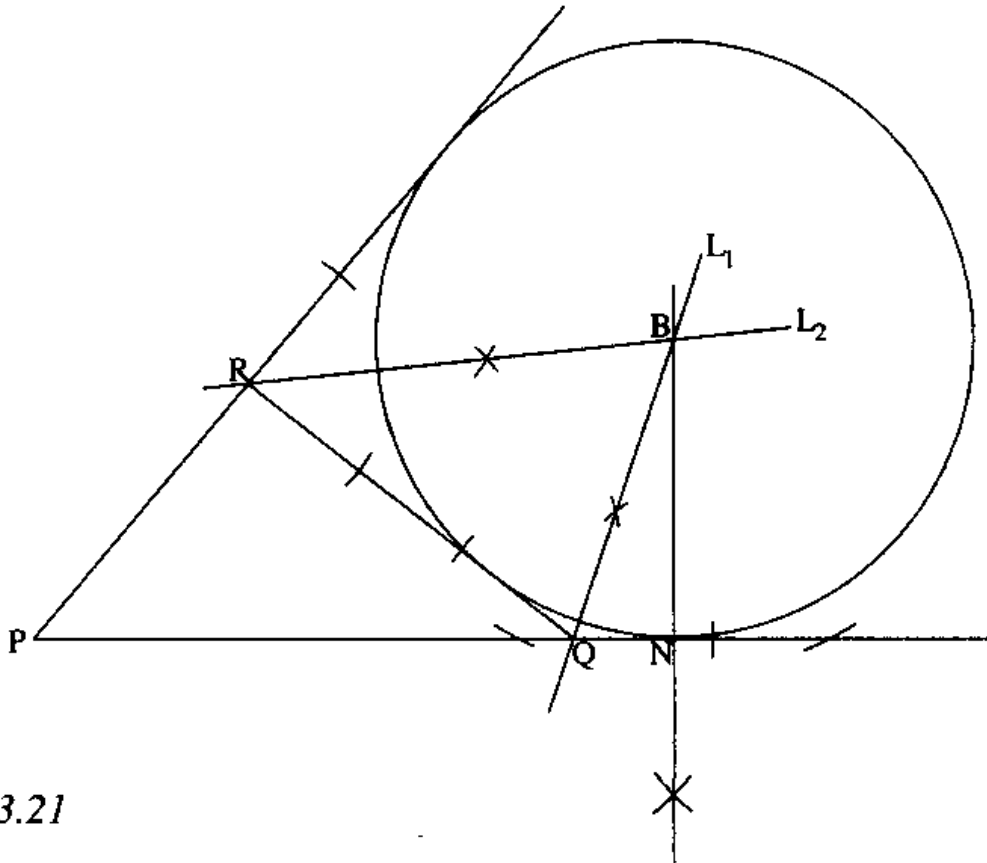


Fig. 3.21

Exercise 3.3

- Triangle ABC is such that $AB = 5$ cm, $BC = 4.5$ cm and $AC = 5.7$ cm. Find a point P equidistant from AC and BC and which is 4 cm from A . Measure CP and angle ACP .
- Draw two lines XY and ZY such that $\angle XYZ = 60^\circ$. Construct a point P , 2 cm from XY and equidistant from XY and YZ . Measure YP .
- Draw a line $AB = 4$ cm. Construct the locus L_1 of points equidistant from A and B . On L_1 , mark a point M , 6 cm from A . Construct the locus L_2 of points equidistant from A and M . If L_1 meets L_2 at R , measure:
 - BR .
 - Angle BRM .
- Two posts L and K are 13 m apart. A water pipe is leaking at a point P , 8 m from L and 14 m from K . Use suitable scale to draw a diagram showing two possible positions of P . Measure $\angle LPK$.

5. PQRS is a rectangle such that $PQ = 8$ cm and $QR = 5$ cm. Draw the rectangle and indicate the locus of a point T within the rectangle which is equidistant from PQ and QR. How many points belong to the locus if:
 - (a) $TR = 4$ cm?
 - (b) $ST = 3$ cm?
6. Two straight water pipes intersect at an angle of 40° . An electric post is equidistant from the pipes and 40 metres from their intersection. Use a suitable scale to draw a diagram indicating all the possible positions of the post.
7. X is a fixed point 3 cm from the straight line YZ. Construct points which are 1 cm from YZ and 3.5 cm from X. Measure the distance between the points.
8. If $YZ = 7$ cm, construct the locus of points X such that the area of triangle XYZ is 14 cm². Use a suitable construction to find the locus of all points Z such that $\angle YXZ = 50^\circ$. Measure XY and XZ.
9. Construct three points A, B and C such that $AB = 3.5$ cm, $AC = 4$ cm and $BC = 4.5$ cm. Find a point P such that it is 3 cm, 2 cm and 2.5 cm from A, B and C respectively. Measure the angle APB.
10. A line AB is 5 cm long. P is a variable point such that the area of triangle APB is 20 cm² and $\angle APB$ is 90° . Draw triangle APB on one side of AB. Measure $\angle ABP$ and $\angle BAP$.
11. The base PQ of triangle PQR is 8 cm long. Construct the triangle if its area is 16 cm² and $\angle QPR = 45^\circ$. Measure PR and QR.
12. Construct triangle PQR such that $PQ = 8.5$ cm, $QR = 6.5$ cm and $PR = 4.5$ cm. Locate the point which is equidistant from P, Q and R. Hence, draw a circle passing through P, Q and R. Measure its radius.
13.
 - (a) Construct an equilateral triangle ABC of side 4 cm.
 - (b) Locate the in-centre and draw the inscribed circle.
 - (c) Draw the escribed circles. What do you notice about the radii of the escribed circles?
14. Construct:
 - (a) an equilateral triangle ABC of side 4 cm.
 - (b) the angle bisector loci of the exterior angles at B and C.
 - (c) the angle bisector locus of angle BAC.What do you notice about the three loci?
15. Construct a rhombus PQRS in which $PQ = QR = 6$ cm and $\angle PQR = 60^\circ$.
 - (a) Measure QS.

- (b) On the same diagram, construct the inscribed circle of triangle PRS.
- (c) Construct the locus of points equidistant from P and R.
- (d) If T is a point on the in-circle in (b) above such that $PT = TS$ and $\angle PTS$ is acute, find PT .

3.4: Loci of Inequalities

An inequality is represented graphically by showing all the points that satisfy it. The intersection of two or more regions of inequalities gives the intersection of their loci.

Example 5

Draw the locus of points (x, y) such that $x + y < 3$, $y - x \leq 4$ and $y > 2$.

Solution

Draw the graphs of $x + y = 3$, $y - x = 4$ and $y = 2$, as shown in figure 3.22.

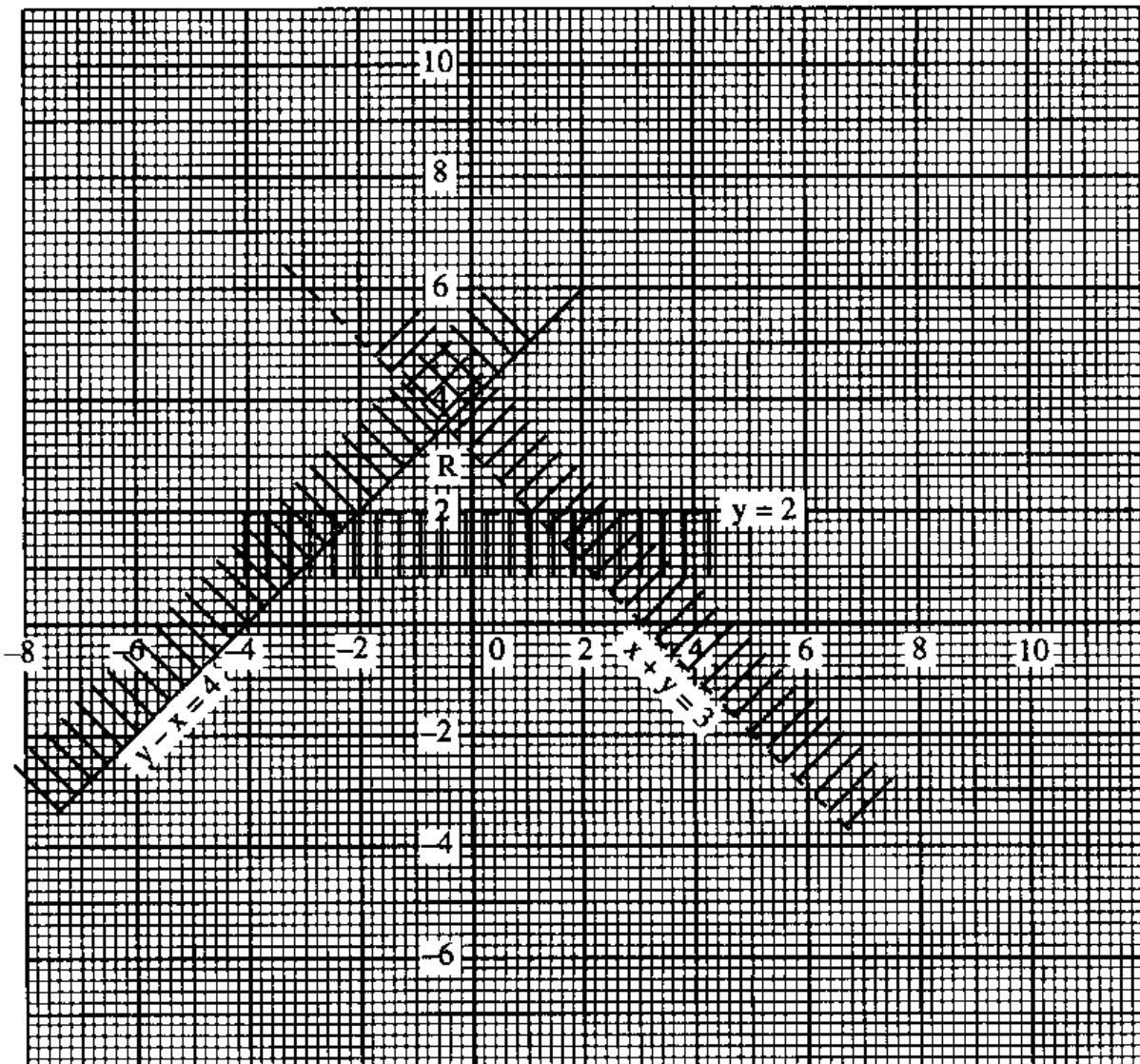


Fig. 3.22

The unwanted regions are usually shaded. In figure 3.22, the unshaded region marked R is the locus of points (x, y) such that $x + y < 3$, $y - x \leq 4$ and $y > 2$. Why are the graphs of $x + y = 3$ and $y = 2$ shown by broken lines and that of $y - x = 4$ by a full line?

Example 6

P is a point inside rectangle ABCD such that $AP \leq PB$ and $\angle DAP \geq \angle BAP$. Show the region in which P lies.

Solution

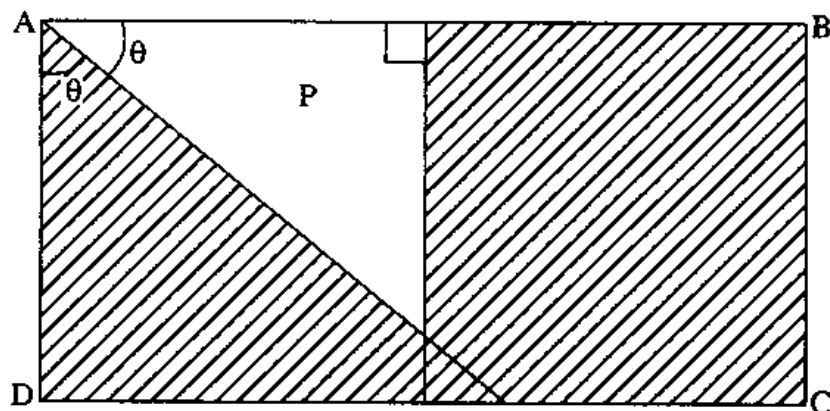


Fig. 3.23

Draw a perpendicular bisector of AB ($AP = PB$) and shade the unwanted region. Bisect $\angle DAB$ ($\angle DAP = \angle BAP$) and shade the unwanted region. P lies in the unshaded region.

Example 7

Draw the locus of a point P which moves that $AP < 3$ cm.

Solution

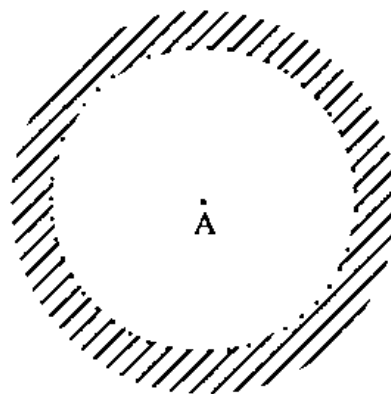


Fig. 3.24

- (i) Draw a circle, centre A and radius 3 cm.
- (ii) Shade the unwanted region.

Exercise 3.4

In questions 1 to 5, draw the locus of points that satisfy the following inequalities simultaneously:

1. $0 \leq y < 3$
 $0 \leq x < 4$
2. $y + 3x \leq 7$
 $y < 4x + 14$
 $y \geq 0$
3. $3x - 2y \leq 10$
 $x \geq 0$
 $y \geq 0$
4. $y - x < 0$
 $x \leq 6$
 $y \geq 0$
5. $y \geq 0$
 $2y \leq x + 4$
 $y \leq 2x + 4$
 $y + 2x \leq 4$
 $2y + x \leq 4$
6. Show on the cartesian plane the locus of points that satisfy the inequality;
 $x^2 + y^2 \leq 2^2$
7. Draw the locus of points that satisfy the inequality;
 $(x + 1)^2 + (y - 2)^2 > 25$.
8. Draw the locus of points that satisfy both the regions $y > -x$ and $y < 4 - x^2$.
9. Draw the locus of points that satisfy both the inequalities $y \leq x$ and $y + x \geq 0$. On your diagram, draw the locus of points which are equidistant from the lines $y = x$ and $y + x = 0$ and is in the wanted region. What is the equation of this line? For what range of values of x is the equation valid?
10. Find the inequality representing the locus of a point P which moves so that its distance from the point (3, 1) is less than 2 units.
11. A rectangle ABCD is such that AB = 9 cm and BC = 5 cm. A variable point P moves inside the rectangle such that $AP \leq PB$ and $AP > 4$ cm. Show the region where P lies. Hence, find the area of this region.
12. Draw the locus of a variable point O such that $2 \text{ cm} < OP \leq 4 \text{ cm}$. Calculate the area of the region where P lies.

3.5: Locus involving Chords

In this section, the following properties of chords of a circle will be used in constructions:

- (i) Perpendicular bisector of any chord passes through the centre of the circle.
- (ii) The perpendicular drawn from a centre of a circle bisects the chord.
- (iii) If chords of a circle are equal, they are equidistant from the centre of the circle, and vice-versa.
- (iv) In figure 3.25, if a chord AB intersects chord CD at O, $AO = x$, $BO = y$, $CO = m$ and $DO = n$, then $m \times n = x \times y$.

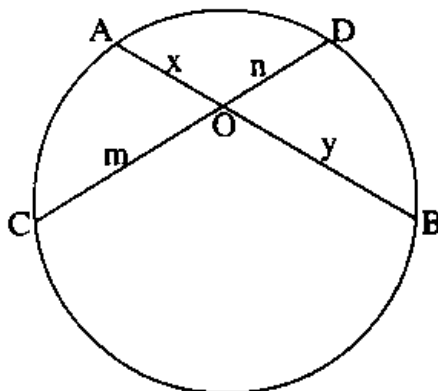


Fig. 3.25

Exercise 3.5

1. A chord AB of a circle is 4 cm. Draw the locus of points equidistant from A and B. Choose one point as a centre to construct the circle through A and B. How many such points can you choose?
2. Point O is the centre of a circle whose radius OR is 3 cm. A chord AB is 2 cm from O and meets OR at N. Construct the chord. What is the relationship between the lengths AN and NB?
3. A circle with centre O, radius 6 cm, has a chord AB 3.5 cm long. Draw the locus of points equidistant from A and B. Measure the distance between O and AB. Use it to construct another chord CD equal in length to AB.
4. Two chords AB and CD intersect at O. If $AO = 2$ cm, $OB = 5$ cm and $CO = 1$ cm, find OD. Hence, construct the two chords and draw the locus of points which pass through A, B, C and D. Calculate the area of the region enclosed by the locus.

Mixed Exercise 1

- Use the unit square to name the transformations represented by each of the following matrices:
(a) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Use the matrix method to solve the following simultaneous equations:
 $2x - y = 3$
 $3x + 2y = 1$
- A transformation is described by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The images of points $P(4, 3)$ and $Q(2, 2)$ under the transformation are $P'(-4, 3)$ and $Q'(-2, 2)$ respectively.
 - Find the values of a , b , c and d .
 - Describe the transformation fully.
- State the locus of points equidistant from the lines $x = 4$ and $x = 6$.
- Describe the locus of each of the following:
 - If P is always equidistant from two points Q and R on a plane.
 - If P moves in three dimensions so that its distance from a fixed point is x cm.
 - If X and Y are two fixed points and P moves in the plane such that $\angle XPY = 90^\circ$.

- Using the data given below, find the mean, median and the mode:

x	0	1	2	3	4	5	6
f	1	3	9	22	18	5	2

- Draw a histogram and a frequency polygon of the frequency distribution shown in the table below:

<i>Class</i>	18-24	25-31	32-38	39-45	46-52	53-59
<i>Frequency</i>	2	7	13	20	12	6

Find the modal class and the mean of the above data.

- Find the image of triangle PQR with vertices at $P(2, 6)$, $Q(2, 2)$ and $R(4, 6)$ under the transformations represented by the following matrices:

(a) $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & 2 \\ -1 & \frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$

9. The vertices of a triangle PQR are at P(0, 0), Q(4, 0) and R(0, 2). Find:
- the co-ordinates of the vertices of $\Delta P'Q'R'$, the image of ΔPQR under the transformation matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.
 - the co-ordinates of the vertices of $\Delta P''Q''R''$, the image of $\Delta P'Q'R'$ under the transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.
 - The matrix that represents the transformation that maps ΔPQR onto $\Delta P''Q''R''$. Describe the transformation fully.
10. A triangle PQR is such that its base QR = 8 cm and the area of the triangle is 20 cm². Find the locus of possible positions of the vertex P
11. An object is given a reflection in y-axis followed by a reflection in the x-axis. Find:
- a single matrix representing the transformation which maps the object to the final image.
 - the single matrix which represents the transformation which maps the final image onto the original object.
12. A triangle PQR with vertices at P(0, 0), Q(6, 0) and R(0, 4) is transformed by the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$. Find:
- the co-ordinates of vertices of the image triangle P'Q'R'.
 - the area of $\Delta P'Q'R'$ using the determinant of the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$
13. A line PQ is 6 cm long. Find the locus of a point R so that $\angle PRQ = 90^\circ$.
14. The image of P(4, 1) is P'(-2, 1) under a reflection.
- Find the equation of the mirror line.
 - If the mirror line in (a) above is rotated through 90° about the point (1, 0), find the equation of its image line.
15. Given that $\mathbf{A} =$, $\mathbf{B} =$ and $\mathbf{C} =$,
evaluate $4\mathbf{AB} - \mathbf{C}^2 + 2\mathbf{BC}$
16. A football league consists of six teams A, B, C, D, E and F. Teams A and F scored a total of 2 and 6 goals respectively. Teams B and C ended in a

tie in terms of total goals scored. Teams D and E also tied in the number of goals. Team D scored a total of 2 goals more than C. If the mean score for the six teams is 4, calculate the total scores of teams B, C, D and E.

17. Using a ruler and a pair of compasses only, construct a triangle PQR in which $PQ = 6$ cm, $QR = 9$ cm and $\angle PQR = 60^\circ$. Measure PR. Construct the circumcircle of triangle PQR. Measure the distance of R from the circumcentre.
18. The table below shows scores of forty pupils in a mathematics test:

50	63	70	30	22	25	82	30
43	47	65	32	27	60	50	52
30	55	29	80	58	46	72	24
35	50	33	36	44	28	63	73
40	42	50	41	80	30	65	75

- (a) Calculate the range of the marks.
 (b) Calculate the mean mark.
 (c) Write down the mode and the median marks.
19. Using a ruler and a pair of compasses only, construct:
 (a) triangle ABC in which $AB = AC = 13$ cm and $BC = 10$ cm.
 (b) an inscribed circle of triangle ABC.
 Measure the distance of the in-centre from point C.
20. In triangle ABC, $AB = 8$ cm, $BC = 5$ cm and angle ABC is 48° . Show all the points which are 2 cm from AB and 6 cm from C.
21. Below is a table showing marks scored in a test by a certain class of students. Draw a cumulative frequency curve for the data:

<i>Class</i>	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
<i>Frequency</i>	6	5	8	10	12	17	20	12	7	3

Use the curve to estimate:

- (a) the median mark.
 (b) the number of candidates who scored between 50 and 65 marks.
 (c) the percentage of candidates who passed if the passmark was 35.
22. A variable point P moves so that it is always equidistant from the lines $x = 0$ and $y = 4$. Find the equation of the locus of P.

23. A and B are two fixed points. Draw the locus of a variable point P which moves such that the angle APB is always:
- 90°
 - 60°

24. The table below shows the length in minutes of calls made through an exchange in one day:

<i>Time</i>	1-3	4-6	7-9	10-12	13-15
<i>Frequency</i>	12	7	6	3	2

Calculate:

- the mean.
 - the median and quartiles.
 - the standard deviation.
25. The masses, to the nearest kg, of 50 adults were recorded as follows:

<i>Mass (kg)</i>	45-50	51-56	57-62	63-68	69-74	75-80
<i>Frequency</i>	2	10	11	20	6	1

Estimate:

- the mean.
 - the median.
 - the interquartile range.
26. The heights, to the nearest cm, of 100 students were recorded as shown below:

<i>Height (cm)</i>	140-145	146-151	152-157	158-163	164-169	170-175	176-181	182-187	188-193	194-199
<i>Frequency</i>	3	4	7	10	10	15	25	16	8	2

Find:

- the mean.
 - the variance.
 - the standard deviation.
27. Draw triangle PQR in which $\angle QPR = 75^\circ$, $PQ = 8$ cm and $PR = 7$ cm. Construct two positions of a point T, equidistant from lines PR and QR, such that the area of a triangle PTQ is 15.2 cm².
28. The following table shows the marks scored in a Mathematics test by 40 students in class 4G:

<i>Mark</i>	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
<i>No. of students</i>	3	5	6	8	7	5	4	2

For the above frequency distribution:

- calculate the mean mark.
- draw the cumulative frequency graph.
- use the graph in (b) to estimate the median.
- find the percentage of the number of pupils who scored between 8 and 29 marks.

29. The table below shows the strength of 100 metal bolts in N/mm^2 :

Strength N/mm^2	4.1-4.3	4.4-4.6	4.7-4.9	5.0-5.2	5.3-5.5	5.6-5.8	5.9-6.1	6.2-6.4	6.5-6.7	6.8-7.0
Frequency	2	3	5	18	23	21	16	8	3	1

Find:

- the mean.
 - the standard deviation.
30. A point P is 65 km from Q on a bearing 068° . Point R is 50 km from P on a bearing 155° . Find:
- the distance of R from Q.
 - the bearing of R from Q.
31. The parallelogram whose co-ordinates are A(1, 1), B(3, 3), C(3, 6) and D(1, 4) is mapped onto Q by a transformation X defined by the matrix . Q is mapped onto R by a transformation represented by . Find:
- the co-ordinates of the vertices of R and its area.
 - the matrix which maps R back onto parallelogram ABCD.
32. Complete the following table:

Class	Midpoint (x)	Frequency (f)	fx
50-54	52	5	—
55-59	57	3	—
—	62	—	434
65-69	—	—	335
—	72	5	—
—	—	—	231
—	82	4	—
		$\Sigma f =$	$\Sigma fx =$

Calculate the mean and standard deviation of the distribution.

33. State the locus of the midpoints of all chords 6 cm long inside a circle of radius 5 cm.

TRIGONOMETRY III

4.1: Some Trigonometric Ratios

In Book Two, we studied the basic trigonometric ratios, i.e., sine, cosine and tangent for a right-angled triangle.

Consider a right-angled triangle OAB with hypotenuse r units, as in figure 4.1.

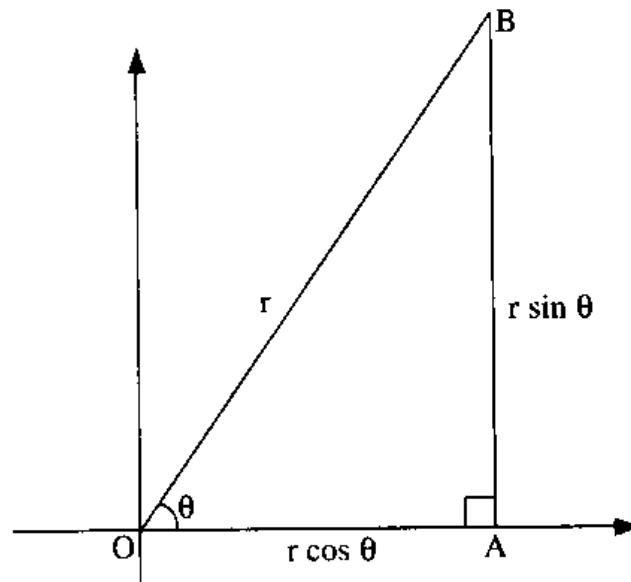


Fig 4.1

$$\text{Then, } \sin \theta = \frac{AB}{r}$$

$$\text{Therefore, } AB = r \sin \theta$$

$$\text{Similarly, } OA = r \cos \theta$$

Since triangle OAB is right-angled;

$$OA^2 + AB^2 = OB^2 \text{ (Pythagoras' theorem)}$$

$$(r \cos \theta)^2 + (r \sin \theta)^2 = r^2 \text{ (substitution)}$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$$

Dividing both sides by r^2 gives;

$$\cos^2 \theta + \sin^2 \theta = 1$$

This is the trigonometric equivalent of the Pythagoras' theorem and is true for all values of θ .

$$\text{Note also from figure 4.1 that } \tan \theta = \frac{\sin \theta}{\cos \theta}$$

Example 1

If $\tan \theta = a$, show that;

$$\frac{\cos \theta \sin^2 \theta + \cos^3 \theta}{\sin \theta} = \frac{1}{a}$$

Solution

Factorising the numerator gives;

$$\frac{\cos \theta (\sin^2 \theta + \cos^2 \theta)}{\sin \theta}$$

Since $\sin^2 \theta + \cos^2 \theta = 1$;

$$\frac{\cos \theta (\sin^2 \theta + \cos^2 \theta)}{\sin \theta} = \frac{\cos \theta (1)}{\sin \theta}$$

But $\frac{\sin \theta}{\cos \theta} = \tan \theta$

Therefore, $\frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} = \frac{1}{a}$

Example 2

Show that $\frac{(1 - \cos \theta)(1 + \cos \theta)}{(1 - \sin \theta)(1 + \sin \theta)} = \tan^2 \theta$

Solution

Removing the brackets from the expression gives;

$$\frac{1 - \cos^2 \theta}{1 - \sin^2 \theta} \quad [\text{Recall } (A - B)(A + B) = A^2 - B^2]$$

Using $\sin^2 \theta + \cos^2 \theta = 1$;

$$1 - \sin^2 \theta = \cos^2 \theta$$

Also;

$$1 - \cos^2 \theta = \sin^2 \theta$$

Therefore $\frac{1 - \cos^2 \theta}{1 - \sin^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta}$
 $= \tan^2 \theta$

Example 3

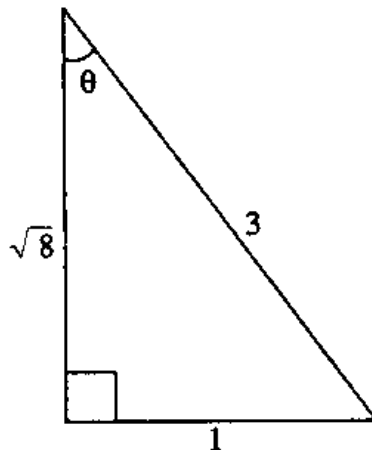
Given that $\sin \theta = \frac{1}{3}$, find without using tables;

(a) $\cos^2 \theta$

- (b) $\tan^2 \theta$
 (c) $\tan^2 \theta + \cos^2 \theta$

Solution

Using the right-angled triangle in figure 4.2;

**Fig 4.2**

$$(a) \quad \cos \theta = \frac{\sqrt{8}}{3}$$

$$\begin{aligned} \therefore \cos^2 \theta &= \left(\frac{\sqrt{8}}{3} \right)^2 \\ &= \frac{8}{9} \end{aligned}$$

$$(b) \quad \tan \theta = \frac{1}{\sqrt{8}}$$

$$\begin{aligned} \therefore \tan^2 \theta &= \left(\frac{1}{\sqrt{8}} \right)^2 \\ &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} (c) \quad \tan^2 \theta + \cos^2 \theta &= \frac{1}{8} + \frac{8}{9} \\ &= 1 \frac{1}{72} \end{aligned}$$

Exercise 4.1

1. Given that $\cos \theta = \frac{1}{4}$ and θ is acute, find without using tables:

- (a) $\sin \theta$

(b) $\tan^2 \theta$

(c) $\frac{1}{\sin^2 \theta}$

2. Simplify $\cos^2 \theta + \cos^2 (90 - \theta)$

In questions 3 to 9, prove the given identities.

3. $\tan \theta + \frac{1}{\tan \theta} = \frac{1}{\sin \theta \cos \theta}$

4. $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$

5. $\sin \theta \cos^2 \theta = \sin \theta - \sin^3 \theta$

6. $\frac{\cos \theta \tan \theta}{\sin \theta} - \cos^2 \theta = \sin^2 \theta$

7. $\frac{\sin^3 \theta}{\cos \theta} = \tan \theta - \cos^2 \theta \tan \theta$

8. $\frac{1}{\cos^2 \theta} - \cos^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta} = \sin^2 \theta$

9. $\frac{\tan^2 \theta + 1}{\tan^2 \theta} = \frac{1}{1 - \cos^2 \theta}$

10. If $x = 2\cos \theta$, show that $\sqrt{\frac{4-x^2}{x}} = \tan \theta$

(Consider positive square roots only)

11. If $y = \cos \theta$, show that;

$$y \sqrt{(2y^2 - 1)} = \sqrt{(2 \sin^2 \theta - 1)(\sin^2 \theta - 1)}$$

12. If $\sin \theta = \frac{2}{5 - \sqrt{2}}$, deduce:

(a) $\cos \theta$

(b) $\tan \theta$

(leave your answer in surd form)

4.2: Waves

Amplitude and Period

The graph in figure 4.3 is a sine wave:

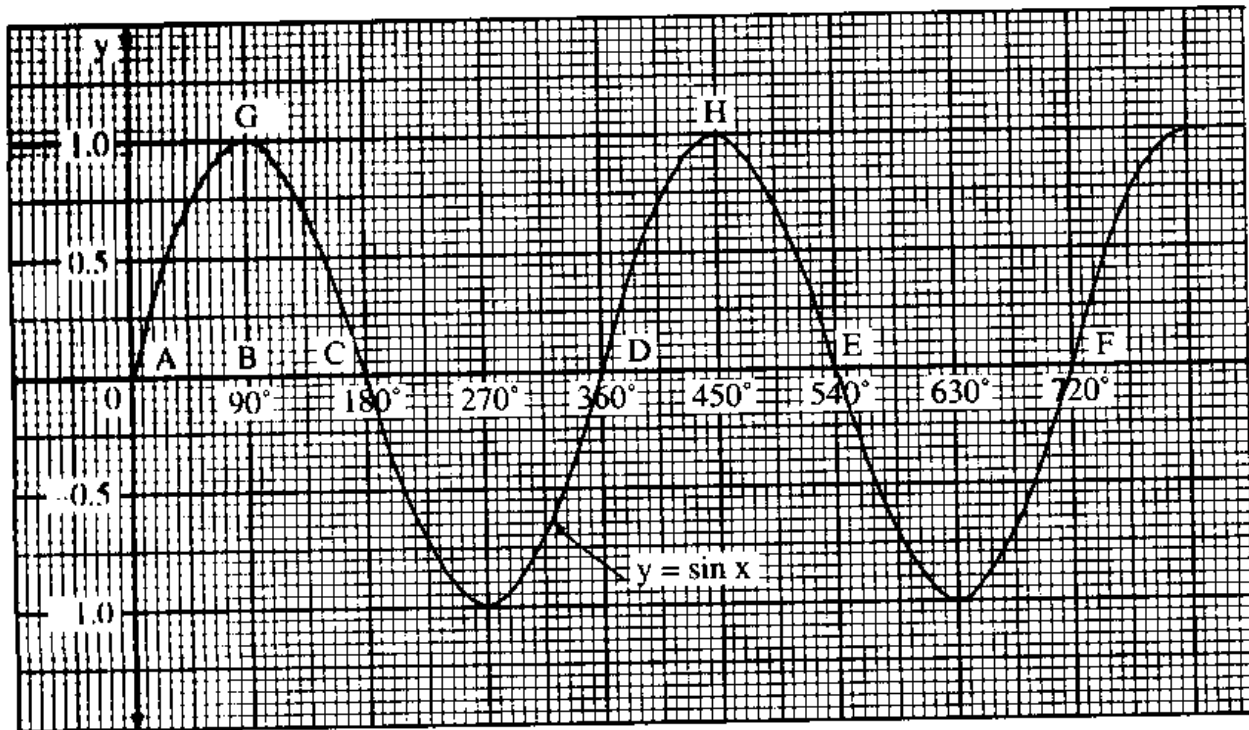


Fig 4.3

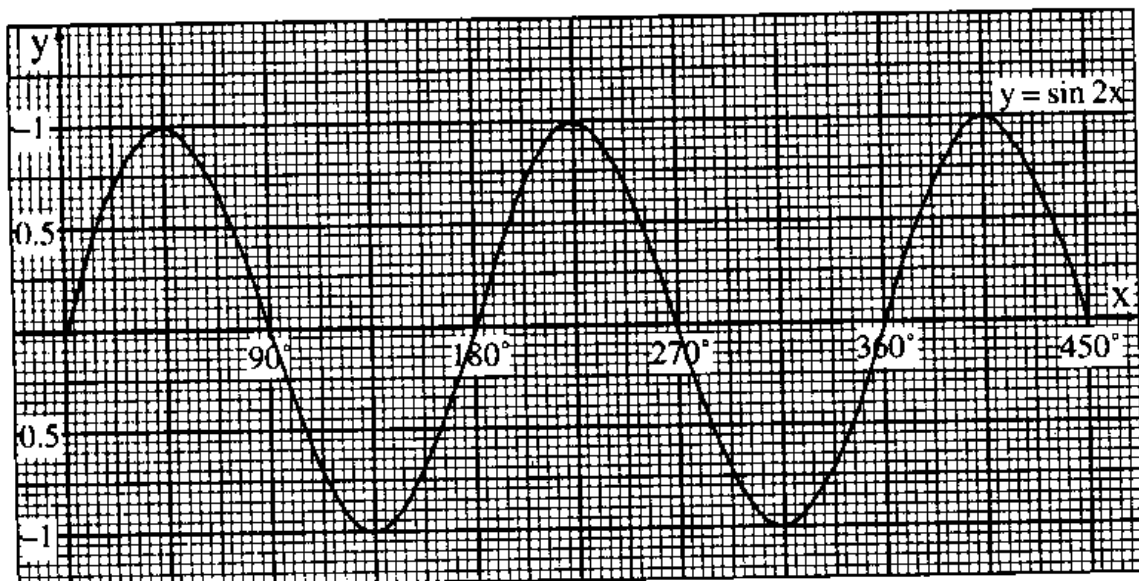
The amplitude of a wave is the maximum displacement of the wave above or below the x-axis. For example, in figure 4.3, the amplitude is 1 unit.

If we start from any point on the graph and follow the graph until it starts to repeat itself, we will have covered one cycle. For example AD and GH are cycles. The cycle in this graph occurs after every 360° . We say that the period of this wave is 360° or 2π radians.

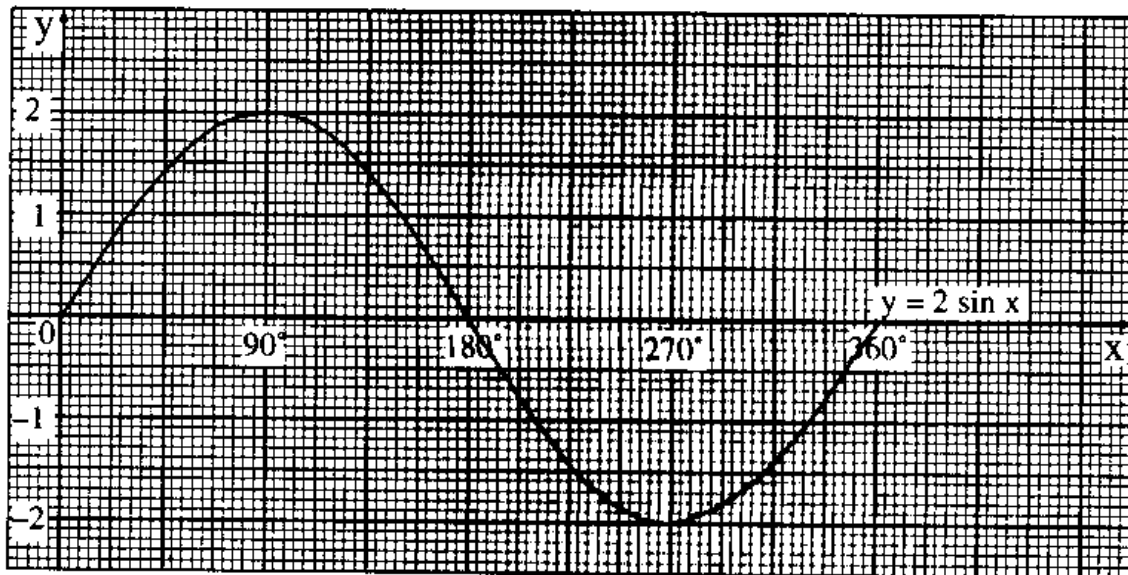
The period of a wave is the interval after which the wave repeats itself.

Exercise 4.2

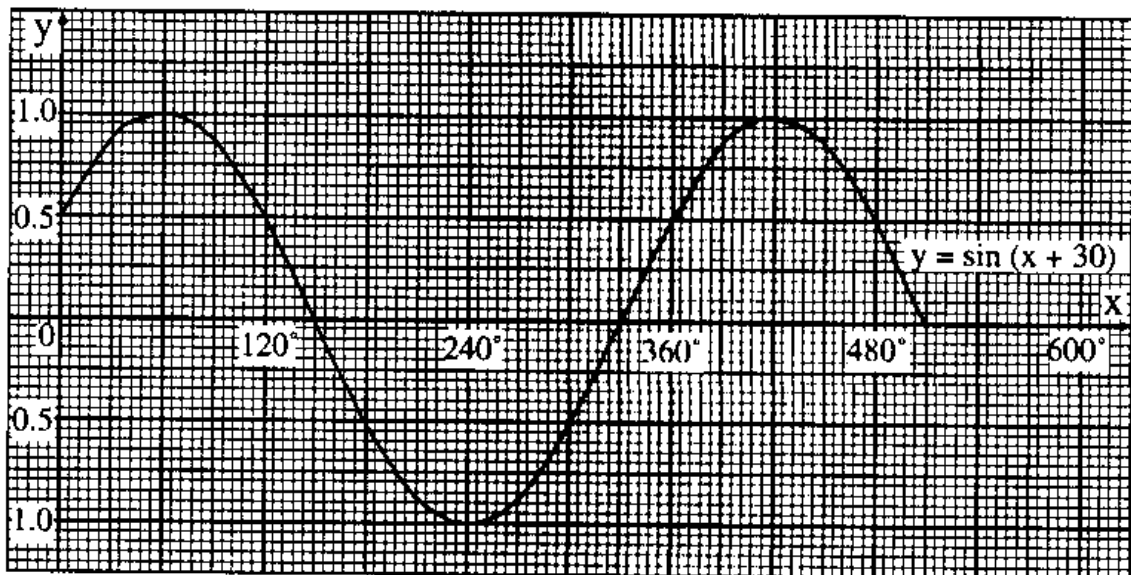
1. For each of the waves in figure 4.4, state the:
 - (a) amplitude.
 - (b) period.
- (i)



(ii)



(iii)



(iv)

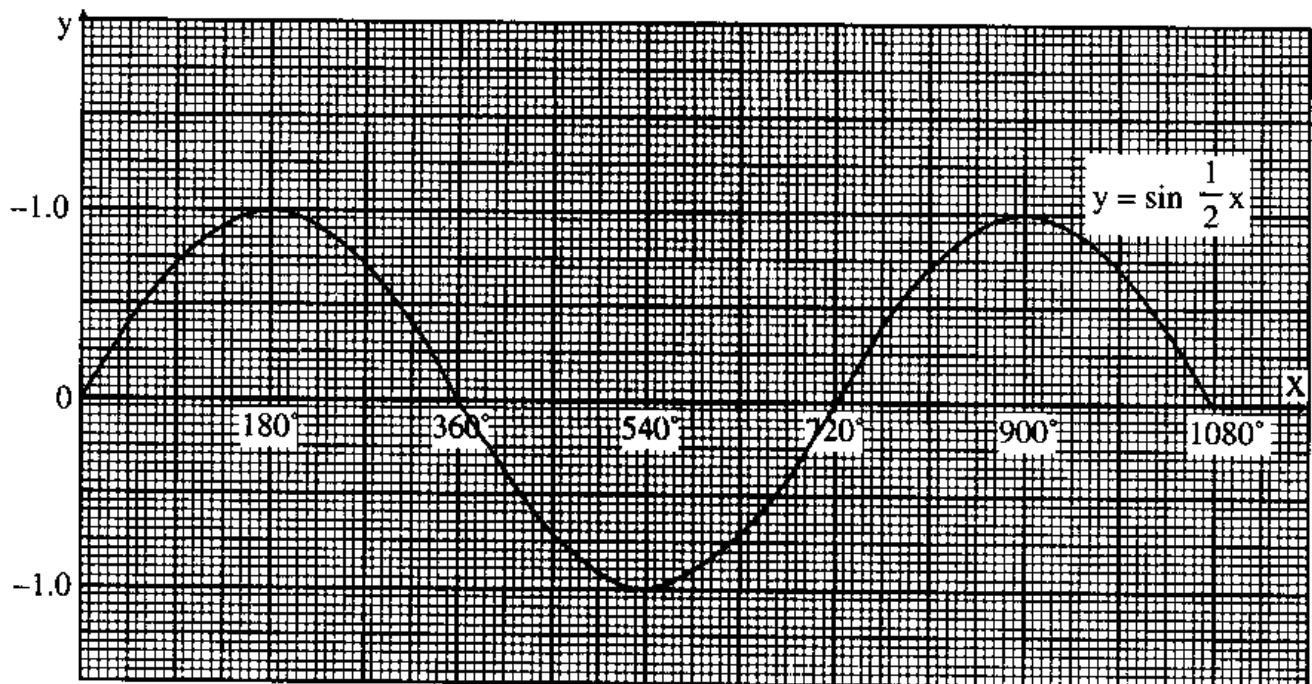


Fig 4.4

2. For each of the following, draw the graph and give its amplitude and period:

- (a) $y = \cos x$ for $0^\circ \leq x \leq 720^\circ$
 (b) $y = 2 \cos x$ for $0^\circ \leq x \leq 540^\circ$
 (c) $y = 3 \sin x$ for $0^\circ \leq x \leq 360^\circ$

Some Transformations of Waves

We can draw the graphs of $y = \sin x$ and $y = 3 \sin x$ on the same axes. The table below gives the corresponding values of $\sin x$ and $3 \sin x$ for $0^\circ \leq x \leq 720^\circ$:

x°	0	30	60	90	120	150	180	210	240	270	300	330	360
$\sin x$	0	0.50	0.87	1.00	0.87	0.50	0	-0.50	-0.87	-1.00	-0.87	-0.50	0
$3 \sin x$	0	1.50	2.61	3.00	2.61	1.50	0	-1.50	-2.61	-3.00	-2.61	-1.50	0

390	420	450	480	510	540	570	600	630	660	690	720
0.5	0.87	1.00	0.87	0.50	0	-0.50	-0.87	-1.00	-0.87	-0.50	0
1.50	2.61	3.00	2.61	1.50	0	-1.50	-2.61	-3.00	-2.61	-1.50	0

When the points are plotted, we obtain the graphs in figure 4.5. The wave $y = 3 \sin x$ can be obtained from $y = \sin x$ by applying a stretch of factor 3 with the x-axis invariant.

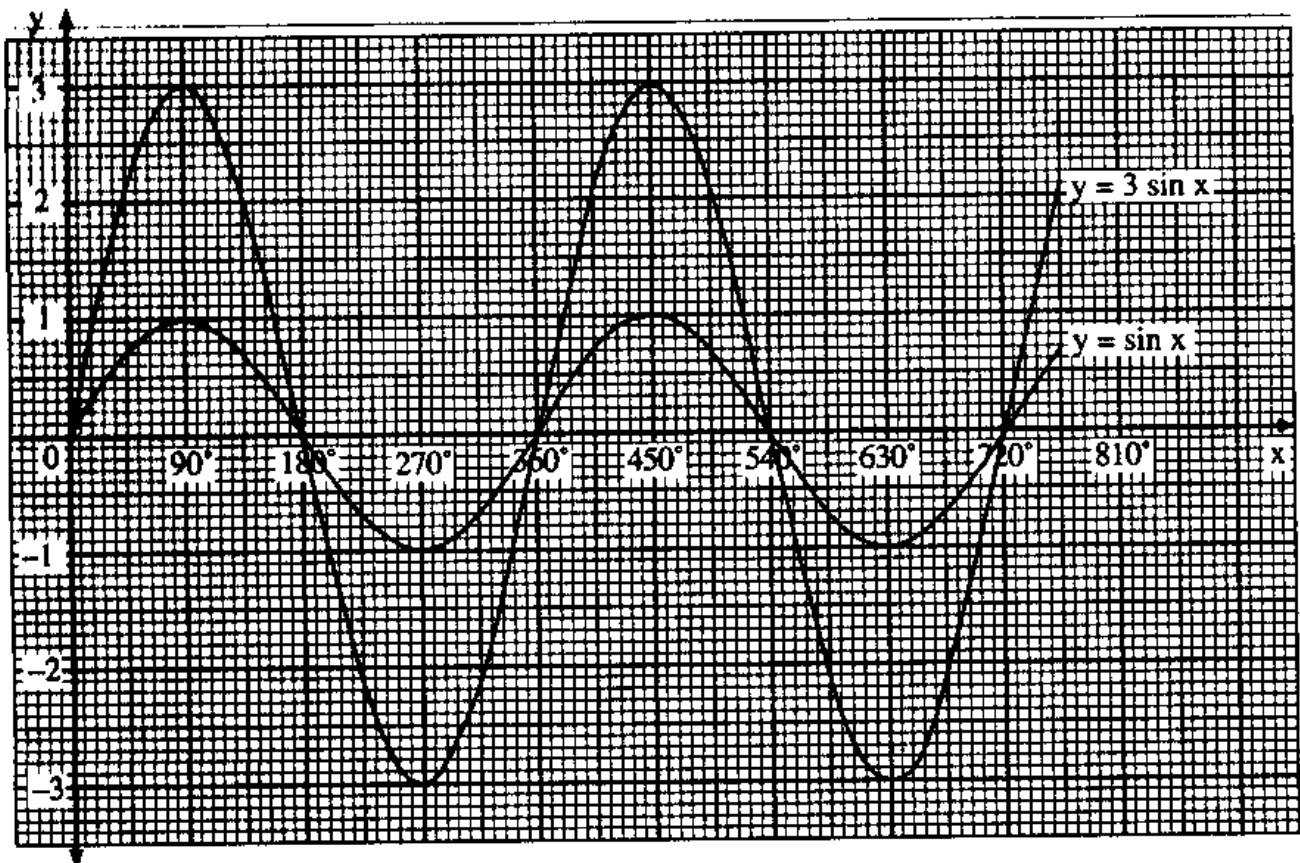


Fig 4.5

The amplitude of $y = 3 \sin x$ is 3, which is three times the amplitude of $y = \sin x$. The periods of the two waves are the same, that is, 360° or 2π .

Let us now compare the waves $y = \cos x$ and $y = \cos \frac{1}{2}x$, shown in figure 4.6. We can obtain $y = \cos \frac{1}{2}x$ from $y = \cos x$ by applying a stretch of factor 2, with y -axis invariant. The amplitudes of the two waves are the same. The period of $y = \cos \frac{1}{2}x$ is 4π , that is, twice the period of $y = \cos x$.

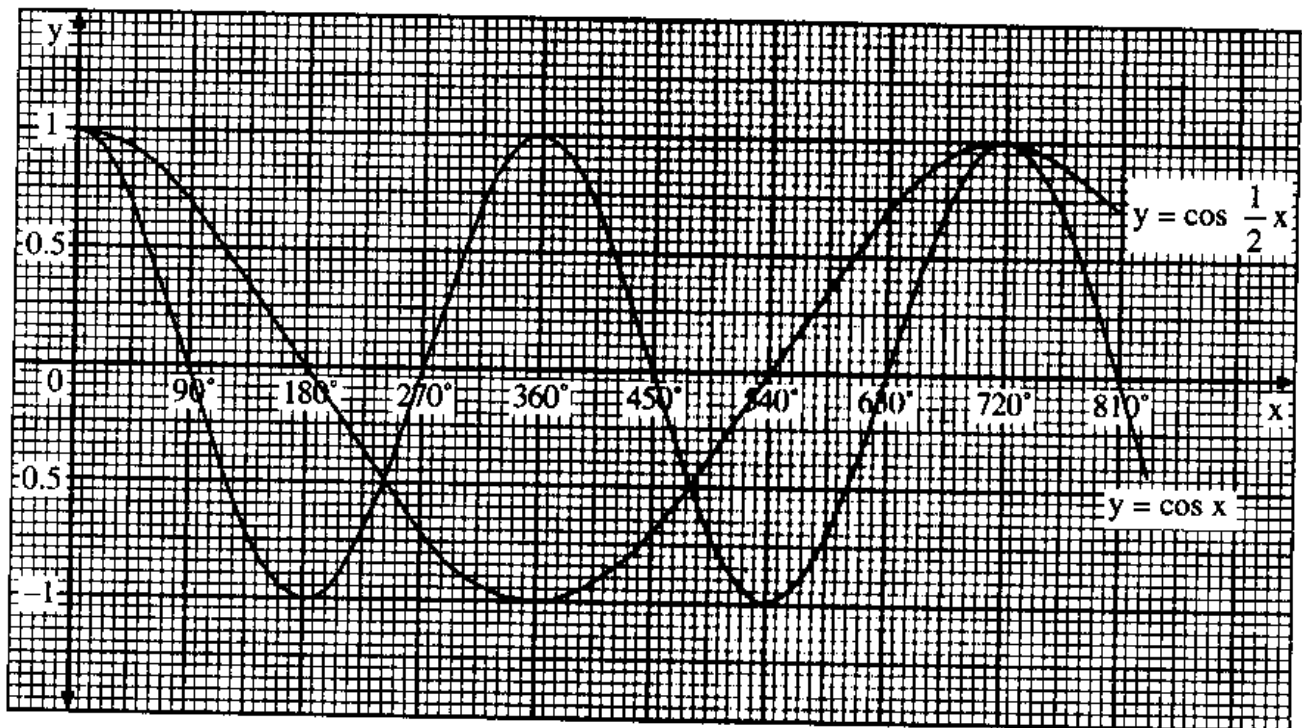


Fig 4.6

Example 4

By drawing the waves for $y = \sin x$ and $y = 2 \sin \frac{1}{2}x$ on the same axes, describe how to obtain $y = 2 \sin \frac{1}{2}x$ from $y = \sin x$. State the period and amplitude of $y = 2 \sin \frac{1}{2}x$.

Solution

Figure 4.7 shows the graphs of $y = \sin x$ and $y = 2 \sin \frac{1}{2}x$.

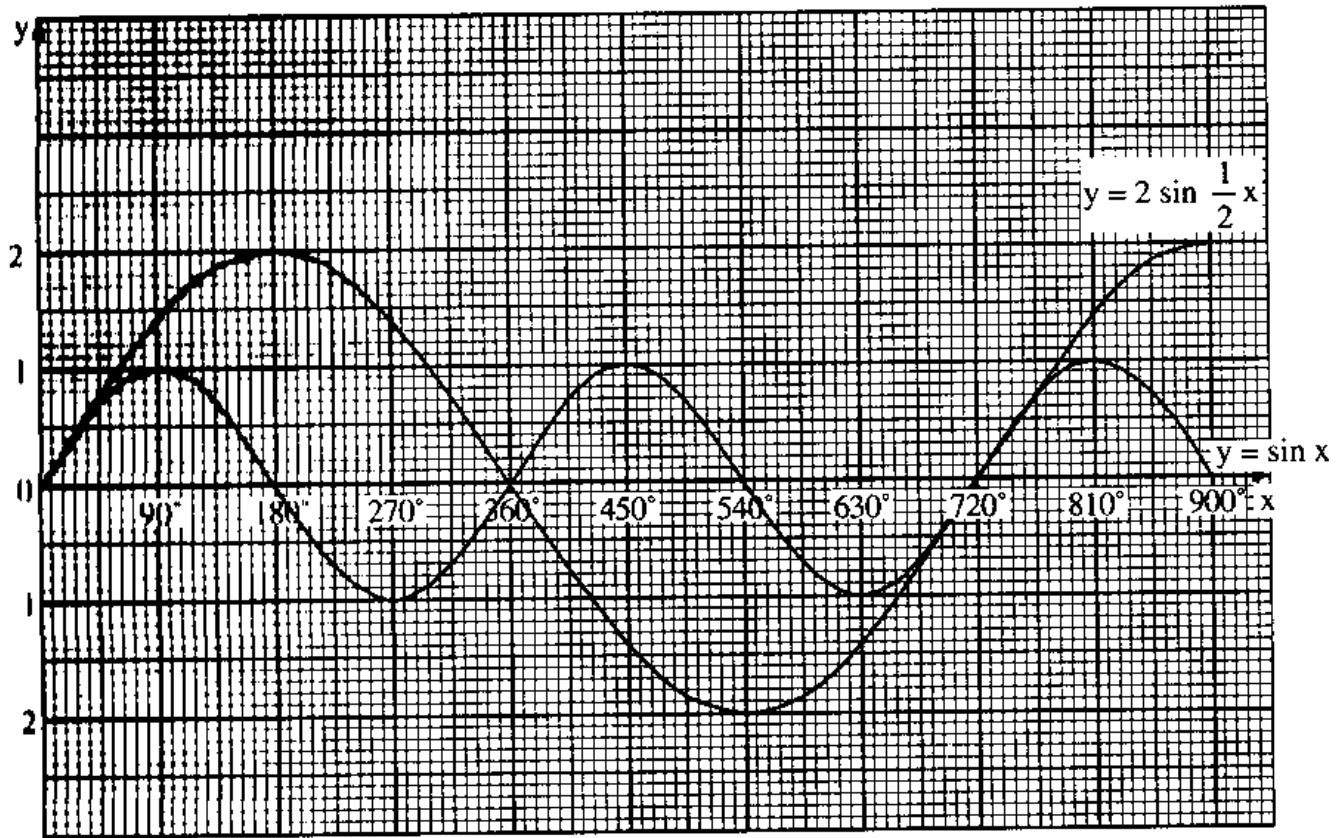


Fig 4.7

$y = 2 \sin \frac{1}{2} x$ can be obtained from $y = \sin x$ by a stretch of scale factor 2 with x -axis invariant followed by a stretch of factor 2 with y -axis invariant. The period for $y = 2 \sin \frac{1}{2} x$ is 720° or 4π radians and its amplitude is 2 units.

We now compare the waves for $y = \sin x$ and $y = \sin(x + 45)^\circ$. Using the table below, we obtain the graphs in figure 4.8.

Table for $y = \sin x$ and $y = \sin(x + 45)^\circ$

x°	0	45	90	135	180	225	270	315	360	405	450	495	540
$(x + 45)^\circ$	45	90	135	180	225	270	315	360	405	450	495	540	585
$\sin x$	0	0.71	1.00	0.71	0	-0.71	-1.00	-0.71	0	0.71	1.00	0.71	0
$\sin(x + 45)^\circ$	0.71	1.00	0.71	0	-0.71	-1.00	-0.71	0	0.71	1.00	0.71	0	-0.71

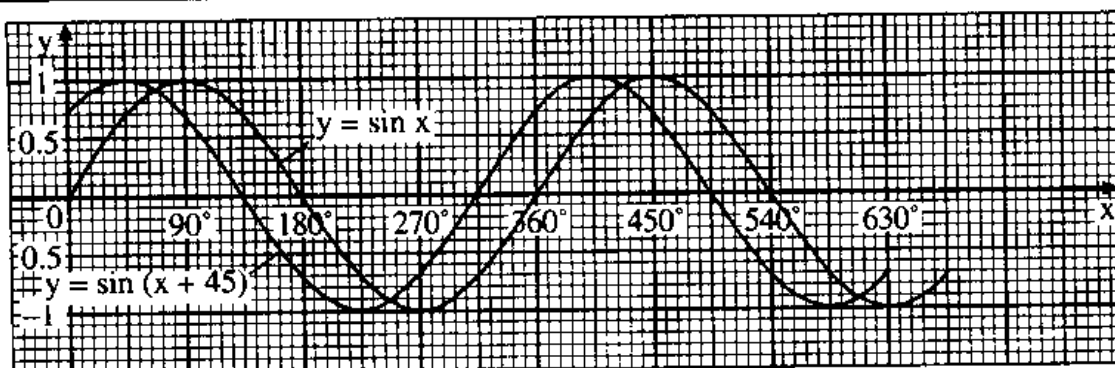


Fig 4.8

Note that the amplitudes and periods of the two waves are the same.

We notice that the wave of $y = \sin x$ leads that of $y = \sin(x + 45)^\circ$ by 45° . $y = \sin(x + 45)^\circ$ can be obtained from $y = \sin x$ by a translation of $\begin{pmatrix} -45 \\ 0 \end{pmatrix}$. We call the angle difference of 45° the **phase angle**.

Generally, in the equation of the form;

$$y = k \sin(bx + \theta) \text{ or} \\ = k \cos(bx + \theta),$$

where k , θ and b are constants:

- (i) the amplitude of the wave is k .
- (ii) the period of the wave is $\left(\frac{360}{b}\right)^\circ$ or $\left(\frac{2\pi}{b}\right)^\circ$
- (iii) the angle θ is the phase angle.

Exercise 4.3

1. Draw on the same axes the graphs of:

- (a) $y = \cos x$ and $y = \cos \frac{1}{3}x$ for $0^\circ \leq x \leq 720^\circ$
- (b) $y = \sin 2x$ and $y = 4 \sin 2x$ for $0 \leq x \leq 360^\circ$
- (c) $y = \sin x$ and $y = \sin(x - 60)^\circ$ for $0 \leq x \leq 540^\circ$

In each case, describe the transformation that maps the first graph onto the second.

2. On the same axes, draw the graphs of $y = \cos x$ and $y = 2 \cos \frac{1}{2}x$. State the amplitude and period of each graph.

3. On the same axes, draw the graphs of $y = 3 \sin \frac{1}{2}x$ and $y = 3 \sin(\frac{1}{2}x - 60)^\circ$.

What transformation would map $y = 3 \sin \frac{1}{2}x$ onto $y = 3 \sin(\frac{1}{2}x - 60)^\circ$?

What is the phase angle?

4. The sine wave $y = \sin x$ was transformed to give the curves described below:

- (a) Amplitude 3, period $2\pi^\circ$.
- (b) Amplitude 2, period 120° .
- (c) Amplitude 1, period $2\pi^\circ$, phase angle $\frac{\pi^\circ}{3}$.
- (d) Amplitude 2, period $2\pi^\circ$, phase angle $\frac{\pi^\circ}{2}$.

Write down their equations.

5. Without drawing, state the relationship between:

- (a) amplitudes.
- (b) periods of the equations of the waves given below:

- (i) $y = \sin 2x$ and $y = \frac{1}{4} \sin 2x$
- (ii) $y = 3 \cos x$ and $y = \frac{3}{2} \cos \frac{1}{2}x$
- (iii) $y = \sin (3x + 60)^\circ$ and $y = 4 \sin (x + 20)^\circ$
- (iv) $y = \cos x$ and $y = k \cos ax$
6. Draw the graph of $V = 2 \sin(\pi t + \frac{\pi}{3})$ for $0 \leq t \leq 2$. Use your graph to determine:
- (a) V when:
- (i) $t = 0.5$
- (ii) $t = 1$
- (b) t when:
- (i) $v = 0.8$
- (ii) $v = -1.5$
7. On the same axes, draw the graphs of $y = \sin x$, $y = \cos x$ and $y = \cos x + \sin x$. From your graphs, deduce the amplitude and period of the wave $y = \cos x + \sin x$.
8. Given the wave equation $y = a \sin (bx + c)$, where a , b and c are constants, deduce;
- (a) the amplitude.
- (b) the period.
- (c) the phase angle.
9. Sketch the displacement-time graphs of $x = 3 \sin (2t + \frac{\pi}{2})$ and $x = 4 \sin (2t - \frac{\pi}{2})$ on the same axes. Write down the amplitudes and periods of the oscillations.

4.3: Trigonometric Equations

You are familiar with algebraic equations in which the solutions are finite. However, in many cases of trigonometric equations, there are an infinite number of roots.

For example, in the equation $\sin \theta = 0$, $\theta = 0^\circ, \pm 180^\circ, \pm 540^\circ$, and so on, indefinitely. We shall therefore specify the range of values for which the roots of a trigonometric equation are required.

Example 4

Solve the following trigonometric equations:

- (a) $\sin 2x = \cos x$, for $0 \leq x \leq 360^\circ$
- (b) $\tan 3x = 2$, for $0 \leq x \leq 360^\circ$

$$(c) \quad 2 \sin \left(x - \frac{\pi}{6}\right)^c = -\frac{\sqrt{3}}{2}, \text{ for } 0 \leq x \leq 2\pi^c$$

Solution

$$(a) \quad \sin 2x = \cos x$$

$$\sin 2x = \sin (90 - x)$$

$$\therefore 2x = 90 - x$$

$$x = 30^\circ$$

For the given range, $x = 30^\circ$ and 150° .

$$(b) \quad \tan 3x = 2$$

From the tables/calculator;

$$3x = 63.43^\circ, 243.43^\circ, 423.43^\circ, 603.43^\circ, 783.43^\circ, 963.43^\circ$$

In the given range;

$$x = 21.14^\circ, 81.14^\circ, 141.14^\circ, 201.14^\circ, 261.14^\circ \text{ and } 321.14^\circ$$

$$(c) \quad 2 \sin \left(x - \frac{\pi}{6}\right) = -\sqrt{3}$$

$$\sin \left(x - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$x - \frac{\pi}{6} = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$x - \frac{\pi}{6} = \frac{4}{3}\pi, \frac{5}{3}\pi$$

$$x = \frac{3}{2}\pi^c, \frac{11}{6}\pi^c$$

Example 5

Draw the graph of $y = 2 \sin x$ for $0 \leq x \leq 360^\circ$. On the same axes, draw the graph of $y = \cos x$. Hence, solve the equation $2 \sin x - \cos x = 0$ for values of x in the above range.

Solution

Figure 4.9 shows the graphs of $y = 2 \sin x$ and $y = \cos x$ for $0 \leq x \leq 360^\circ$.

Consider the equation;

$$2 \sin x - \cos x = 0$$

Then, $2 \sin x = \cos x$

Thus, the solution to this equation will be the values of x at the intersections of the two curves. Thus, $x = 27^\circ$ or 205.5° .

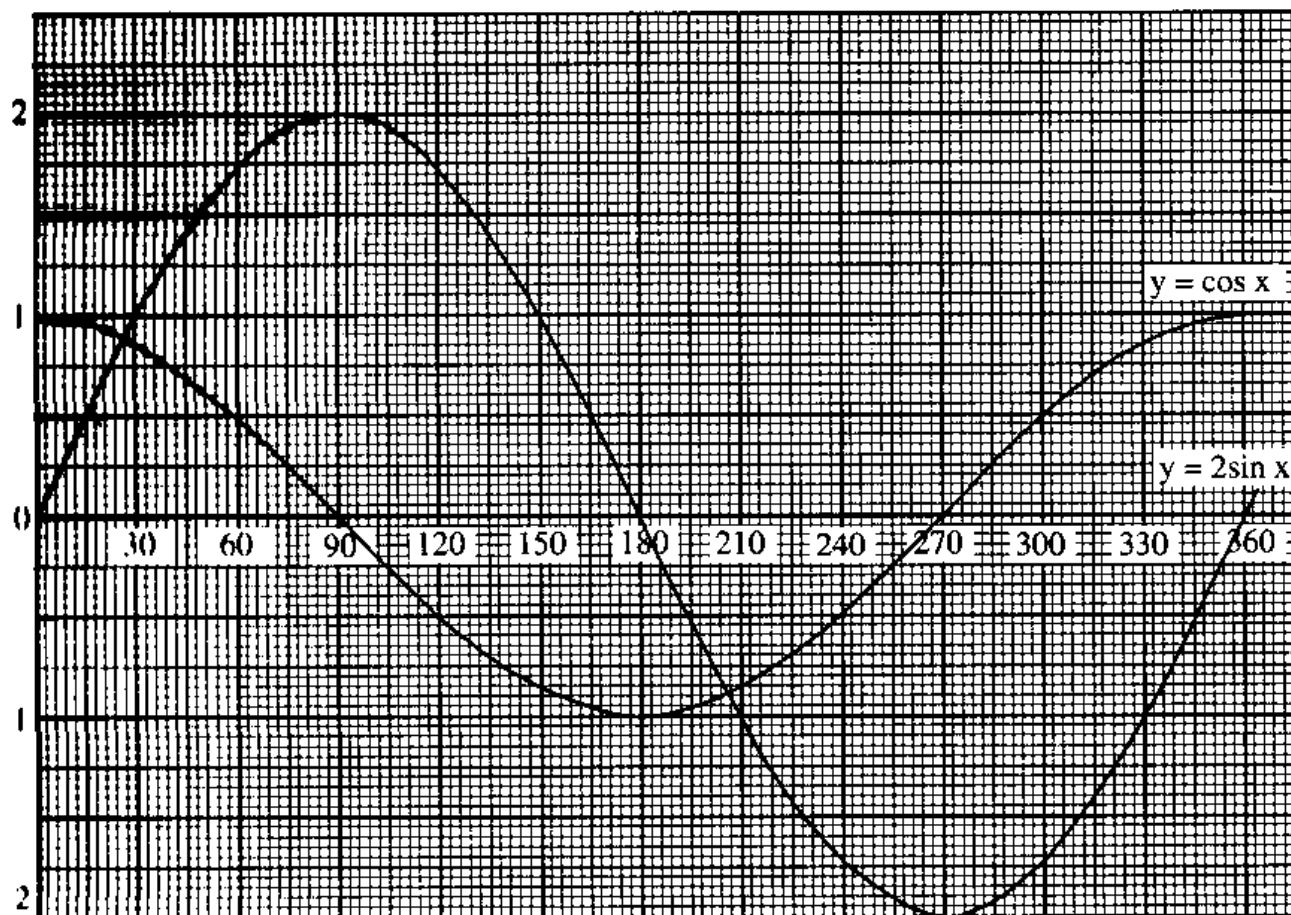


Fig 4.9

Exercise 4.4

- Solve the following equations, giving your answers in radians for $0 \leq x \leq 2\pi$:
 - $\cos 2x = \sin x$
 - $\tan x - \sqrt{3} = 0$
 - $2 \cos 2(x + 30) = 1$
 - $\tan\left(\frac{\pi}{6} + x\right) = 1$
- Solve the following equations, giving your answers in degrees, for $0 \leq x \leq 360^\circ$:
 - $2 \sin(x - 30)^\circ = \tan 45^\circ$
 - $2 \sin(x + 30)^\circ = -0.7986$
 - $2 \cos 2x = 1.1756$
- Solve the following equations, giving your answers in radians, for $0^\circ \leq \theta \leq 2\pi$:
 - $2 \sin \theta = 1$
 - $\cos \theta = \frac{\sqrt{3}}{2}$
 - $\frac{1}{2} \tan \theta = \frac{\sqrt{3}}{2}$
 - $\cos 2\theta = \frac{1}{2}$
 - $\cos \theta = 0$
- Solve the following equations, giving your answers in degrees for $0^\circ \leq x \leq 360^\circ$:
 - $\sin^4 x - \cos^4 x = 0$
 - $4 - 4 \cos^2 x = 4 \sin x - 1$

5. By substituting $y = \sin^2 2x$, solve the equation $5 \cos^2 2x = \sin^4 2x$, for $0 \leq x \leq 360^\circ$.
6. Draw the graphs of $y = 3 \sin x$ and $y = \cos 2x$ for $0^\circ \leq x \leq 360^\circ$ on the same axis and hence solve the equation $3 \sin x - \cos 2x = 0$.
7. Draw the graphs of $y = \tan x$ and $y = \cos x$ for $-\pi \leq x \leq \pi$ on the same axes and use your graphs to solve $\tan x - \cos x = 0$.
8. Draw the graph of $y = \sin (2x + 30)^\circ$ for $0^\circ \leq x \leq 360^\circ$. By including a straight line, use your graph to solve:
 - (a) $\sin (2x + 30) = 0.8$
 - (b) $\sin (2x + 30) - \frac{1}{2} = 0$
 - (c) $3 \sin (2x + 30) \geq 2$
9. Draw the graphs of $y = \sin 3x$ and $y = \cos \frac{2}{3}x$ on the same axis for $0^\circ \leq x \leq 360^\circ$. Use your graph to determine the value of x for which $\sin 3x = \cos \frac{2}{3}x$ within the given interval.

THREE DIMENSIONAL GEOMETRY

5.1: Geometrical Properties of Common Solids

A geometrical figure having length only is said to be in one dimension. Objects in one dimension are best represented by a line. For example, the edge AB of the cuboid shown in figure 5.1 is of one dimension.

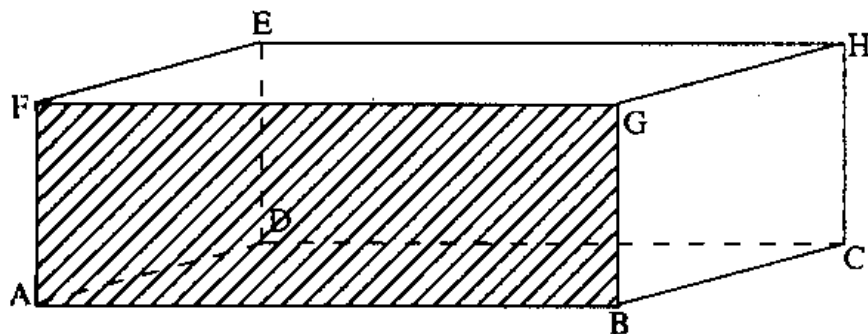


Fig. 5.1

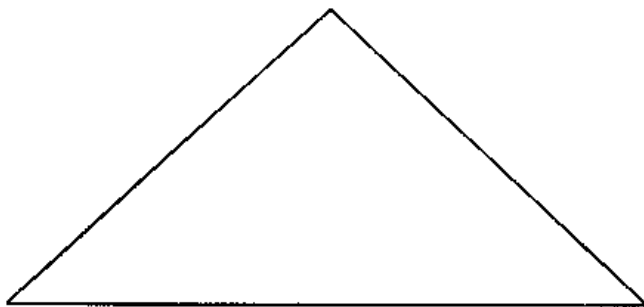
A figure having area but not volume is said to be two dimensional. For example, the shaded face ABGF of the cuboid in figure 5.1 is two dimensional (a plane).

A figure having volume is said to be three dimensional. All solids are three dimensional.

A solid can be considered in terms of its vertices (points), edges (lines) and faces (planes). In this topic, we shall study the geometrical relationship between lines and planes in three dimensions.

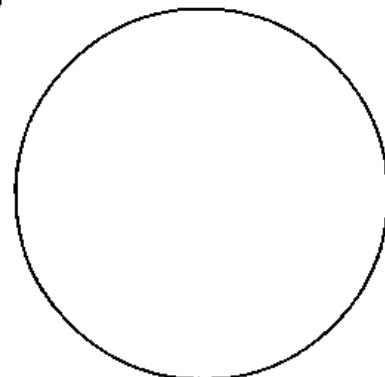
State the dimensions of each of the following figures:

(i)

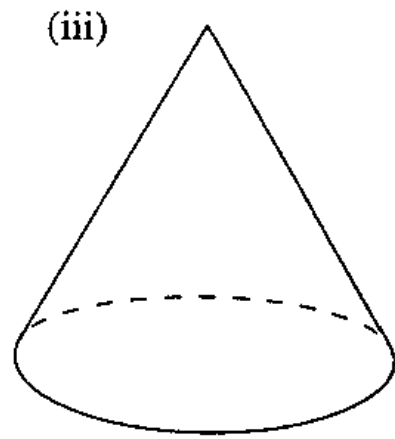


Triangle

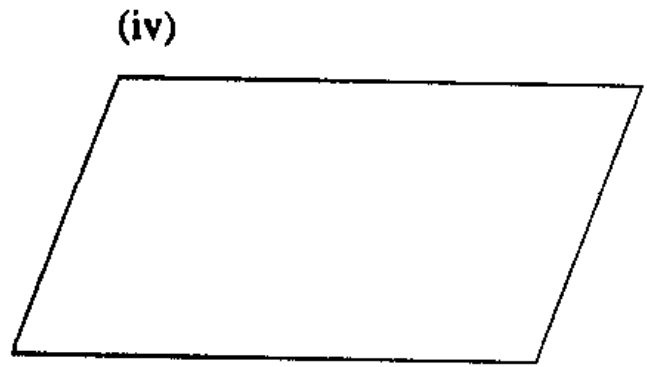
(ii)



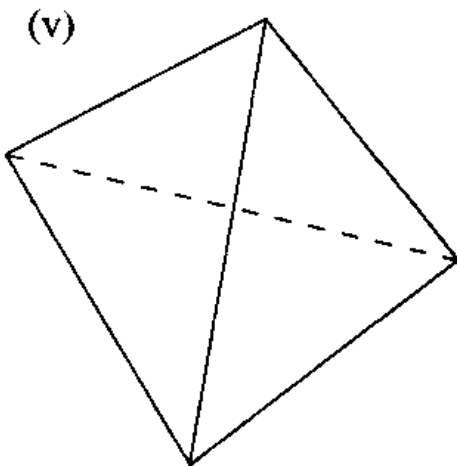
Circle



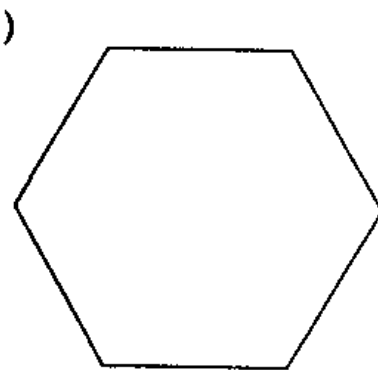
Cone



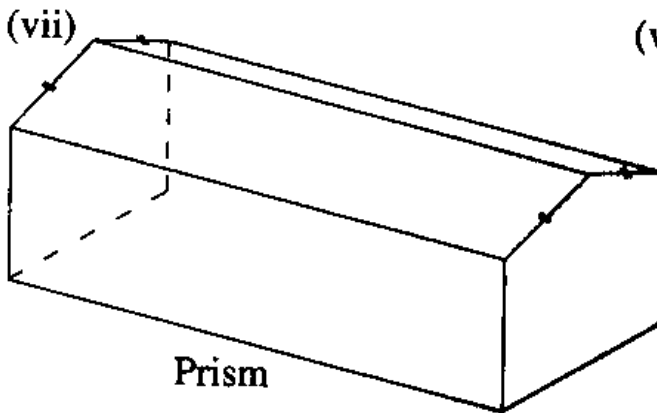
Parallelogram



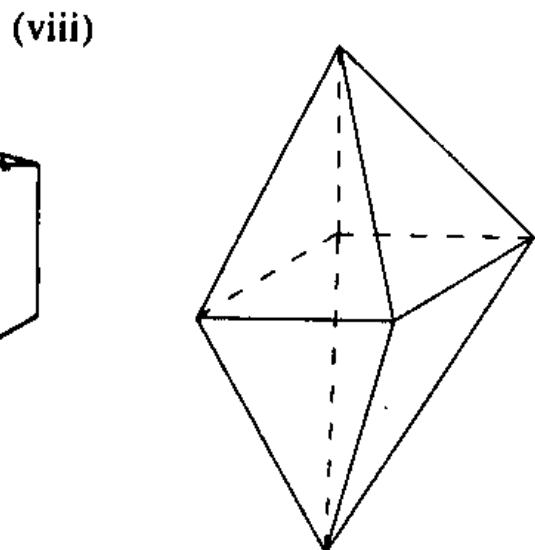
Tetrahedron



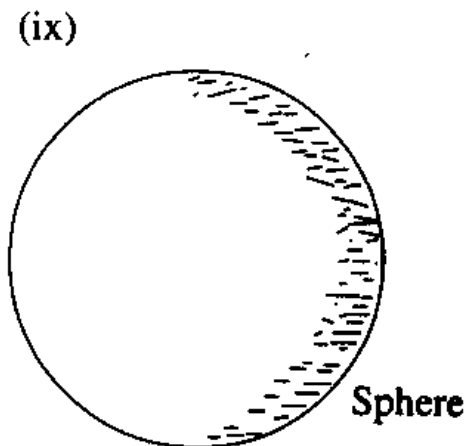
Regular hexagon



Prism



Octahedron



Sphere

Fig. 5.2

~~For each of the three dimensional solids in figure 5.2, state the number of:~~

- (i) edges. (ii) faces.
 (iii) planes of symmetry. (iv) axis and order of rotational symmetry.

5.2: Angle between a Line and a Plane

Figure 5.3 shows a piece of wire PQ held against a horizontal plane ABCD at Q when the sun is vertically above the plane.

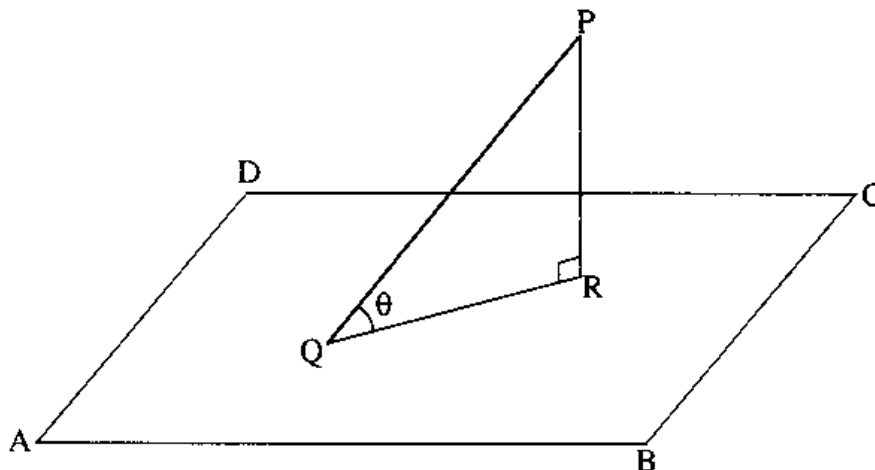


Fig. 5.3

The shadow QR of the wire will fall on the plane. We call the shadow the **projection** of the wire on the plane. QR is the projection of QP on the plane ABCD. The point P is projected onto point R. Note that PR is perpendicular to QR and the plane ABCD.

The angle between the lines QP and QR is taken to be the angle between the line QP and the plane ABCD.

In general, **the angle between a line and a plane is the angle between the line and its projection on the plane.**

Example 1

Figure 5.4 shows a right pyramid with a rectangular base ABCD. L is the midpoint of BC and O the centre of the base.

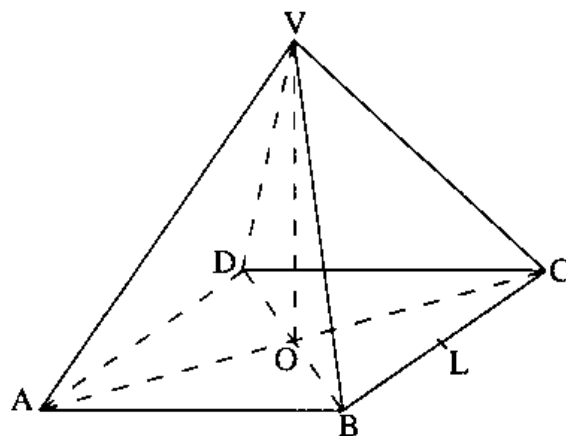


Fig. 5.4

- (a) State the projection of the following on the plane ABCD:
 (i) VD (ii) VA (iii) VB (iv) VC (v) VO
- (b) State the projection of:
 (i) VB on the plane VAC.
 (ii) VC on the plane VBD.
- (c) Name two lines in the figure that are perpendicular to VO.
- (d) Name planes which are perpendicular to BD.
- (e) Name the angle between VC and the plane ABCD.
- (f) Name the angle between the plane VCB and the line VO.

Solution

- (a) (i) OD (ii) OA (iii) OB (iv) OC (v) Point O
- (b) (i) VO (ii) VO
- (c) AC and BD. Note that any line on the plane ABCD is perpendicular to VO.
- (d) VAC, VBD and VOL
- (e) $\angle VCO$
- (f) $\angle OVL$

Example 2

Figure 5.5 shows a rectangular tank of length 8 m, width 6 m and height 4 m. SB is a diagonal of the tank and T is the midpoint of SR. Calculate:

- (a) the length SB.
 (b) the angle between SB and the plane CDSR.
 (c) the length AT.
 (d) angle ATD.

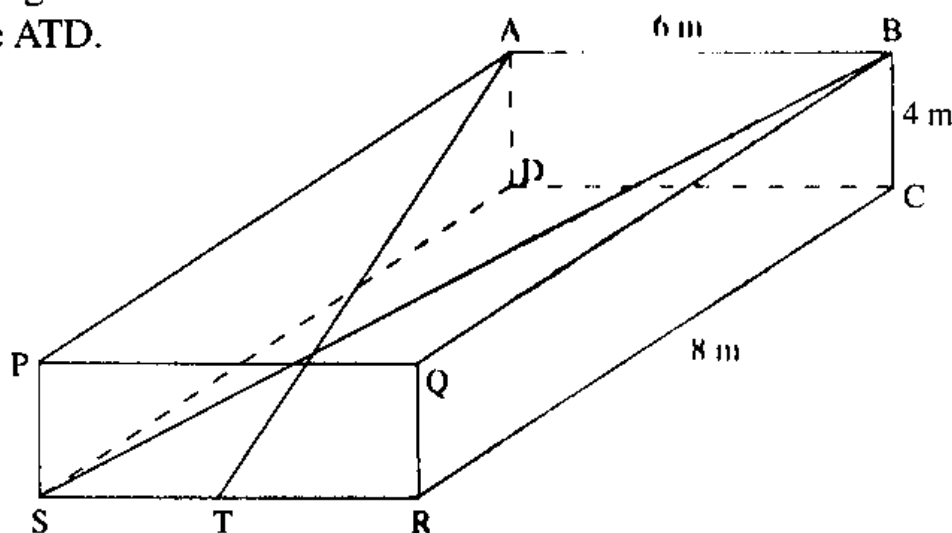


Fig. 5.5

Solution

- (a) Consider the right-angled triangles SRC and SCB in the figure. The triangles are redrawn in figure 5.6 (a) and (b).

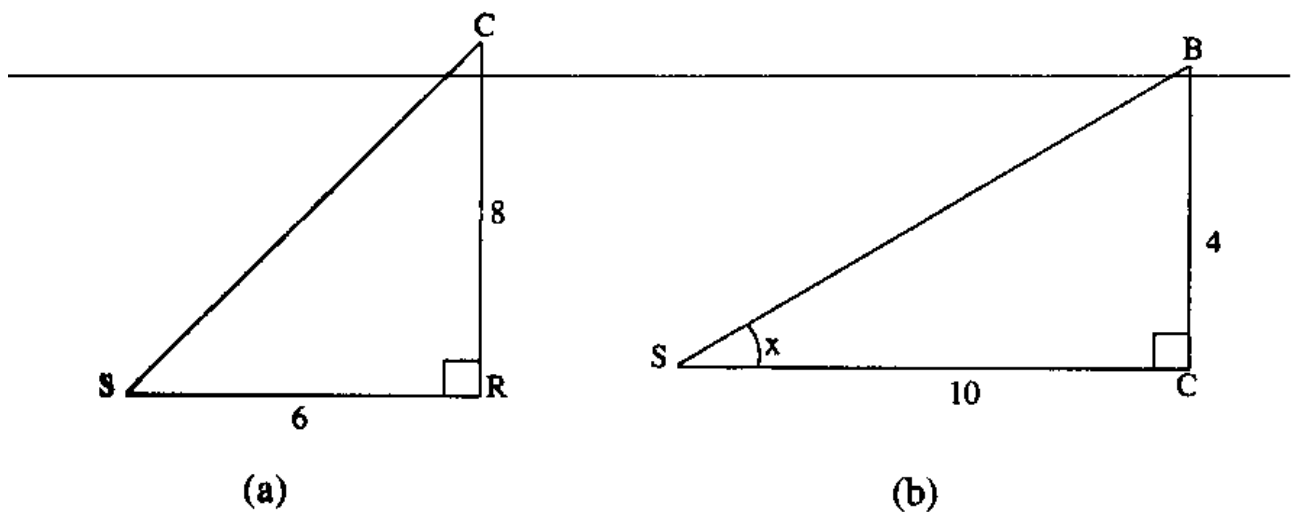


Fig. 5.6

In triangle SRC;

$$(SC)^2 = (SR)^2 + (RC)^2$$

$$\begin{aligned} \text{Therefore, } SC^2 &= 6^2 + 8^2 \\ &= 100 \end{aligned}$$

$$\begin{aligned} \text{Therefore, } SC &= \sqrt{100} \text{ m} \\ &= 10 \text{ m} \end{aligned}$$

In triangle SCB;

$$\begin{aligned} SB^2 &= 10^2 + 4^2 \\ &= 116 \end{aligned}$$

$$\begin{aligned} \text{Thus, } SB &= \sqrt{116} \text{ m} \\ &= 10.77 \text{ m} \end{aligned}$$

- (b) SC is the projection of SB on the plane CDSR. Therefore, the angle between SB and the plane CDSR is angle BSC. If angle BSC = x° , then;

$$\begin{aligned} \tan x &= \frac{4}{10} \\ &= 0.4 \end{aligned}$$

$$\tan^{-1}(0.4) = 21.8^\circ$$

Therefore, angle BSC = 21.8°

- (c) The triangles in figure 5.7 (a) and (b) are used to calculate AT and angle ATD.

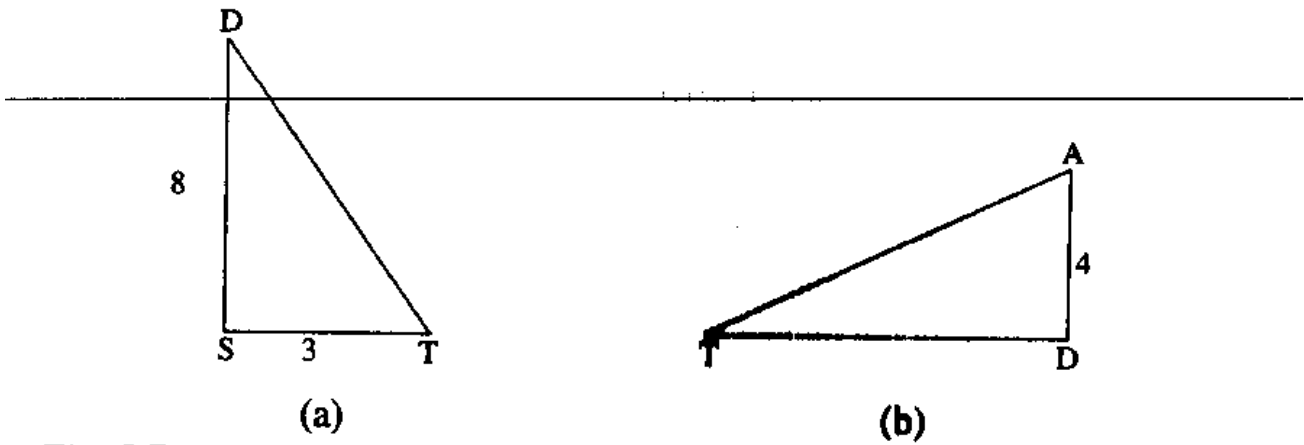


Fig. 5.7

In ΔSTD ;

$$(TD)^2 = 3^2 + 8^2$$

$$= 73$$

$$TD = \sqrt{73}$$

$$= 8.544 \text{ m}$$

In ΔTDA ;

$$(AT)^2 = 73 + 16$$

$$AT = \sqrt{89}$$

$$= 9.434 \text{ m}$$

(d) In ΔATD , let $\angle ATD = y^\circ$

$$\text{Then, } \tan y = \frac{4}{8.544}$$

$$= 0.4682$$

$$\tan^{-1}(0.4682) = 25.09^\circ$$

$$\text{Therefore, } y = 25.09^\circ$$

Exercise 5.1

Figure 5.8 shows a cuboid. $PO = 2.5 \text{ cm}$, $ON = 4 \text{ cm}$ and $PQ = 3 \text{ cm}$. Use the figure to answer questions 1 to 3.

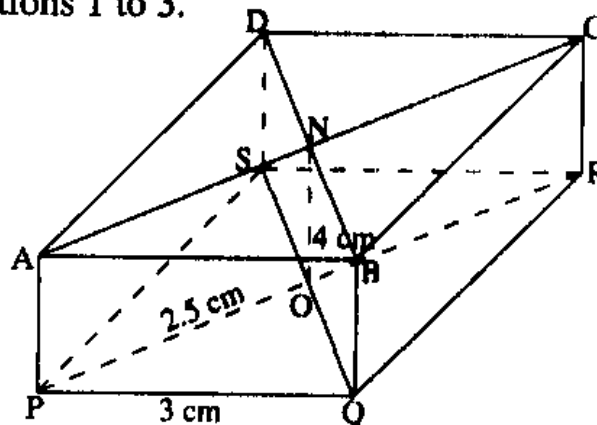


Fig. 5.8

1. (a) Name the projection of:
- PC on the plane PQRS.
 - AC on the plane PQRS.
 - AR on the plane PQRS.
 - AR on the plane ABCD.
 - AR on the plane SRCD.
 - OA on the plane QBDS.
 - ON on the plane ABCD.
- (b) Find the angle between each line and each plane in (a) above.
2. (a) Find the length of:
- QR
 - NP
 - DQ
 - CP
- (b) Calculate the angle between the line:
- NP and the plane PQRS.
 - CP and the plane PQRS.
 - CQ and the plane DCRS.
 - OD and the plane ABCD.
 - PN and the plane BQSD.
3. Find the angles PQR, ANB, NQO and AOC.
4. A right circular cone has a base radius of 6 cm and a vertical height of 8 cm. Calculate:
- the angle the slanting edge makes with the base of the cone.
 - the length of the slanting edge of the cone.
5. Figure 5.9 shows a right pyramid on a rectangular base. AB is twice BC. $TC = TB = TA = TD = 17$ cm and $TO = 15$ cm.

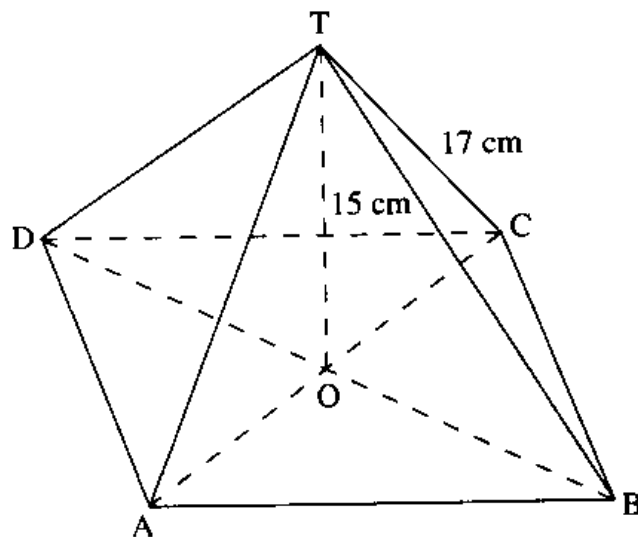


Fig. 5.9

Calculate:

- the length AB.
 - the angle between TC and the plane ABCD.
 - the angle between TD and the plane TAC.
 - the volume of the pyramid.
6. Figure 5.10 shows a cuboid PQRSNMLK:

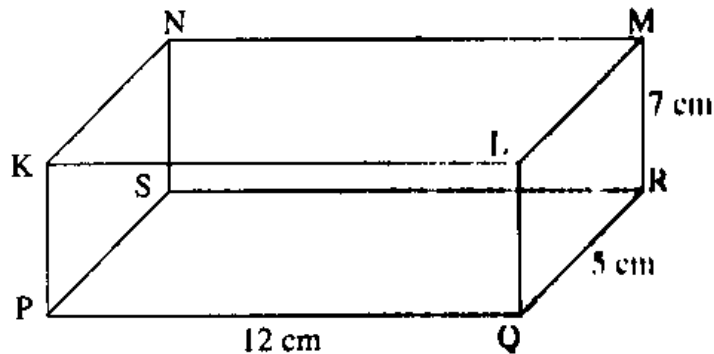


Fig. 5.10

$SP = 5$ cm, $PQ = 12$ cm and $SN = 7$ cm. Find:

- SQ.
 - the shortest distance from P to line SQ.
 - the angle between the line NQ and the plane PQRS.
7. Figure 5.11 shows a cuboid ABCDHGFE:

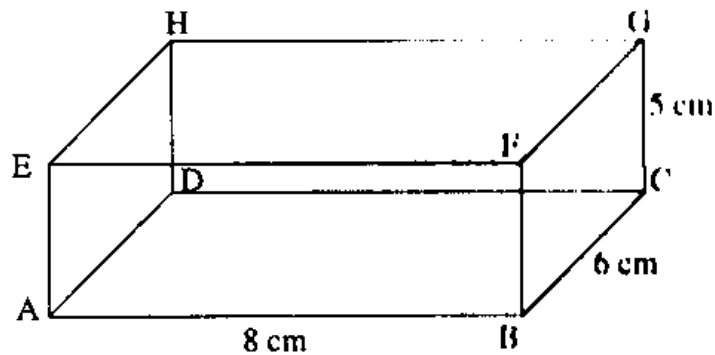


Fig. 5.11

$AB = 8$ cm, $BC = 6$ cm and $CG = 5$ cm. Calculate:

- BD.
 - DF.
 - the angle between DF and the plane ABCD.
8. Figure 5.12 shows a tent with a rectangular base ABCD. $AB = 4$ m, $BC = 3$ m and PC is a vertical post 2.5 m tall. AP, BP and DP are steel bars inclined to the base:

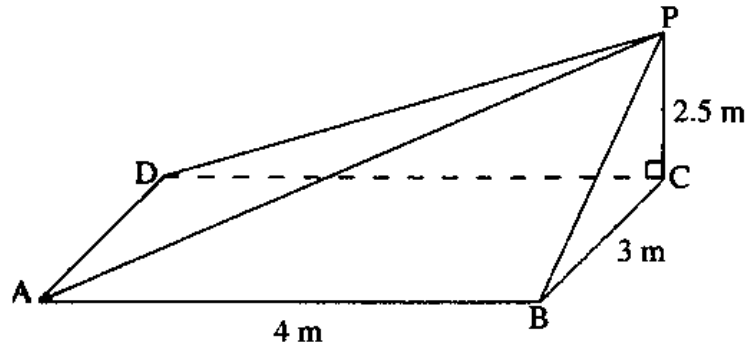


Fig. 5.12

Calculate:

- the lengths of the steel bars AP, BP and DP.
- the angle between the steel bar AP and the base ABCD.

9. Figure 5.13 represents a wedge ABCDEF. $AB = 5$ cm, $BC = 12$ cm and $\angle FBC = 32^\circ$:

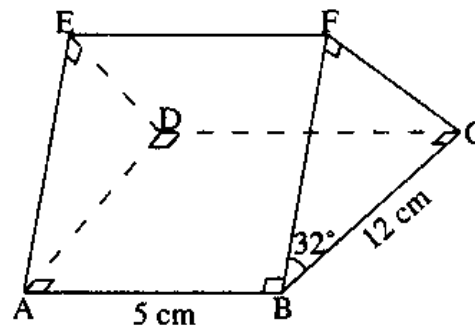


Fig. 5.13

Calculate:

- FC.
- BD.
- the angle between line EB and plane ABCD.

10. Figure 5.14 shows a solid PQRS. $PQ = 30$ cm, $QR = 40$ cm, $\angle SPR = 46^\circ$ and $\angle SRP = \angle PQR = \angle SRQ = 90^\circ$.

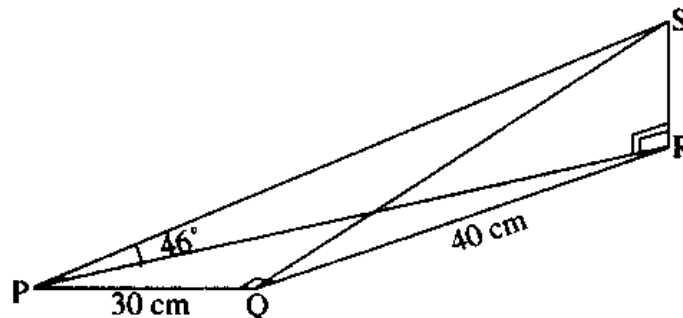


Fig. 5.14

Find:

- PR
- SR
- angle SQR

11. In an isosceles triangle PQR , $PR = QR = 10$ cm and $PQ = 12$ cm. M is the midpoint of PQ and a line NM is vertical to the plane PQR . If $NM = 15$ cm, calculate:
- the lengths of NP and NR .
 - the angle between NR and plane PQR .
12. Figure 5.15 shows the model of a hut with $HG = GF = 10$ cm and $FB = 6$ cm. The four slanting edges of the roof are each 12 cm long:

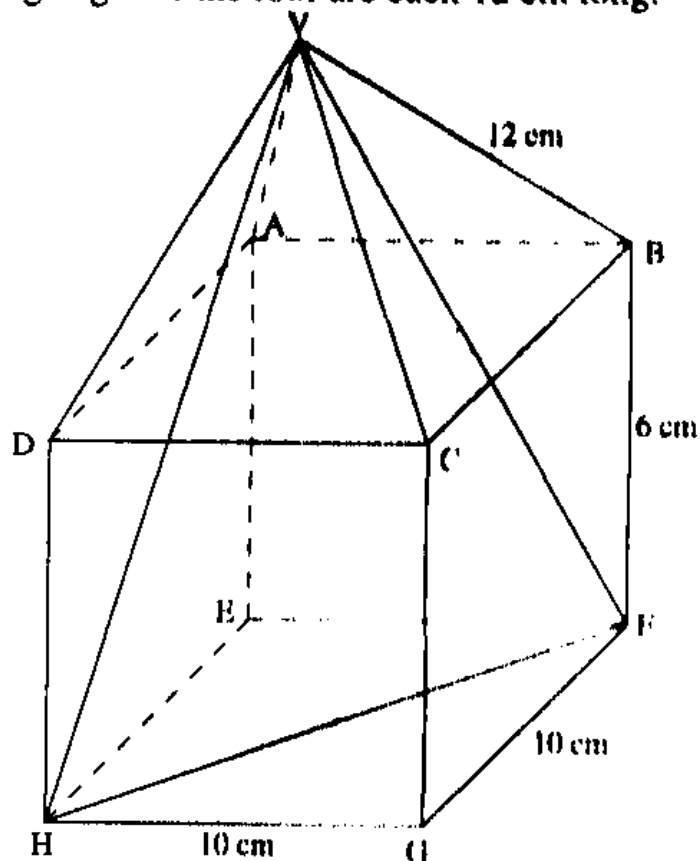


Fig. 5.15

Calculate:

- DF .
- angle VHF .
- the length of the projection of line VH on the plane $EFGH$.
- the height of the model hut.
- VH .
- the angle DF makes with plane $ABCD$.
- the volume of the model hut.

5.3: Angle Between Two Planes

Any two planes are either parallel or intersect in a straight line.

Figure 5.16 shows two planes $PQRS$ and $PQTU$ intersecting along a straight line PQ .

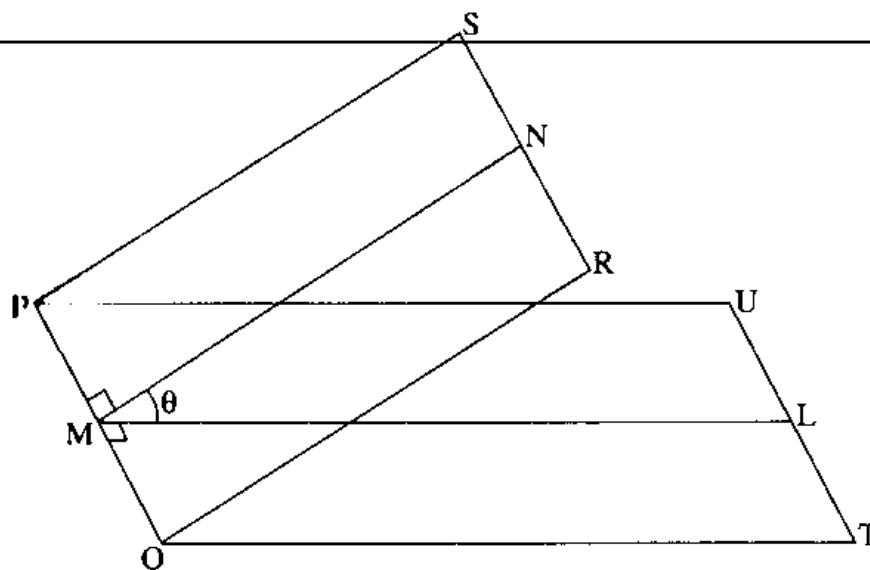


Fig. 5.16

NM is on PQRS and LM on PQTU. NM and LM are perpendicular to PQ at M. Angle θ between the lines NM and LM is the angle between the two planes PQRS and PQTU.

In general, **the angle between two planes is the angle between two lines, one on each plane, which are perpendicular to the line of intersection at a point.**

Example 3

Figure 5.17 is a cuboid ABCDSPQR:

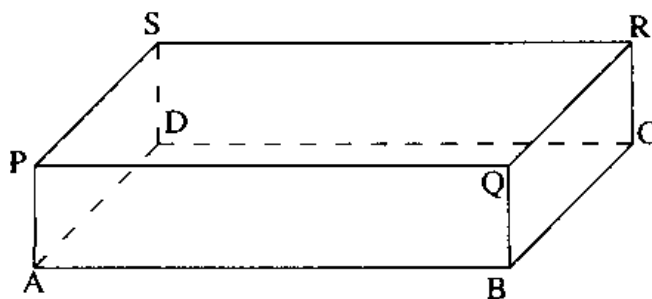


Fig. 5.17

Name one angle between the planes:

- ABCD and QBCR.
- ABCD and AQRD.
- ABCD and ABRS.

Solution

(a)

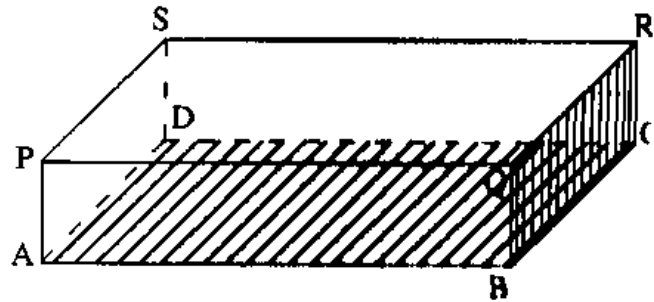


Fig. 5.18

The line of intersection of the two planes is BC . Lines QB and AB meet at B on BC and are perpendicular to BC . Similarly, DC and RC meet at C and are perpendicular to BC . Therefore the angle between the planes $ABCD$ and $QBCR$ is angle ABQ or DCR .

(b) The planes $ABCD$ and $AQRD$ are shaded in figure 5.19. The line of intersection of $ABCD$ and $AQRD$ is AD . Lines AQ and BA meet at A on AD and are perpendicular to AD . Similarly, DC and DR meet at D on AD and are perpendicular to AD . Therefore, the angle between the planes $ABCD$ and $AQRD$ is $\angle QAB$ or $\angle CDR$.

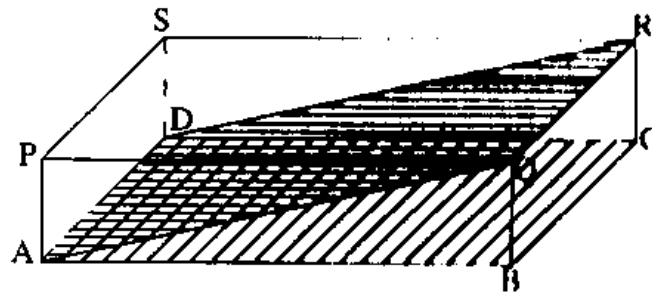


Fig. 5.19

(c) The planes $ABCD$ and $ABRS$ are shaded in figure 5.20:

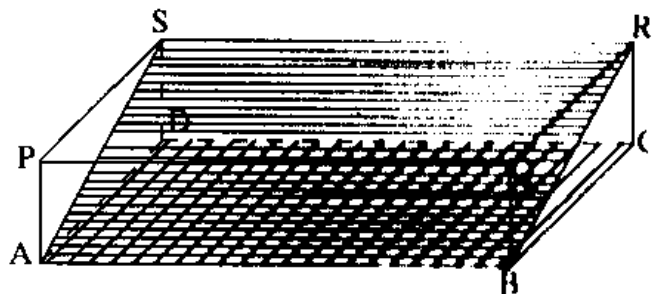


Fig. 5.20

The angle between the planes $ABCD$ and $ABRS$ is angle RBC or SAD .

Example 4

In figure 5.21, PQRS is a regular tetrahedron of side 4 cm and M is the midpoint of RS:

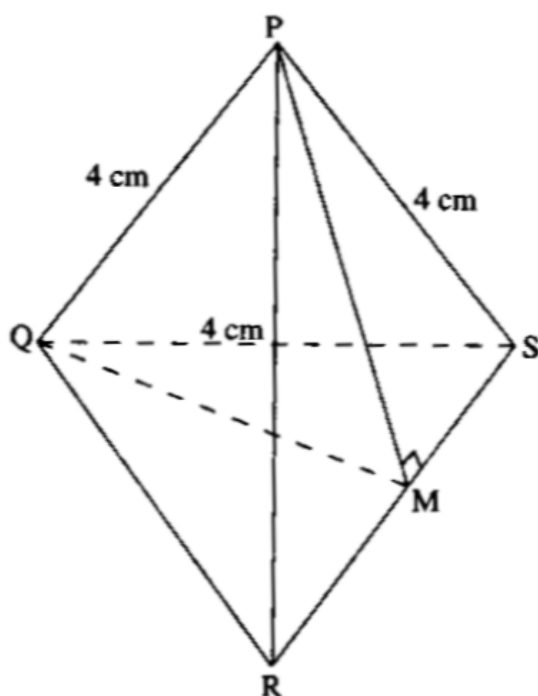


Fig. 5.21

- (a) Show that PM is $2\sqrt{3}$ cm long, and that ΔPMQ is isosceles.
 (b) Calculate the angle between planes PSR and QRS.
 (c) Calculate the angle between line PQ and plane QRS.

Solution

- (a) Triangle PRS in figure 5.21 is equilateral. Since M is the midpoint of RS, PM is a perpendicular bisector, see figure 5.22.

$$\begin{aligned} PM^2 &= 4^2 - 2^2 \\ &= 12 \end{aligned}$$

$$\begin{aligned} PM &= \sqrt{12} \text{ cm} \\ &= \sqrt{4 \times 3} \text{ cm} \\ &= 2\sqrt{3} \text{ cm} \end{aligned}$$

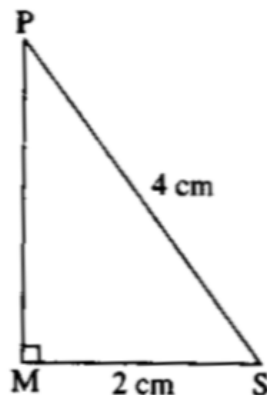


Fig. 5.22

Similarly, ΔMQR is right-angled at M, as in figure 5.23.

$$\begin{aligned} QM^2 &= 4^2 - 2^2 \\ &= 12 \end{aligned}$$

$$\begin{aligned} QM &= \sqrt{12} \text{ cm} \\ &= 2\sqrt{3} \text{ cm} \end{aligned}$$

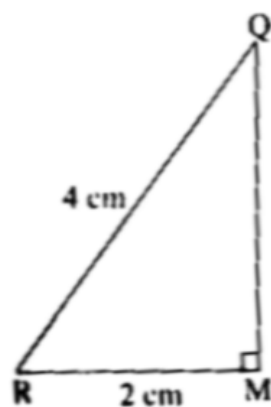


Fig. 5.23

Since $PM = QM = 2\sqrt{3}$ cm, ΔPMQ is isosceles.

- (b) The required angle is $\angle PMQ$ as in figure 5.24 (a). Using cosine rule;

$$\begin{aligned} 4^2 &= (2\sqrt{3})^2 + (2\sqrt{3})^2 - 2(2\sqrt{3})(2\sqrt{3}) \cos m \\ 16 &= 12 + 12 - 2 \times 12 \cos m \\ &= 24 - 24 \cos m \end{aligned}$$

$$\begin{aligned} \cos m &= \frac{24 - 16}{24} \\ &= 0.3333 \end{aligned}$$

Therefore, $m = 70.53^\circ$

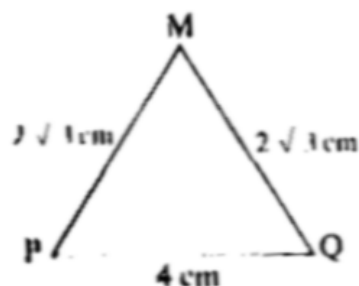


Fig. 5.24

Alternatively;

In figure 5.25, T is the midpoint of PQ;

Let $\angle PMT$ be x° . Then;

$$\begin{aligned} \sin x &= \frac{2}{2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \\ &= 0.5774 \end{aligned}$$

$$x = 35.27^\circ$$

$$\begin{aligned} \text{Therefore, } \angle PMQ &= 2 \times 35.27 \\ &= 70.54^\circ \end{aligned}$$

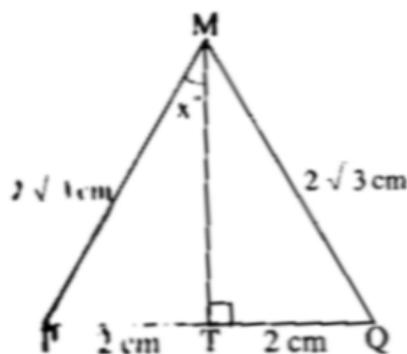


Fig. 5.24

(c) The required angle is $\angle PQM$, as shown in figure 5.26.

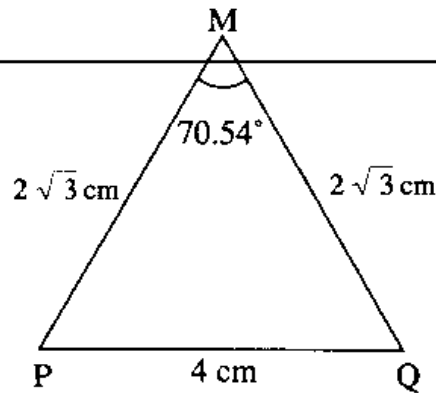


Fig. 5.26

Since $\triangle PMQ$ is isosceles with $\angle PMQ = 70.54^\circ$;

$$\begin{aligned}\angle PQM &= \frac{1}{2}(180 - 70.54) \\ &= \frac{1}{2}(109.46) \\ &= 54.73^\circ\end{aligned}$$

5.4: Skew Lines

In figure 5.27, the lines ML and PQ do not meet and are not parallel. These lines do not lie on the same plane. Such lines are called **skew lines**.

The angle between skew lines is found by translating one of the lines to the plane containing the other.

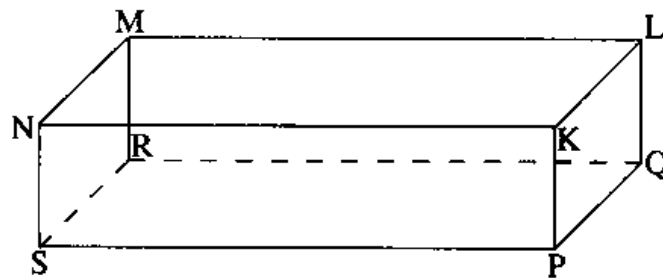


Fig. 5.27

For example, to get the angle between ML and PQ we proceed as follows:

Either:

(i) Translate line PQ to the plane $KLMN$ to obtain KL . Then, the angle between the two lines is $\angle KLM$, which is 90° .

or

(ii) Translate line ML to the plane $PQRS$ to get RQ . Then the angle between the two lines is $\angle PQR$, which is 90° .

Use figure 5.27 to name:

(i) lines that are skew to line NK.

(ii) two angles equal to the angle between SP and NL.

Exercise 5.2

1. From the cuboid **PQRSTU** in figure 5.28, give examples of the following:

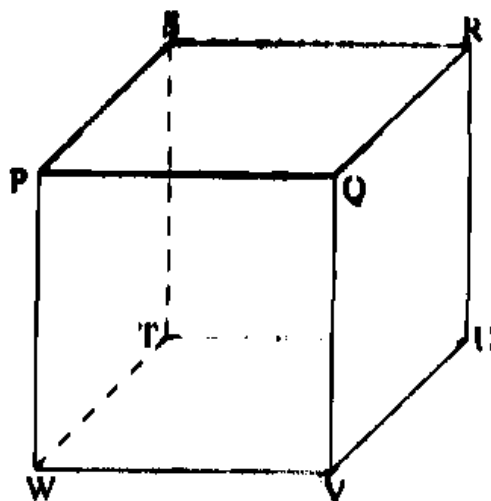


Fig. 5.28

- Two parallel lines.
 - Two intersecting lines.
 - Two parallel planes.
 - Two intersecting planes.
 - Three planes passing through one point.
 - Two skew lines which are perpendicular.
 - Two skew lines which are not perpendicular.
 - Three planes which intersect in a line.
 - Three lines that would form an isosceles triangle.
 - Planes that would intersect to form a tetrahedron.
2. Figure 5.29 shows a right pyramid with a rectangular base ABCD. $TD = TA = TB = TC = 7$ cm, $AB = 12$ cm and $BC = 5$ cm:

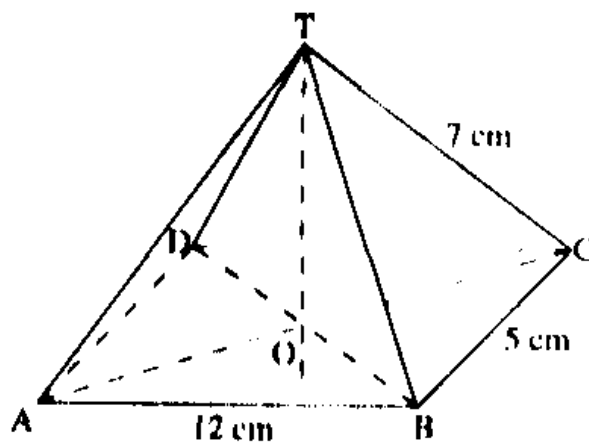


Fig. 5.29

- (a) Name the line of intersection of the planes:
 (i) ACT and DBT.
 (ii) ABCD and TBC.
- (b) Find the size of the angle between:
 (i) ACT and DBT.
 (ii) ABCD and TBC.

3. Figure 5.30 shows a framework of a cube PQRSWTUV of side 8 cm.

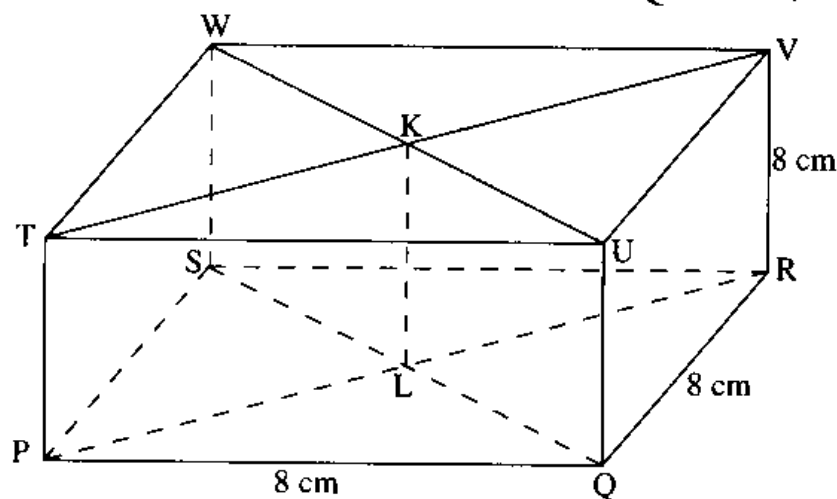


Fig. 5.30

- (a) Name the lines of intersection of the planes:
 (i) QSWU and PRVT.
 (ii) PSWT and TUVW.
- (b) Give the size of angles between each pair of planes in (a).
- (c) Find the angle between the lines:
 (i) WV and PS.
 (ii) TQ and PW.
- (d) Name an angle between the planes PUVS and PQRS.
4. Figure 5.31 shows a prism whose cross-section is an isosceles triangle ABC. The base CB is 6 cm and the height is 4 cm. Find the angle between the planes ABFE and BCGF, and its tangent.

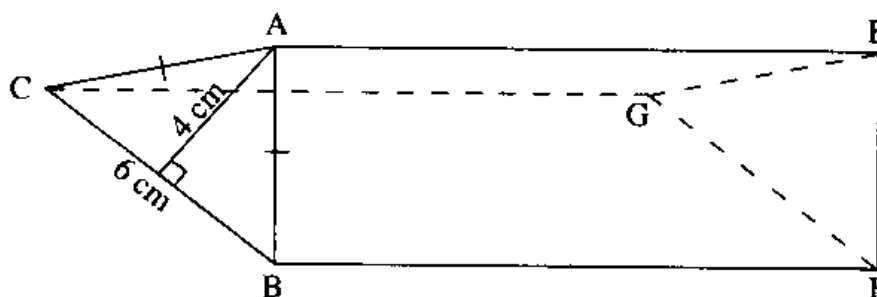


Fig. 5.31

5. Figure 5.32 shows the net of a model of a triangular prism. If the net is folded to make the prism, find:
- the size of the angle between the planes marked D and A.
 - the size of the angle between the planes marked A and C. Calculate its tangent.

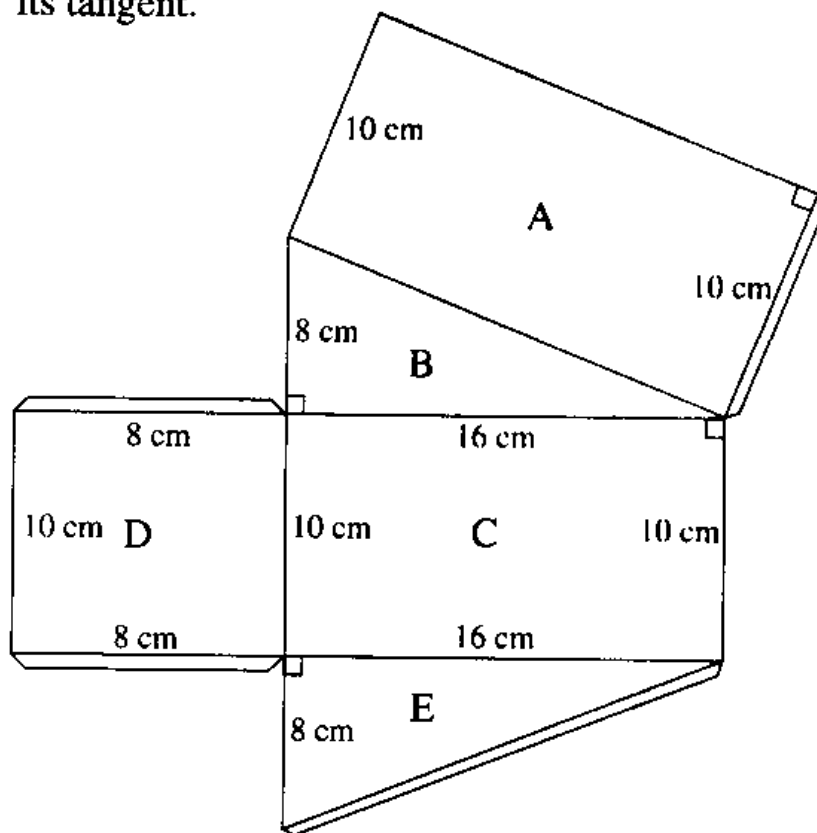


Fig. 5.32

6. Figure 5.33 shows a right pyramid $VPQRS$ on a rectangular base $PQRS$:

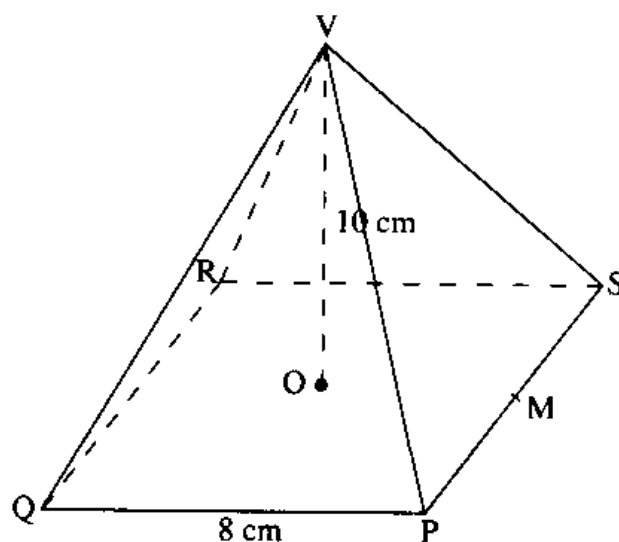


Fig. 5.33

If M is the midpoint of SP , $PS = 6$ cm, $PQ = 8$ cm and the height OV of the pyramid is 10 cm:

(a) find:

(i) $\angle VMP$.

(ii) $\angle SQV$.

(b) Calculate the angle between the planes VSP and PQRS.

7. Figure 5.34 shows a wedge ABCDEF with face FCDE \perp ABCF. BC = 6 cm, DC = 8 cm and ED = 12 cm. Find:

(a) the angle between the planes ABDE and FCDE.

(b) the angle between the line AD and the plane EFCD.

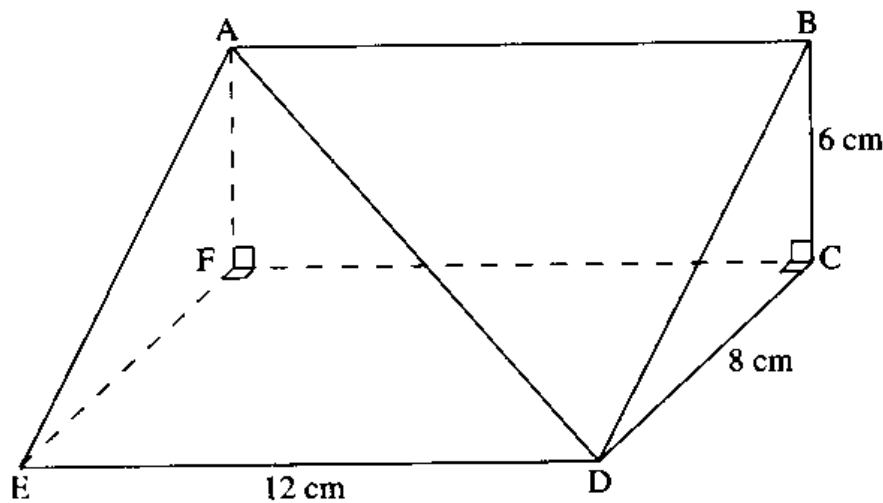


Fig. 5.34

8. Figure 5.35 shows a wedge ABCDEF. AD = DE = BC = CF = 30 cm. $\angle FBC = \angle EAD = 45^\circ$ and AB = 40 cm. Find the angle between:

(a) the line AF and the plane ABCD.

(b) the lines FA and DC.

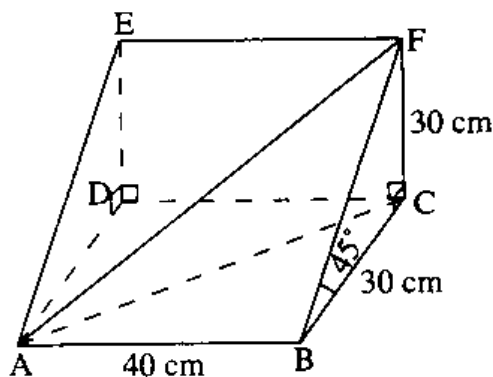


Fig. 5.35

9. Figure 5.36 is a solid in which base ABCD is a rhombus. AC = 32 cm, BD = 24 cm and CE = 24 cm.

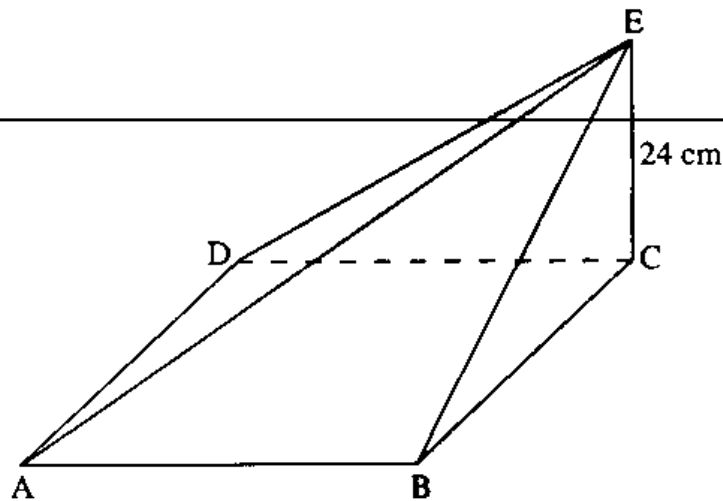


Fig. 5.36

Calculate the angles between the planes:

- (a) EBD and ABCD.
- (b) ECB and ECD.

10. Figure 5.37 shows a cuboid in which $PQ = 8$ cm, $QR = 12$ cm and $CR = 4$ cm. M and N are midpoints of QR and AD respectively.

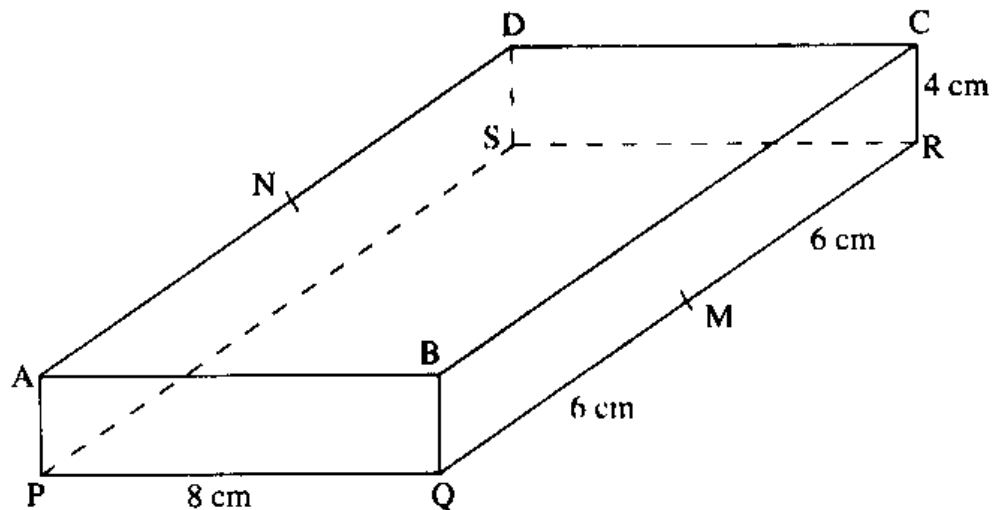


Fig. 5.37

- (a) Find the length:
 - (i) SM.
 - (ii) CM.
 - (b) Calculate the angle the line:
 - (i) DM makes with the plane PQRS.
 - (ii) DM makes with the plane BQRC.
 - (iii) AM makes with the plane DCRS.
11. Figure 5.38 shows a prism whose cross-section is a regular pentagon of side 5 cm. The length of the prism is 18 cm.

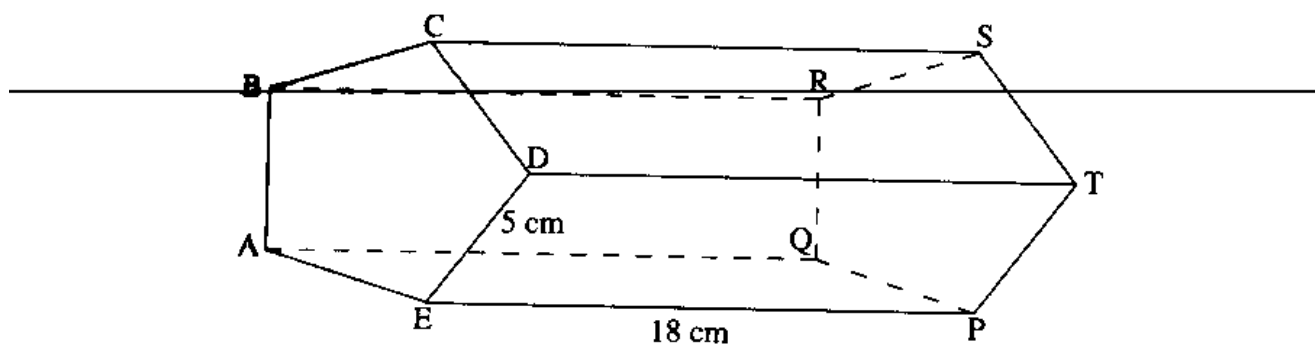


Fig. 5.38

Calculate the angle between the planes:

- AEPQ and ADTQ.
- AEPQ and ACSQ.
- ACSQ and ADTQ.
- ECSP and BDTR.

12. Figure 5.39 shows a frustum ABCDEFGH of a right pyramid. $AB = 9$ cm, $BC = 12$ cm, $FG = 6$ cm, $GH = 8$ cm and the height of the frustum is 10 cm.

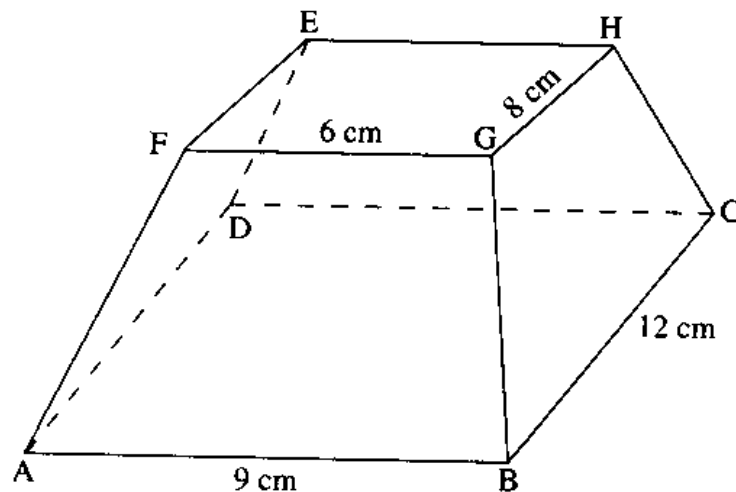


Fig. 5.39

- Find the height of the pyramid.
- Find the length of:
 - AC.
 - AH.
- Calculate the angle between:
 - line AH and the plane ABCD.
 - the planes ABHE and ABCD.

Chapter Six

LONGITUDES AND LATITUDES

6.1: Great and Small Circles

Figures 6.1 (a) and (b) show two equal spheres:

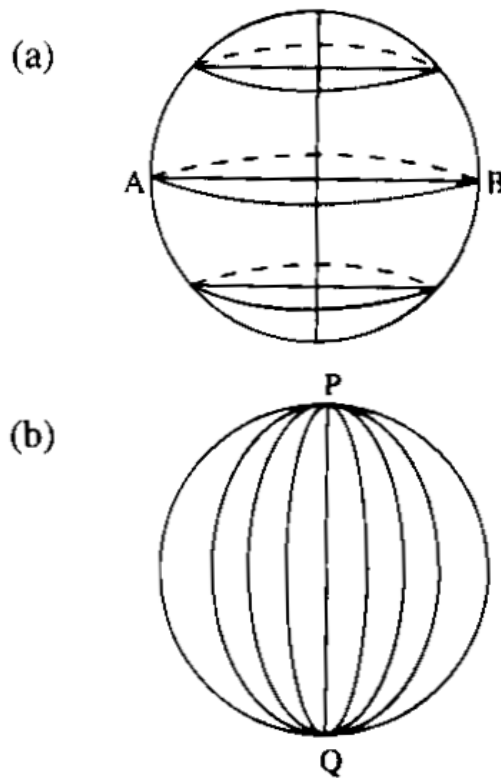


Fig. 6.1

Notice that circles drawn in figure 6.1 (a) are of different sizes. The largest of them with diameter AB divides the sphere into two equal hemispheres. This largest circle is called a **great circle**. All the others are called **small circles**.

The circles drawn as in figure 6.1 (b) all pass through points P and Q. Each of them divides the sphere into two equal hemispheres. Therefore, they are all great circles. The radius of any great circle is the radius of the sphere.

Latitudes

Any circle whose plane is perpendicular to the axis of the earth, such as those of figure 6.2, is called a **latitude**. The assumption made is that the earth is a perfect sphere.

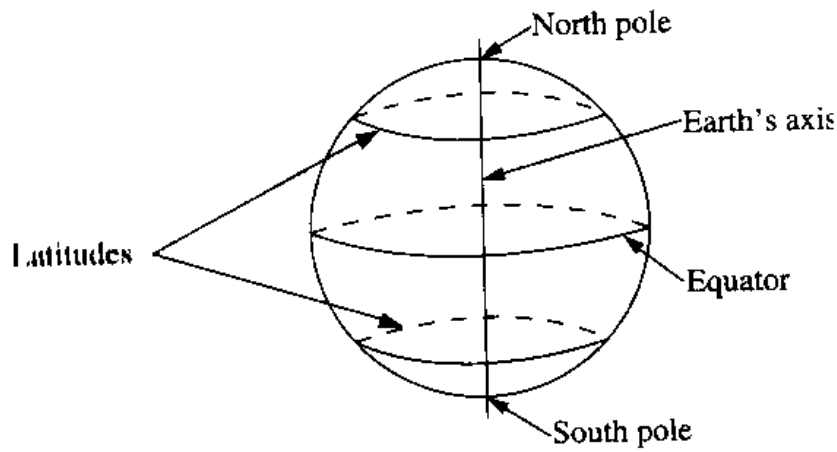


Fig. 6.2

The equator is the latitude that divides the earth into two equal parts. It is the only great circle among the latitudes. All the other latitudes are parallel to the equator and are measured in degrees north or south of it. Thus, the equator is the reference point for all the other latitudes.

In figure 6.3, A is a point on the surface of the earth north of the equator while C is a point on the surface of the earth south of the equator. The angle θ subtended by the arc AB at the centre of the earth is the latitude of the circle passing through point A north and parallel to the equator.

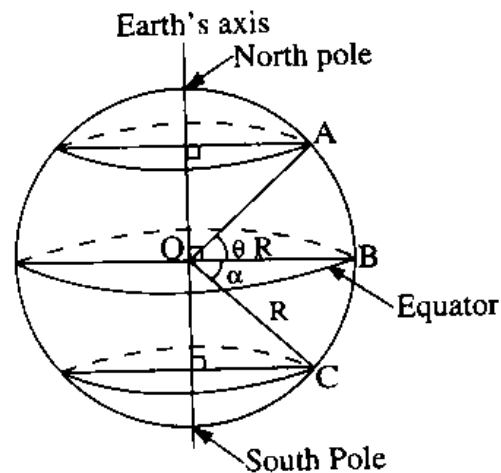


Fig. 6.3

Similarly, the angle α is the latitude of the circle through point C south and parallel to the equator.

Note that the maximum angle of latitude is 90° north or south of the equator.

Longitudes

In figure 6.4, the circles passing through the north and south poles are great circles, usually called lines of **longitudes (meridians)**.

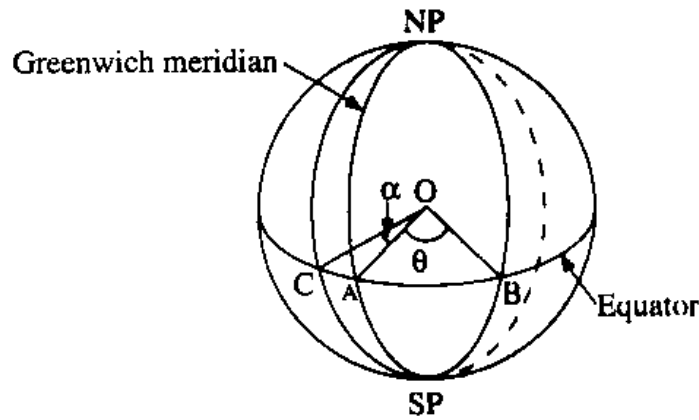


Fig. 6.4

The meridian which passes through Greenwich in London is called the **Greenwich Meridian** or **Prime Meridian**. It is used as the reference meridian from which positions of other meridians on the earth's surface are measured in degrees, east or west of it.

The angle subtended at the centre of the earth (on the plane of equator) by an arc of the equator between the Greenwich Meridian and a given meridian is the longitude of that meridian.

For example, in figure 6.4, the longitude of the meridian from North Pole through C to South Pole is α west of the Greenwich Meridian. Similarly, the longitude of the meridian through B is θ east of the Greenwich Meridian.

If we view the earth from the north pole, the longitudes of various positions of points would be as shown in figure 6.5 below:

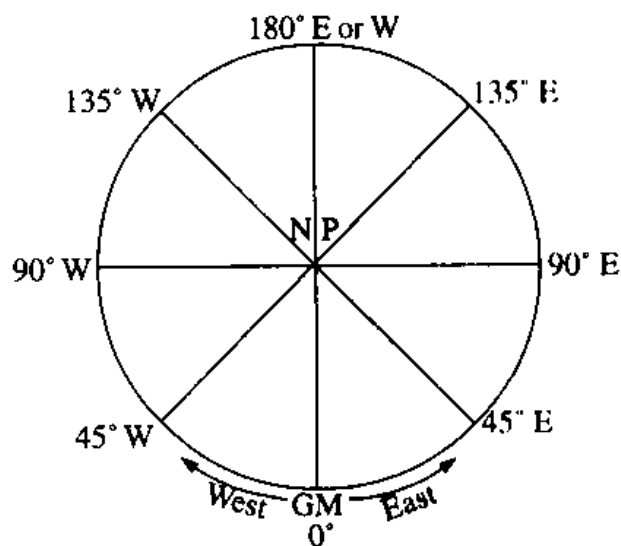


Fig. 6.5

Note:

- (i) The meridian directly opposite the Greenwich is 180° E or W.
- (ii) The maximum angle of longitude is 180° E or W.

6.2: Position of a Place on the Earth's Surface

Any position on the earth's surface can be defined by its latitude and longitude, given by the ordered pair (latitude, longitude). For example, the position of Nairobi on the earth's surface is (1° S, 37° E).

Example 1

Two points A and B on the earth's surface are at opposite ends of a diameter through the centre of the earth. If the position of A is (40° N, 42° E), what is the position of B?

Solution

Figure 6.6 shows the view of the earth from the north pole marked N:

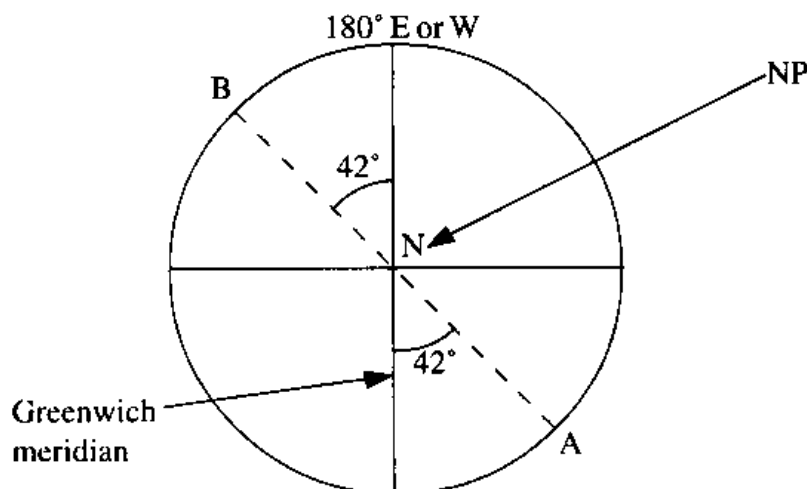


Fig. 6.6

The angle between the Greenwich Meridian and AN is 42° , so the angle between BN and the longitude 180° E or W is 42° . The longitude of point B is $(180 - 42)^\circ$ West, but its latitude does not change (40° N). Therefore, B is (40° N, 138° W).

Note:

- (i) If P is θ north of the equator and Q is α south of the equator, then the difference in latitude between them is given by $(\theta + \alpha)$, see figure 6.7 (a).
Angle POQ = $\theta + \alpha$
- (ii) If P and Q are on the same side of the equator, then the difference in latitude is $(\theta - \alpha)$ see figure 6.7 (b). Angle POQ = $\theta - \alpha$.

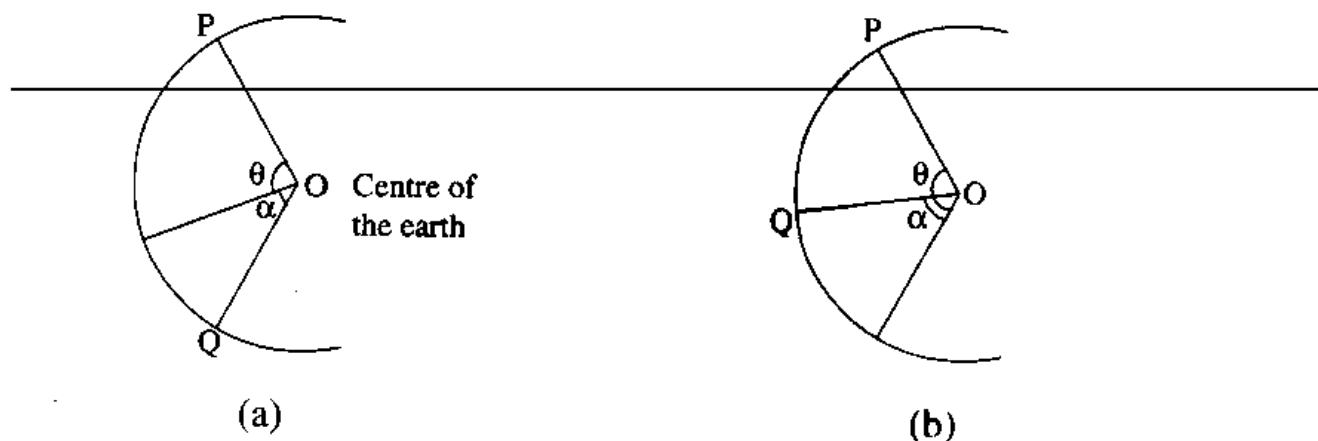


Fig. 6.7

Exercise 6.1

Refer to figure 6.8 to answer questions 1 to 3.

1. Use latitudes and longitudes to give the positions of the points A, B, C, D and E:

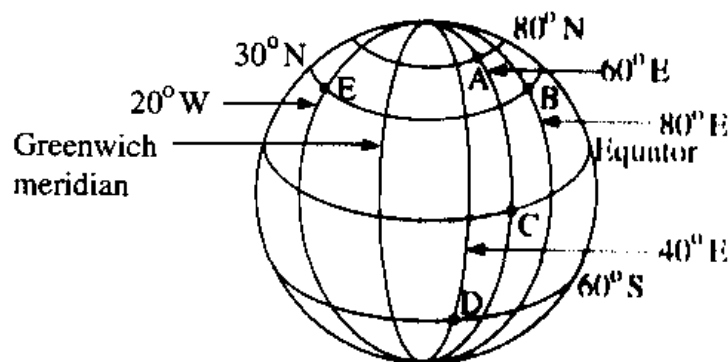


Fig. 6.8

2. Find the difference in longitude between:
 - (a) A and B.
 - (b) E and D.
 - (c) E and B.
 - (d) D and C.
 - (e) D and B.
3. Find the difference in latitude between:
 - (a) A and C.
 - (b) E and A.
 - (c) D and A.
 - (d) B and D.
4. On a map of Kenya whose longitudes and latitudes are indicated, give the name of a town whose position is (3°S, 40°E).
5. When can two parallels of latitude have the same radius?

6. ~~From a map of Kenya, give the latitude and longitude of Nanyuki. From the same map, name the lake on which the point $(4^{\circ}\text{N}, 36^{\circ})$ lies. What is the difference in latitude between this point on the lake and Nanyuki?~~
7. Figure 6.9 represents the earth as a sphere, with N and S the poles and EADB the equator. NGAS represents the Greenwich Meridian:

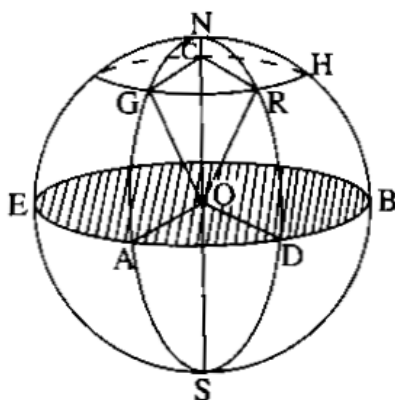


Fig. 6.9

State the angles equal to:

- (a) DOR.
(b) AOD.

6.3: Distances on the Surface of the Earth

Distance along a Great Circle

Consider arc AB along the equator, which subtends an angle θ at the centre O of the earth, see figure 6.10:

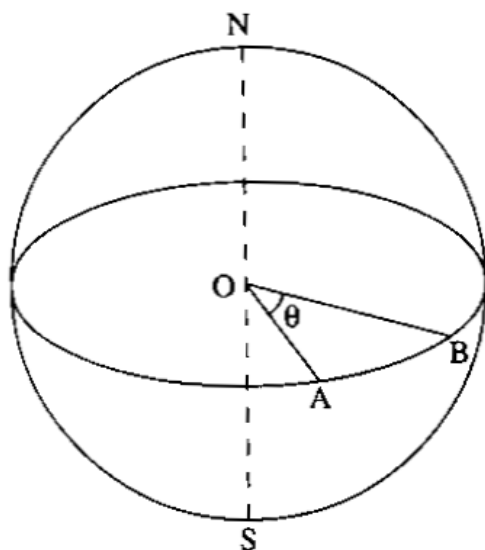


Fig. 6.10

Similarly, in figure 6.11, arc AB on a longitude subtends an angle θ at the centre O of the earth.

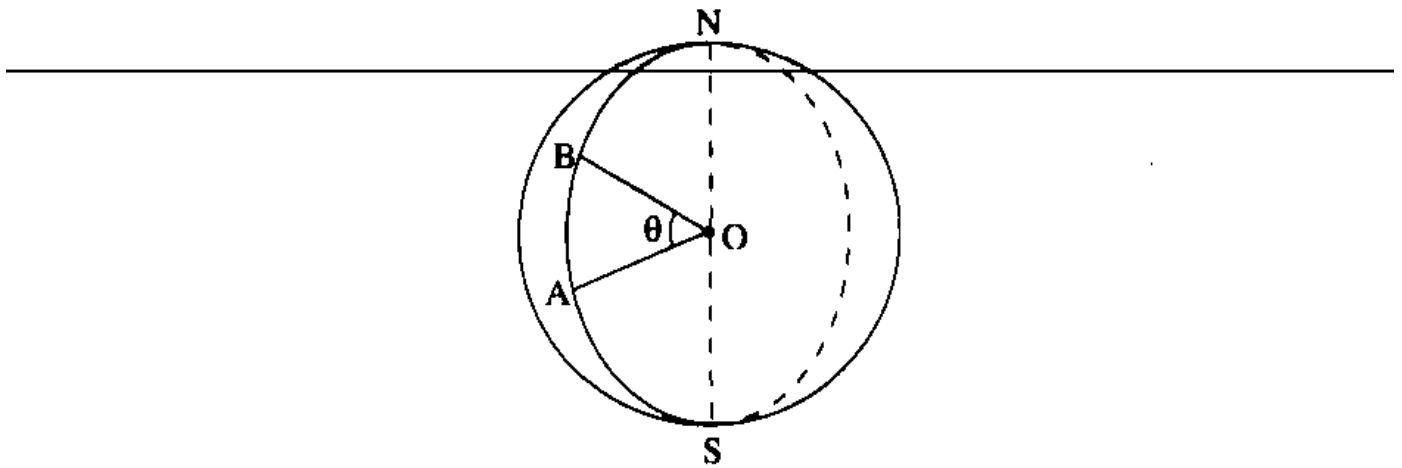


Fig. 6.11

If $\theta = 1^\circ$ in either figure 6.10 or 6.11, the length of the arc is approximately 60 nautical miles. So, the length of an arc of a great circle subtending an angle $1'$ (one minute) at the centre of the earth is 1 nautical mile (nm), see figure 6.12.

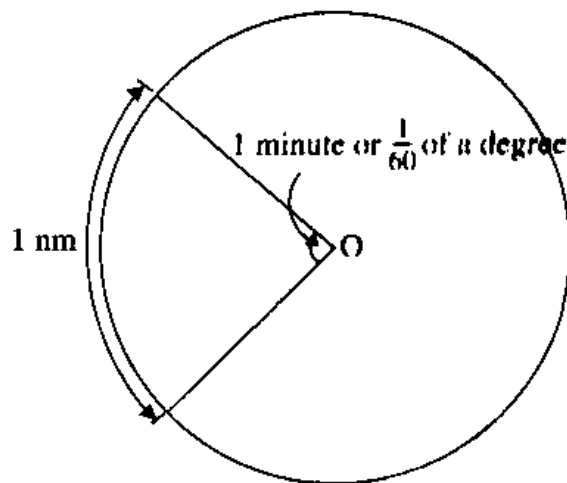


Fig. 6.12

A nautical mile is the standard international unit for measuring distances travelled by ships and aeroplanes.

$$1 \text{ nautical mile (nm)} = 1.853 \text{ km}$$

Example 2

Find the distance between points P (40°N , 50°E) and Q ($20^\circ 30'\text{S}$, 50°E) and express it in:

(a) nm.

(b) km.

(Take radius of the earth to be 6 370 km)

Solution

Figures 6.13(a) and (b) show the positions of P and Q on the same longitude:

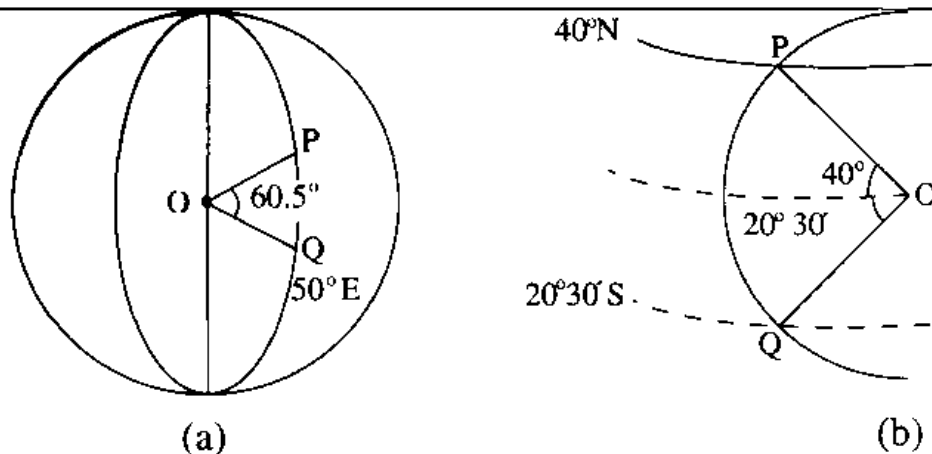


Fig. 6.13

The two points are on the same great circle 50°E .

(a) Angle subtended at the centre is $40^\circ + 20.5^\circ = 60.5^\circ$

1" is subtended by 60 nm

60.5° is subtended by; $60 \times 60.5 = 3\,630$ nm

(b) The radius of the earth is 6 370 km.

Therefore, the circumference of the earth along a great circle is;

$$2\pi r = 6\,370 \times 2 \times \frac{22}{7}$$

Angle between the points is 60.5° . Therefore, we find the length of an arch of a circle which subtends an angle of 60.5° at the centre.

$$360^\circ \text{ is subtended by arc whose length is } 6\,370 \times 2 \times \frac{22}{7}$$

$$\text{Therefore, } 60.5^\circ \text{ is subtended by; } \frac{60.5}{360} \times 6\,370 \times 2 \times \frac{22}{7} = 6\,729 \text{ km}$$

Example 3

Find the distance between points A($0^\circ, 30^\circ\text{E}$) and B($0^\circ, 50^\circ\text{E}$) and express it in:

(a) nm.

(b) km.

(Take the radius of the earth to be 6 370 km)

Solution

Figure 6.14 shows the two points A and B:

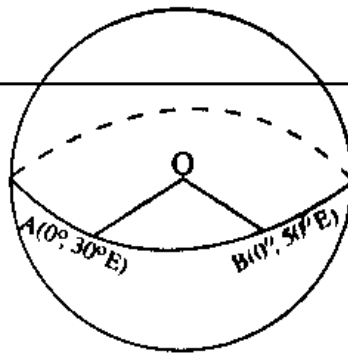


Fig. 6.14

- (a) The two points lie on the equator, which is a **great circle**. Therefore, we are calculating distance along a **great circle**.

Angle between points A and B is $(50^\circ - 30^\circ) = 20^\circ$

1° is subtended by an arc of 60 nm

20° is subtended by; $60 \times 20 = 1\,200$ nm

- (b) Distance in km $= \frac{20}{360} \times 6370 \times 2 \times \frac{22}{7}$
 $= 2\,224$ km

Generally, if an arc of a great circle subtends an angle θ at the centre of the earth, the arc's length is $(60 \times \theta)$ nautical miles.

Distance along a Small Circle (Circle of Latitude)

In figure 6.15, ABC is a small circle, centre X and radius r cm. PQST is a great circle, centre O, radius R cm. The angle θ is between the two radii (OC and OT).

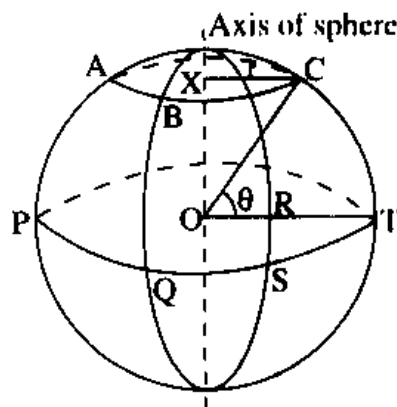


Fig. 6.15

From the figure, XC is parallel to OT. Therefore, angle COT = angle XCO = θ (alt. angles)

Angle $CXO = 90^\circ$ (radius XC is perpendicular to axis of sphere)

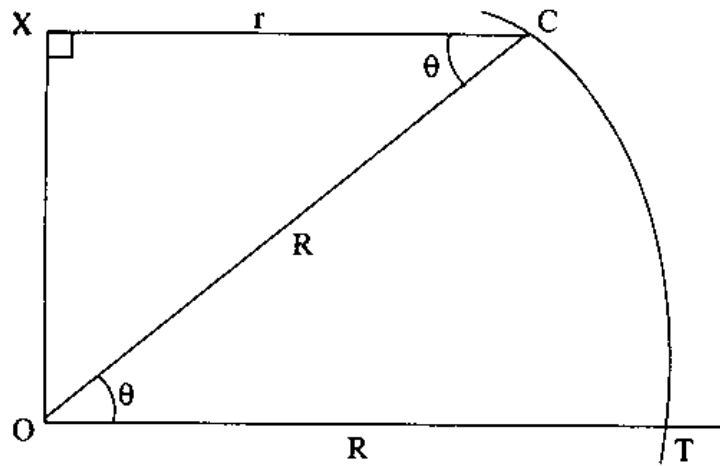


Fig. 6.16

Thus, from the right-angled triangle OXC (see figure 6.16);

$$\cos \theta = \frac{r}{R}$$

Therefore, $r = R \cos \theta$

This expression can be used to calculate the distance between any two points along the small circle ABC , centre X and radius r .

Example 4

Figure 6.17 shows three points A , B and C on the surface of a sphere. If $O_1A = 3$ cm, $OB = 5$ cm and angle $BOC = 36.9^\circ$, find:

- the angle x .
- the distance O_2C .

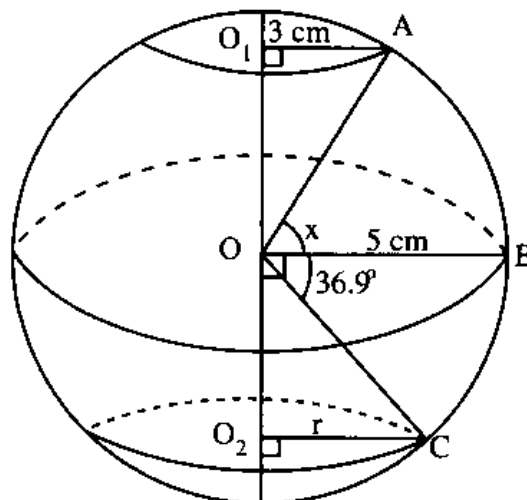


Fig. 6.17

Solution

(a) Consider figure 6.18 below:

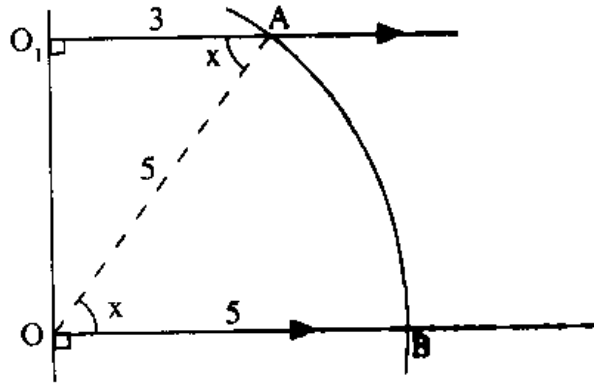


Fig. 6.18

Triangle AO_1O is right-angled at O_1 , $OA = OB$ (radius of sphere) and $\angle O_1AO = \angle AOB = x$ (alternate angles).

$$\begin{aligned}\text{Therefore, } \cos x &= \frac{3}{5} \\ &= 0.6\end{aligned}$$

$$\begin{aligned}x &= \cos^{-1} 0.6 \\ &= 53.1^\circ\end{aligned}$$

(b) In figure 6.19, let $O_2C = r$ cm.

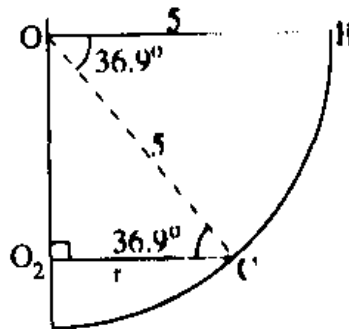


Fig. 6.19

Triangle OCO_2 is right-angled at O_2 .

Angle $OCO_2 = \text{angle } COB = 36.9^\circ$ (alternate angles)

$OB = OC = 5$ cm (radius of sphere).

$$\text{Therefore, } \cos 36.9^\circ = \frac{r}{5}$$

$$\begin{aligned}r &= 5 \cos 36.9 \\ &= 5 \times 0.7997 \\ &= 3.999 \text{ cm}\end{aligned}$$

We have seen that an angle of 1° at the centre of a great circle, radius R is subtended by 60 nm.

Now, consider a circle of latitude θ and radius r . An angle of 1° at the centre of the circle will be subtended by an arc of length;

$$\frac{1}{R} \times 60 \text{ nm}$$

If θ is the angle of latitude, then $r = R \cos \theta$

Therefore, 1° is subtended by; $\frac{R \cos \theta \times 60}{R} = 60 \cos \theta$

In general, if the angle at the centre of circle of latitude θ is α , then the length of its arc is $60 \alpha \cos \theta$ nm, where α is the angle between the longitudes along the same latitude.

Example 5

Find the distance in kilometres and nautical miles between two points, $P(30^\circ\text{N } 45^\circ\text{E})$ and $Q(30^\circ\text{N } 60^\circ\text{W})$.

Solution

Figure 6.20 (a) shows the position of P and Q on the surface of the earth while figure 6.20 (b) shows their relative positions on the small circle. C is the centre of the circle of latitude 30°N with radius r .

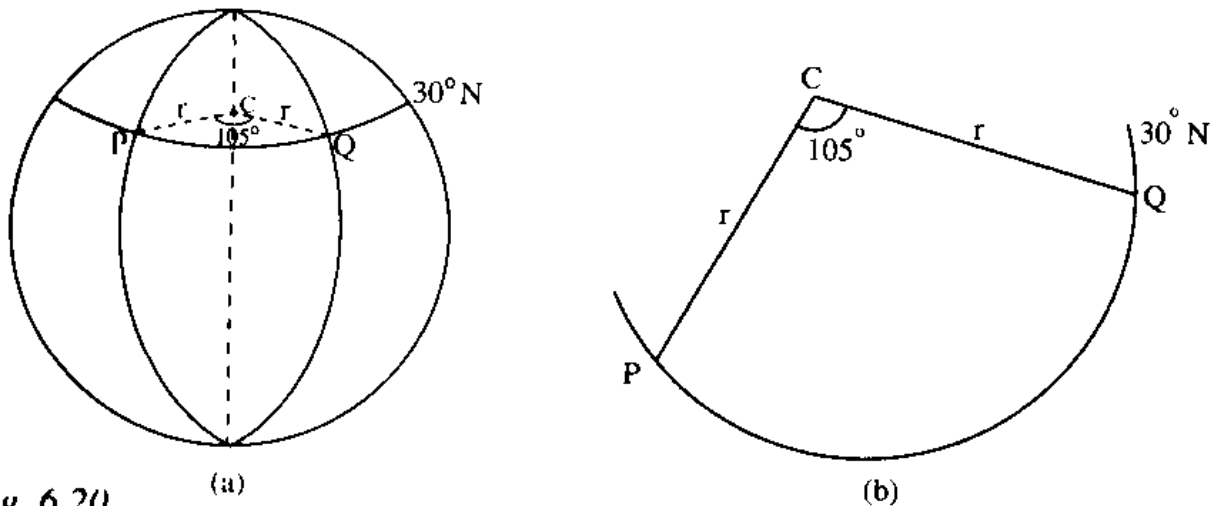


Fig. 6.20

The angle subtended by the arc PQ centre C is $45^\circ + 60^\circ = 105^\circ$. So, the length

$$\begin{aligned} \text{of } PQ &= \frac{105}{360} \times 2\pi r \\ &= \frac{105}{360} \times 2\pi R \cos 30^\circ \text{ km} \\ &= \frac{105}{360} \times 2 \times \frac{22}{7} \times 6\,370 \cos 30^\circ \text{ km} \\ &= 10\,113 \text{ km} \end{aligned}$$

$$\begin{aligned} \text{The length of PQ in nautical miles} &= 60 \times 105 \cos 30^\circ \text{ nm} \\ &= 60 \times 105 \times 0.8660 \text{ nm} \\ &= 5\,456 \text{ nm} \end{aligned}$$

Example 6

Find the distance between points A (50°S , 25°E) and B (50°S , 140°E) in:

- (a) km.
(b) nm.

(Take the radius of the earth to be 6 370 km and $\pi = \frac{22}{7}$)

Solution

- (a) In figure 6.21, O is the centre of circle of latitude 50°S on which arch AB lies. The radius r of the circle about 50°S is given by $R \cos 50^\circ$. The angle subtended by arc AB at O is $140^\circ - 25^\circ = 115^\circ$.

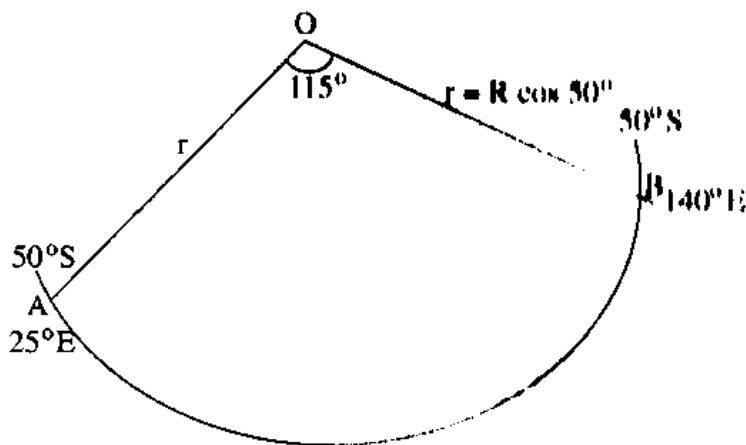


Fig. 6.21

$$\begin{aligned} \text{Arc AB} &= \frac{115}{360} \times 2\pi r \\ &= \frac{115}{360} \times 2\pi R \cos 50^\circ \\ &= \frac{115}{360} \times 2 \times \frac{22}{7} \times 6\,370 \cos 50^\circ \\ &= 8\,222 \text{ km} \end{aligned}$$

- (b) Arc AB = $60 \times 115 \cos 50^\circ$ nm
 $= 60 \times 115 \times 0.6428$ nm
 $= 4\,435$ nm

Shortest Distance Between two points on Earth's Surface**Example 7**

P and Q are two points on latitude 50°N . They lie on longitudes 48°W and 132°E respectively. Find the distance from P to Q:

- (a) along a parallel of latitude.
 (b) along a great circle.

Solution

The positions of P and Q on earth's surface are shown in figure 6.22.

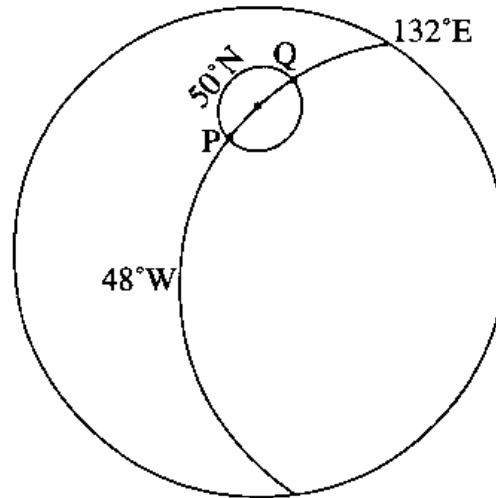


Fig. 6.22

- (a) The length of the circle parallel of latitude 50°N is $2\pi r$ km, which is $2\pi R \cos 50^\circ$ km. The difference in longitude between P and Q is, $132^\circ + 48^\circ = 180^\circ$

$$\begin{aligned} PQ &= \frac{180}{360} \times 2 \times \frac{22}{7} \times 6\,370 \cos 50^\circ \\ &= 12\,869 \text{ km} \end{aligned}$$

- (b) The required great circle passes via the North Pole. Therefore, the angle subtended at the centre by the arc PNQ is; $180^\circ - 2 \times 50^\circ = 80^\circ$, as in figure 6.23.

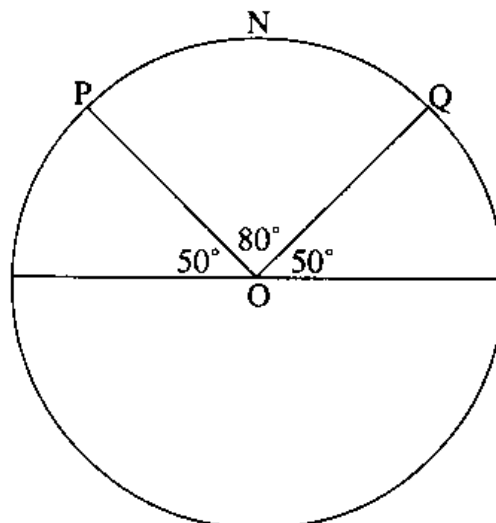


Fig. 6.23

$$\begin{aligned} \text{Therefore, arc PNQ} &= \frac{80}{360} \times 2\pi R \\ &= \frac{80}{360} \times 2 \times \frac{22}{7} \times 6\,370 \text{ km} \\ &= 8\,898 \text{ km} \end{aligned}$$

From this example, we notice that the distance between two points on the earth's surface along a great circle is shorter than the distance between them along a small circle.

In general, the shortest distance between two points on the earth's surface is that along a great circle.

Exercise 6.2

(In this exercise, take radius of the earth to be 6 370 km and $\pi = \frac{22}{7}$)

- Calculate the distance between the points P(40°N, 15°W) and Q(40°N, 60°W) in:
 - km.
 - nm.
- Find the circumference of the latitude 70°S in:
 - km.
 - nm.
- Find the circumference of the Tropic of Cancer (23 $\frac{1}{2}$ °N) in:
 - km.
 - nm.
- Find the length of the parallel of latitude through Nairobi (1°S, 37°E) in:
 - km.
 - nm.
- Find the distance in nautical miles between points A(45°N, 35°W) and B(45°N, 100°E) along the parallel of latitude.
- Find the latitude south of the equator such that when one travels along the latitude for 80 nm, a change of 2° in longitude is made.
- Find the distance between the points P(35°N, 50°E) and Q(26° 30'S, 50°E) on the earth's surface in:
 - km.
 - nm.
- Find the distance along a circle of latitude between A(55°N, 35°E) and B(55°N, 70°E) on the earth's surface in:
 - km.
 - nm.

9. A ship sails 450 nm due East of a point T (40°N , 60°E). Find the new longitude.
10. Find the distance between the points A(50°N , 60°E) and B(50°N , 30°W) in:
 (a) km.
 (b) nm.
11. Figure 6.24 shows the points K, L, M and N on the earth's surface. K and L are on latitude $P^{\circ}\text{S}$. N and M are on the equator such that N is due north of K and M is due north of L. If $NM = 2 KL$, find P.

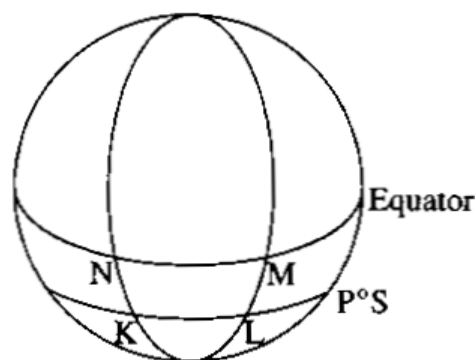


Fig. 6.24

12. A spherical globe is of radius 20 cm. Find the shortest distance along its equator between longitudes 15°W and 120°E .
13. Figure 6.25 shows points A(40°N , 30°W), B(40°N , 30°E), C(40°S , 30°E) and D(40°S , 30°W):

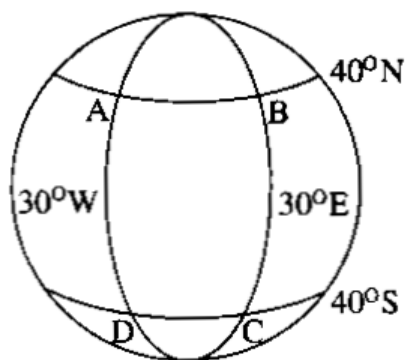


Fig. 6.25

Calculate the perimeter of ABCD.

14. Find the distance between the points A(30°N , 40°E) and B(24°S , 40°E).
15. Figure 6.26 shows points P and Q on latitude $y^{\circ}\text{S}$. They also lie on the longitudes 120°W and 50°E respectively. If the distance between them is 2 890 nm, calculate y.

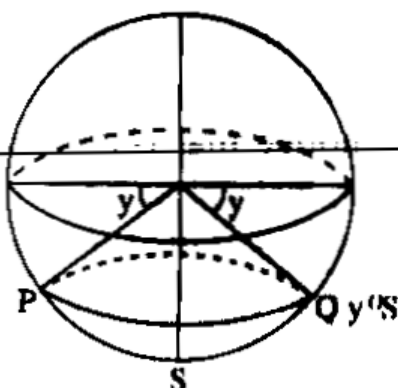


Fig. 6.26

16. Two points P and Q on the surface of the earth are on the same latitude. The difference between their longitude is 72° . If the distance between them along the latitude is 1 280 km, find the latitude of P and Q.
17. A point B is due east of A along the equator. The distance between A and B is 1 040 km. If the longitude of A is 15°E , calculate the longitude of B.
18. Find the shortest distance between two points P(50°N , 70°W) and Q(50°N , 110°E) on the earth's surface in kilometres.

6.4: Longitude and Time

The earth rotates through 360° about its axis every 24 hours in a west-east direction. This means that for every 1° change in longitude there is a corresponding change in time of 4 minutes, or there is a difference of 1 hour between two meridians 15° apart.

All places on the same meridian have the same local time. Local time at Greenwich is called Greenwich Mean Time, abbreviated GMT.

All meridians to the west of Greenwich Meridian have sunrise after the meridian and their local times are behind GMT. Likewise, all meridians to the east of Greenwich Meridian have sunrise before that meridian and their local times are ahead of GMT. Since the earth rotates from west to east, any point P is ahead in time of another point Q if P is east of Q on the earth's surface.

Example 8

Find the local time of Nairobi (1°S , 37°E), when the local time of Manderu (4°N , 42°E) is 3.00 p.m.

Solution

The difference in longitude between Manderu and Nairobi is $(42^\circ - 37^\circ) = 5^\circ$, that is Manderu is 5° East of Nairobi. Therefore their local times differ by;
 $4 \times 5 \text{ min} = 20 \text{ min}$.

Since Nairobi is in the west of Mandera, we subtract 20 minutes from 3.00 p.m. This gives local time for Nairobi as 2.40 p.m.

From example 8, you notice that Nairobi and Mandera would have different local times. However, Nairobi and Mandera are in the same world time-zone. So they use the same time called standard time.

The world has 24 time zones. This facilitates the use of the same standard time within a given zone.

Example 9

If the local time of London (52°N , 0°) is 12.00 noon, find the local time of Nairobi (1°S , 37°E).

Solution

Difference in longitude is $(37^{\circ} + 0^{\circ}) = 37^{\circ}$

So, difference in time is $4 \times 37 \text{ min} = 148 \text{ min}$
 $= 2 \text{ hrs } 28 \text{ min.}$

Therefore, local time of Nairobi is 2 hours 28 minutes ahead that of London, that is, 2.28 p.m.

Example 10

If the local time of a point A(0° , 170°E) is 12.30 a.m. on Monday, find the local time of a point B(0° , 170°W).

Solution

Difference in longitude between A and B is $170^{\circ} + 170^{\circ} = 340^{\circ}$

In time is $4 \times 340 = 1\,360 \text{ min}$
 $= 22 \text{ hrs } 40 \text{ min.}$

Therefore, local time of point B is 22 hours 40 minutes behind Monday 12.30 p.m., that is, Sunday 1.50 a.m.

6.5: Speed

A speed of 1 nautical mile per hour is called a **knot**. This unit of speed is used by airmen and sailors.

Example 11

A ship leaves Mombasa (4°S , 39°E) and sails due east for 98 hours to a point K (4°S , 80°E) in the Indian Ocean. Calculate its average speed in:

- km/h.
- knots.

Solution

- (a) The length x of the arc from Mombasa to the point K in the ocean.

$$\begin{aligned} x &= \frac{41}{360} \times 2\pi r \\ &= \frac{41}{360} \times 2\pi R \cos 4^\circ \text{ km} \\ &= \frac{41}{360} \times 2 \times \frac{22}{7} \times 6\,370 \cos 4^\circ \text{ km} \\ &= 4\,549 \text{ km} \end{aligned}$$

$$\begin{aligned} \text{Therefore, speed,} &= \frac{4\,549}{98} \text{ km/h} \\ &= 46.41 \text{ km/h} \end{aligned}$$

- (b) The length x of the arc from Mombasa to the point K in the ocean in nautical miles

$$\begin{aligned} x &= 60 \times 41 \times \cos 4^\circ \text{ nm} \\ &= 60 \times 41 \times 0.9976 \text{ nm} \\ &= 2\,454 \text{ nm} \end{aligned}$$

$$\begin{aligned} \text{Therefore, speed} &= \frac{2\,454}{98} \text{ knots} \\ &= 25.04 \text{ knots} \end{aligned}$$

Exercise 6.3

(Take the radius of the earth to be 6 370 km)

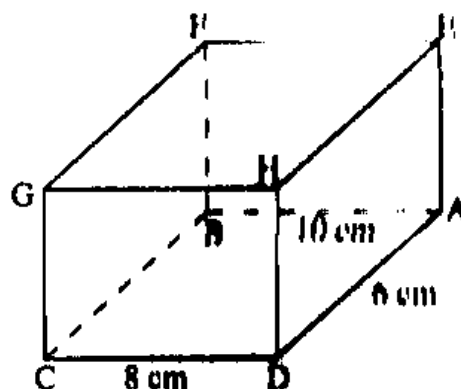
- Find the speed in knots for the given distances and the time taken:
 - 100 nm, 2 hours.
 - 50 nm, $\frac{1}{2}$ hour.
 - 9 840 nm, 7 hours.
 - 0.5 nm, 0.25 hours.
 - Find the distance covered in each of the following cases:
 - 20 knots for 6 hours.
 - 150 knots for 12 hours.
 - 40.6 knots for 8 hours.
- Find the local time at the following points when it is 1300 H in London on Monday:

(a) (1°S, 37°E)	(b) (0°, 9°E)	(c) (41°N, 74°W)
(d) (51°N, 7°E)	(e) (1°N, 103°E)	(f) (60°N, 40°W)
(g) (41°N, 4°W)	(h) (35°N, 105°W)	(i) (20°S, 24°40'W)
- An object moves from (30.5°N, 0°) to (30.5°N, 45°E) in 3 hours. Calculate the speed of the object in:
 - km/h.
 - knots.

4. A point Q lies on the Tropic of Capricorn (23.5°S). Calculate the speed of Q due to the rotation of the earth in:
 - (a) knots.
 - (b) km/h.
5. A weather forecaster reports that the centre of a cyclone at (30°N , 120°W) is moving due north at 24 knots. How long will it take to reach a point (45°N , 120°W)?
6. Points X and Y lie on the equator and their longitudes differ by 10° . An aircraft takes 3 hours to fly between the two points. Calculate its speed in:
 - (a) knots.
 - (b) km/h.
7. Two aircrafts A and B took off at the same time on Monday from Jomo Kenyatta International Airport (1°S , 37°E) at 11.00 p.m. Aircraft A flew due east while B flew due west. If they met again after 18 hours at (1°S , 117°W), calculate:
 - (a) their respective speeds.
 - (b) the time they met again.
8. Two places X and Y are 960 km apart and X is due South of Y. If the latitude of X is 40°N , find the latitude of Y.
9. An aeroplane leaves point A(40°N , 78°W) and flies due West at 600 km/h. If it travels for 3 hours 20 minutes:
 - (a) how far has it travelled?
 - (b) what is the longitude of its position?
10. A ship in distress relays a signal and its position as (45°S , 30°W). Two rescue vessels at A(45°S , 35°W) and B(52°S , 30°W) respectively picked the signal and started moving immediately towards the scene. If A steams due east at 18 knots and B due north at 24 knots, calculate the time taken by each to arrive at the scene.
11. A plane leaves point A(84°S , 10°E) and flies due North for $6\frac{1}{2}$ hours at a speed of 468 knots. It reaches a point B($\alpha^{\circ}\text{S}$, 10°E) and then flies west to point X for 1 hour 20 minutes at the same speed. Find the value of α and the longitude of X.

Mixed Exercise 2

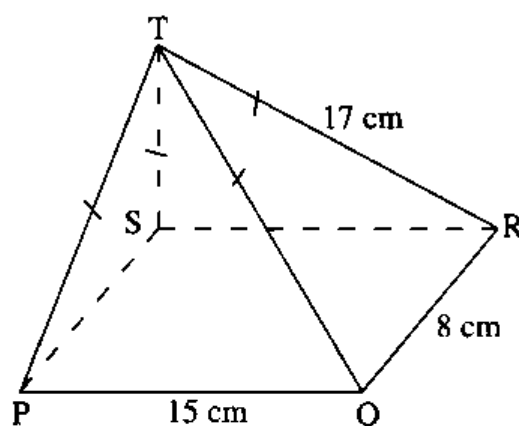
- Write down the first five terms in the expansion of the following expressions in ascending powers of x :
 - $(x + 1)^{12}$
 - $(1 - x)^{13}$
 - $(2 + x)^n$
 - $(2 - \frac{1}{2}x)^9$
- Solve for x in the equation $6 \sin \frac{1}{2}x = 2 \tan \frac{1}{2}x \cos \frac{1}{2}x$ for $0 \leq x \leq 2\pi$.
- Show the region satisfied by the inequalities $y > 2$, $x \geq 0$ and $y \leq 8 - 2x$.
- If $\sin 2\theta = 2\sin \theta \cos \theta$, prove that $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$.
- The figure below shows a cuboid $ABCDEFGH$ in which $CD = 8$ cm, $AD = 6$ cm and $DH = 10$ cm. Find:
 - DF .
 - $\angle BDF$.



- Two towns X and Y on latitude $15^\circ S$ differ in longitude by $36^\circ 12'$. Find the distance between them along their parallel of latitude. (Take radius of the earth to be 6370 km).
- On the same axes, draw the graphs of $y = 3\sin(2x + 30^\circ)$ and $y = \sin 2x$. From your graph, deduce:
 - the amplitude of each wave,
 - the period of each wave,
 - which wave leads the other, and by how much.
- Find the coefficient of x^3 in the expansion of $(2 - 3x)^n$.
- Obtain the first four terms of the expansion of $(1 + \frac{x}{12})^6$ in ascending powers of x . Use your expansion to evaluate $(\frac{3}{4})^6$.

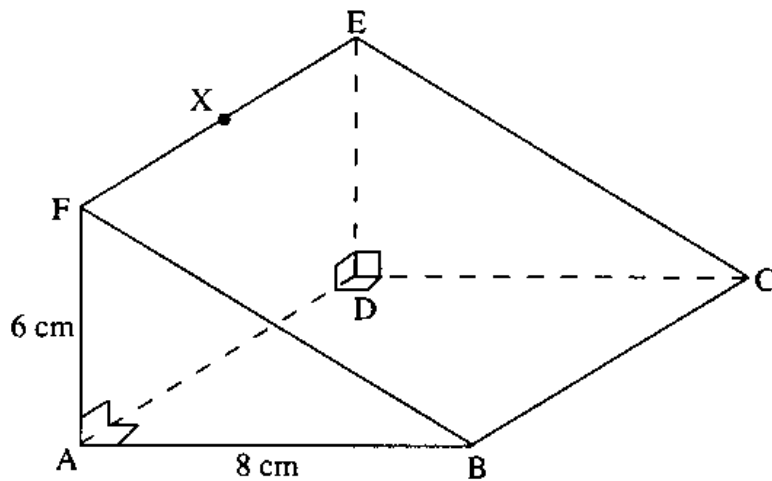
10. On the same axes, draw the graphs of $y = \cos(\theta + 30^\circ)$ and $y = \sin 2\theta$ for $0^\circ \leq \theta \leq 90^\circ$. From your graphs, determine the value of θ for which $\cos(\theta + 30^\circ) = \sin 2\theta$.
11. Evaluate without using tables;
$$\frac{\tan 60^\circ \sin 45^\circ \cos 45^\circ}{\tan 30^\circ \sin 60^\circ}$$
12. A ship leaves port (50°N , 120°W) and sails due west at an average speed of 30 knots. How many nautical miles has it travelled after sailing continuously for 3 days, and what is its latitude then?
(Take the radius of the earth to be 6 370 km).
13. The graph of $y = \cos x$ is transformed so that its image has the equation $y = 3\cos(2x + 30^\circ)$. Describe all the transformations that have taken place.
14. Find two values of θ for which $12 \sin \theta - 1 = 5$, for $0^\circ \leq \theta \leq 360^\circ$.
15. Solve the equation $8 = 3\tan \theta$ for $0^\circ \leq \theta \leq 360^\circ$.
16. Expand:
(a) $(4x - y)^5$
(b) $(1 + \frac{1}{4}x)^6$
17. Find the coefficient of the term in $x^3 y^3$ in the expansion $(2x - 3y)^6$.
18. Use the binominal expansion to evaluate $(5.998)^4$ correct to 6 decimal places.
19. Using the same scale and axis, draw the graph of $y = 1 - \cos 2x$ and $y = 3 \sin x$ for $0^\circ \leq x \leq 360^\circ$. Hence, solve the equation $\cos 2x + 3 \sin x = 1$.
20. Sketch the curve $y = \frac{1}{2} \sin(x + 60)$ for the interval $0^\circ \leq x \leq 360^\circ$ and state:
(a) the amplitude.
(b) the period of the curve.
21. Solve for x if $2\cos x + \sin \frac{1}{2}x + 1 = 0$, for $0 \leq x \leq 360^\circ$.
(Hint: $\cos 2A = \cos^2 A - \sin^2 A$.)
22. Without using tables, deduce the value of $\sin x$ and $\cos x$ if $\tan x$ is:
(a) $\frac{3}{4}$ (b) $\frac{11}{12}$ (c) $-\frac{15}{8}$

23. Use the binomial expansion to evaluate:
- $(0.998)^8$, correct to 3 decimal places.
 - $(1.993)^{10}$, correct to 3 decimal places.
 - $(1.01)^6$, correct to 3 decimal places.
24. (a) Expand $(3x - \frac{1}{3x})^8$ in ascending powers of x .
- (b) Simplify $(2 + \sqrt{3})^3 - (2 - \sqrt{3})^3$.
- (c) Find the sixth term in the expansion of $(\sqrt{2} - \sqrt{3})^{10}$.
25. In the figure below, $TS = TP = TQ = TR = 17$ cm, $PQ = 15$ cm and $QR = 8$ cm:



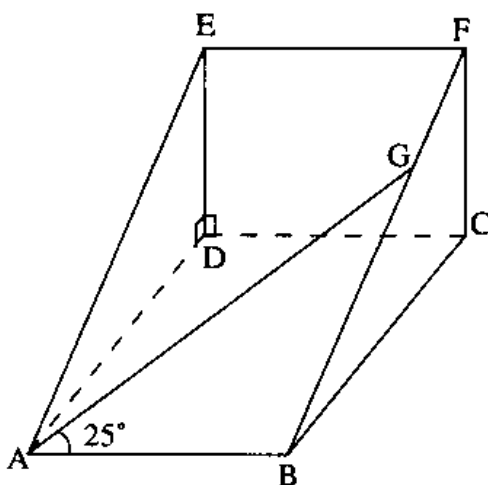
- Find: (a) the angle between the line TP and the plane $PQRS$.
 (b) the angle between the planes TPS and TRQ .
26. The positions of points A and B are $(25^\circ\text{N}, 45^\circ\text{E})$ and $(25^\circ\text{N}, 63^\circ\text{E})$ respectively. Calculate the distance between the points along the latitude.
27. Given that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, find:
- $\tan A$, if $\tan(A + 45^\circ) = 2$.
 - $\tan(A + B)$, if $\tan A = \frac{\sqrt{3}}{2}$ and $\tan B = \frac{2}{\sqrt{5}}$, leaving your answer in surd form.
28. Use the expansion given in question 27 above:
- to show that $\tan(A + B + C) = \frac{44}{23}$, if $\tan A = \frac{2}{5}$, $\tan B = \frac{1}{2}$ and $\tan C = \frac{1}{4}$.
 - to calculate the acute angle between the lines $y = \frac{1}{2}x$ and $y = 2x$.

29. The figure below shows a prism in which $FA = 6$ cm and $AB = 8$ cm. X is a point on the edge FE such that $\angle FBX = 45^\circ$. Calculate the angle between BX and the plane $ABCD$.



30. Solve for x in $2 \cos x = \sin^2 x + 2$, for $0 \leq x \leq 360^\circ$.
31. Four dice are tossed at random. Find the probability that:
- the total score is 4.
 - the total score is 10.
 - the total score is 15.
32. Solve x in $\cos 3x = \sin(x + 20)$, for $0 \leq x \leq 360^\circ$.
33. A tight belt passes over two pulleys of diameter 1.5 m and 2.0 m with their centres 2.5 m apart. Calculate the length of the belt.
34. Find the first four terms in the expansion of $(1 - 5x)^5 (1 + x)^3$ in ascending powers of x .
35. Calculate the distance between the points $P(11^\circ\text{N}, 32^\circ\text{E})$ and $Q(39^\circ\text{N}, 32^\circ\text{E})$ along the great circle:
- in kilometres.
 - in nautical miles.
- (Take the radius of the earth to be 6 370 km).
36. Two gear wheels are to be fixed such that the distance between their centres is 80 cm. If the diameters of the two gears are 60 cm and 20 cm respectively, calculate the length of the drive-chain required to connect the wheels.
37. In expansion of $(2x - 5)^{10}$, find the ratio of the coefficient of the term in x^6 to that of the term in x^8 .

38. The positions of two towns are $(40^\circ\text{N}, 60^\circ\text{E})$ and $(40^\circ\text{N}, 180^\circ\text{E})$. Find the speed of an aircraft in kilometres per hour if it takes 10 hours to cover the distance between the two towns.
(Take the radius of the earth to be 6 370 km).
39. The figure below shows a wedge ABCDEF in which $AB = EF = DC = 150$ cm and $ED = FC = 50$ cm. F divides BG externally in the ratio 5 : 2. AG makes an angle of 25° with the edge AB. Find the angle between the planes ABFE and ABCD.



Chapter Seven

LINEAR PROGRAMMING

7.1: Forming Linear Inequalities

In Book Two, we dealt with inequalities. In this section, we shall form inequalities to represent given conditions involving real life situations.

Example 1

Yona is five years younger than his sister. The sum of their ages is less than 36 years. If Yona's age is x years, form all the inequalities in x for this situation.

Solution

The age of Yona's sister is $x + 5$ years.

Therefore, the sum of their ages is;

$$x + (x + 5) \text{ years}$$

Thus;

$$2x + 5 < 36$$

$$2x < 31$$

$$x < 15.5$$

$$x > 0 \text{ (age is always positive).}$$

Example 2

The perimeter of a rectangle whose length is x cm and width y cm is not more than 10 cm. Form all the inequalities connecting x and y .

Solution

The perimeter of the rectangle is $2(x + y)$ cm.

Therefore, $2(x + y) \leq 10$

$$x + y \leq 5$$

Since the dimensions of a rectangle are positive, we have $x > 0$ and $y > 0$

Exercise 7.1

1. Stephen's salary is twice as much as his wife's. The wife's salary is in excess of sh. 4 000. The sum of their monthly salaries is not more than sh. 16 000. If the wife's salary is sh. b , form all the inequalities in b to represent the information.

2. In a Physics test, George scored more marks than John. The two together scored more than 100 marks, with John's not less than 20. If the marks scored by George and John were x and y respectively, form all the inequalities representing this situation.
3. Juma is 2 cm taller, but 3 kg lighter than Kundu. The sum of their heights is not less than 240 cm, and the sum of their weights is less than 150 kg. If Kundu is x cm tall and weighs y kg, form all the inequalities in x and y to represent the information.
4. Chesang has 10 more goats than Walumbe. Altogether, they own not less than 60 goats, with Walumbe's goats numbering more than a third of the total. By choosing a letter to represent the number of Walumbe's goats, write down all inequalities for the information.
5. The sum of two integers 'a' and 'b' is less than five times their difference. Their product is not less than 17. If a is greater than b, form all the inequalities representing the situation.
6. Miriam wishes to buy x bowls and y plates. The cost of a bowl is sh. 20 and that of a plate sh. 48. She has sh. 384 to spend on not more than 7 items (bowls and plates). Form all the inequalities representing the situation.
7. A factory produces at most 400 items in a day. The non-defective items sell at sh. 100 each while the defective ones are sold at a discount of 40%, giving a day's sales of at least sh. 30 000. If in a day x non-defective and y defective items are sold, form all the inequalities to represent the situation.

7.2: Solutions of Linear Inequalities

Problems involving a number of inequalities are solved by first manipulating each of the given situations separately.

Example 3

Mary is twice as old as her sister Jane, who is not less than five years old. The sum of their ages is not more than 30 years. Given that Jane is x years old, form all the inequalities in x to represent the above information and hence find the range of values of x satisfying these inequalities.

Solution

Since Jane's age is x years, Mary's age is $2x$ years.

$$\begin{aligned}
 x &\geq 5 \dots\dots\dots (1) \text{ (Jane's age is not less than 5 years)} \\
 x + 2x &\leq 30 \dots\dots\dots \text{ (Sum of their ages is not more than 30 years).} \\
 3x &\leq 30 \\
 x &\leq 10 \dots\dots\dots (2) \\
 \text{Combining (1) and (2), we get;} \\
 5 &\leq x \leq 10.
 \end{aligned}$$

Example 4

The length of a room is 4 metres more than its width. The width is greater than 5 m while the perimeter of the room is not greater than 36 m. Given that the width of the room is b m, find the range of values of b satisfying all the inequalities representing the given situation.

Solution

Since the width is b m, the length is $b + 4$ m.

$$\begin{aligned}
 P &= 2(b + 4 + b) \\
 &= (4b + 8 \text{ m})
 \end{aligned}$$

$$\therefore 4b + 8 \leq 36$$

$$4b \leq 28$$

$$b \leq 7 \dots\dots\dots (1)$$

$$\text{But } b > 5 \text{ (the width is greater than 5 m) } \dots\dots\dots (2)$$

From (1) and (2), $5 < b \leq 7$.

7.3: Solution by Graphing

Solutions to inequalities formed to represent given conditions can be determined by graphing the inequalities and then reading off appropriate values (possible solutions).

Example 5

A student wishes to purchase not less than 10 items comprising books and pens only. A book costs sh. 20 and a pen sh. 10. If the student has sh. 220 to spend, form all possible inequalities from the given conditions and graph them clearly, indicating the possible solutions.

Solution

Let the number of books be x and the number of pens y . Then, the inequalities are:

$$(i) \quad x + y \geq 10 \text{ (the items bought to be at least ten)}$$

$$(ii) \quad 20x + 10y \leq 220 \text{ (only sh. 220 is available).}$$

This simplifies to $2x + y \leq 22$

(iii) $x > 0$ and $y > 0$ (number of items bought cannot be negative)

The graph in figure 7.1 represents the inequalities:

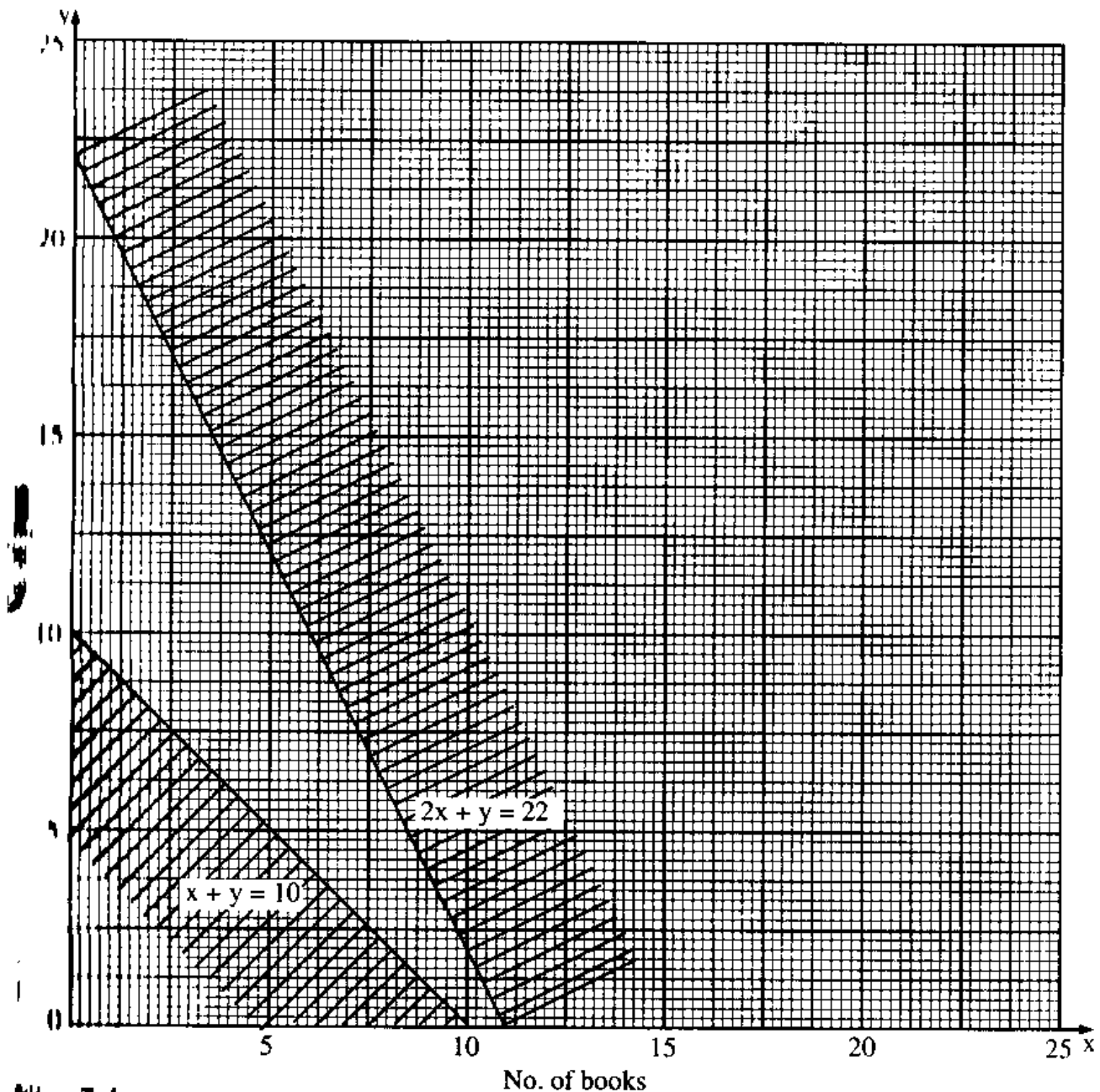


Fig. 7.1

All the points in the unshaded region represent possible solutions. A point with co-ordinates (x, y) represents x books and y pens. For example, the point $(3, 10)$ means 3 books and 10 pens could be bought by the student.

Example 6

A car hire firm lets x Toyota and y Subaru cars in a day. At most, 8 cars are hired per day for a total charge of more than sh. 18 000. The hire charge per day is sh. 2 000 and sh. 3 000 for a Toyota and Subaru car respectively. Form all the

possible inequalities for this situation and graph them. From the graph, list the possible solutions.

Solution

The inequalities are:

- (i) $x + y \leq 8$ (at most 8 cars are hired per day)
- (ii) $2\,000x + 3\,000y > 18\,000$ (the amount is greater than sh. 18 000), which simplifies to $2x + 3y > 18$.
- (iii) $x \geq 0, y \geq 0$ (number of cars is non-negative)

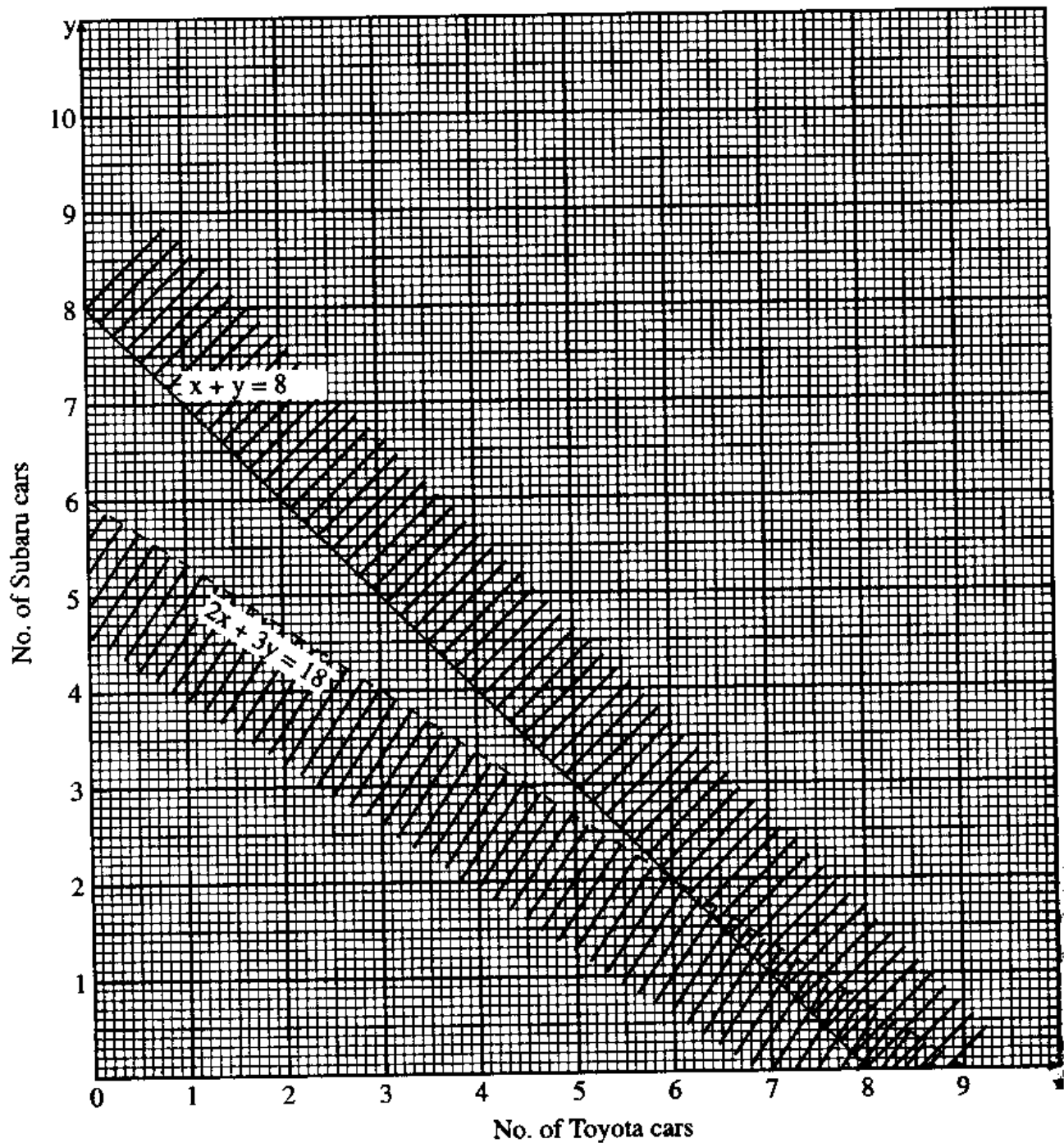


Fig. 7.2

The possible solutions read from the graph are given in table 7.1 below.

Table 7.1

<i>No. of Toyota cars</i>	<i>No. of Subaru cars</i>
0	8
0	7
1	7
1	6
2	6
2	5
3	5
4	4
5	3

Exercise 7.2

1. A firm is to hire two employees. One of them is to earn twice as much as the other, and their total salary should not be more than sh. 39 000. Find the maximum salary that can be paid to the less paid employee.
2. The sum of two digits is more than 10. The second digit is twice the first digit and the first digit is less than 6. Taking the first digit as x , form all inequalities in x representing the given situation and find the range of values for x .
3. Innah owns more than fifty animals (goats and sheep only). There are twenty more goats than sheep, which are not more than thirty in number. By choosing a letter to represent the number of sheep, write down all the inequalities for the information and hence find the range of solutions satisfying the same.
In questions 4 to 13, determine the solution by drawing graphs.
4. A class teacher may spend at most sh. 10 000 to purchase exercise books and class readers for his class. He has to purchase at least 400 exercise books and not more than 20 class-readers.
An exercise book costs sh. 10 while a class-reader costs sh. 100. Determine the possible purchases, given that exercise books are bought in multiples of 100 while class readers are bought in multiples of ten.
5. A farmer wants to keep more than 45 animals (goats and sheep only) on a 0.6 hectare grazing land. A goat requires 0.15 of a hectare and a sheep 0.1 of a hectare for grazing. Find the possible number of goats and sheep the farmer can keep.

6. A headmaster intends to spend not more than sh. 8 000 in transporting 70 students to a football match using two *matatus*, A and B. A has a passenger capacity of ten and B a capacity of 28. The cost per trip for A is sh. 1 200 and for B sh. 2 000. If A makes less than 5 trips, find all the possible solutions for the number of trips made by each *matatu*.
7. In a Mathematics test, the number of correct answers given by Jane are more than 10 but less than the correct answers given by Clara. The sum of the marks scored by the two girls is less than 150, while the difference between the marks is less than 20. If each correct answer is awarded 5 marks, find all the possible numbers of correct answers for each girl.
8. A tailor needs 5 m of cotton material and 3 m of woollen material to make a suit selling at sh. 2 400. He needs 2 m of cotton material and 4 m of woollen material to make a dress selling at sh. 900. If he has less than 90 m cotton material and more than 120 m of woollen material, form all the possible inequalities and use them to find all the possible amounts of money he can get.
9. A company makes less than 30 telephone calls a day. The average charge for a trunk call is sh. 12 and the charge for a local call is sh. 8. A minimum of 12 local calls are made per day. The number of trunk calls is less than half the number of local calls. Find all the possible daily expenditure on telephone.
10. A man has sh. 400 to spend on exercise books and pencils for his children. The cost of a pencil is sh. 5 and that of an exercise book sh. 20. He must buy more than 10 exercise books and the number of pencils should not be more than $\frac{1}{3}$ of the number of exercise books bought. Find the possible expenditure on the items.
11. A farmer wishes to fence off a rectangular enclosure from his land. The width is not to be less than 30 m and the perimeter is at most 150 m. The distance between adjacent fencing poles is to be 1 m all around. A corner pole cost sh. 50 and the other poles along the sides cost sh. 45 each. Determine all possible expenditure on fencing poles. If further the fence is to be a square, find all the possible expenditure.

12. A co-operative society has to transport 200 churns of milk. A pick-up and a tractor with a trailer are available. The pick-up can ferry 12 churns per trip and the tractor 20 per trip. The tractor can make a maximum of 5 trips. The total number of trips must not exceed 18. A tractor costs sh. 60 per trip and the pick-up sh. 100 per trip. Find the possible number of trips the two vehicles should make to transport the milk and the corresponding costs.
13. In order to pay workers' wages, an accountant needs sh. 5 and sh. 10 shilling coins only, besides currency notes. He needs at least forty 5-shilling pieces. The number of ten-shilling coins must at least be double the number of five-shilling coins. The sum of money in coins must not exceed sh. 1 500. Find all the possible combinations of sh. 5 and sh. 10 coins the accountant needs.

7.4: Optimisation

Solutions to problems in the last section consisted of several possibilities. In this section, we learn how to identify from the possible solutions one or more that will best meet set requirements of the problem. Let us consider the following problem.

A contractor intends to transport 1 000 bags of cement using a lorry and a pick-up. The lorry can carry a maximum of 80 bags while the pick-up can carry a maximum of 20 bags. The pick-up has to make more than twice the number of trips the lorry makes and the total number of trips has to be less than 30. The cost per trip is sh. 2 000 for the lorry and sh. 900 for the pick up. Find the minimum expenditure.

If we let x and y be the number of trips made by the lorry and the pick-up respectively, then the conditions are given by the following inequalities;

- (i) $x + y < 30$
- (ii) $80x + 20y \geq 1\,000$, which simplifies to $4x + y \geq 50$
- (iii) $y > 2x$
- (iv) $x > 0$

The total cost of transporting the cement is given by sh. $2\,000x + 900y$. This is called the **objective function**. Figure 7.3 shows the graph for the above inequalities:

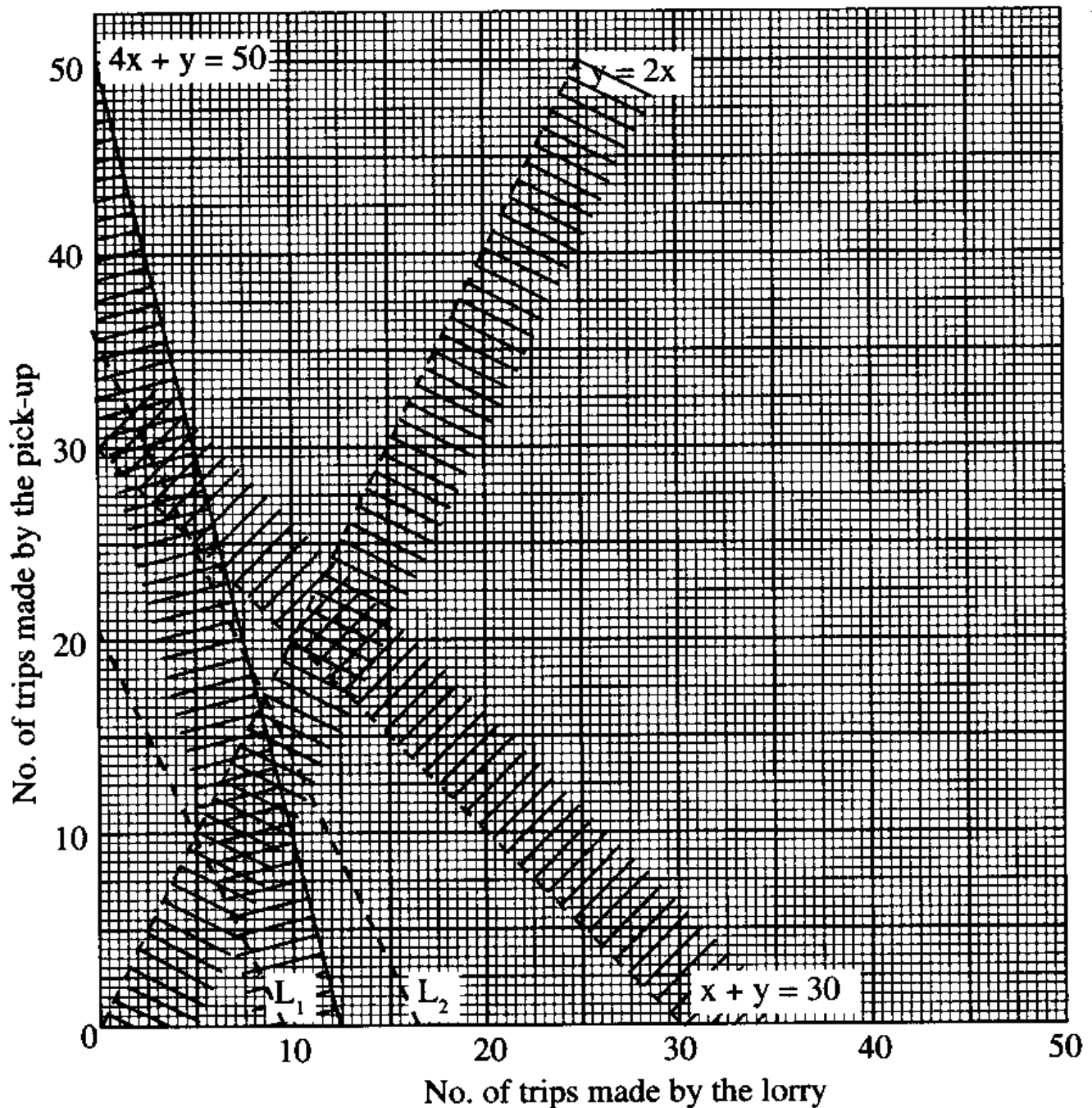


Fig. 7.3

From the graph, we can identify 7 possibilities;
 (7, 22), (8, 18), (8, 19), (8, 20), (8, 21), (9, 19), (9, 20).

Note that in this case, co-ordinates stand for the number of trips. For example (7, 22) means 7 trips by the lorry and 22 trips by the pick-up. Therefore, the possible amount of money in shillings to be spent by the contractor can be calculated as follows:

- (i) $(2\,000 \times 7) + (900 \times 22) = 33\,800$ (ii) $(2\,000 \times 8) + (900 \times 18) = 32\,200$
 (iii) $(2\,000 \times 8) + (900 \times 19) = 33\,100$ (iv) $(2\,000 \times 8) + (900 \times 20) = 34\,000$
 (v) $(2\,000 \times 8) + (900 \times 21) = 34\,900$ (vi) $(2\,000 \times 9) + (900 \times 19) = 35\,100$
 (vii) $(2\,000 \times 9) + (900 \times 20) = 36\,000$

We note from the calculation that the least amount the contractor would spend is sh. 32 200. This is when the lorry makes 8 trips and the pick-up 18 trips.

When the possibilities are many, this method of determining the solution by calculation becomes tedious. An alternative method is therefore needed. This involves drawing the graph of the function we wish to maximise or minimise, the objective function. This function is usually of the form $ax + by$, where a and b are constants.

In the example above, we want to find the minimum value of the function $2\,000x + 900y$. We let $2\,000x + 900y = k$, where k is a constant. For this, we use the graph in figure 7.3. We choose a convenient point (x, y) to give the value of k , preferably close to the region of the possibilities. For example, the point $(5, 10)$ was chosen to give an initial value of k .

Thus, $2\,000x + 900y = 19\,000$.

We now draw the line $2\,000x + 900y = 19\,000$. Such a line is referred to as a **search line**.

Using a ruler and a set square, slide the set square keeping one edge parallel to l_1 until the edge strikes the feasible point nearest l_1 (see the dotted line l_2). From the graph, this point is $(8, 18)$, which gives the minimum expenditure as we have seen earlier. The feasible point furthest from the line l_1 gives the maximum value of the objective function.

The determination of the minimum or the maximum value of the objective function $ax + by$ is known as **optimisation**.

In general, obtaining solutions to problems of the type discussed in this section involves:

- (i) forming the inequalities satisfying given conditions.
- (ii) formulating the objective function.
- (iii) graphing the inequalities.
- (iv) optimising the objective function.

The whole of this process is called **linear programming**.

Exercise 7.3

Use the graphical method in this exercise.

1. Maximise the value of $4x + 3y$, subject to:
 $x \leq 5$, $x + y \leq 6$, $y \leq 4$
2. Minimise the value of $3x + 5y$, subject to:
 $x \geq 4$, $y \geq 5$ and $x - y \leq 6$
3. Maximise the value of $3x + 5y$, if;
 $x - 3y + 5 \leq 0$, $2x + y \leq 20$ and $3x + 2y \geq 25$, $x \geq 0$

4. Maximise the value of $6x - 5y$ subject to the conditions:
 $2x + y \geq 10$, $x - y \leq 3$, $4x - y \leq 20$, $3x + 5y \leq 70$ and $2x - y \geq 4$
5. A shopkeeper deals in cameras and briefcases. He has been able to establish from experience that in order to maintain his sales, at least 120 items must be kept on display all the time. Of these items, at least 30 should be cameras and at least 60 briefcases. If on average the cost of display for each camera is K£40 and that of a briefcase is K£25, how many cameras and briefcases should be displayed to minimise the cost of display?
6. Mrs Njoroge is preparing cakes for sale. She has 80 eggs and 10 cups of sugar. She makes two types of cake. Type A requires 6 eggs and 2 cups of sugar while type B requires 12 eggs and $\frac{3}{4}$ cup of sugar. If a type A cake sells at sh. 25 while a type B cake sells at sh. 20, how many of each should she bake to realise the highest possible sales?
7. A farmer has 100 metres of fencing wire and plans to make a rectangular pen beside a wall, using the wall as one of the longer sides of the pen. The width should be at least 5 m. If the length of the pen is to be at most more than $1\frac{1}{2}$ times the width and the dimensions in metres are to be integers, find the maximum possible value of length plus width for the pen.
8. Wanjiku is preparing for a birthday party. She wants to buy some bottles of juice at sh. 15 each and some cakes at sh. 10 each. She has only sh. 300 to spend. If she wants to buy as many bottles of juice as possible and at least 15 cakes, find the maximum number of bottles of juice she can buy.
9. A carpenter makes two kinds of stools, round and rectangular ones. To make each round stool requires 6 man-hours whereas a rectangular stool requires 3 man-hours. The cost of material for a round stool is sh. 120 and that for a rectangular one is sh. 100. The profit that can be made on a round stool is sh. 80 and on a rectangular one is sh. 60. The carpenter has to abide by the following conditions:
- A contract to supply 15 round stools and 10 rectangular stools per week has to be fulfilled.
 - Only 240 man-hours are available in a week.
 - His total weekly cost of materials for all stools must not exceed sh. 6 000.

Find the number of stools of each type which should be made each week in order that the profit will be maximised.

10. A baby food manufacturer wishes to mix two brands of food so that the vitamin content per kilogram of the mixture is at least 18 units of vitamin A, 14 units of vitamin B, 20 units of vitamin C and 24 units of vitamin D. The vitamin content per kilogram of each brand is shown in table 7.2 below.

Table 7.2

	<i>Vitamin content</i>			
<i>Vitamin</i>	A	B	C	D
<i>Brand 1</i>	4	2	2	2
<i>Brand 2</i>	2	2	4	6

If Brand 1 costs sh. 10 per kilogram and Brand 2 sh. 14 per kilogram, find the minimum cost per kilogram of such a mixture.

DIFFERENTIATION

8.1: Average and Instantaneous rates of Change

In chapter 15 of Book Three, we learnt that the gradient of a curve at a point is given by the gradient of the tangent to the curve at that point. Thus, the gradient of a curve changes from point to point.

Consider the curve $y = x^2$:

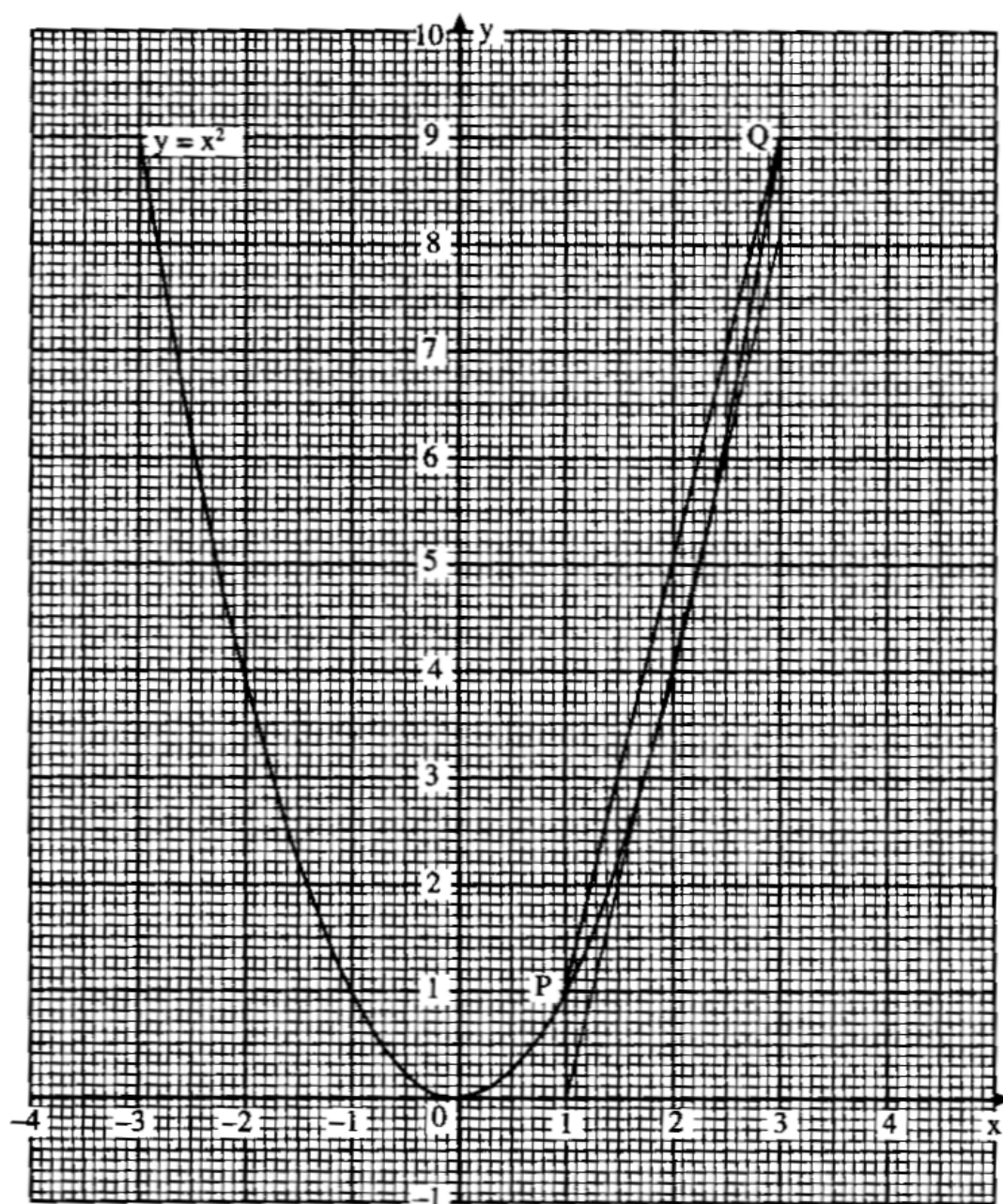


Fig. 8.1

The average rate of change between $x = 1$ and $x = 3$ is given by the gradient of

$$\text{PQ, which is } \frac{9 - 1}{3 - 1} = \frac{8}{2} \\ = 4$$

The rate of change of the curve at $x = 2$ is found by drawing the tangent at $x = 2$ and establishing its gradient. You need to get another point on the tangent, e.g., $(3, 8.1)$, see figure 8.1.

$$\text{The gradient of the tangent} = \frac{8.1 - 4}{3 - 2} \\ = \frac{4.1}{1} \\ = 4.1$$

Clearly, this method only gives an approximate value.

Draw the graph of $y = x^2$. Use the graph to find:

- (i) the average rate of change from $x = 0$ and $x = 4$.
- (ii) the instantaneous change at the point $(2.5, 6.5)$.

8.2: Gradient of a Curve at a Point

Let us consider the gradient of the curve $y = x^2$ at the point $(1, 1)$. The curve is shown in figure 8.2:

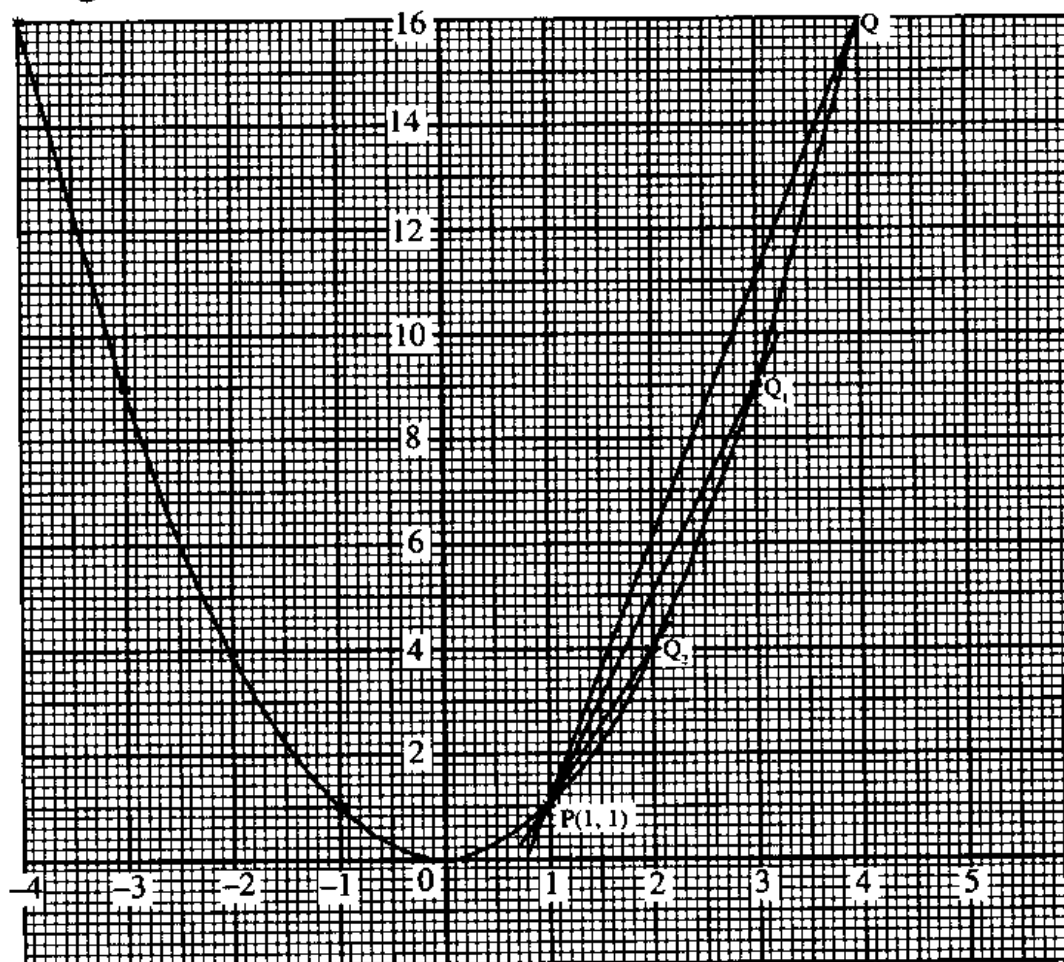


Fig. 8.2

We can approximate the gradient of the curve at (1, 1) by taking the gradient of the chord PQ as Q moves closer to P along the curve. The table below shows the gradient of PQ for different positions of Q.

Table 8.1

<i>Position of Q (x, y)</i>	<i>Change in x</i>	<i>Change in y</i>	<i>Gradient</i>
(2, 4)	1	3	$\frac{3}{1} = 3$
(1.5, 2.25)	0.5	1.25	$\frac{1.25}{0.5} = 2.5$
(1.1, 1.21)	0.1	0.21	$\frac{0.21}{0.1} = 2.1$
(1.01, 1.0201)	0.01	0.0201	$\frac{0.0201}{0.01} = 2.01$
(1.001, 1.002001)	0.001	0.002001	2.001
(1.0001, 1.00020001)	0.0001	0.00020001	2.0001

From the table, we notice that as Q moves closer to P:

- (i) the change in x becomes smaller.
- (ii) the chord PQ tends to be the tangent to the curve at P.
- (iii) the gradient of PQ tends to 2.

Thus, we say that the gradient of the curve $y = x^2$ at the point (1, 1) is 2.

Use a similar approach to find the gradient of the curve $y = x^2$ at the points:

- (i) (2, 4)
- (ii) (3, 9)

8.3: Gradient of $y = x^n$ (where n is a Positive Integer)

Suppose we want to find the gradient of the curve $y = x^2$ at a general point (x, y). We note that a general point on the curve $y = x^2$ will have co-ordinates of the form (x, x^2). The gradient of the curve $y = x^2$ at a general point (x, y) can be established as explained below.

Take a small change in x, say h. This gives us a new point on the curve with co-ordinates [(x + h), (x + h)²], as shown in figure 8.3.

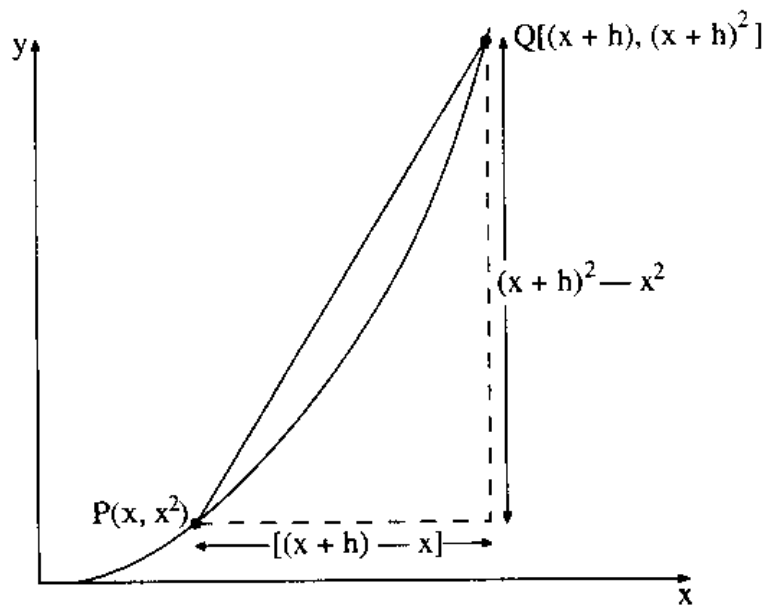


Fig 8.3

$$\begin{aligned}
 \text{The gradient of PQ} &= \frac{\text{change in } y}{\text{change in } x} \\
 &= \frac{(x+h)^2 - x^2}{(x+h) - x} \\
 &= \frac{x^2 + 2xh + h^2 - x^2}{x+h-x} \\
 &= \frac{2xh + h^2}{h} \\
 &= 2x + h
 \end{aligned}$$

By moving Q as close to P as possible, h becomes sufficiently small to be ignored. Thus, $2x + h$ becomes $2x$.

Therefore, at a general point (x, y) on the curve $y = x^2$, the gradient is $2x$. $2x$ is called the gradient function of the curve $y = x^2$. We can use the gradient function to determine the gradient of the curve at any point on the curve. For example, the gradient at:

- (i) $(1, 1)$ is $2 \times 1 = 2$.
- (ii) $(3, 9)$ is $2 \times 3 = 6$.
- (iii) $(4, 6)$ is $2 \times 4 = 8$.
- (iv) $(2, 4)$ is $2 \times 2 = 4$.

Similarly, find the gradient of the curve $y = x^2$, when:

- (i) $x = -3$ (ii) $x = 0$ (iii) $y = 9$
- (iv) $y = 25$ (v) $y = 81$ (vi) $x = 5$

We can use the same procedure to determine the gradient function of $y = x^3$. In figure 8.4, P and Q are two general points on the curve $y = x^3$, with co-ordinates (x, x^3) and $[(x+h), (x+h)^3]$ respectively.

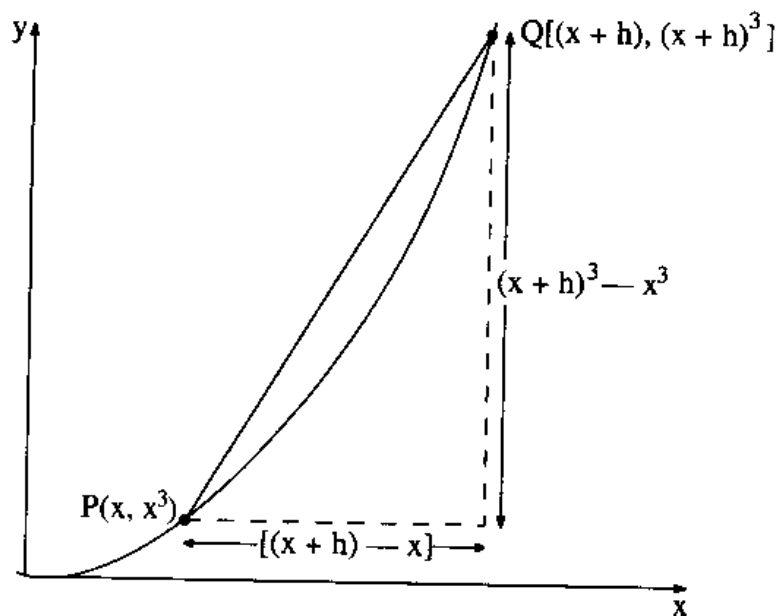


Fig. 8.4

$$\begin{aligned}
 \text{Gradient of PQ} &= \frac{(x+h)^3 - x^3}{(x+h) - x} \\
 &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{x+h-x} \\
 &= \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= 3x^2 + 3xh + h^2
 \end{aligned}$$

As Q moves closer and closer to P, h becomes smaller and smaller and hence terms containing h can be ignored. Thus, $3x^2 + 3xh + h^2$ becomes $3x^2$. Therefore, the gradient function of $y = x^3$ is $3x^2$.

Use the same procedure to show that the gradient function of $y = x^4$ is $4x^3$. Below is a summary of the results as we have found so far.

Table 8.2

Function	Gradient function
$y = x^2$	$2x$
$y = x^3$	$3x^2$
$y = x^4$	$4x^3$

In general, the gradient function of $y = x^n$ is given by nx^{n-1} , where n is a positive integer. The gradient function is called the **derived function** or the **derivative** and the process of obtaining it is called **differentiation**. For example, when we differentiate the function $y = x^5$, we get $5x^4$.

Note:

It can be proved that the rule, the derivative of x^n is nx^{n-1} , holds for any positive integer, negative integer or a fraction.

Differentiate each of the following:

- (i) x^6 (ii) x^9 (iii) x^{20} (iv) x^{-3} (v) $x^{\frac{1}{2}}$ (vi) $x^{-\frac{1}{2}}$

8.4: Delta Notation (Δ)

We have used h to represent a small increase in x . A small increase is usually represented using a Greek letter Δ (delta). The lower case (small letter) for the letter is δ . A small increase in x is denoted by Δx (read as 'delta x '). Similarly, a corresponding change in y is denoted by Δy or δy .

Let us now consider the points $P(x, y)$ and $Q[(x + \Delta x), (y + \Delta y)]$ on the curve $y = x^2$, as in figure 8.5.

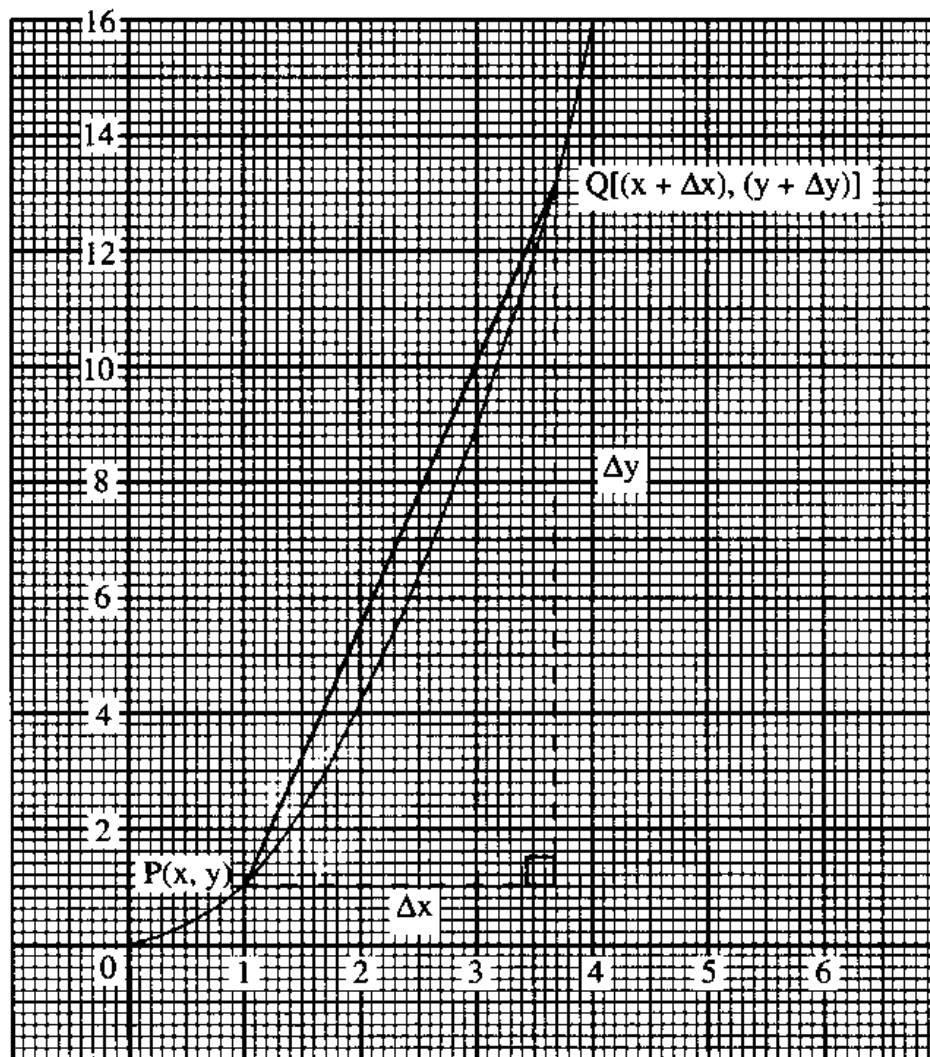


Fig. 8.5

Note that Δx is a single quantity and not a product of Δ and x . Similarly, Δy is a single quantity.

On the curve $y = x^2$, the co-ordinates of P and Q are as shown in figure 8.5.

$$\begin{aligned} \text{Therefore, the gradient of PQ, } \frac{\Delta y}{\Delta x} &= \frac{(x + \Delta x)^2 - x^2}{(x + \Delta x) - x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{x + \Delta x - x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= 2x + \Delta x \end{aligned}$$

As Δx tends to zero;

(i) Δx can be ignored.

(ii) $\frac{\Delta y}{\Delta x}$ gives the derivative, which is denoted by $\frac{dy}{dx}$

$$\text{Thus, } \frac{dy}{dx} = 2x$$

When we find $\frac{dy}{dx}$, we say we are differentiating with respect to x .

For example, given $y = x^4$;

$$\text{then, } \frac{dy}{dx} = 4x^3.$$

We sometimes use y' to denote the derivative.

Consider the function $y = ax^2$, where a is a constant. Take two general points on the curve $P(x, ax^2)$ and $Q[(x + \Delta x), a(x + \Delta x)^2]$, where Δx is a small increment in x .

$$\begin{aligned} \text{The gradient of PQ, } \frac{\Delta y}{\Delta x} &= \frac{a(x + \Delta x)^2 - ax^2}{(x + \Delta x) - x} \\ &= \frac{a[x^2 + 2x\Delta x + (\Delta x)^2] - ax^2}{x + \Delta x - x} \\ &= \frac{ax^2 + 2ax\Delta x + a(\Delta x)^2 - ax^2}{x + \Delta x - x} \\ &= \frac{2ax\Delta x + a(\Delta x)^2}{\Delta x} \\ &= 2ax + a\Delta x. \end{aligned}$$

As Δx tends to zero (i.e., becomes smaller and smaller), the terms containing Δx can be ignored and $\frac{\Delta y}{\Delta x}$ tends to $\frac{dy}{dx}$. Therefore, the derivative $\left(\frac{dy}{dx}\right)$ of $y = ax^2$ is $2ax$.

Similarly, the derivative of $y = ax^3$ can be established as follows.

Take two general points on the curve $y = ax^3$, i.e., (x, ax^3) and $[(x + \Delta x), a(x + \Delta x)^3]$.
The gradient of line joining the two points is;

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{a(x + \Delta x)^3 - ax^3}{(x + \Delta x) - x} \\ &= \frac{a[x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] - ax^3}{(x + \Delta x) - x} \\ &= \frac{ax^3 + 3ax^2\Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 - ax^3}{x + \Delta x - x} \\ &= \frac{3ax^2\Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3}{\Delta x} \\ &= 3ax^2 + 3ax\Delta x + a(\Delta x)^2 \end{aligned}$$

As Δx tends to zero, terms containing Δx can be ignored.

Therefore, $\frac{dy}{dx} = 3ax^2$

Use a similar method to show that the gradient functions of $y = 5x^2$ and $y = 6x^3$ are $10x$ and $18x^2$ respectively.

In general, **the derivative of $y = ax^n$ is nax^{n-1} .**

Example 1

Differentiate each of the following functions:

(a) $y = 4x^3$ (b) $y = -\frac{1}{2}x^{12}$ (c) $y = \frac{2}{3}x^{\frac{3}{4}}$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{dy}{dx} &= 3 \times 4x^{3-1} & \text{(b)} \quad \frac{dy}{dx} &= 12 \times -\frac{1}{2}x^{12-1} & \text{(c)} \quad \frac{dy}{dx} &= \frac{3}{4} \times \frac{2}{3}x^{\frac{3}{4}-1} \\ &= 12x^2 & &= -6x^{11} & &= \frac{1}{2}x^{-\frac{1}{4}} \end{aligned}$$

Differentiate each of the following functions:

(i) $y = 8x^5$ (ii) $y = -4x^6$ (iii) $y = \frac{1}{2}x^4$
(iv) $y = -\frac{1}{8}x^{12}$ (v) $y = -\frac{1}{5}x^{10}$ (vi) $y = -\frac{1}{5}x^{\frac{1}{3}}$

Find a general expression for the gradient function of a straight line whose equation is of the form $y = mx$, where m is a constant.

You should find that the gradient function is m . We also note that the gradient of a straight line whose equation is of the form $y = c$, where c is a constant, is always zero because it is parallel to the x -axis.

This same result could have been obtained by writing $y = c$ as $y = cx^0$, which gives a gradient function $0 \cdot cx^{-1} = 0$.

8.5: The Derivative of a Polynomial

A polynomial in x is an expression of the form;

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n;$$

where n is a positive integer and $a_0, a_1, a_2, \dots, a_n$ are constants. For example, $2x^3 + 4x^2 - 5x - 7$ is a polynomial.

Let us consider the derivative of $y = 4x^2 + 3x + 2$.

Two general points on the curve are $(x, 4x^2 + 3x + 2)$ and $[(x + \Delta x), 4(x + \Delta x)^2 + 3(x + \Delta x) + 2]$, where Δx is a small increment in x . The gradient of the line joining the two points is given as;

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{[4(x + \Delta x)^2 + 3(x + \Delta x) + 2 - (4x^2 + 3x + 2)]}{(x + \Delta x) - x} \\ &= \frac{4x^2 + 8x\Delta x + 4(\Delta x)^2 + 3x + 3\Delta x + 2 - 4x^2 - 3x - 2}{x + \Delta x - x} \\ &= \frac{8x\Delta x + 4(\Delta x)^2 + 3\Delta x}{\Delta x} \\ &= 8x + 4\Delta x + 3 \end{aligned}$$

As Δx tends to zero, $4\Delta x$ can be ignored while $\frac{\Delta y}{\Delta x}$ tends to $\frac{dy}{dx}$.

Therefore, $\frac{dy}{dx} = 8x + 3$

Using a similar method, we obtain the following:

<i>Polynomial</i>	<i>Gradient function</i>
$4x^2 + 3x + 12$	$8x + 3$
$x^3 - 2x^2 + 7x - 4$	$3x^2 - 4x + 7$
$3x^4 + x^3 - x$	$12x^3 + 3x^2 - 1$

Note that the gradient function of each of the polynomials could be obtained by differentiating it term by term.

In general, **the derivative of the sum of a number of terms is obtained by differentiating each term in turn.**

Example 2

Differentiate $y = (x + 2)(x - 1)$ with respect to x .

Solution

Expanding the expression, we get $y = x^2 + x - 2$

Therefore, $\frac{dy}{dx} = 2x + 1$

Example 3

Find the derived function of $y = \frac{x^3 + x^2 + x}{x}$

Solution

$$y = \frac{x^3 + x^2 + x}{x}$$

Simplifying the expression, we have;

$$y = x^2 + x + 1$$

Therefore, $\frac{dy}{dx} = 2x + 1$

Example 4

Find the derived function of each of the following:

(a) $S = 2t^3 - 3t^2 + 4t$

(b) $V = \frac{4}{3}\pi r^3$

(c) $A = v^2 - 2v + 10$

Solution

(a) $S = 2t^3 - 3t^2 + 4t$

$$\frac{dS}{dt} = 6t^2 - 6t + 4$$

(b) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

(c) $A = v^2 - 2v + 10$

$$\frac{dA}{dv} = 2v - 2$$

Exercise 8.1

1. By taking a small change in x (Δx), deduce the derivative of:

(a) $y = 3x^2$ (b) $y = 2x - 2$ (c) $y = 2x^3 - 3x + 4$

2. Write down the gradient functions of the following:

(a) $y = x^5$ (b) $y = x^8$ (c) $y = 4x$ (d) $y = 9x + 5$

(e) $y = 7x - 8$ (f) $y = -\frac{1}{2}x + \frac{4}{5}$ (g) $y = 8$ (h) $y = -12$

(i) $y = -\frac{1}{3}$ (j) $y = -5x$

3. Differentiate the following with respect to x :

(a) $y = 3x^4 - 2$

(b) $y = x^4 - 3x^2 + 7$

(c) $y = 5x^4 - 2x^5 - 7x^3$

(d) $y = -\frac{1}{2}x^{20} - \frac{1}{9}x^9 + 20$

(e) $y = 4x^7 + \frac{1}{7}x^6 - x^3$

(f) $y = 3x - \frac{1}{2}$

(g) $y = x^4 + 3x$

(h) $y = \frac{1}{2}x - x^3$

4. Find the derived function $\left(\frac{dy}{dx}\right)$ if:

(a) $y = 3x^2 - 7x + 10$

(b) $y = 7x^2 - \frac{1}{2}x$

(c) $y = x^4 - 4x^3 + \frac{1}{2}x^2$

(d) $y = 3x + 7$

(e) $y = 7$

(f) $y = -\frac{1}{10}$

5. Find y' if:

(a) $y = \frac{1}{10}x^5 + 7x^3 - 2x^2$

(b) $y = x^3 - x^2 - x - 1$

(c) $y = 4t^4 - \frac{1}{6}t^2 + 7t + 10$

(d) $y = 9t^5 + t^3 - 3t - \frac{1}{2}$

(e) $y = 20r^3 + \frac{1}{4}r^4$

(f) $y = \frac{1}{24}r^{24} - \frac{1}{40}r^{10} - 9$

6. Find the gradient of the given curves at the indicated points:

(a) $y = x^2 - 3x$ (2, -2)

(b) $y = -x^2$ (0, 0)

(c) $y = x^4 - 3x^2 + 6x$ (1, 4)

(d) $y = x^5 - 7x + 3$ (0, 3)

(e) $y = x^2 - 4$ (-4, 12)

7. At what point is the gradient of:

(a) $y = x^2 + 3x + 2$ equal to 9?

(b) $y = \frac{2}{3}x^3 + x^2$ equal to 4?

(c) $y = \frac{1}{4}x^4 - 2x^3 + \frac{5}{2}x^2$ equal to 0?

8. Find $\frac{dy}{dx}$ given that:

(a) $y = \frac{x^2 + 4x + 4}{x + 2}$

(b) $y = (x - 3)(x + 1)(x - 1)$

(c) $y = \frac{x^4 - 2x^3 + 3x^2 + 2x - 5}{x^3}$

(d) $y = \frac{(x - 5)^2}{5x}$

(e) $y = x(5x - 2) + 3(x + 2)^2$

(f) $y = \frac{x(x^2 - 1)}{x + 1}$

(g) $y = x^{-2}(x - x^3)$

(h) $y = \sqrt{x}(x^{\frac{2}{3}} - x^{\frac{1}{2}})$

(i) $y = \frac{(x^2 + x)(x^{-2} + 1)}{x^2}$

8.6: Equations of Tangents and Normals to a Curve

We have seen that the gradient of a curve is the same as the gradient of the tangent to the curve at that point. We can use this idea to find the equation of the tangent to a curve at a given point.

Example 5

Find the equation of the tangent to the curve:

$$y = x^3 + 2x + 1 \text{ at } (1, 4)$$

Solution

$$y = x^3 + 2x + 1$$

$$\frac{dy}{dx} = 3x^2 + 2$$

At the point (1, 4), the gradient is $3 \times 1^2 + 2 = 5$

We want the equation of a straight line through (1, 4), whose gradient is 5.

$$\text{Thus, } \frac{y - 4}{x - 1} = 5$$

$$y - 4 = 5x - 5$$

$$y = 5x - 1$$

Therefore, the equation of the tangent to the curve is $y = 5x - 1$.

A normal to a curve at a point is a line perpendicular to the tangent to the curve at the given point. In Example 5, the gradient of the tangent to the curve at (1, 4) is 5. Thus, the gradient of the normal to the curve at this point is $-\frac{1}{5}$.

Therefore, equation of the normal is;

$$\frac{y - 4}{x - 1} = -\frac{1}{5}$$

$$5(y - 4) = -1(x - 1)$$

$$y = \frac{-x + 21}{5}$$

Example 6

Find the equation of the normal to the curve $y = x^3 - 2x - 1$ at (1, -2)

Solution

$$y = x^3 - 2x - 1$$

$$\frac{dy}{dx} = 3x^2 - 2$$

At the point $(1, -2)$, gradient of the tangent line is 1. Therefore, the gradient of the normal is -1 . The required equation is;

$$\frac{y - (-2)}{x - 1} = -1$$

$$\frac{y + 2}{x - 1} = -1$$

$$y + 2 = -x + 1$$

$$y = -x - 1$$

The equation of the normal is $y = -x - 1$

Exercise 8.2

1. Find the equation of the tangent to the given curve at the indicated point:

(a) $y = x^3 + x^2 + 1$ $(1, 3)$

(b) $y = x^3 + 3x^2 - 3$ $(x = 2)$

(c) $y = x^3 + 6x^2 - 3x + 1$ $(x = 0)$

(d) $y = x^4 - 3x^2 + 2x$ $(x = 1)$

2. Find the equation of the normal to the given curves in question 1 above at the given points.

3. Find the equation of the tangent to the curve $y = x^3$ at the point $(1, 1)$.

4. What is the equation of the normal to the curve $y = x^2 - 1$ at the point $(2, 3)$?

5. Given the curve $y = x^3 - 3x - 1$, find the equation of:

(a) the tangent, and,

(b) the normal at $(1, -3)$.

6. Find the equation of the tangent and normal to the curve $y = x^3 - 2x^2 - 1$ when $y = -1$.

7. Determine the point on the curve $y = \frac{1}{2}x^2 + 4$ at which the gradient is 8. Hence, find the equation of the normal to the curve at this point.

8. Find the point on the curve $y = x^2 - 3x + 7$ at which the gradient is 5, and the equation of the tangent and normal at this point.

8.7: Stationary Points

Consider the curve $y = x^2 + 2x + 3$.

We have, $\frac{dy}{dx} = 2x + 2$

The table below gives the values of the gradient function at different points:

x	-4	-3	-2	-1	0	1	2	3
Gradient at x	-6	-4	-2	0	2	4	6	8

Figure 8.6 shows the sketch of the curve $y = x^2 + 2x + 3$ and the sign of the values of its gradient at different positions.

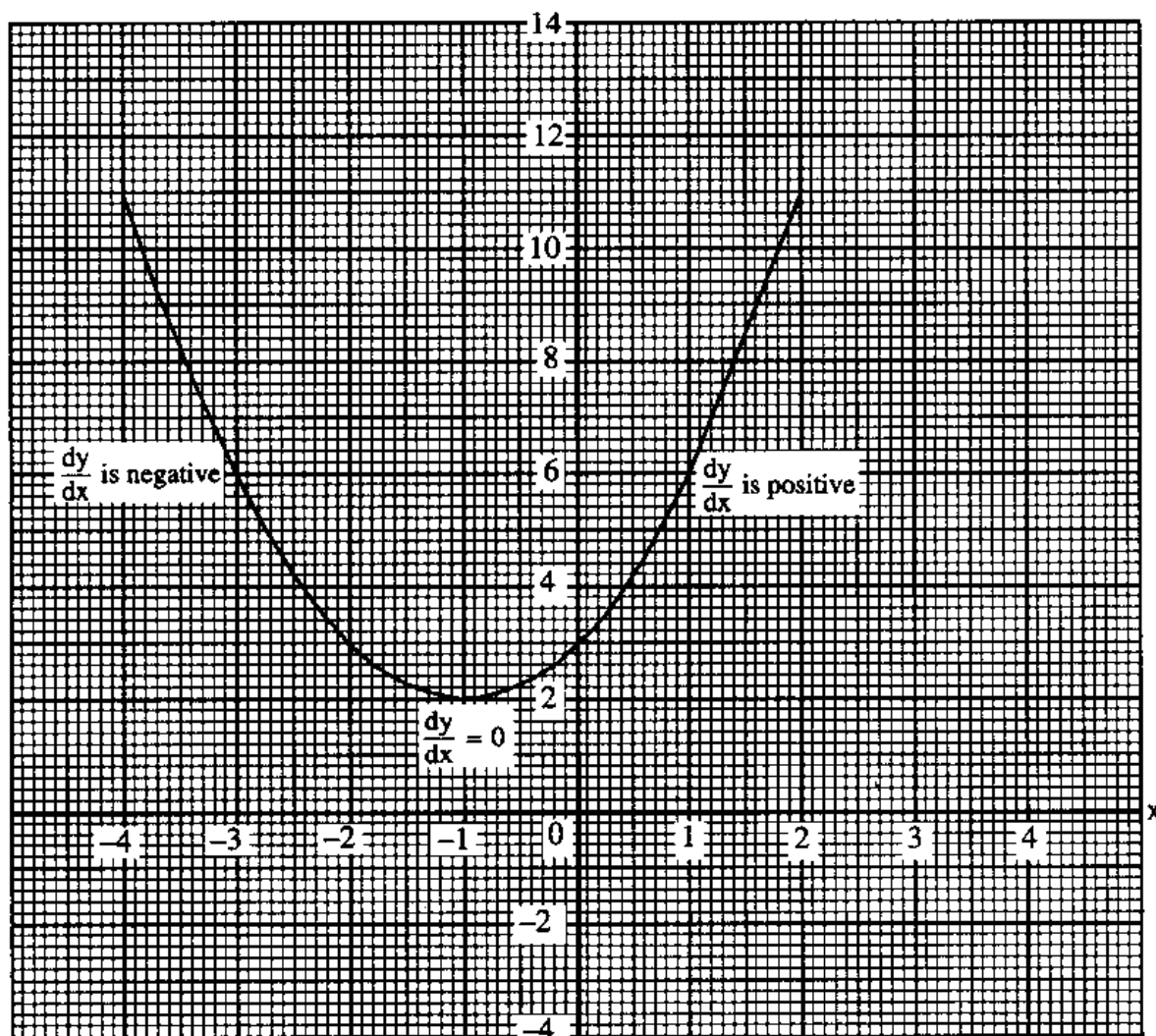


Fig. 8.6

We notice that:

- (i) the gradient at $x = -1$ is zero.
- (ii) the gradient to the left of $x = -1$ is negative.
- (iii) the gradient to the right of $x = -1$ is positive.

The gradient of the curve as you move from left to the right of point $(-1, 2)$ changes sign from negative through zero to positive. Such a point is called a **minimum point**.

Using the same procedure as in the example above, we obtain a sketch of $y = -x^2 + 4x + 2$, as shown in figure 8.7.

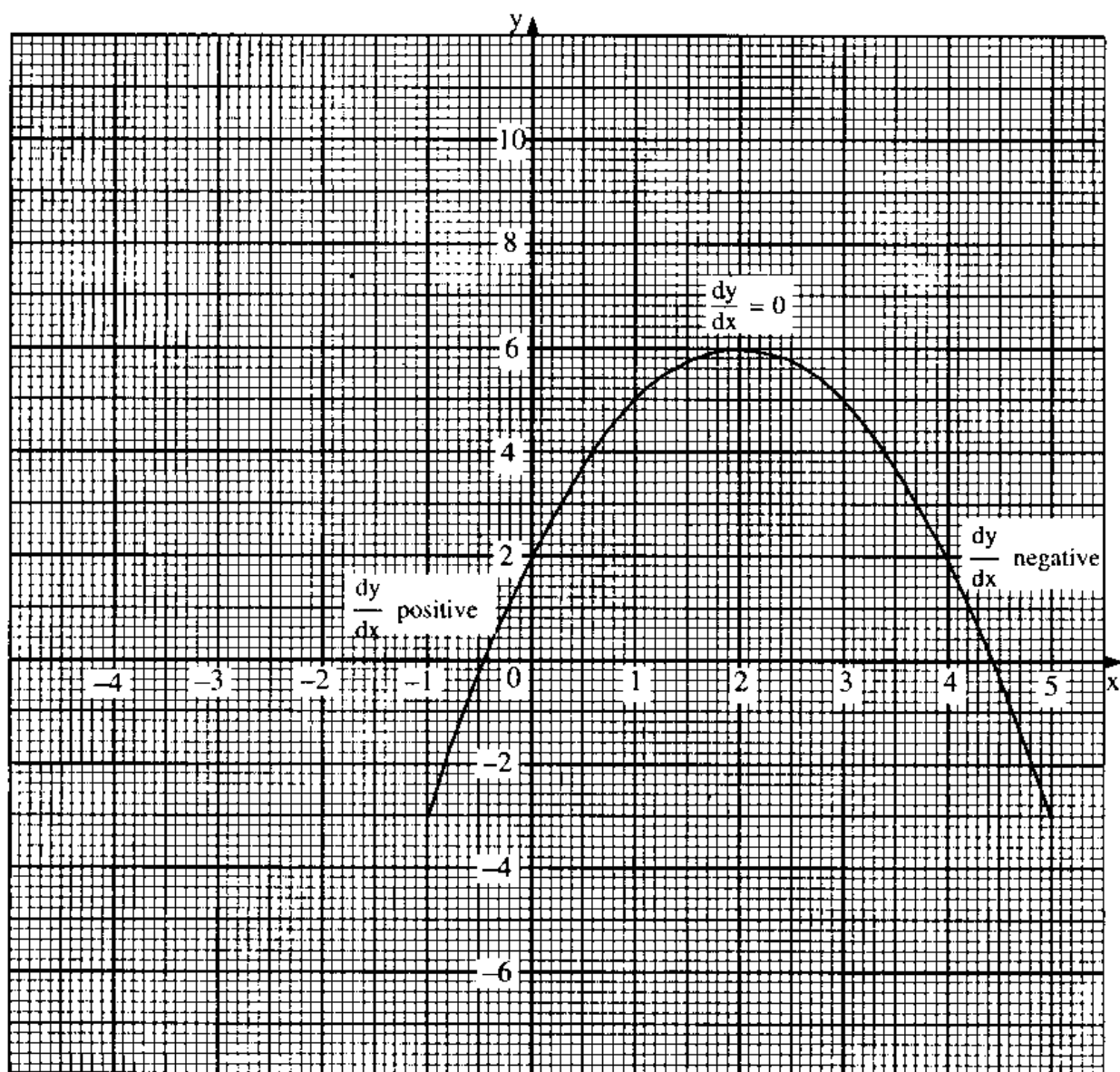


Fig. 8.7

We notice that:

- (i) the gradient is zero at $x = 2$.
- (ii) the gradient to the left of $x = 2$ is positive.
- (iii) the gradient to the right of $x = 2$ is negative.

A point such as $(2, 6)$ at which the gradient changes from positive through zero to negative is called **maximum point**. Maximum and minimum points are called **turning points**.

Figure 8.8 shows the sketch of the curve $y = x^3 - 4$.

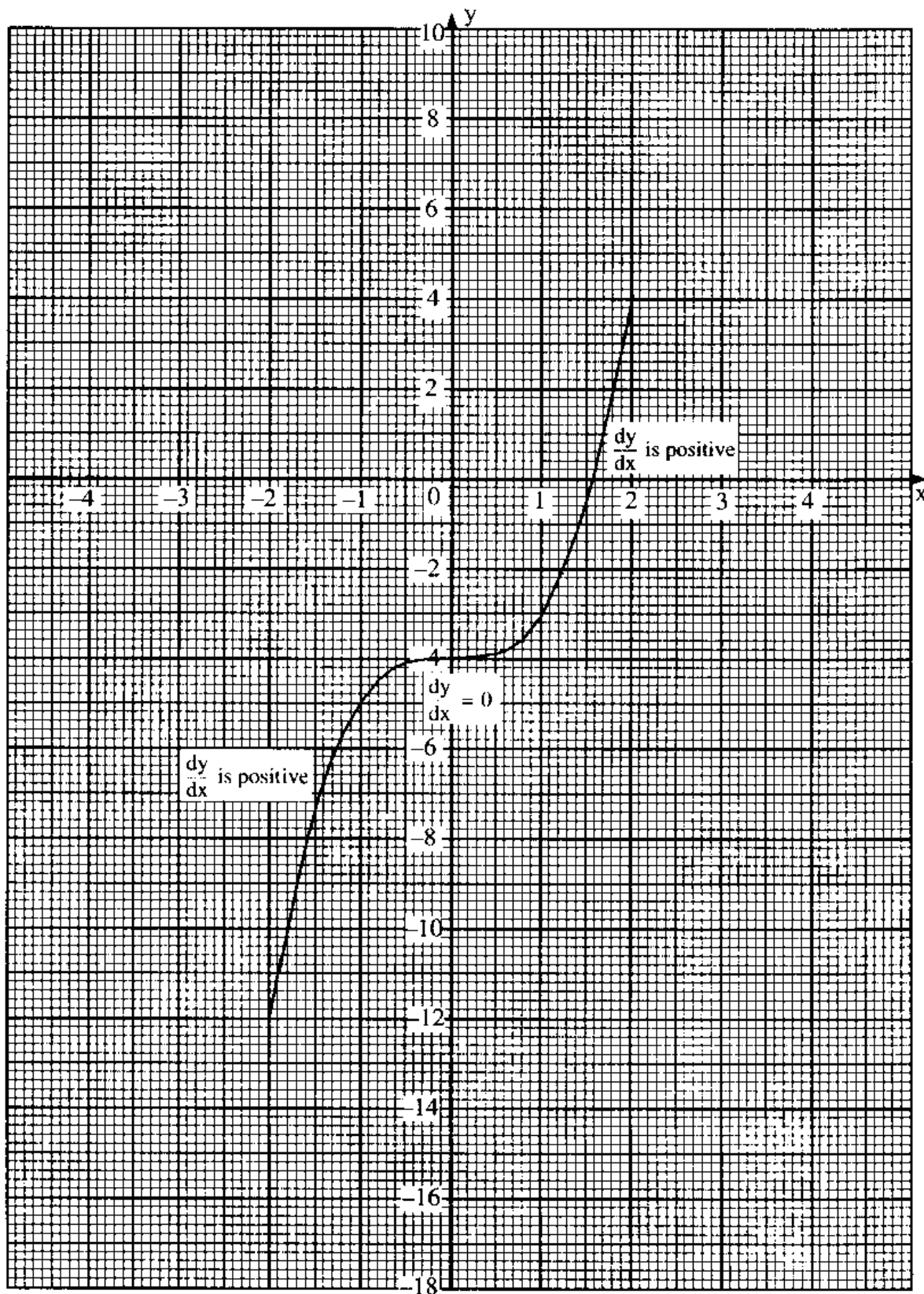


Fig. 8.8

The gradient of the curve is zero at (0, -4). To the left and right of this point, the gradient is positive.

A point at which the gradient changes from positive through zero to

positive or from negative zero to negative is called a **point of inflection**. The point at which the gradient is zero is called a **stationary point**.

The table below can be used as a summary on how to identify the type of stationary point.

<i>Type of stationary point</i>	<i>Gradient to the left</i>	<i>Gradient at stationary point</i>	<i>Gradient to the right</i>	<i>Diagrammatic representation</i>
Maximum	+ve	0	-ve	
Minimum	-ve	0	+ve	
Inflection	+ve	0	+ve	
Inflection	-ve	0	-ve	

Example 7

Identify the stationary points on the curve $y = x^3 - 3x + 2$. For each point, determine whether it is a maximum, minimum or a point of inflection.

Solution

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

At stationary point, $\frac{dy}{dx} = 0$

$$\text{Thus, } 3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$3(x + 1)(x - 1) = 0$$

$$x = -1 \text{ or } x = 1$$

$$\text{When } x = -1, y = 4$$

$$\text{When } x = 1, y = 0$$


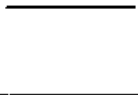
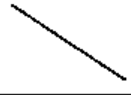
Therefore, stationary points are $(-1, 4)$ and $(1, 0)$.

Consider the sign of gradient to the left and right of $x = 1$.

x	0	1	2
$\frac{dy}{dx}$	-3	0	9
<i>Diagrammatic representation</i>			

Therefore, (1, 0) is a minimum point.

Similarly, sign of gradient to the left and right of $x = -1$ gives;

x	-2	-1	0
$\frac{dy}{dx}$	9	0	-3
<i>Diagrammatic representation</i>			

Therefore (-1, 4) is a maximum point.

Example 8

Identify the stationary points on the curve $y = 1 + 4x^3 - x^4$. Determine the nature of each stationary point.

Solution

$$y = 1 + 4x^3 - x^4$$

$$\frac{dy}{dx} = 12x^2 - 4x^3$$

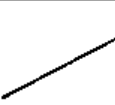
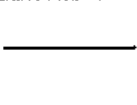
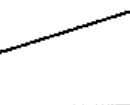
At stationary points, $\frac{dy}{dx} = 0$

$$12x^2 - 4x^3 = 0$$


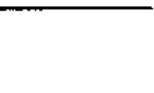
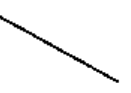
$$4x^2(3 - x) = 0$$

$$x = 0 \text{ or } x = 3$$

Stationary points are (0, 1) and (3, 28).

x	-1	0	1
$\frac{dy}{dx}$	16	0	8
<i>Diagrammatic representation</i>			

Therefore, (0, 1) is a point of inflection.

x	1	3	4
$\frac{dy}{dx}$	8	0	-64
<i>Diagrammatic representation</i>			

Therefore, (3, 28) is a maximum point.

Exercise 8.3

- Identify the stationary points for the following curves. Determine whether each point is a maximum, minimum or a point of inflection:
 - $y = 2x^3 + 5$
 - $y = 2x^2 - 4x + 3$
 - $y = (3 - 2x)(x + 5)$
 - $y = 2x^2 - x^4 + 2$
 - $y = x^5 - 9x^3 + 3$
- Find the stationary points on the following curves. Determine the nature of each point:
 - $y = 4x^3 + 9x^2 - 30x + 10$
 - $y = x^4 - 8x^3 - 3x^2 + 36$
 - $y = 4x^3 - 27x^2 + 24x + 12$
 - $y = 8x^3 + 15x^2 - 18x + 12$
- Show that the derivative of $y = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 + 7$ is $x^2(x + 1)^2$.
Show that when $x = 0$ and $x = -1$, the curve has points of inflection.

8.8: Curve Sketching

A sketch of a curve shows the general shape and any important features of the curve.

Example 9

Sketch the curve $y = (x - 1)(x + 3)^2$.

Solution

Determining where the curve cuts the y-axis and x-axis.

When x is zero, y is -9 . Therefore the curve cuts y-axis at $(0, -9)$.

When y is zero, $x = 1$ or -3 .

Therefore, the curve cuts the x-axis at $(1, 0)$ and $(-3, 0)$.

Expanding the given expression, we get $y = x^3 + 5x^2 + 3x - 9$

$$\frac{dy}{dx} = 3x^2 + 10x + 3$$

At any stationary point, $\frac{dy}{dx} = 0$

$$\text{Therefore, } 3x^2 + 10x + 3 = 0$$

$$x = -3$$



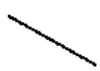


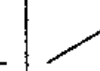
$$\text{or } x = -\frac{1}{3}$$

$$\text{When } x = -3, y = 0$$

$$\text{When } x = -\frac{1}{3}, y = -9\frac{13}{27}$$

Thus, the stationary points are $(-3, 0)$ and $(-\frac{1}{3}, -9\frac{13}{27})$

The sign of the gradient to the left and to the right of each stationary point is shown in the table below. Figure 8.9 shows the sketch.

x	-4	-3	-2	-1	$-\frac{1}{3}$	0
$\frac{dy}{dx}$ at x	11	0	-5	-4	0	3
Diagrammatic representation						

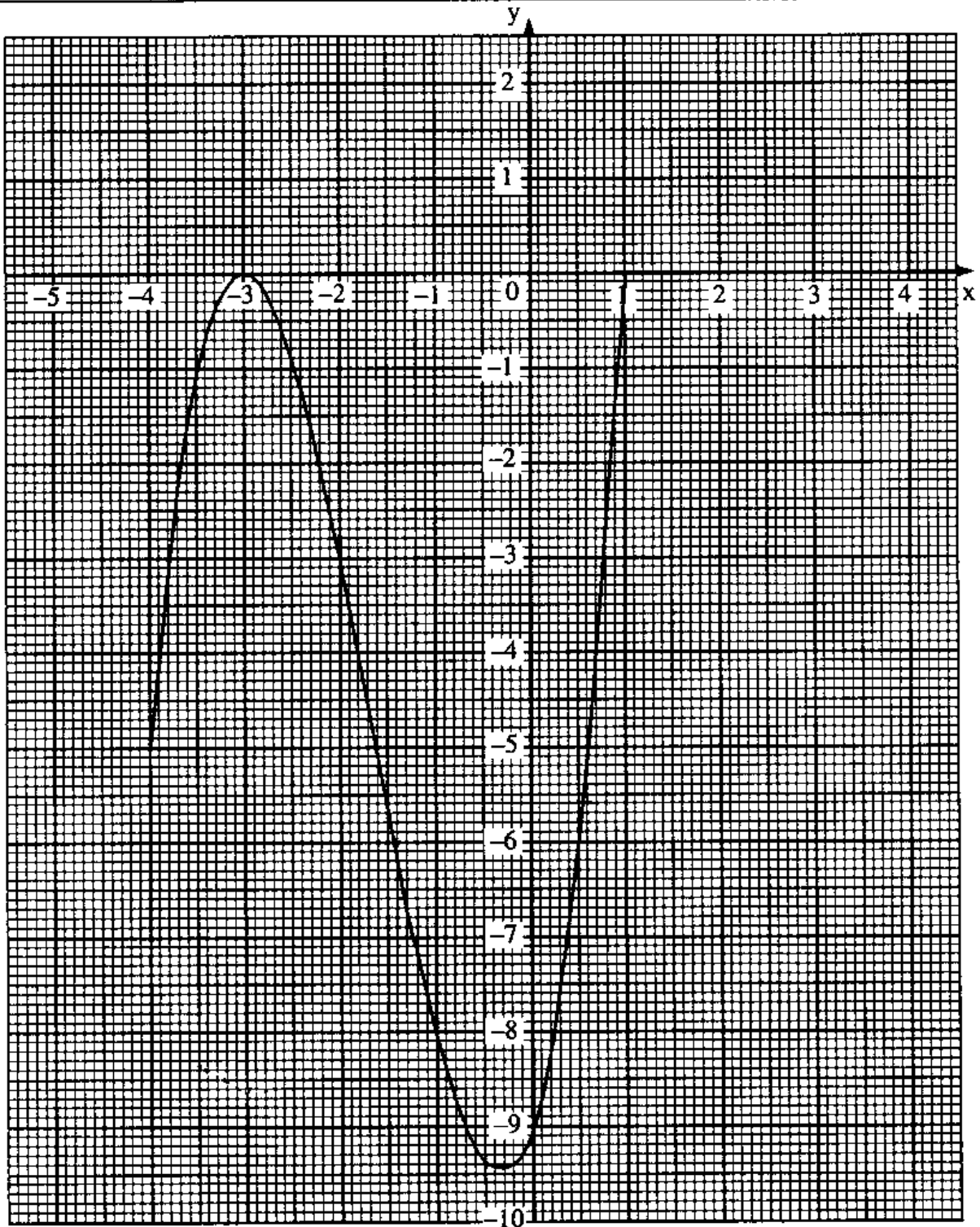


Fig. 8.9

Exercise 8.4

1. Sketch the following curves:

(a) $y = x^2 - 6x$

(b) $y = x^2 + 6x + 9$

(c) $y = (x + 2)\left(\frac{1}{3}x^2 + \frac{4}{3}x + \frac{1}{3}\right)$

(d) $y = (x + 1)\left(\frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}\right)$

(e) $y = (x - 2)(4x^2 + 20x + 16)$

(f) $y = (x - 3)(2x^2 - 3x + 1)$

(g) $y = (x + 5)(x^2 - 5x + 6)$

(h) $y = x^4 - 16x^2$

8.9: Application of Differentiation in Calculation of Velocity and Acceleration**Velocity**

We saw in Book Three that the gradient of a displacement-time graph gives velocity.

Consider a car whose displacement from a fixed point after a time t seconds is given by the equation, $S = 20t + 4$. Figure 8.10 shows a graph of displacement(S) against time (t).

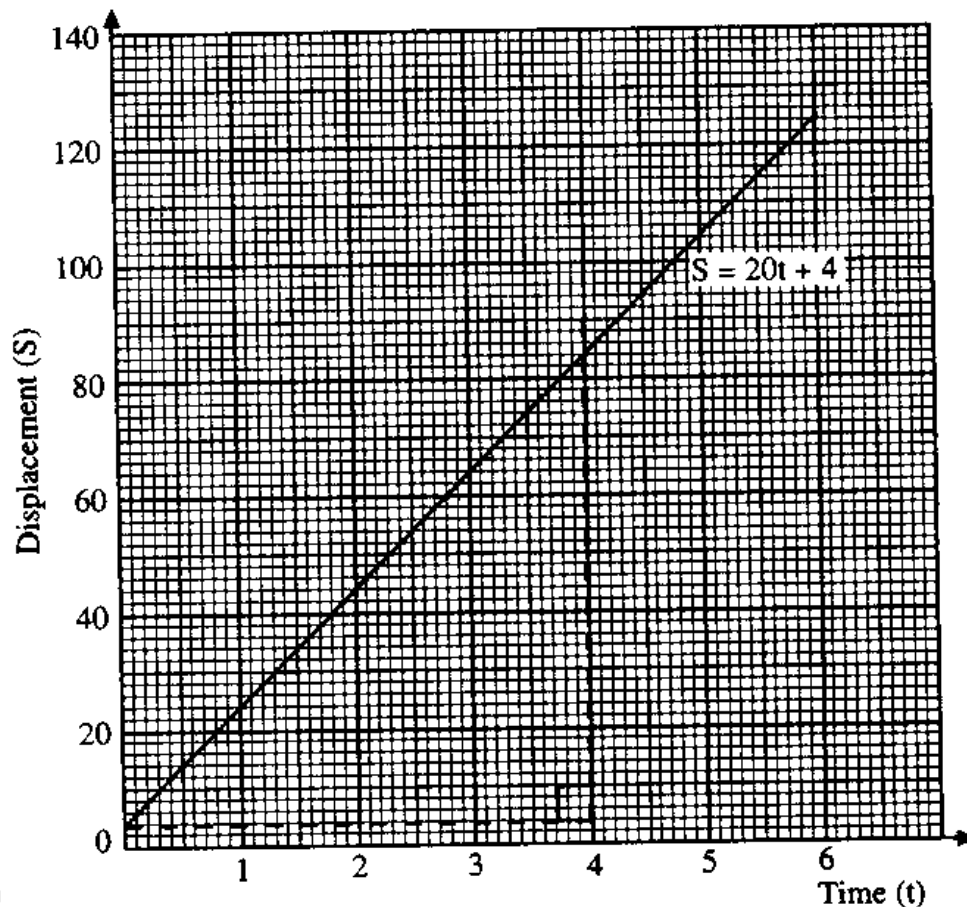


Fig. 8.10

$$\begin{aligned} \text{Gradient} &= \frac{84 - 4}{4 - 0} \\ &= \frac{80}{4} \\ &= 20 \end{aligned}$$

The gradient represents the velocity of the car, and is constant. The same can be obtained by differentiating $S = 20t + 4$ with respect to t .

$$\frac{dS}{dt} = 20$$

In general, if displacement (S) is expressed in terms of time (t), then the velocity is $v = \frac{dS}{dt}$.

Example 10

The displacement, S metres, covered by a moving particle after time, t seconds, is given by $S = 2t^3 + 4t^2 - 8t + 3$. Find:

- (a) velocity at:
- (i) $t = 2$.
 - (ii) $t = 3$.
- (b) instant at which the particle is at rest.

Solution

$$S = 2t^3 + 4t^2 - 8t + 3.$$

The gradient function is given by;

$$\begin{aligned} v &= \frac{dS}{dt} \\ &= 6t^2 + 8t - 8 \end{aligned}$$

- (a) Velocity

(i) at $t = 2$ is;

$$\begin{aligned} v &= 6 \times 2^2 + 8 \times 2 - 8 \\ &= 24 + 16 - 8 \\ &= 32 \text{ m/s} \end{aligned}$$

(ii) $t = 3$ is;

$$\begin{aligned} v &= 6 \times 3^2 + 8 \times 3 - 8 \\ &= 54 + 24 - 8 \\ &= 70 \text{ m/s} \end{aligned}$$

- (b) The particle is at rest when v is zero.

$$\text{Therefore, } 6t^2 + 8t - 8 = 0$$

$$2(3t^2 + 4t - 4) = 0$$

$$2(3t - 2)(t + 2) = 0$$

$$t = \frac{2}{3} \text{ or } t = -2.$$

It is not possible to have $t = -2$.

The particle is at rest at $t = \frac{2}{3}$ seconds.

Acceleration

Figure 8.12 shows the velocity-time graph, $v = 4t + 2$, of a particle.

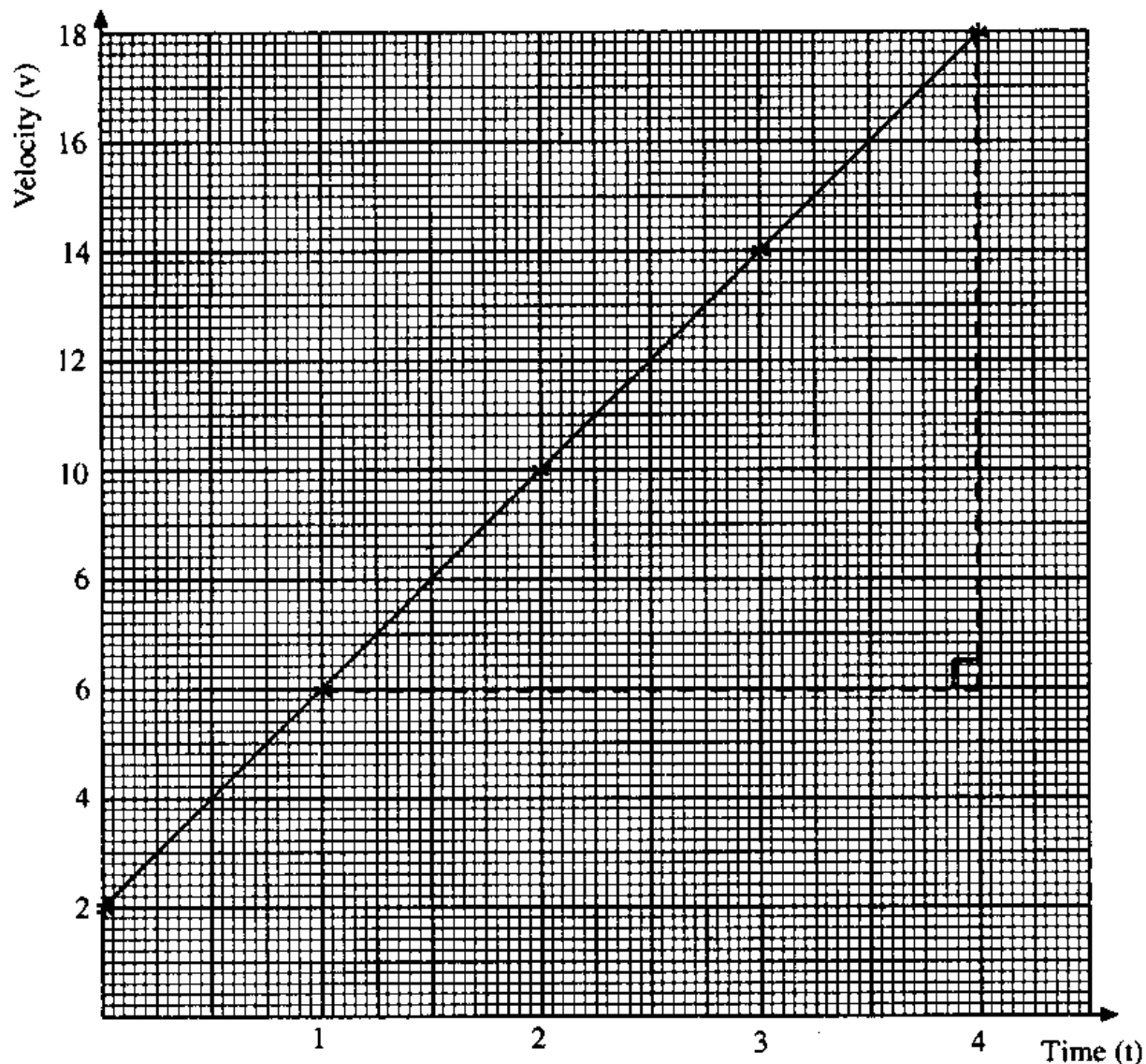


Fig. 8.12

$$\begin{aligned} \text{The gradient of the graph} &= \frac{18 - 6}{4 - 1} = \frac{12}{3} \\ &= 4 \end{aligned}$$

This gives the acceleration.

We can also obtain acceleration by differentiating an equation relating velocity to time. If velocity v , is expressed in terms of time, t , then the acceleration, a , is given by $a = \frac{dv}{dt}$.

Example 11

A particle moves in a straight line such that its velocity $v \text{ ms}^{-1}$ after t seconds is given by $v = 3 + 10t - t^2$

Find:

- (a) the acceleration at:
 (i) $t = 1$ sec (ii) $t = 3$ sec.
 (b) the instant at which the acceleration is zero.

Solution

(a) $v = 3 + 10t - t^2$

$$a = \frac{dv}{dt} = 10 - 2t$$

(i) at $t = 1$ sec, $a = 10 - 2 \times 1$
 $= 8 \text{ ms}^{-2}$

at $t = 3$ sec, $a = 10 - 2 \times 3$
 $= 4 \text{ ms}^{-2}$

- (b) Acceleration is zero when $\frac{dv}{dt} = 0$.
 Therefore, $10 - 2t = 0$
 $t = 5$ seconds

Exercise 8.5

- A stone is thrown vertically upwards such that its height S metres above the ground after t seconds is given by $S = 40t - 10t^2$. Find:
 - its height above the ground after 2 seconds.
 - its velocity after 1 second.
 - the instant at which the stone is 30 m high.
- In each of the following velocity equations, find the acceleration at the times $t = 1$ and $t = 2$.

(a) $v = 5 + 5t - t^2$	(b) $v = 4t^3 - 16t + 3$
(c) $v = 5t^2 - t + 3$	(d) $v = 10t - 5t^2$
(e) $v = 7t^2 + 10t - 3$	
- The velocity $v \text{ ms}^{-1}$ of a particle moving in a straight line after t seconds is given by $v = t - 7t^2$, where t is the time in seconds. Calculate:
 - the acceleration:
 - at $t = 2$.
 - at $t = 5$.
 - the instant at which the acceleration is zero.
- The velocity $v \text{ ms}^{-1}$ of an object after t seconds is given by $v = 5t^2 - 10t$. Find at what instant:
 - the velocity is zero.
 - the acceleration is zero.

5. A particle moves along a straight line such that its displacement S metres from a given point is $S = t^4 + 3t^2 + 4$, where t is time in seconds. Determine its displacement, velocity and acceleration when:
- $t = 0$.
 - $t = 1$.
 - $t = 4$.
6. A particle moves in a straight line such that after t seconds its displacement S metres from a fixed point is given by $S = 4t - 2t^2 - 5t^3$. Find:
- the velocity at $t = 3$.
 - time when the velocity is zero.
 - the acceleration at $t = 2$.
7. A ball is kicked upwards and after t seconds its height h metres is given by $h = 1 + 60t - 18t^2$. Find the height, velocity and acceleration of the ball when:
- $t = 1$
 - $t = 2$
 - $t = 3$
 - $t = 1\frac{2}{3}$

8.10: Maxima and Minima

Example 12

A farmer has 100 m of wire-mesh to fence a rectangular enclosure. What is the greatest area he can enclose with the wire-mesh?

Solution

Let the length of the enclosure be x m. Then, the width is $\frac{100 - 2x}{2}$ m = $(50 - x)$ m

Dimensions can be represented as shown in figure 8.13 below:

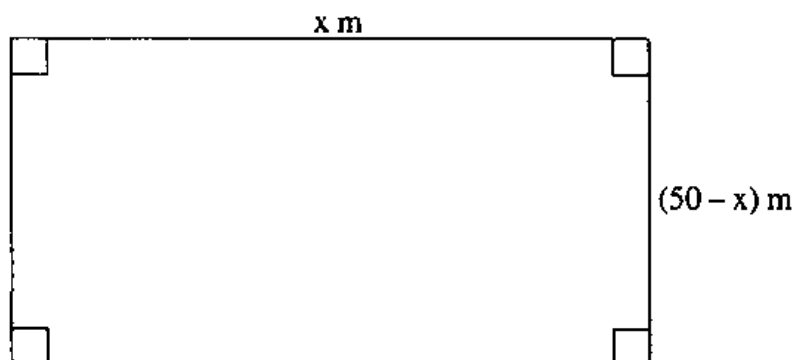


Fig. 8.13

Then, the area A of the rectangle is given by;

$$\begin{aligned} A &= x(50 - x) \text{ m}^2 \\ &= 50x - x^2 \text{ m}^2 \end{aligned}$$

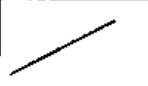
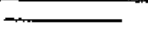
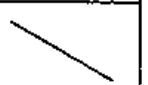
For maximum or minimum area,

$$\frac{dA}{dx} = 0$$

$$\text{Thus, } 50 - 2x = 0$$

$$x = 25$$

The table below helps us to determine if the value of x obtained corresponds to a maximum or minimum area.

x	24	25	26
$\frac{dA}{dx}$	+ve	0	-ve
<i>Diagrammatic representation</i>			

Therefore, area is maximum when $x = 25$ m.

$$\begin{aligned} \text{That is, } A &= 50 \times 25 - (25)^2 \\ &= 625 \text{ m}^2 \end{aligned}$$

Example 13

A closed cylindrical metal tin is to have a capacity of 250π ml. If the area of the metal used is to be a minimum, what should the radius of the tin be?

Solution

Let the total surface area of the cylinder be A cm², radius r cm and height h cm.

$$\text{Then, } A = 2\pi r^2 + 2\pi rh$$

$$\begin{aligned} \text{Volume} &= \pi r^2 h \\ &= 250\pi \text{ cm}^3 \end{aligned}$$

$$\pi r^2 h = 250\pi$$

$$\begin{aligned} \text{Making } h \text{ the subject, } h &= \frac{250\pi}{\pi r^2} \\ &= \frac{250}{r^2} \end{aligned}$$

Put $h = \frac{250}{r^2}$ in the expression for surface area to get;

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r \cdot \frac{250}{r^2} \\ &= 2\pi r^2 + 500\pi r^{-1} \end{aligned}$$

$$\frac{dA}{dr} = 4\pi r - 500\pi r^{-2}$$

For minimum surface area, $\frac{dA}{dr} = 0$.

$$4\pi r - \frac{500\pi}{r^2} = 0.$$

$$4\pi r^3 - 500\pi = 0.$$

$$4r^3 = 500$$

$$r^3 = \frac{500}{4} = 125$$

$$r = \sqrt[3]{125}$$

$$= 5$$

Therefore, the area is minimum when $r = 5$ cm.

Exercise 8.6

- The height S after t seconds of an object thrown vertically upwards is given by $S = 4.9t - t^2$. Find the maximum possible height it reaches.
- A piece of wire 18 cm long is to be bent to form a rectangle. If its length is x cm, obtain an expression for its area in terms of x . Hence, calculate the dimensions of the rectangle with maximum area from the expression.
- A man has 75π cm² sheet of metal and wishes to make an open cylindrical container with it. Find the dimensions of the tin for which the volume will be maximum.
- The distance S of an object from a fixed point is given by $S = \frac{1}{3}t^3 - \frac{3}{2}t^2$. Show that the velocity is minimum at $t = \frac{3}{2}$ and find this velocity.
- Onyango wishes to fence a rectangular research plot using 100 m of wire. One end of the plot has a wall already erected. Calculate the maximum area he can fence.
- A rectangle is inscribed in a right-angled triangle ABC. Show that the area of the largest such rectangle is half the area of the triangle.
- Figure 8.14 is a cuboid with dimensions as shown. All measurements are in centimetres. Find an expression for its volume in terms of x . Show that the value of x which gives the minimum volume is $\frac{-2 + \sqrt{13}}{3}$ cm.

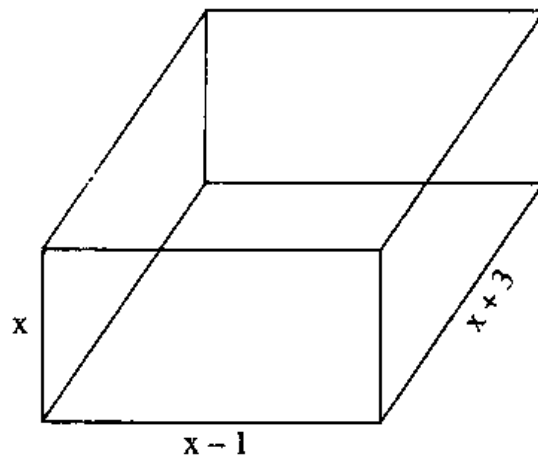


Fig. 8.14

8. A piece of wire 8 m long is to be cut into two parts. If the parts are bent to form a square and a circle respectively, find the radius of the circle if the sum of their areas is minimum.
9. Find the base radius of a cylindrical hole with maximum volume which can be drilled into a cone of height 16 cm and radius 12 cm as shown in figure 8.15.

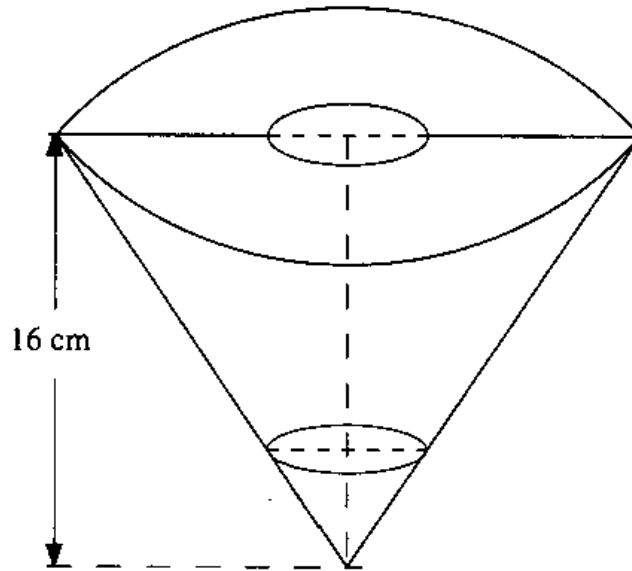


Fig. 8.15

10. A square sheet of metal has an area of 100 m^2 . An open rectangular tank is to be made by cutting equal squares from the corners and bending the sides up. Find the height of the tank for which the volume will be maximum.

AREA APPROXIMATION

9.1: Introduction

Shapes in many real life situations are mainly irregular. Examples of irregular shapes include land masses, lakes, oceans, forests, leaves, e.t.c. Surveyors, architects, engineers and other scientists strive to find areas of irregular shapes in their fields of study.

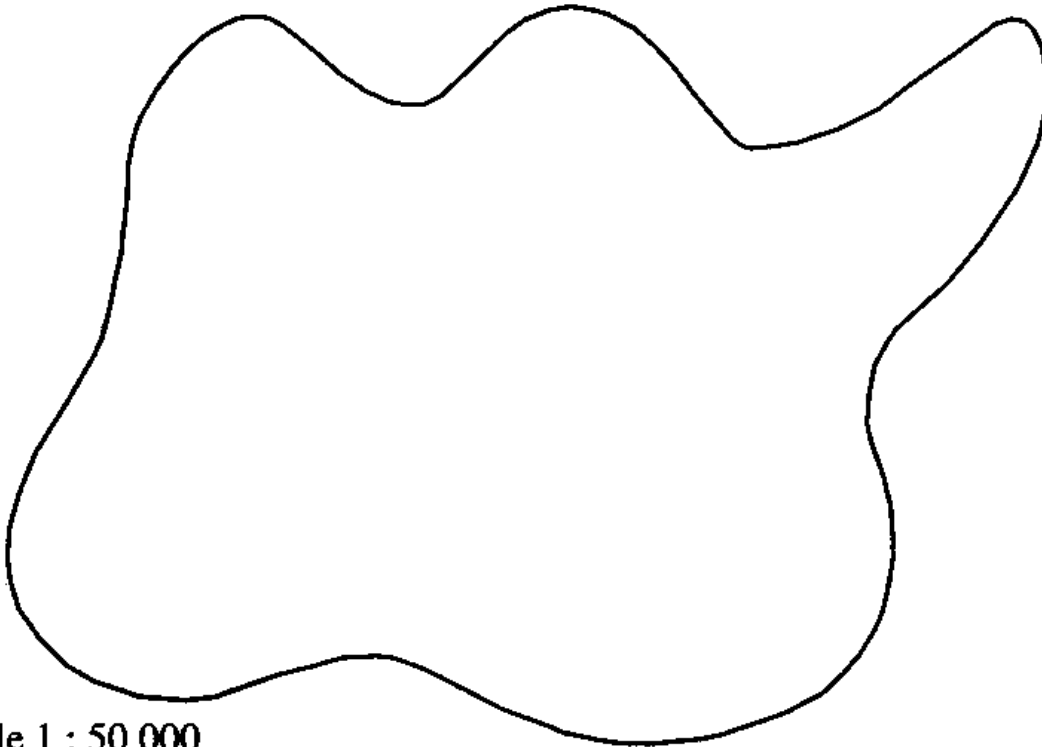
In most cases, we approximate areas of irregular shapes (surfaces). In Book One, we estimated area of irregular shapes using:

- (i) unit squares of a grid which cover the required area.
- (ii) triangulation method.

In this chapter, we shall still apply the counting of squares technique and also deal with other methods of estimating areas of irregular figures.

9.2: Using Counting Technique to Approximate Area

We need to find the area of the irregular land mass shown in figure 9.1, from a map whose scale is 1 : 50 000.



Scale 1 : 50 000

Fig. 9.1

The following steps should be followed:

- (i) Copy the outline of the region on a tracing paper.

- (ii) Put the tracing in (i) on a one centimetre square grid, as shown in figure 9.2 below.

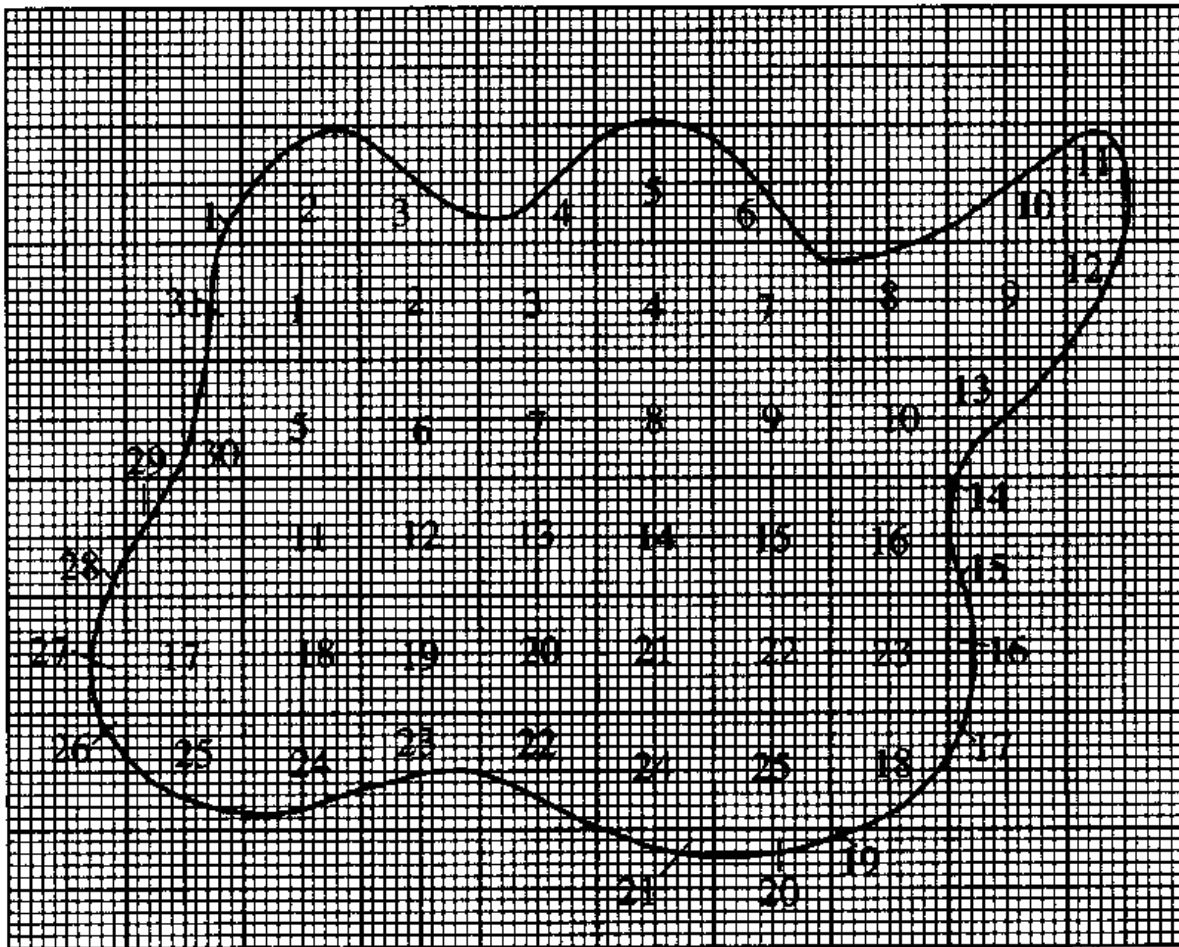


Fig. 9.2

- (iii) Count all the whole squares fully enclosed within the region.
 (iv) Count all the partially enclosed squares and take them as half square centimetre each.
 (v) The sum of (iii) and (iv) gives an estimate of the area of the land mass in square centimetres.
 (vi) From figure 9.2;
 number of complete squares = 25
 number of part squares = 31

$$\begin{aligned} \text{Therefore, total number of squares} &= 25 + \frac{31}{2} \\ &= 25 + 15.5 \\ &= 40.5 \end{aligned}$$

The area of the land mass on the paper is therefore 40.5 cm².
 The actual area is calculated as follows:

Scale on map is 1 : 50 000.

1 cm on the map represents 50 000 centimetres or 0.5 km on the ground.

1 cm² represents (0.5)² km² = 0.25 km²

Therefore, the approximate actual area of the land mass is;

$$\begin{aligned} 40.5 \times 0.25 \text{ km}^2 &= 10.125 \text{ km}^2 \\ &= 10.13 \text{ km}^2 \end{aligned}$$

Alternatively, the irregular surface can be subdivided into convenient shapes, e.g., rectangles, triangles, e.t.c. and the areas of the individual shapes added, as shown in figure 9.3 below.

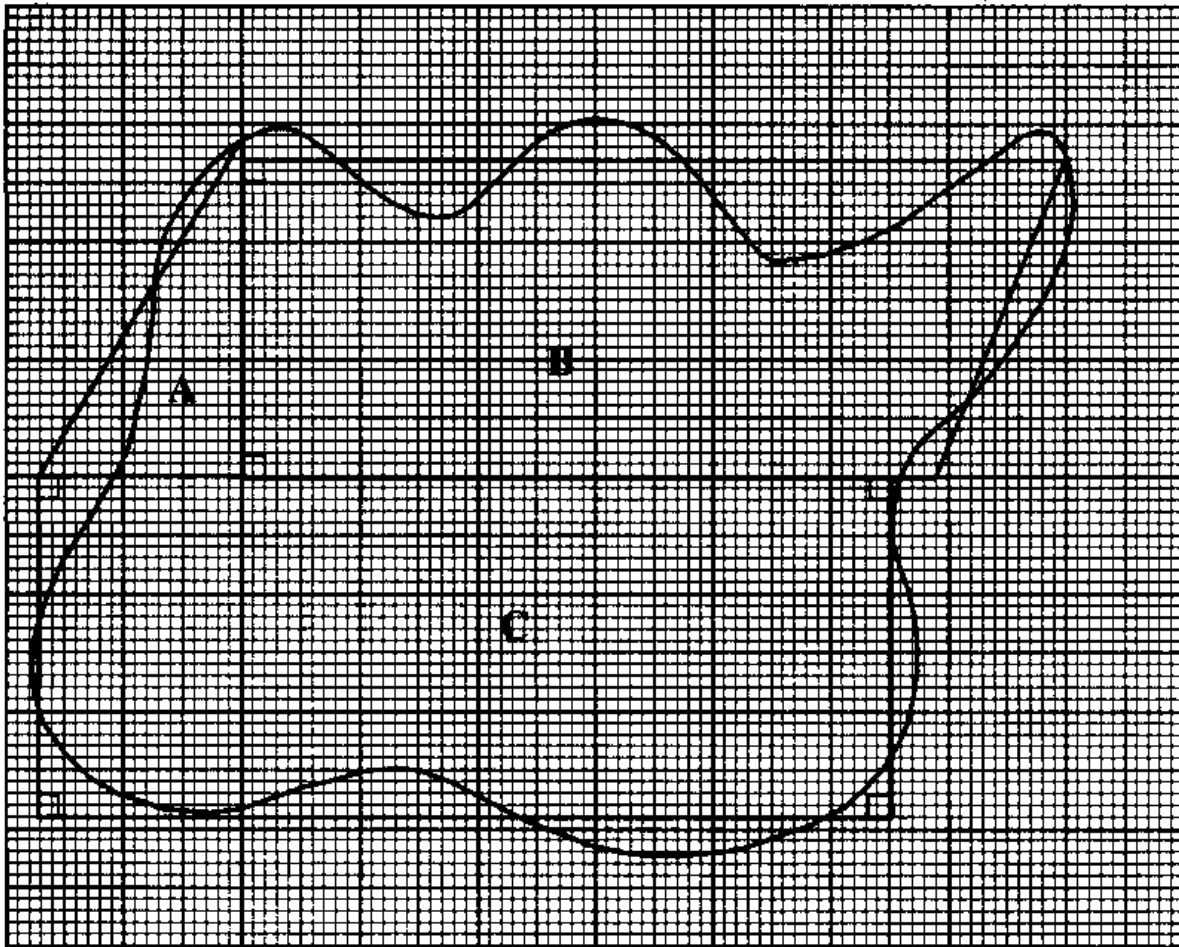


Fig. 9.3

Area of the land mass = area of triangle A + area of trapezium B + area of rectangle C.

$$\begin{aligned} \text{Area of triangle A} &= \frac{1}{2} \times 1.7 \times 2.9 \\ &= 2.465 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of trapezium B} &= \frac{1}{2} \times 2.7 \times (5.9 + 7.0) \text{ cm}^2 \\ &= 17.415 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of rectangle C} &= 7.2 \times 2.9 \text{ cm}^2 \\ &= 20.880 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned}\text{Total area} &= (2.465 + 17.415 + 20.880) \text{ cm}^2 \\ &= 40.76 \text{ cm}^2\end{aligned}$$

Therefore, the approximate area of the actual land mass is;
 $(40.76 \times 0.25) \text{ km}^2 = 10.19 \text{ km}^2$

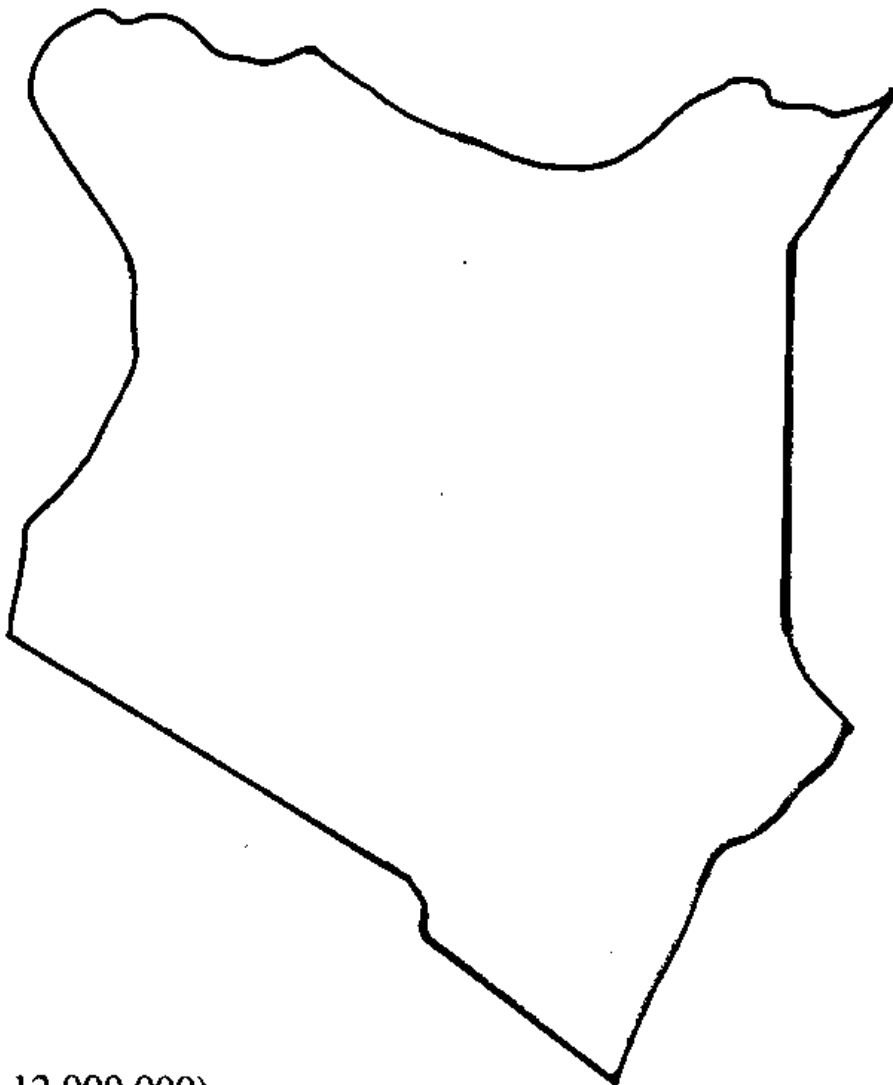
Note:

The smaller the subdivisions, the greater the accuracy in approximating area.

Exercise 9.1

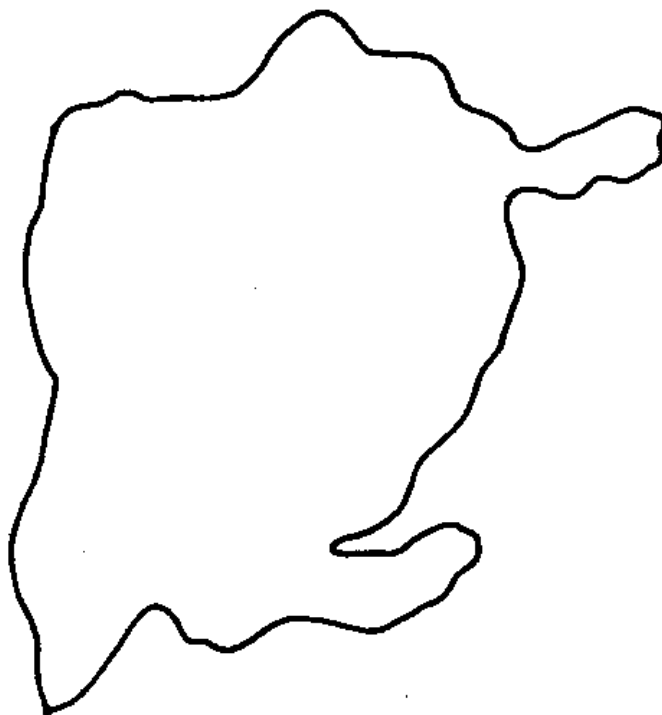
1. Copy the following figures whose scales are indicated and use a grid of 1 cm squares to estimate their actual areas in km^2 .

(a)



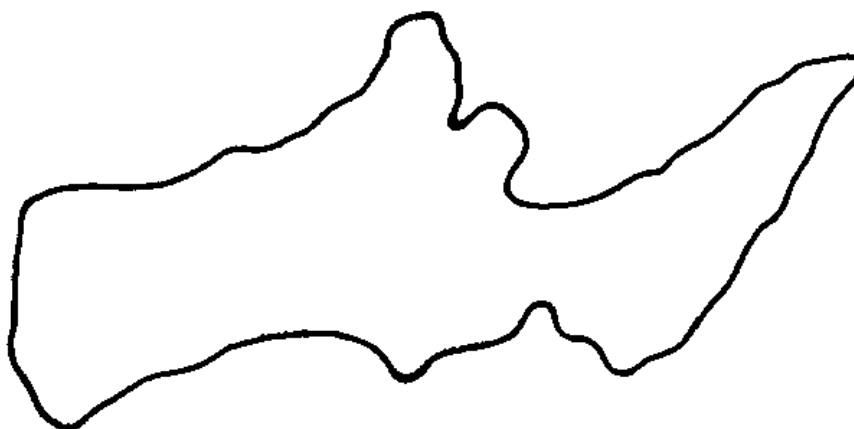
(Scale 1 : 12 000 000)

(b)



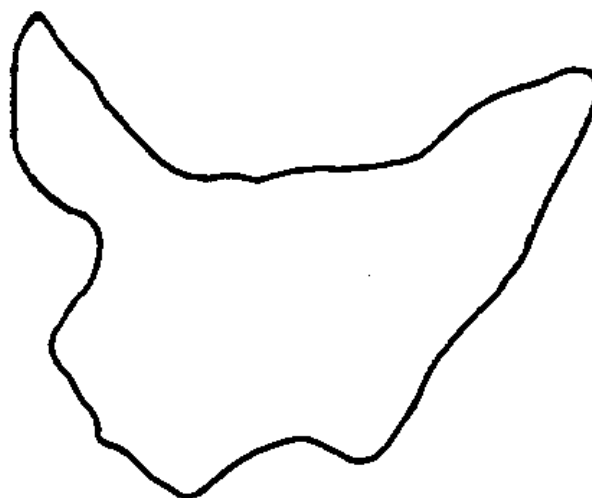
(Scale 1 : 6 000 000)

(c)



(Scale 1 : 12 000 000)

(d)



(Scale 1 : 2 400 000)

Fig. 9.4

9.3: Approximating Area by Trapezium Method

Suppose we wish to find the area of the region shown in figure 9.5.

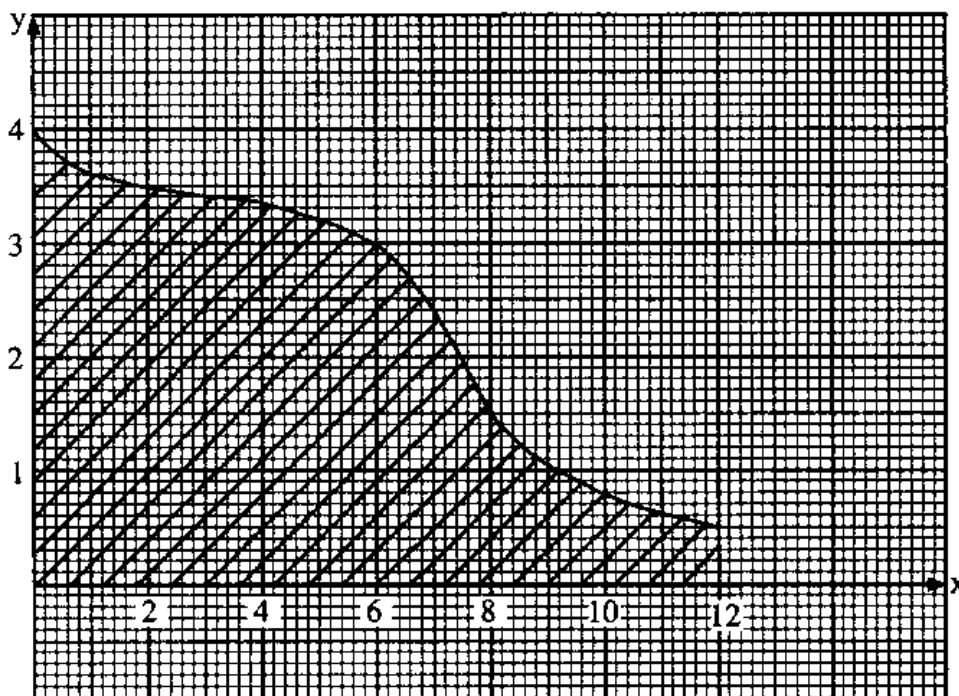


Fig. 9.5

The region may be divided into six trapezia of uniform as shown in figure 9.6.

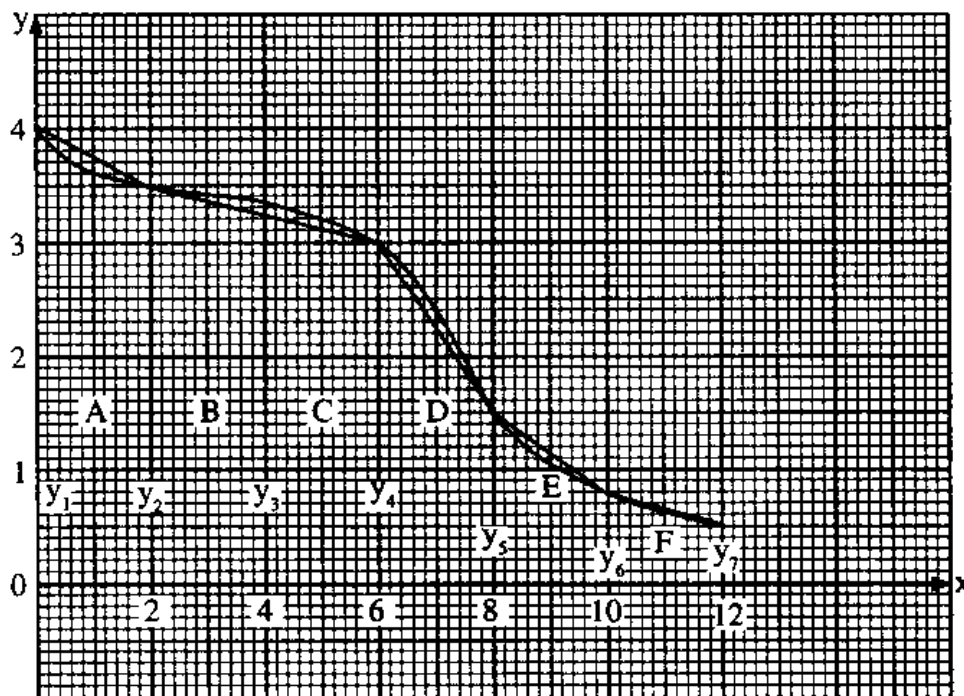


Fig. 9.6

The area of the region is approximately equal to the sum of the areas of the six trapezia.

Note that the width of each trapezium is 2 cm, and 4 and 3.5 are the lengths of the parallel sides of the first trapezium.

$$\text{The area of trapezium A} = \frac{1}{2} \times 2(4 + 3.5) = 7.5 \text{ cm}^2$$

$$\begin{aligned} \text{Similarly, area of trapezium B} &= \frac{1}{2} \times 2(3.5 + 3.2) \\ &= 6.7 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of trapezium C} &= \frac{1}{2} \times 2(3.2 + 3) \\ &= 6.2 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of trapezium D} &= \frac{1}{2} \times 2(3 + 1.5) \\ &= 4.5 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of trapezium E} &= \frac{1}{2} \times 2(1.5 + 0.8) \\ &= 2.3 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of trapezium F} &= \frac{1}{2} \times 2(0.8 + 0.5) \\ &= 1.3 \text{ cm}^2 \end{aligned}$$

Therefore, the total area of the region is:

$$(7.5 + 6.7 + 6.2 + 4.5 + 2.3 + 1.3) \text{ cm}^2 = 28.5 \text{ cm}^2$$

If the lengths of the parallel sides of the trapezia (ordinates) are $y_1, y_2, y_3, y_4, y_5, y_6$ and y_7 , as shown in figure 9.6, the total area of the region is;

$$\begin{aligned} &\frac{1}{2} \times 2(y_1 + y_2) + \frac{1}{2} \times 2(y_2 + y_3) + \frac{1}{2} \times 2(y_3 + y_4) + \frac{1}{2} \times 2(y_4 + y_5) + \frac{1}{2} \times 2(y_5 + y_6) \\ &+ \frac{1}{2} \times 2(y_6 + y_7). \\ &= \frac{1}{2} \times 2\{(y_1 + y_2 + y_2 + y_3 + y_3 + y_4 + y_4 + y_5 + y_5 + y_6 + y_6 + y_7)\}. \end{aligned}$$

Note that except for the first and the last lengths, each of the other lengths is counted twice. Therefore, the expression for the area simplifies to:

$$\frac{1}{2} \times 2 \{(y_1 + y_7) + 2(y_2 + y_3 + y_4 + y_5 + y_6)\}$$

In general, the approximate area of a region using trapezium method is given by:

$$A = \frac{1}{2} h \{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})\};$$

where h is the uniform width of each trapezium, y_0 and y_n are the first and last lengths respectively. This method of approximating areas of irregular shapes is called **trapezium rule**.

It is important to note that a better approximation to the area is obtained by making the width smaller, i.e., dividing the region into more trapezia.

Example 1

A car starts from rest and its velocity is measured every second for 6 seconds, see table below.

Time (t)	0	1	2	3	4	5	6
Velocity v (ms^{-1})	0	12	24	35	41	45	47

Use the trapezium rule to calculate the distance travelled between $t = 1$ and $t = 6$.

Solution

In general, the area under velocity-time graph represents the distance covered between the given times. The graph of the velocity against time is shown in figure 9.7.

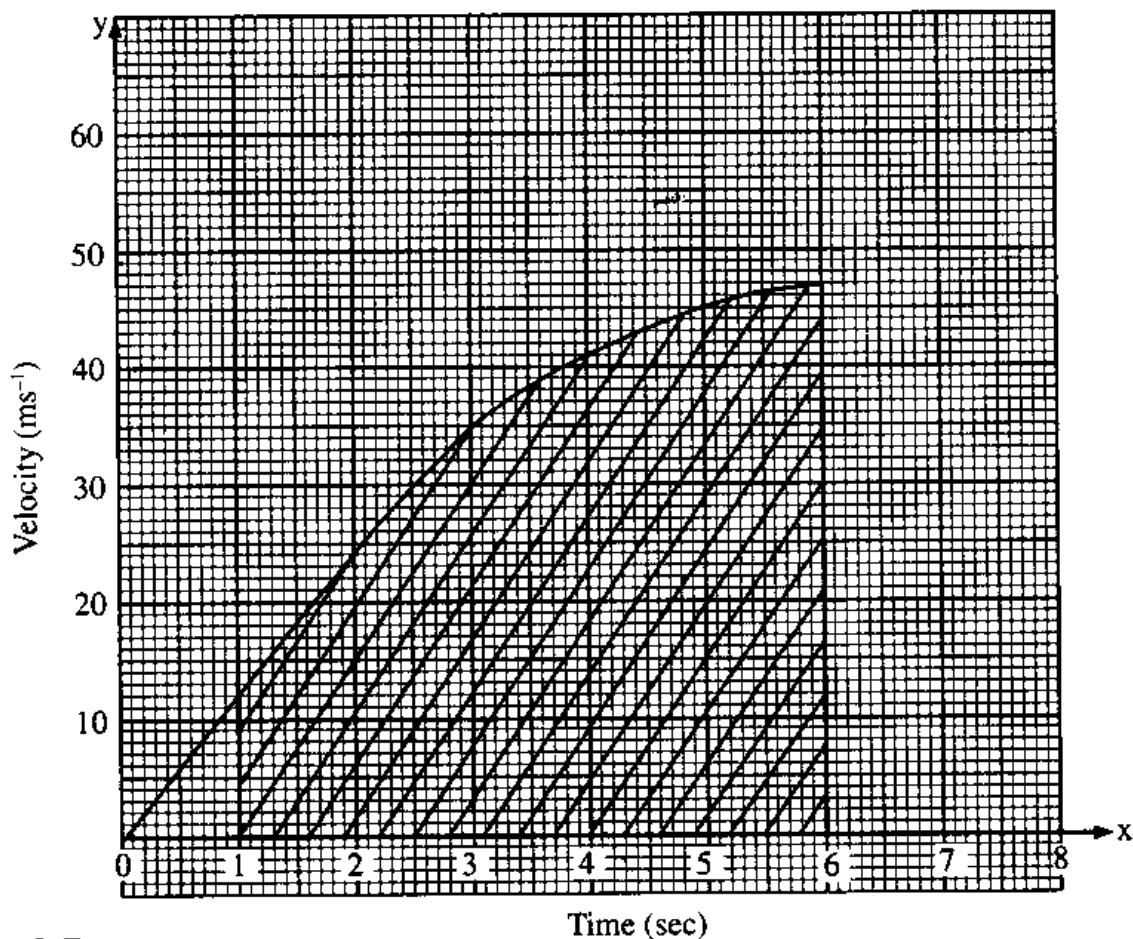


Fig. 9.7

To find the required displacement, we find the area of the region bounded by the graph, $t = 1$ and $t = 6$, as follows;

We choose to divide the required area into five trapezia, each of width 1 unit.
Using the trapezium rule;

$$A = \frac{h}{2} \{(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)\}$$

$$\begin{aligned} \text{The required displacement} &= \frac{1}{2} \times 1 \{(12 + 47) + 2(24 + 35 + 41 + 45)\} \\ &= \frac{1}{2} (59 + 2 \times 145) \\ &= 174.5 \text{ m} \end{aligned}$$

Note that the displacement travelled can also be calculated without drawing the graph by reading off lengths from the table.

Example 2

Estimate the area bounded by the curve $y = \frac{1}{2}x^2 + 5$, the x-axis, the line $x = 1$ and $x = 5$ using the trapezium rule.

Solution

To plot the graph $y = \frac{1}{2}x^2 + 5$, make a table of values of x and the corresponding values of y as follows:

x	0	1	2	3	4	5
$y = \frac{1}{2}x^2 + 5$	5	5.5	7	9.5	13	17.5

The graph is shown in figure 9.8.

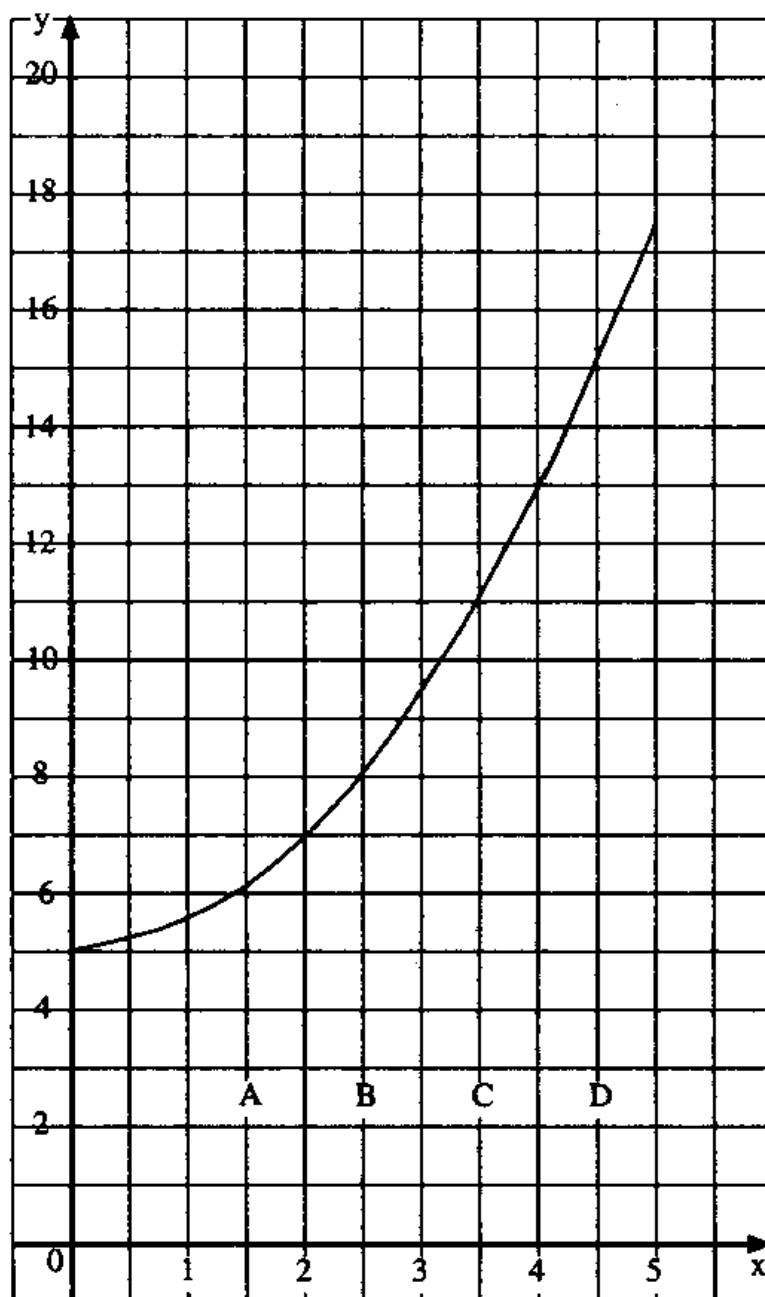


Fig. 9.8

By taking the width of each trapezium to be 1 unit, we get 4 trapezia, A, B, C and D as shown in the figure. The area under the curve is approximately;

$$\begin{aligned} \frac{1}{2} \times 1 \{(y_1 + y_5) + 2(y_2 + y_3 + y_4)\} &= \frac{1}{2} (5.5 + 17.5) + 2(7 + 9.5 + 13) \text{ sq. units} \\ &= \frac{1}{2} (23.0 + 59) \text{ sq. units} \\ &= 41 \text{ sq. units.} \end{aligned}$$

Exercise 9.2

1. Use trapezium rule to estimate the area of the region bounded by the graphs of the following equations, the x-axis and the indicated lines:
 - (a) $y = x^2$, $x = 0$ and $x = 2$ (use 4 trapezia).

- (b) $y = x^2 + 2$, $x = -2$ and $x = 2$ (use 9 ordinates)
- (c) $y = \frac{6}{x}$, $x = 2$ and $x = 4$ (use 5 ordinates)
2. Draw a semicircle of radius 3.5 cm on a graph paper.
- (a) Find its exact area, using $\pi = \frac{22}{7}$.
- (b) Estimate its area to 2 d.p. using trapezium rule with 7 trapezia.
- (c) Find the percentage error in the area.
3. Plot the graph of $y = x^2 - 10x + 25$ and use the trapezium rule to estimate the area enclosed by the curve, the x-axis, the line $x = 1$ and $x = 5$ (use 4 trapezia).
4. Figure 9.9 is a shaded area of land which is to be preserved for social amenities in a village:

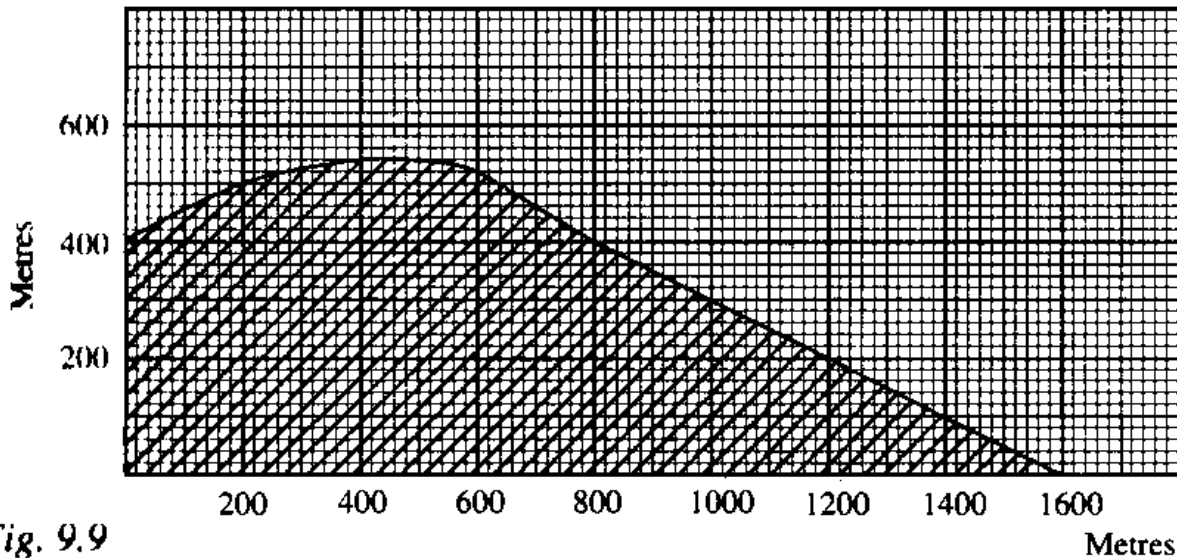


Fig. 9.9

Copy the figure and use the trapezium rule to estimate the area of the land in hectares (use 8 strips).

5. The speed in m/s at different times for a falling stone is shown in the table below:

<i>Time (sec)</i>	0	1	2	3	4	5
<i>Speed (m/s)</i>	0	9.8	19.4	20.2	40.2	50.2

Use the trapezium rule to estimate the distance covered.

6. The table below shows the volume of petrol used by a car and the rate of petrol consumption during a journey:

<i>Volume of petrol (litres) (x)</i>	1	2	2.5	4	5	6
<i>Rate of petrol consumption (km/litre) (y)</i>	7	7.8	8	7	6	4.8

Use the trapezium rule to estimate the distance travelled by the car.

7. The volume V in m^3 and pressure P in Nm^{-2} for a given mass of a certain gas at a constant temperature are related by the formula $PV = 20$. Plot a graph of P against V and, using the trapezium rule with four equal intervals, estimate the area under the curve from $V = 4$ to $V = 8$.
8. Find the area enclosed by the curve $y = -x^2 + 2x + 8$ and the positive x and y axes using:
 - (a) 4 trapezia.
 - (b) 8 trapezia.
9. During a survey, offsets were taken at intervals on one side along a straight line PQ which cut off the irregular side of a piece of land. The table below shows perpendicular offsets at different intervals along PQ . Use the trapezium rule to find the area of the irregular portion of land.

<i>Metres along PQ</i>	0	30	50	60	80	90	100
<i>Offset in metres</i>	0	15	35	55	30	25	0

10. A horizontal ditch 1.5 m wide is dug in uneven ground and the vertical depth of the ditch in cm at 4 m intervals is 30, 40, 50, 60, 45, 25 and 10. Use the trapezium rule to find the volume of earth removed.
11. Copy and complete the table for values of x and y where $y = \sin 2x$.

<i>x (rads)</i>	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
<i>2x (rads)</i>	—	—	$\frac{\pi}{2}$	—	—
<i>sin 2x</i>	0	—	—	—	0

Plot the graph of $y = \sin 2x$ for values of x between $x = 0$ and $x = 90^\circ$. Use trapezium rule with four equal intervals to estimate the area.

12. Plot the graph of $y = \frac{1}{2}x^2 + 1$ between $x = -4$ and $x = 4$. Estimate the area enclosed by the curve, the lines $y = 1.5$ and $y = 5$ and the x -axis by trapezium rule.

9.4: The Mid-ordinate Rule

Consider figure 9.10 in which the area OPQR is to be estimated:

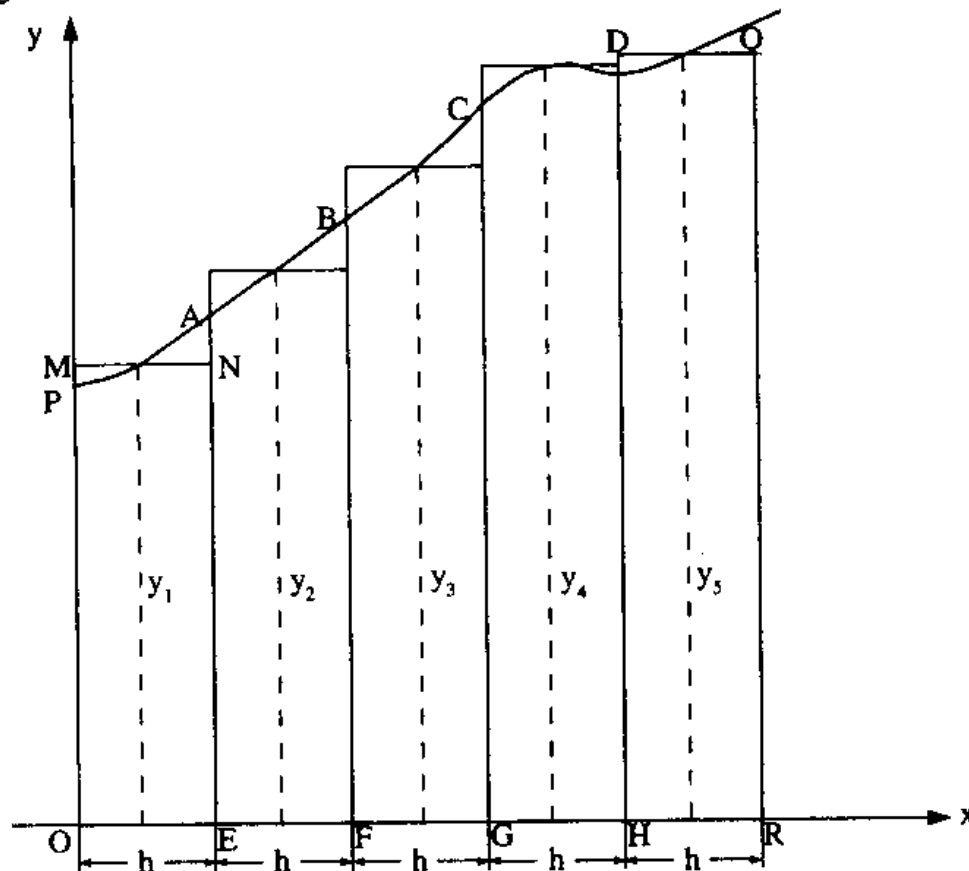


Fig. 9.10

The area of OPQR is estimated as follows:

- (i) Divide the base OR into a number of strips, each of width h . In this case, 5 strips have been chosen, where $h = \frac{\text{length of base OR}}{\text{number of strips}}$
- (ii) From the midpoints of OE, EF, FG, GH and HR, draw vertical lines (mid-ordinates) to meet the curve PQ as in the figure.
- (iii) Label the mid-ordinates so drawn y_1, y_2, y_3, y_4 and y_5 .
- (iv) We take the area of each trapezium to be equal to the area of a rectangle whose width is the length of interval (h) and the length is the value of mid-ordinates. Therefore, the area of the region OPQR is given by;

$$\begin{aligned} A &= (y_1 \times h) + (y_2 \times h) + (y_3 \times h) + (y_4 \times h) + (y_5 \times h) \\ &= h(y_1 + y_2 + y_3 + y_4 + y_5) \end{aligned}$$

This is the mid-ordinate rule.

Generally, the mid-ordinate rule for approximating areas of irregular shapes is given by; **Area = (width of interval) \times (sum of mid-ordinates)**

Example 3

Estimate the area of a semicircle of radius 4 cm using the mid-ordinate rule with four equal strips, each of width 2 cm.

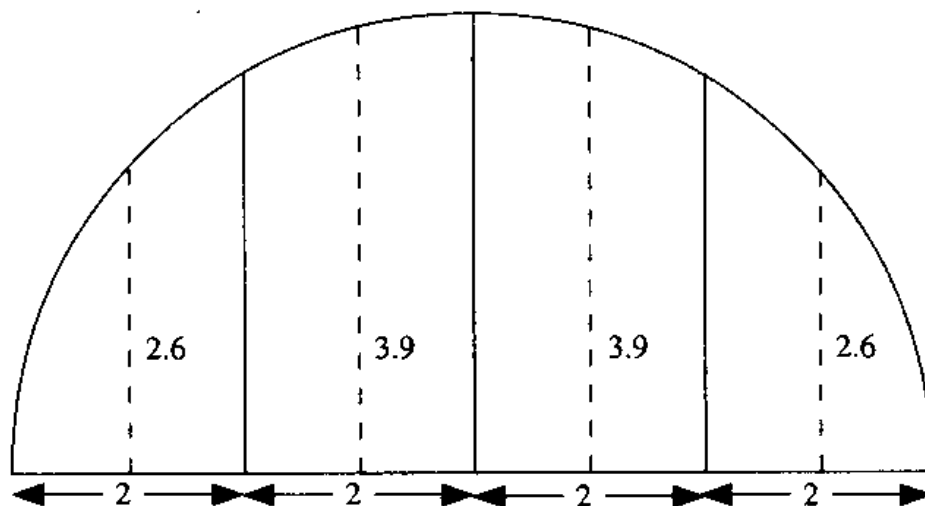


Fig. 9.11

Solution

Figure 9.11 shows a semicircle of radius 4 cm divided into 4 equal strips, each of width 2 cm. The dotted lines are the mid-ordinates whose lengths are measured.

By the mid-ordinate rule;

$$\begin{aligned} \text{area} &= h(y_1 + y_2 + y_3 + y_4) \\ &= 2(2.6 + 3.9 + 3.9 + 2.6) \\ &= 2 \times 13 \\ &= 26 \text{ cm}^2 \end{aligned}$$

Note that the actual area is $\frac{\pi r^2}{2} = \frac{3.142 \times 4^2}{2} = 25.14 \text{ cm}^2$ (4 s.f.)

Example 4

Estimate the area enclosed by the curve $y = \frac{1}{2}x^2 + 1$, $x = 0$, $x = 3$ and the x-axis, using the mid-ordinate rule.

Solution

The graph of the curve $y = \frac{1}{2}x^2 + 1$ is shown in figure 9.12. We choose to take 3 strips. The dotted lines are the mid-ordinates and the width of each of the 3 strips is 1 unit.

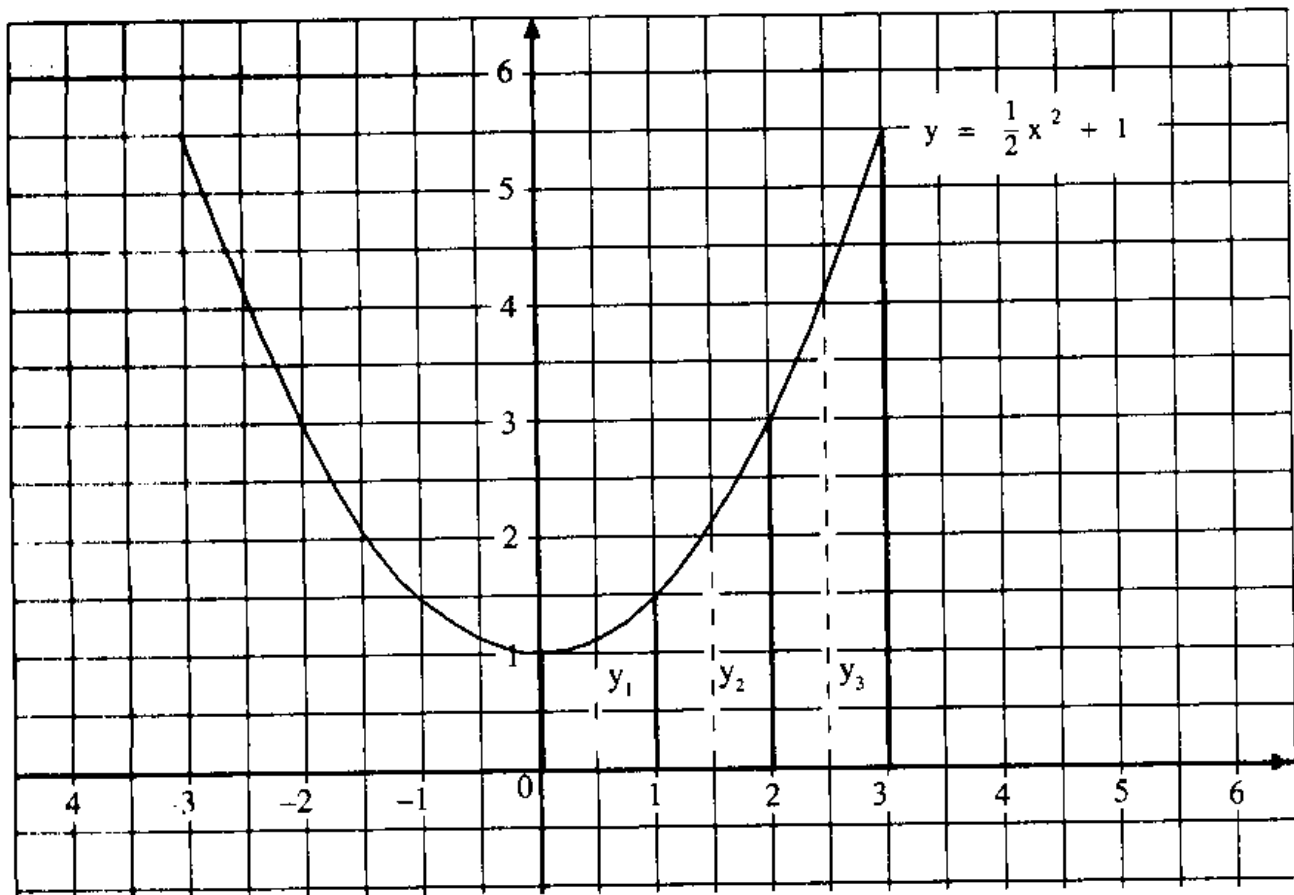


Fig. 9.12

By calculation, y_1 , y_2 and y_3 are obtained from the equation;

$$y = \frac{1}{2}x^2 + 1$$

$$\begin{aligned} \text{When } x = 0.5, y_1 &= \frac{1}{2} \times (0.5)^2 + 1 \\ &= 1.125 \end{aligned}$$

$$\begin{aligned} \text{When } x = 1.5, y_2 &= \frac{1}{2} \times (1.5)^2 + 1 \\ &= 2.125 \end{aligned}$$

$$\begin{aligned} \text{When } x = 2.5, y_3 &= \frac{1}{2} \times (2.5)^2 + 1 \\ &= 4.125 \end{aligned}$$

Using the mid-ordinate rule the area required is;

$$\begin{aligned} A &= 1(y_1 + y_2 + y_3) \\ &= 1(1.125 + 2.125 + 4.125) \\ &= 7.375 \text{ square units} \end{aligned}$$

Estimate the area of the same region by dividing the region into 6 equal strips.

Note:

- (i) The values of mid-ordinates can be obtained from the graph if it is accurately drawn.

- (ii) As in the trapezium rule, the greater the number of strips, the better the approximation.

Exercise 9.3

1. Draw the graph of $y = 2x^2 + 6$. Use the mid-ordinate rule with four strips to estimate the area enclosed by the curve, the lines $x = 1$ and $x = 5$ and the x -axis.
2. Use the mid-ordinate rule to find the area enclosed by $y = -x^2 + 2x + 8$, the y -axis and the x -axis using four strips.
3. In an isosceles triangle ABC, $AB = 10$ cm and $BC = 12$ cm. Using BC as the base, draw perpendicular lines at intervals of 2 cm. By using the mid-ordinate rule, estimate the area of the triangle.
4. Draw the graph of $y = \frac{1}{2}x^2 - 2$ from $x = -4$ to $x = 4$. Estimate the area enclosed by the curve, the lines $x = -2$, $x = 4$ and the x -axis by using the mid-ordinate rule with six strips.
5. Estimate the area between the curve $y = 10 \cos x$, and the x -axis for the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ using mid-ordinate rule with 8 strips.
6. Figure 9.13 shows a plot of land OABC drawn to scale. The plot is divided into five strips of equal width. Use the mid-ordinate rule to estimate its area in hectares.

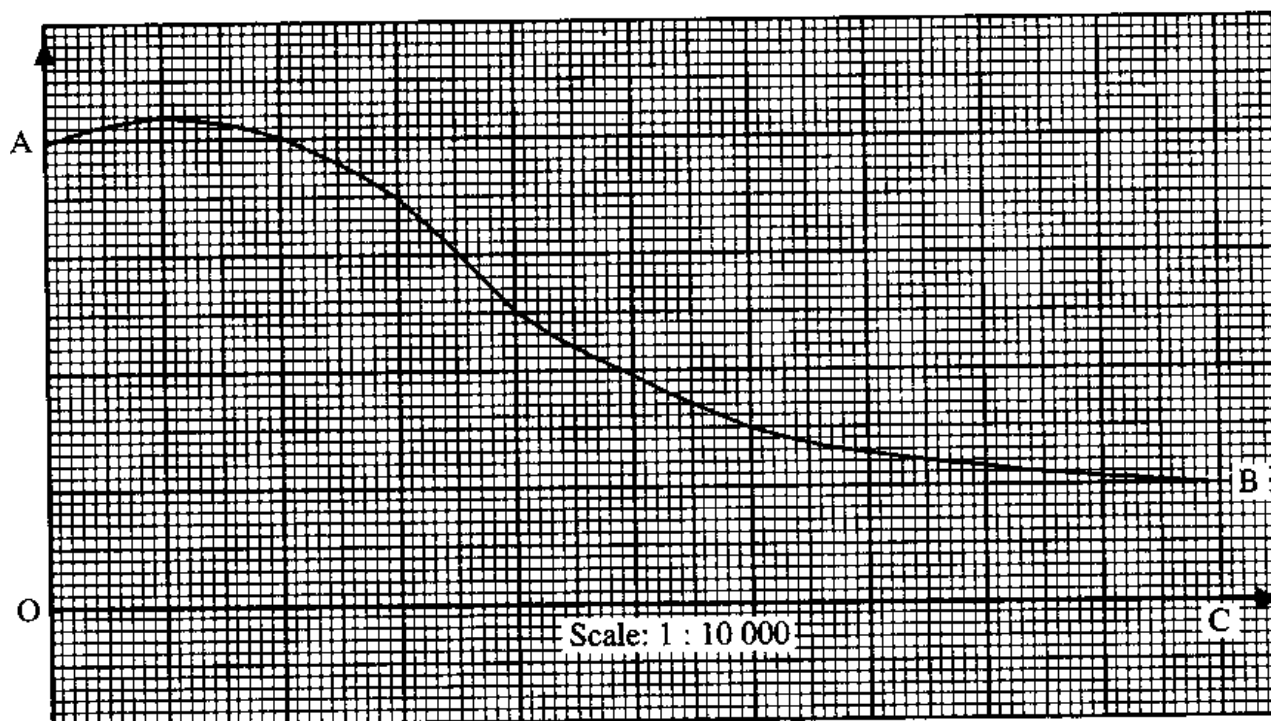


Fig. 9.13

7. The speed of a car during the first minutes after starting is given in the table below:

<i>Time t (s)</i>	10	20	30	40	50	60
<i>Speed v (ms^{-1})</i>	3.7	8.6	15.0	19.6	22.9	24.8

Taking scales of 5 seconds and $2\ ms^{-1}$ to 1 cm, draw the speed-time graph. Using four strips of equal width between $t = 20$ and $t = 50$, use the mid-ordinate rule to estimate the distance covered by the car between the 20th and 50th seconds.

8. The depth of a water pipe below the ground level varies as follows:

<i>Horizontal distance (m)</i>	0	10	20	30	40	50	60
<i>Depth underground (m)</i>	0.20	0.25	0.28	0.30	0.32	0.30	0.29

Plot the graph showing the variation of depth with length. If the width of the trench to be dug to expose the pipe is 0.5 m wide, use the mid-ordinate rule to estimate the volume of the earth to be dug.

9. The table below gives the variation of depth of a cable below the ground:

<i>Horizontal distance (m)</i>	0	45	90	135	180	225
<i>Depth underground (m)</i>	2.2	2.4	2.9	3.2	3.3	2.4

Plot a graph of variation of depth with length. If the width of the trench to be dug to expose the cable is 50 cm, use the mid-ordinate rule to calculate the volume of the earth to be dug in cubic metres.

10. Depths of a creek 35 m wide are found by sounding across the creek at equal intervals, as shown in the table below:

<i>Depth (m)</i>	0	3.2	3.6	4.0	4.2	3.9	3.4	0
------------------	---	-----	-----	-----	-----	-----	-----	---

Use the mid-ordinate rule to calculate the cross-sectional area of the creek at that point of the creek.

11. The table below gives the values of an alternating current at equal intervals of 2.0 milliseconds (ms):

<i>Time (ms)</i>	0	2.0	4.0	6.0	8.0	10.0	12.0
<i>Current (amperes)</i>	0	1.6	3.6	5.1	3.8	3.2	0

Plot a graph of current against time and use the mid-ordinate rule to estimate the area between the curve and the time-axis.

Chapter Ten

INTEGRATION

10.1: Reverse Differentiation

In Chapter 8, we saw how to obtain a gradient function from a given function. In this chapter, we look into the reverse process, that is, how to get a function from a gradient function.

Consider the lines in figure 10.1 below:

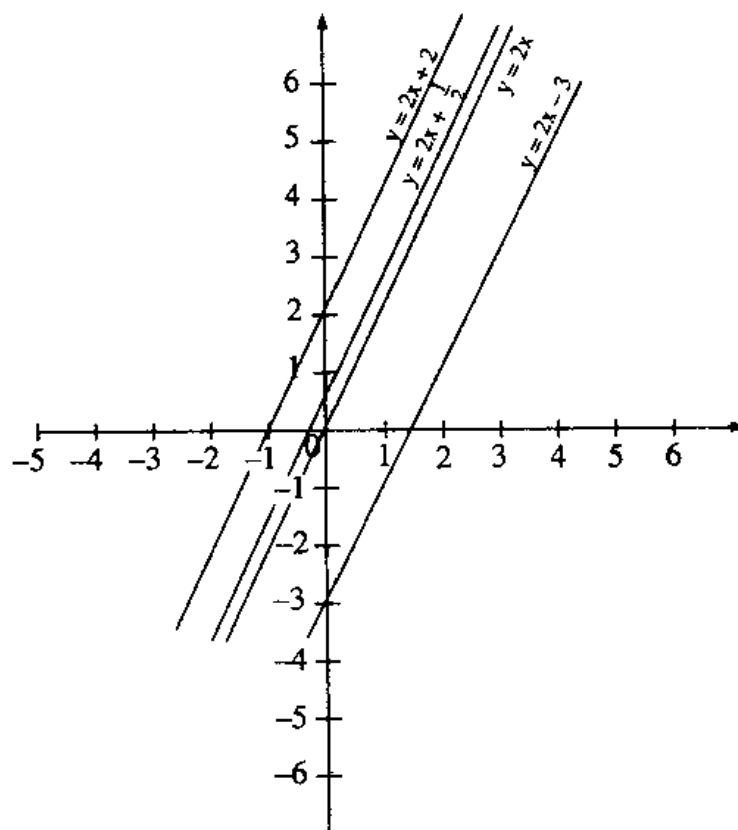


Fig. 10.1

The lines all have a gradient of 2. Thus, their gradient function is $\frac{dy}{dx} = 2$.

Generally, the lines whose gradient is 2 are of the form $y = 2x + c$, where c is a constant, also referred to as the y -intercept. So, for any given gradient, an infinite number of lines can be drawn. Their relative positions can be distinguished by the value of c . In each of the following, give three possible equations of a line

whose derived function $\frac{dy}{dx}$ is:

(i) $\frac{1}{2}$

(ii) -2

(iii) $-\frac{4}{3}$

(iv) 6

You notice that the equations are of the form:

(i) $y = \frac{1}{2}x + c$ (ii) $y = -2x + c$ (iii) $y = -\frac{4}{3}x + c$ (iv) $y = 6x + c$, where c is a constant in each case.

We know from Chapter 8 that the gradient function is not always a constant.

For example, if $\frac{dy}{dx} = 2x$, then this must have come from the functions of the form $y = x^2 + c$, where c is a constant.

Example 1

Find y if $\frac{dy}{dx}$ is:

(a) $3x^2$ (b) $4x^3$ (c) $\frac{1}{2}x$ (d) x^3

Solution

(a) $\frac{dy}{dx} = 3x^2$

Then, $y = x^3 + c$

(b) $\frac{dy}{dx} = 4x^3$

Then, $y = x^4 + c$

(c) $\frac{dy}{dx} = \frac{1}{2}x$

Then, $y = \frac{x^2}{4} + c$

(d) $\frac{dy}{dx} = x^3$

Then, $y = \frac{x^4}{4} + c$

The process of finding functions from their gradient (derived) functions is called **integration**. The table below shows some gradient functions and their corresponding functions.

$\frac{dy}{dx}$ (gradient function)	Function (y)
x^2	$\frac{x^3}{3} + c$
x^3	$\frac{x^4}{4} + c$
x^4	$\frac{x^5}{5} + c$
$2x^3$	$\frac{2x^4}{4} + c$
$\frac{1}{2}x^5$	$\frac{x^6}{2 \times 6} + c$

We notice that to integrate, we reverse the rule for differentiation. In differentiation, we multiply by the power of x and reduce the power by 1. In integration therefore, we increase the power of x by one and divide by the new power.

In general, if $\frac{dy}{dx} = x^n$, then $y = \frac{x^{n+1}}{n+1} + c$, where c is a constant and $n \neq -1$.
 Since c can take any value, we call it an arbitrary constant.

Example 2

Integrate the following expressions:

- (a) $2x^5$ (b) x^{-2} (c) $5x^3 - 2x + 4$ (d) $3x^{-2} + 5x$ (e) $x^{\frac{1}{3}}$

Solution

(a) If $\frac{dy}{dx} = 2x^5$;

$$\begin{aligned} \text{then, } y &= \frac{2x^{5+1}}{5+1} + c \\ &= \frac{2x^6}{6} + c \\ &= \frac{x^6}{3} + c \end{aligned}$$

(b) If $\frac{dy}{dx} = x^{-2}$;

$$\begin{aligned} \text{then, } y &= \frac{x^{(-2+1)}}{(-2+1)} + c \\ &= \frac{x^{-1}}{-1} + c \\ &= -x^{-1} + c \end{aligned}$$

(c) If $\frac{dy}{dx} = 5x^3 - 2x + 4$

$$\begin{aligned} \text{then } y &= \frac{5x^{(3+1)}}{(3+1)} - \frac{2x^{(1+1)}}{(1+1)} + \frac{4x^{(0+1)}}{(0+1)} + c \\ &= \frac{5}{4}x^4 - \frac{2}{2}x^2 + 4x + c \\ &= \frac{5}{4}x^4 - x^2 + 4x + c \end{aligned}$$

(d) If $\frac{dy}{dx} = 3x^{-2} + 5x$;

$$\begin{aligned} \text{then, } y &= \frac{3x^{(-2+1)}}{(-2+1)} + \frac{5x^{(1+1)}}{(1+1)} + c \\ &= \frac{3x^{-1}}{-1} + \frac{5x^2}{2} + c \\ &= -3x^{-1} + \frac{5x^2}{2} + c \end{aligned}$$

(e) If $\frac{dy}{dx} = x^{\frac{1}{3}}$,

$$\begin{aligned}\text{Then, } y &= \frac{x^{\left(\frac{1}{3}+1\right)}}{\left(\frac{1}{3}+1\right)} + c \\ &= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c \\ &= \frac{3}{4}x^{\frac{4}{3}} + c\end{aligned}$$

Exercise 10.1

1. Integrate each of the following expressions:

- | | | |
|-------------------------------------|--|-----------------------------|
| (a) $3x$ | (b) 7 | (c) $2x + 4$ |
| (d) $x^2 + 2x$ | (e) x^5 | (f) x^7 |
| (g) x^{-5} | (h) $-3x^4$ | (i) $-5x^{-7}$ |
| (j) $5x^3 - 4x^2$ | (k) $3x^4 - x^{-2} + 3$ | (l) $9x^3 - 2x^{-4} + 3x^2$ |
| (m) $8x^3 + 3x^2 + 4x + 3$ | (n) $\frac{1}{x^2}$ | (p) $\frac{1-x^4}{x^2-1}$ |
| (q) $\frac{1}{x^3} - \frac{1}{x^4}$ | (r) $4x^{\frac{1}{2}} + \frac{1}{2}x^4 + 3x^2$ | |

2. Find y , given that $\frac{dy}{dx}$ is:

- | | | |
|-----------------|--|------------------------|
| (a) $4ax^3$ | (b) $\frac{x^3 - 4x + 4}{x^3}$ | (c) $(1 - 4x)(1 + 4x)$ |
| (d) $(x + 3)^2$ | (e) $(1 + \frac{1}{x})(\frac{1}{x} - 1)$ | |

3. Integrate each of the following:

- | | |
|---|---|
| (a) $\frac{dy}{dx} = 3x^2 - \frac{1}{2}x + 4$ | (b) $\frac{dy}{dx} = 4x^3 + 3x^2 + 2x - 2$ |
| (c) $\frac{dy}{dx} = x^{-3} + x^{-2} + 1$ | (d) $\frac{dy}{dx} = \frac{3}{x^2} + \frac{2}{x^3} + \frac{1}{x^4} + 3$ |
| (e) $\frac{dy}{dx} = \frac{9}{x^3} + \frac{8}{x^2} + 6$ | (f) $\frac{dy}{dx} = \frac{4}{x^4} - \frac{3}{x^3} + \frac{2}{x^2} - 1$ |

4. Find the general equation of a curve whose gradient function is:

- | | |
|-------------------------|---------------------------------|
| (a) $x^2 + 4x$ | (b) $3x^3 - \frac{5x^2}{2} + 7$ |
| (c) $\frac{1}{x^2} + 5$ | (d) $5\sqrt{x} + 3x^2 + 2$ |

$$(e) \quad 3x^{\frac{1}{3}} + 5x^{\frac{1}{4}} - 7x^3 + 9 \qquad (f) \quad \frac{8x^{\frac{1}{2}} + x^{\frac{1}{4}}}{2x^{\frac{1}{4}}}$$

$$(g) \quad 13x^{\frac{1}{5}} + \frac{1}{3x^5} + 8$$

We can arrive at a particular equation from a general one by finding the arbitrary constant if additional information is given.

Example 3

Find the equation of a curve whose gradient function is $\frac{dy}{dx} = 2x + 3$ and passes through $(0, 1)$.

Solution

Since $\frac{dy}{dx} = 2x + 3$, the general equation is $y = x^2 + 3x + c$. The curve passes through $(0, 1)$. Substituting these values in the general equation, we get;

$$1 = 0 + 0 + c$$

$$1 = c$$

Hence, the particular equation is $y = x^2 + 3x + 1$

Example 4

Find V in terms of h if $\frac{dV}{dh} = 3h^2 + 4$ and $V = 9$ when $h = 1$.

Solution

The general solution is;

$$V = \frac{3h^3}{3} + 4h + c$$

$$= h^3 + 4h + c$$

$V = 9$ when $h = 1$. Therefore;

$$9 = 1^3 + 4 + c$$

$$9 = 5 + c$$

$$4 = c$$

Hence, the particular solution is;

$$V = h^3 + 4h + 4$$

Exercise 10.1 (continued)

5. If $\frac{dA}{dr} = 4r^3 + 3r^2 + 4$, find A in terms of r , given that when $r = 1$, $A = 19$.

6. A curve passes through $(1, -1)$. If the gradient function is $3x^2 + 2$, find its equation.
7. A curve whose gradient function is $\frac{dy}{dx} = x^2 - x + \frac{1}{4}$ has its stationary point at $y = \frac{1}{6}$. Find its equation.
8. Given that $\frac{dr}{dt} = (2t + 1)(t^2 - 1)$ and $r = 1$ when $t = 0$, find:
 (a) an expression for r in terms of t .
 (b) the value of r when $t = 3$.

10.2: Definite and Indefinite Integrals

In Chapter 9, we saw that to improve the accuracy of the area under a given curve, we reduce the width of each trapezium. In this section, we shall learn how to find the exact area. Suppose we wish to estimate the area shaded beneath the curve shown in figure 10.2.

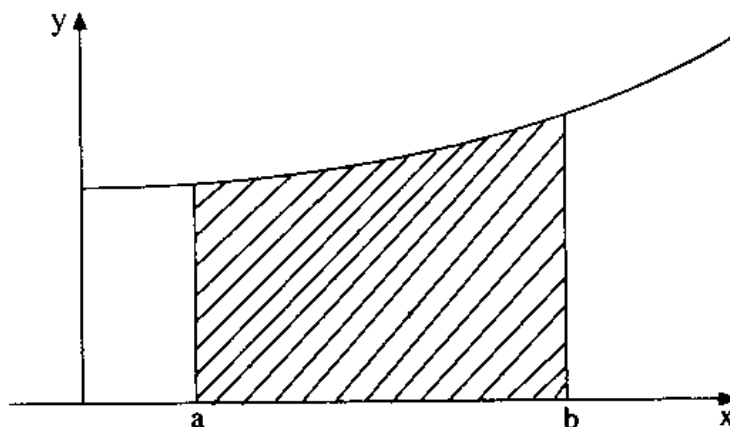
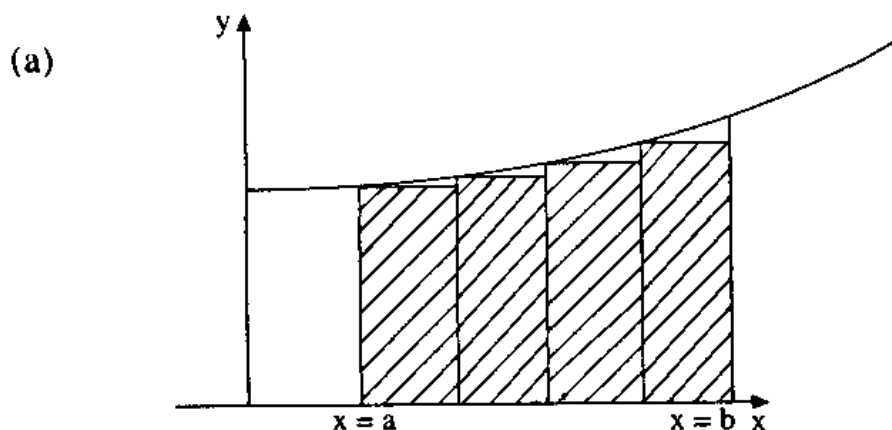


Fig. 10.2

The area can be subdivided into rectangular strips in two different ways, as shown in figure 10.3.



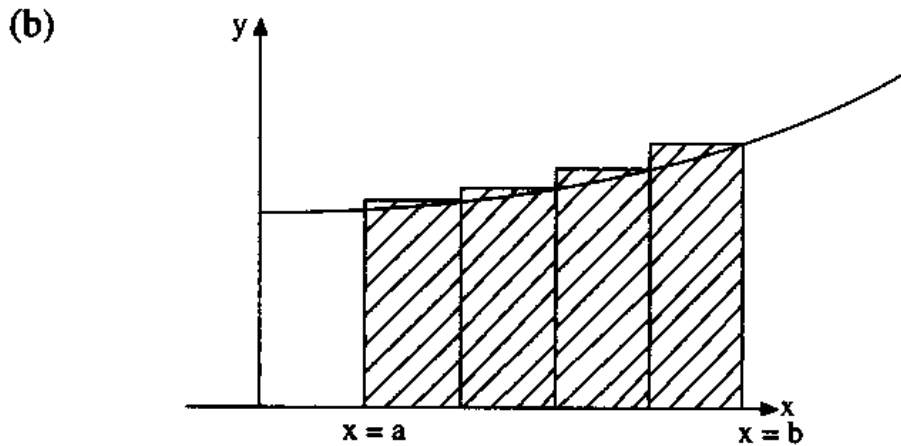


Fig. 10.3

The shaded areas in figure 10.3 (a) and (b) show, respectively, an underestimated and an overestimated area under the curve.

The actual area lies between the underestimated and overestimated area. In each of the two cases, the accuracy of the area can be improved by increasing the number of rectangular strips between $x = a$ and $x = b$. Figure 10.4 is a typical strip extracted from figure 10.3 (a).

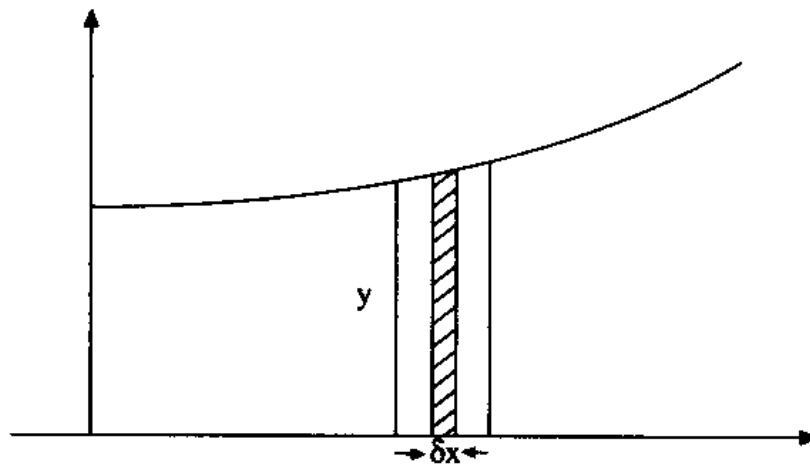


Fig. 10.4

Notice that we can choose δx small enough to reduce the length of our rectangular strip to y . The area of this strip will be $y\delta x$. The area beneath the curve is the sum of all areas of such strips that can be fitted into the region.

Thus, area beneath the curve between $x = a$ and $x = b$ is given by;

$$\text{Area} = \sum_{x=a}^{x=b} y\delta x$$

The exact area is the limiting value of the sum as δx tends to zero and is written as;

$$\int_a^b y \delta x$$

The symbol \int is an instruction to integrate.

Thus, $\int y dx$ means integrate the expression for y with respect to x .

The expression $\int_a^b \delta x$, where a and b are limits, is called a definite integral. 'a' is called the lower limit and 'b' the upper limit. Without limits, the expression is called an indefinite integral.

Consider the definite integral $\int_2^6 (2x^2 + 3) dx$.

The procedure of evaluating it is as follows:

- (i) Integrate $2x^2 + 3$ with respect to x , giving $\frac{2}{3}x^3 + 3x + c$.
- (ii) Place the integral in square brackets and insert the limits, thus

$$\left[\frac{2}{3}x^3 + 3x + c \right]_2^6$$

- (iii) Substitute the limits;

$$x = 6 \text{ gives } \frac{2 \times 6^3}{3} + 3 \times 6 + c = 162 + c$$

$$x = 2 \text{ gives } \frac{2 \times 2^3}{3} + 3 \times 2 + c = \frac{34}{3} + c$$

- (iv) Subtract the result of the lower limit from that of the upper limit, that is;

$$(162 + c) - \left(\frac{34}{3} + c \right) = 150 \frac{2}{3}$$

Notice that the constant c cancels in the subtraction.

The steps can be summarised as;

$$\begin{aligned} \int_2^6 (2x^2 + 3) dx &= \left[\frac{2}{3}x^3 + 3x \right]_2^6 \\ &= \left[\frac{2 \times 6^3}{3} + (3 \times 6) \right] - \left[\frac{2 \times 2^3}{3} + 3 \times 2 \right] \\ &= 150 \frac{2}{3} \end{aligned}$$

Example 5

(a) Find the indefinite integral:

(i) $\int(x^2 + 1) dx$

(ii) $\int(x^3 + 4x) dx$

(b) Evaluate:

(i) $\int_0^1(x^4 - 5) dx$

(ii) $\int_{-1}^2(-x^3 + 5x - 2) dx$

(iii) $\int_2^3(3x^2 - 4x + 5) dx$

Solution

(a) (i) $\int(x^2 + 1) dx = \frac{x^3}{3} + x + c$

(ii) $\int(x^3 + 4x) dx = \frac{x^4}{4} + 2x^2 + c$

(b) (i)
$$\begin{aligned}\int_0^1(x^4 - 5)dx &= \left[\frac{x^5}{5} - 5x\right]_0^1 \\ &= \left(\frac{1}{5} - 5\right) - \left(\frac{0}{5} - 0\right) \\ &= -4\frac{4}{5}\end{aligned}$$

(ii)
$$\begin{aligned}\int_{-1}^2(-x^3 + 5x - 2)dx &= \left[\frac{-x^4}{4} + \frac{5x^2}{2} - 2x\right]_{-1}^2 \\ &= (-4 + 10 - 4) - \left(-\frac{1}{4} + \frac{5}{2} - 2\right) \\ &= 2 - 4\frac{1}{4} \\ &= -2\frac{1}{4}\end{aligned}$$

(iii)
$$\begin{aligned}\int_2^3(3x^2 - 4x + 5)dx &= \left[\frac{3x^3}{3} - \frac{4x^2}{2} + 5x\right]_2^3 \\ &= (27 - 18 + 15) - (8 - 8 + 10) \\ &= 14\end{aligned}$$

Exercise 10.2

1. Evaluate:

(a) $\left[5x^2 + 3x - 4\right]_2^5$

(b) $\left[2x^3 + 3x^2 - x\right]_{-3}^2$

(c) $\left[x^3 - 6x\right]_{-5}^0$

(d) $\left[9x^2 + c\right]_1^7$

2. Find:

(a) $\int (3x^2 - 4x + 2) dx$

(b) $\int (4 - x^3) dx$

(c) $\int (x^4 + 4x^3 - 9x^2 + x) dx$

(d) $\int (t^2 - 2t + 1) dt$

(e) $\int (t^3 - 2t^2 + 3t - 4) dt$

3. Evaluate the following:

(a) $\int_2^4 (x^3 + 6x^2 - 4x + 1) dx$

(b) $\int_0^5 (4t^3 - t^2 + 2t - 1) dt$

(c) $\int_1^6 (2u^3 - \frac{1}{2}u^2 + 4) du$

(d) $\int_0^1 (4t^3 - 2t - 2) dt$

(e) $\int_{-2}^0 (x^2 + 2x) dx$

(f) $\int_1^5 (x^2 - 6x + 5) dx$

10.3: Area Under a Curve

In Section 9.3, we found that the area of the curve $y = \frac{1}{2}x^2 + 5$ between the limits $x = 1$ and $x = 5$ by the trapezium rule to be 41 square units. We now find the exact area using integration. The area is given by;

$$\begin{aligned} A &= \int_1^5 \left(\frac{1}{2}x^2 + 5 \right) dx \\ &= \left[\frac{1}{6}x^3 + 5x \right]_1^5 \\ &= \left[\frac{5^3}{6} + (5 \times 5) \right] - \left[\frac{1}{6} + (5 \times 1) \right] \\ &= 40\frac{2}{3} \text{ square units} \end{aligned}$$

Example 6

Find the exact area enclosed by the curve $y = x^2$, the x-axis, the lines $x = 2$ and $x = 4$.

Solution

The area is shown in figure 10.5.

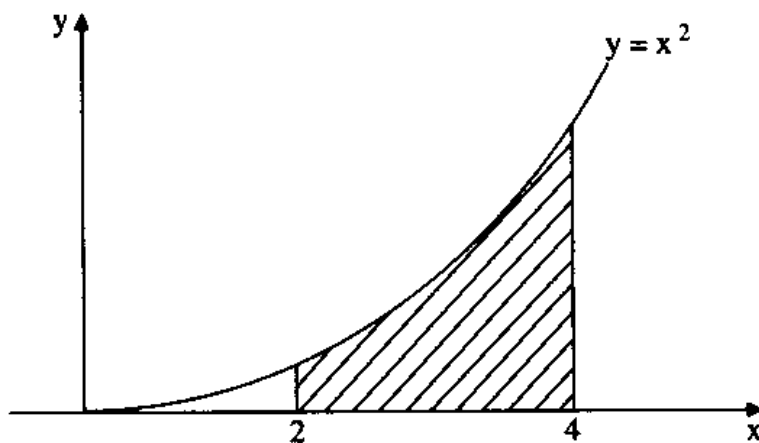


Fig. 10.5

The area is given by;

$$\begin{aligned} \int_2^4 x^2 dx &= \left[\frac{1}{3}x^3 \right]_2^4 \\ &= \frac{64}{3} - \frac{8}{3} \\ &= 18\frac{2}{3} \text{ square units} \end{aligned}$$

Example 7

Find the area of the region bounded by the curve $y = x^3 - 3x^2 + 2x$, the x -axis, $x = 1$ and $x = 2$.

Solution

The required area is shown in figure 10.6.

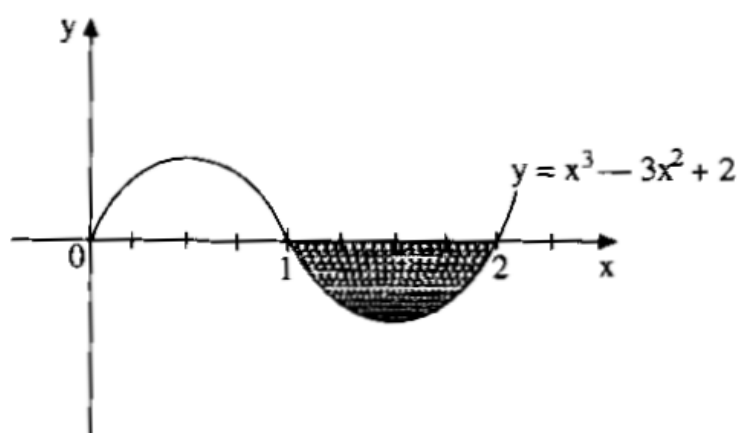


Fig. 10.6

The area is given by;

$$\begin{aligned} \int_1^2 (x^3 - 3x^2 + 2x) dx &= \left[\frac{x^4}{4} - x^3 + x^2 \right]_1^2 \\ &= (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1 \right) \\ &= 0 - \frac{1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

The negative sign shows that the area is below the x -axis. We disregard the negative sign and give it as positive. In this case, the answer is $\frac{1}{4}$ square units.

Example 8

Find the area enclosed by the curve $y = x^2 - 10x + 9$, the x -axis and the lines $x = 4$ and $x = 10$.

Solution

The required area is shaded as shown in figure 10.7.

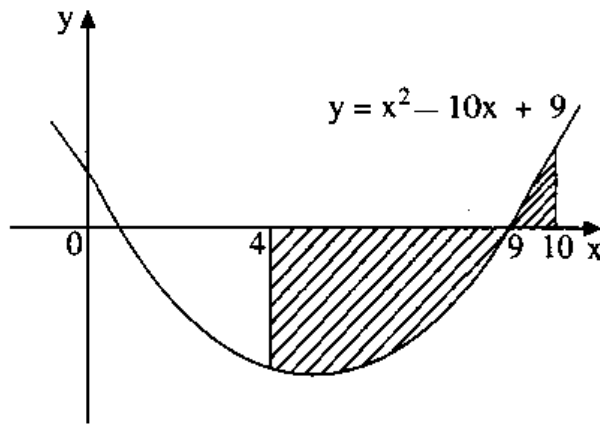


Fig. 10.7

The area is the sum of the area below and above the x-axis.

$$\begin{aligned}
 \text{Area} &= \int_4^9 (x^2 - 10x + 9) dx + \int_9^{10} (x^2 - 10x + 9) dx \\
 &= \left[\frac{x^3}{3} - 5x^2 + 9x \right]_4^9 + \left[\frac{x^3}{3} - 5x^2 + 9x \right]_9^{10} \\
 &= [(243 - 405 + 81) - (\frac{64}{3} - 80 + 36)] + [\frac{1000}{3} - 500 + 90] - (243 - 405 + 81) \\
 &= -58\frac{1}{3} + 4\frac{1}{3} \\
 &= 58\frac{1}{3} + 4\frac{1}{3} \text{ (Drop negative sign for area under x-axis)} \\
 &= 62\frac{2}{3} \text{ square units}
 \end{aligned}$$

Example 9

Find the area enclosed by the curve $y = 9x - x^2$ and the line $y = x$.

Solution

The required area is shown in figure 10.8.

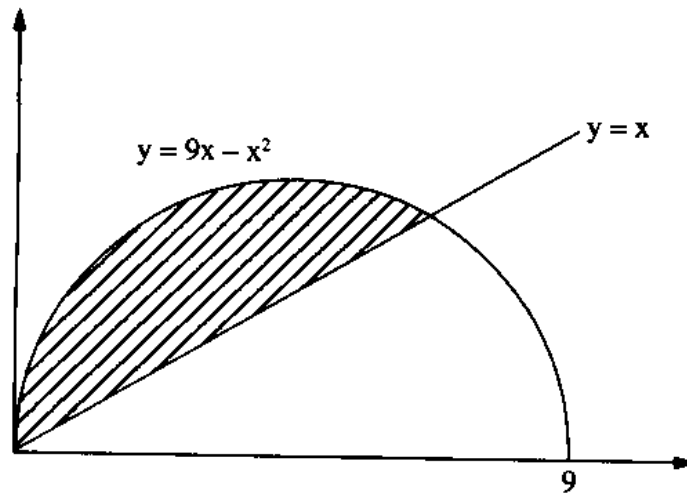


Fig. 10.8

To find the limits of integration, we must find the x co-ordinates of the points of intersection when;

$$x = 9x - x^2$$

$$\Rightarrow 0 = 8x - x^2$$

$$0 = x(8 - x)$$

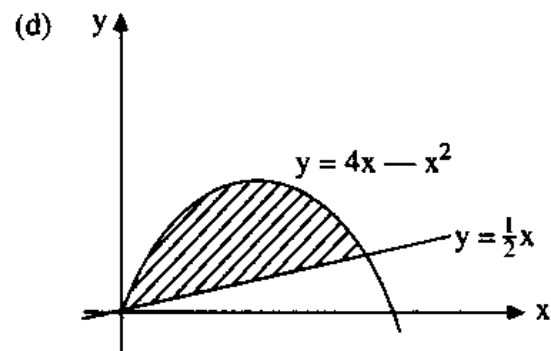
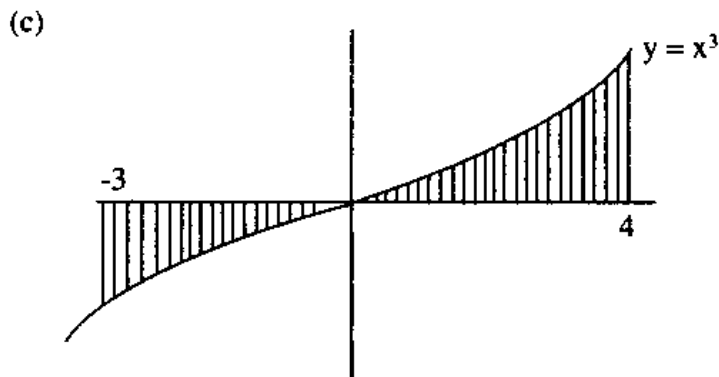
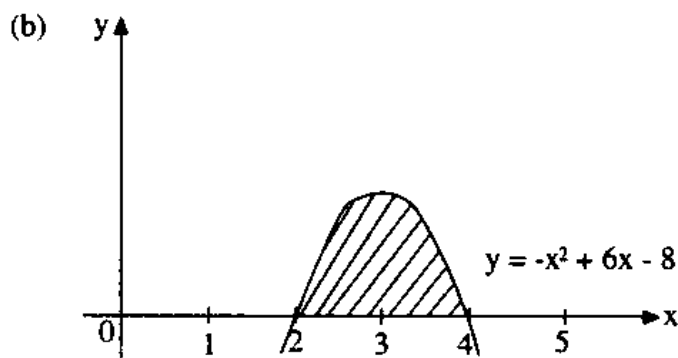
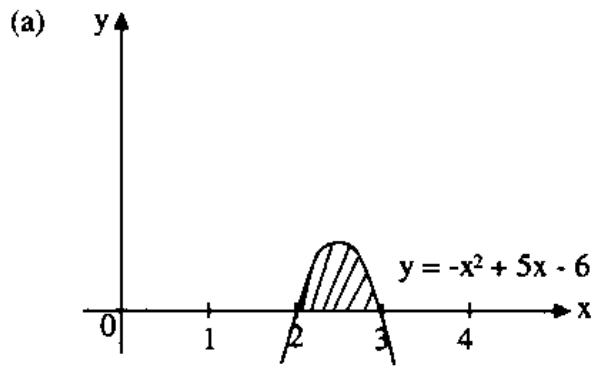
$$x = 0 \text{ or } x = 8$$

The required area is found by subtracting area under $y = x$ from area under $y = 9x - x^2$.

$$\begin{aligned} \text{The required area} &= \int_0^8 (9x - x^2) dx - \int_0^8 x dx \\ &= \left[\frac{9x^2}{2} - \frac{x^3}{3} \right]_0^8 - \left[\frac{x^2}{2} \right]_0^8 \\ &= 117\frac{1}{3} - 32 \\ &= 85\frac{1}{3} \text{ square units} \end{aligned}$$

Exercise 10.3

1. In each of the following, find the shaded area:



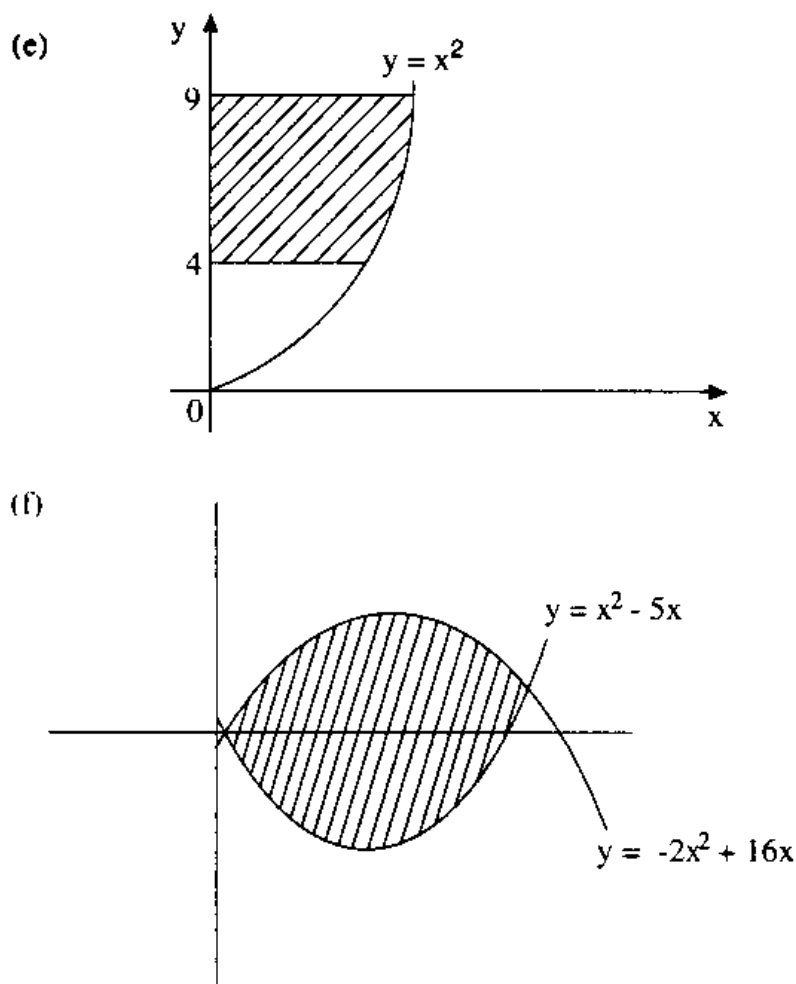


Fig. 10.9

2. Find the area enclosed by the x-axis and the curve $y = 3x^2 + 8x$.
3. Find the area enclosed by the curve $y = x^2 - 4x + 3$, the x-axis and the y-axis.
4. Determine the area bounded by the curve $y = x^2 - 4$, the x-axis and the line $x = 3$.
5. Find the area enclosed by the x-axis and the curve:
 - (a) $y = x^3 - 5x^2 + 4x$
 - (b) $y = (x + 3)(x^2 + 2x + 1)$
 - (c) $y = (x + 2)(x^2 - 4x + 4)$
 - (d) $y = (x - 1)(x + 1)(x + 2)$
 - (e) $y = x(x - 1)(x + 1)$
6. Find the area enclosed by the curve $y = x^2 - 7x$ and line $y = x$.
7. Calculate the area beneath the curve $y = x^2 - 5x + 6$ bounded by the lines:
 - (a) $x = 2$ and $x = 3$.
 - (b) $x = 2$ and $x = 4$.
 - (c) $x = 2\frac{1}{2}$ and $x = 4$.

For part (c) above, compare the area with the value for the integral

$$\int_{2.5}^4 (x^2 - 5x + 6) dx. \text{ Why are the two values different?}$$

10.4: Application in Kinematics

In Chapter 8, we saw that the derivative of displacement S with respect to time t gives velocity v , while the derivative of velocity with respect to time gives acceleration, a . It then follows that we integrate:

- (i) velocity with respect to time to get displacement.
- (ii) acceleration with respect to time to get velocity.

Example 10

A particle moves in a straight line through a fixed point O with velocity $(4 - t)$ m/s. Find an expression for its displacement S from this point, given that $S = 4$ when $t = 0$.

Solution

$$\text{Since } \frac{dS}{dt} = 4 - t;$$

$$S = 4t - \frac{t^2}{2} + c$$

Substituting $S = 4$, $t = 0$ to get c ;

$$4 = 4 \times 0 - \frac{0^2}{2} + c$$

$$4 = c$$

$$\text{Therefore, } S = 4t - \frac{t^2}{2} + 4$$

Example 11

A ball is thrown upwards with a velocity of 40 m/s.

- (a) Determine an expression in terms of t for:
 - (i) its velocity.
 - (ii) its height above the point of projection.
- (b) Find the velocity and height after:
 - (i) 2 seconds.
 - (ii) 5 seconds.
 - (iii) 8 seconds.
- (c) Find the maximum height attained by the ball. (Take acceleration due to gravity to be 10 m/s^2)

Solution

(a) In this case, $\frac{dv}{dt} = -10$ (since the ball is projected upwards)

$$\text{Therefore, } v = -10t + c$$

$$\text{When } t = 0, v = 40 \text{ m/s}$$

$$\text{Therefore, } 40 = 0 + c$$

$$40 = c$$

(i) The expression for velocity is $v = 40 - 10t$.

(ii) Since $\frac{ds}{dt} = v = 40 - 10t$;

$$S = 40t - 5t^2 + c$$

$$\text{When } t = 0, s = 0$$

$$c = 0$$

The expression for displacement is;

$$S = 40t - 5t^2$$

(b) Since $v = 40 - 10t$

(i) When $t = 2$

$$v = 40 - 10(2)$$

$$= 40 - 20$$

$$= 20 \text{ m/s}$$

$$S = 40t - 5t^2$$

$$= 40(2) - 5(2)^2$$

$$= 80 - 20$$

$$= 60 \text{ m}$$

(ii) When $t = 5$

$$v = 40 - 10(5)$$

$$= -10 \text{ m/s}$$

$$S = 40(5) - 5(5)^2$$

$$= 200 - 125$$

$$= 75 \text{ m}$$

(iii) When $t = 8$

$$v = 40 - 10(8)$$

$$= -40 \text{ m/s}$$

$$S = 40(8) - 5(8)^2$$

$$= 320 - 320$$

$$= 0$$

(c) Maximum height is attained when $v = 0$.

$$\text{Thus, } 40 - 10t = 0$$

$$t = 4$$

$$\text{Maximum height } S = 160 - 80$$

$$= 80 \text{ m}$$

Example 12

The velocity v of a particle is 4 m/s. Given that $S = 5$ when $t = 2$ seconds:

- (a) find the expression of displacement in terms of time.
 (b) find the:
 (i) distance moved by the particle during the fifth second.
 (ii) distance moved by the particle between $t = 1$ and $t = 3$.

Solution

(a) $\frac{dS}{dt} = 4$

$$S = 4t + c$$

Since $S = 5$ m when $t = 2$;

$$5 = 4(2) + c$$

$$5 - 8 = c$$

$$-3 = c$$

$$\text{Thus, } S = 4t - 3$$

(b) (i) $[4t - 3]_4^5 = [(20 - 3) - (16 - 3)]$
 $= 17 - 13$
 $= 4$ m

(ii) $[4t - 3]_1^3 = [(12 - 3) - (4 - 3)]$
 $= 9 - 1$
 $= 8$ m

Exercise 10.4

- In each of the questions (a) to (f), the velocity v ms^{-1} of a particle is given after t seconds. Find the distance moved in the time interval stated. In each case, $S = 0$ when $t = 0$.
 - $v = 2t^2$, from $t = 0$ to $t = 10$.
 - $v = t^2 + 2$, from $t = 0$ to $t = 4$.
 - $v = t^3 - 15t + 3$, from $t = 1$ to $t = 3$.
 - $v = t^2 + 2t + 4$, from $t = 0$ to $t = 5$.
 - $v = 5t(t - 3) + 2$, from $t = 1$ to $t = 4$.
 - $v = 3 + 2t - 4t^2$, from $t = 0$ to $t = 8$.
- In parts (a) to (f) below, the acceleration of a particle in ms^{-2} is given. Find the velocity v and calculate its value at the stated time. In each case, $v = 0$ when $t = 0$:
 - $a = 8t$, $t = 5$
 - $a = 2t^2 + t - 4$, $t = 4$

- (c) $a = 3 - 2t$, $t = 3$
 (d) $a = 16t^2(t + 1)$, $t = 2$
 (e) $a = t^3(t^2 + 2t - 1)$, $t = 1$
 (f) $a = 3t(3t + 2)^2$, $t = 3$
3. The initial velocity of a particle is given by the expression $(9t^2 + 3)$ m/s. Find the expression for distance and hence find the distance covered by the particle during the 4th second.
4. A particle moving in a straight line starts at a fixed point O with a velocity of 20 ms^{-1} . After t seconds, its acceleration $a \text{ ms}^{-2}$ is given by $a = 5 - 3t$. Find:
 (a) its velocity at $t = 3$.
 (b) the distance covered when the particle first comes to rest.
5. A particle moving at a speed of 20 ms^{-1} is subjected to a deceleration of $6t \text{ ms}^{-2}$, where t is measured from the instant it begins to slow down. Show that the velocity is zero after $\sqrt{\frac{20}{3}}$ seconds.
6. A particle moves from rest with an acceleration of $t(2 - t) \text{ ms}^{-2}$. After how long is its velocity zero again? Show that the distance covered by the particle after this time is $\frac{9}{4}$ metres.
7. A particle moves along a straight line with a constant acceleration. At $t = 0$, its velocity is u and it is at a fixed point O. If v is the velocity after t seconds, show that:
 (a) $v = u + at$
 (b) $S = ut + \frac{1}{2}at^2$ (where S is the displacement after t seconds).
8. An object is dropped from a height of 20 metres above the ground. Determine:
 (a) the time it takes to hit the ground.
 (b) the velocity at which it hits the ground. (Take acceleration due to gravity to be 10 m/s^2 , $v = u + at$ and $S = ut + \frac{1}{2}at^2$).
9. A stone is thrown vertically downwards from the top of a cliff at 24 ms^{-1} . Taking the acceleration due to gravity to be 10 ms^{-2} , find an expression for its velocity and position after t seconds.
10. A ball is kicked vertically upwards from a point 0.5 m above the ground at a velocity of 16 ms^{-1} . Assuming that acceleration due to gravity is 10 ms^{-2} , determine:
 (a) an expression for its velocity t seconds later.
 (b) an expression for its height above the ground t seconds later.
 (c) the maximum height reached by the ball.

Mixed Exercise 3

1. Evaluate:

(a) $\int_1^3 (x^2 - 1) dx$ (b) $\int_0^4 (14 - x^2) dx$ (c) $\int_1^6 (x^2 - 12x + 10) dx$

2. The curve $y = ax^2 + bx + c$ passes through the origin and has a minimum point at $(-2, -4)$. Determine the values of a , b and c .

3. (a) Evaluate $\int_0^1 (y_1 - y_2) dx$, given that $y_1 = x^{\frac{1}{2}}$ and $y_2 = x^2$.

(b) The gradient of a curve at any point is $x - 1$. If the curve passes through the point $(0, 1)$, determine the equation of the curve.

4. The displacement S metres of a particle after t seconds is given by $S = 40t^3 - t^2 + 3t + 3$. Find its velocity and acceleration when $t = 2$.

5. Differentiate the following:

(a) $5x^8 + \frac{1}{4}x^5 - x^4$

(b) $-\frac{1}{3}x^6 - \frac{1}{6}x^{12} + 3$

6. Find the gradient of the curve $y = x^5 - 3x^2 + 5x$ at the point $(1, 3)$.

7. Evaluate:

(a) $\int_{-2}^1 x^3 dx$ (b) $\int_{-2}^{-1} x^{-2} dx$ (c) $\int_{-2}^{-1} (1 - x)^3 dx$

8. S , v , a and t represent displacement, velocity, acceleration and time respectively of a particle starting from rest. Obtain expressions for the values indicated in brackets against the following expressions.

(a) $v = \frac{1}{2}t - 1$ (a , S)

(b) $a = 2t + 3$ (S)

(c) $S = 5t^3 + 3t^2 - 2$ (a)

(d) $v = t^3 - 3t^2 + 3t - 1$ (a , S)

9. In each of the following functions, find $\frac{dy}{dx}$:

(a) $y = 2x^2 + 3x + 4$

(b) $y = x^3 + 4x^2 - 3x + 2$

(c) $y = x^4 + 5x^3 - 2x^2 + x - 3$

10. The height of an object thrown vertically upwards is given by the equation $S = 12t^2 - 2t + 1$. Calculate the maximum height reached before it starts falling freely.
11. Each expression below represents distance covered by a moving particle. Obtain the corresponding expression for the acceleration:
 (a) $S = t^3 - 5t^2 + 3t + 5$ (b) $S = 4t^3 - 3t^2 + t - 4$
 (c) $S = \frac{1}{2}t^3 - 3t^2 - 5t$ (d) $S = \frac{1}{4}t^5 - 5t^6 + t^3 + t^2 - 3$
12. A curve passes through point $(1, -5)$. If its gradient function is given by the equation $3x^2 - x + 1$, find its equation.
13. For each of the expressions below, find the stationary points and state whether they are minimum, maximum or inflection, as the case may be:
 (a) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$ (b) $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$
 (c) $y = 6 - 5x + x^2$ (d) $y = 5x - 6 - x^2$ (e) $y = \frac{1}{3}x^3 + 2x^2$
14. Draw the region given by the following inequalities: $x \leq 15$, $y \geq 25$, $6x + y \geq 60$, $3y + 4x \leq 180$. Find the minimum value of the expression $2x + y$ in this region if x and y are integers.
15. Integrate the following with respect to x :
 (a) $y = x^2 + 3x^{-3} + 4x - 4$
 (b) $y = \frac{3}{x^7} + 4x^4 - \frac{5}{x^5} + 6x^6$
 (c) $y = \frac{2}{x^2} - \frac{3}{x^4} + \frac{4}{x^5} - \frac{5}{x^6}$
16. Differentiate the following with respect to x :
 (a) $y = 2x^3 - x^2 + 3x - 10$ (b) $y = (x^2 + 1)(2x - 1)$
 (c) $y = 5x^2 - 2x + 1$ (d) $y = \frac{6x^3 + 14x^2 - 12x}{(3x - 2)}$
17. A particle moves along a straight line OX. Its distance, S metres, after leaving O, t seconds later, is given by $S = \frac{1}{6}t^3 - \frac{1}{2}t^2 + 3$. Calculate the time it attains the maximum velocity. Find also the velocity and acceleration after 2 seconds.
18. Evaluate the following definite integrals:
 (a) $\int_1^2 (x^2 + 2x + 1) dx$ (b) $\int_0^4 (x^3 - 3x^2 + 2x - 1) dx$

$$(c) \int_5^8 \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + 4 \right) dx \quad (d) \int_a^b (4x^2 + 5x) dx$$

19. Integrate the following with respect to x :

$$(a) 1 - 2x - 4x^{\frac{5}{2}} \quad (b) \frac{1}{3}x^2 - \frac{1}{2}x + 4$$

$$(c) \frac{2x^2}{3} - \frac{x}{5} + \frac{1}{4} \quad (d) x^7$$

20. The acceleration of a particle is given as 4 ms^{-2} . Find an equation for its velocity v and the displacement S at any time t , given that when $t = 0$, $v = 5 \text{ ms}^{-1}$ and $S = 0$.

21. Find the area of the region enclosed by the curve $y = 16 - x^2$ and the x -axis.

22. Find the approximate area enclosed by the x -axis and the curve $y = 9 - x^2$ between $x = -4$ and $x = 4$, using:

(a) trapezium rule with 8 trapezia.

(b) mid-ordinate rule with 12 strips.

23. Find an equation of the tangent to curve $y = 2x^3 + x^2 + 3x - 1$ at the point $(1, -5)$, expressing your answer in the form $y = mx + c$.

24. Plot the curve $y = x^2 + 2$ for $0 \leq x \leq 3$. Approximate the area under this curve using the trapezium rule with three trapezia. Use integration to find the exact area and hence find the percentage error in using the trapezium rule.

25. If $x + y \leq 8$, $3x + y \leq 12$, $x + 3y < 12$, find:

(i) the maximum value of $x + y$.

(ii) the minimum value of $2x + y$.

26. Find gradient functions of the curves:

$$(a) y = 2x^2 + 3x + 1 \quad (b) y = 3x^2 - x - 1$$

Find the points on the curves at which the gradient is 11.

27. Draw the curve $y = 3x^2 + 8$ between $x = 1$ and $x = 5$. Estimate the area enclosed by the curve, the lines $x = 1$, $x = 5$ and the x -axis by the use of the mid-ordinate rule with 8 strips.

28. A butcher wishes to buy goats. A she-goat costs sh. 900 while a he-goat costs sh. 1 500. He has space to keep at most 20 goats, and at most sh. 21 000 to spend. He makes a profit of sh. 200 on each she-goat and sh. 280 on each he-goat. How many goats of each sex should he buy to get maximum profit?

29. Find the area enclosed by the curve $y = x^2 - 5x + 4$, the x-axis, $x = 3$ and $y = -2$.
30. Find an equation of the tangent to the curve $y = 3x^3 + \frac{1}{2}x^2 - x + 4$ at the point $(2, 28)$. Deduce the equation of the normal to the tangent at the same point.
31. Use trapezium rule to find the area bounded by the curve $y = \frac{1}{1+x}$, $x = 0$ and $x = 5$. Use strips of unit length.
32. Draw the graph of $y = \frac{1}{2}x^2 + 3$ from $x = -2$ to $x = 4$. Find the area enclosed by the curve, the lines $x = -2$, $x = 4$ and the x-axis.
33. Find the area of the largest piece of a rectangular ground that can be enclosed by 2 km of fencing, if part of an existing wall is used on one side.
34. Calculate the area between the line $y = 2 - 3x$ and the curve $y = 3x^2 + 2x$.
35. (a) Expand $(2t - 1)(t - 1)(t - 2)$.
(b) A particle moves in a straight line such that its distance from a fixed point A after time t seconds is given by $S = 2t^3 - 7t^2 + 7t - 2$. Find:
(i) the times at which the particle is at A.
(ii) the distance from A, the velocity and the acceleration of the particle at $t = 0$ and $t = 2$.
(iii) the times at which the particle is instantaneously at rest, leaving your answers in surd form.
36. Find the area under the graph $y = 4x^3 + 1$ between $x = 0$ and $x = 2$:
(a) (i) by using the trapezium rule and taking intervals of 0.5 of a unit.
(ii) by integration.
(b) Express the error in (a) as a percentage of the area obtained in (b).
37. The velocity $v \text{ ms}^{-1}$ of a particle after t seconds is given by $v = 5t^2 - 2t$. Find:
(a) the times at which the particle is instantaneously at rest and the acceleration during each of these times.
(b) the distance covered during the third second.
38. Using trapezoidal rule and strips of unit width, estimate the area under the curve $y = \frac{3}{x}$ between the lines $x = 1$ and $x = 7$.
39. Plot the curve given by the points in the table below. Use the trapezium rule to find an approximate area under the curve and the lines between $x = 0$ and $x = 6$ (use six trapezia).

x	0	1	2	3	4	5	6
y	0.5	0.33	0.25	0.2	0.16	0.14	0.13

40. The motion of a point Q moving along a straight line may be described by the equation $S = -t^3 + 10t^2 + 8t$, where S is displacement in metres and t is time in seconds. Calculate:
- the maximum velocity of the motion.
 - the acceleration of the motion after 3 seconds.
 - the time at which the velocity is zero.

Revision Exercises

Revision Exercise 1

- Find the size of each angle of a triangle PQR in which PQ = 9 cm, QR = 12 cm and RP = 6 cm.
- R** represents a half-turn about the point (2, 3). **H** represents a half-turn about the point (-2, -1). Find a single transformation equivalent to **HR** and state its inverse.
- For each of the following wave functions, state the period and amplitude:
 - $y = 2 \sin x$
 - $y = \sin 2x$
 - $y = \frac{1}{3} \sin x$
 - $y = \frac{1}{3} \sin 3x$
 - $y = \sin \frac{1}{2}x$
 - $y = \sin \frac{1}{3}x$
- Solve the following pairs of simultaneous equations graphically:
 - $y + \frac{1}{3}x = 2$
 - $3y + 4x = 8$
 - $\frac{1}{2}y + 2x = 7$
 - $\frac{5}{2}y - x = 5$

$\frac{1}{3}y + x = 6$ $4y + 3x = 6$ $y - \frac{1}{2}x = 5$ $5y + 4x = 10$
- A card is drawn at random from a normal pack of playing cards. What is the probability of drawing:
 - a black card?
 - a red queen?
 - an ace?
 - the queen of spades?
- A pencil PQ is 20 cm long. The end P is placed on a horizontal flat surface such that PQ makes an angle of 47° with the surface. Calculate how far above the surface the point Q is.
- Solve the following inequalities and show your solutions on a number line:
 - $3x - 5 > 5$
 - $2x + \frac{1}{2} < 3$
 - $-4x - 6 > 7$
- Evaluate each of the following:
 - $\frac{7\frac{1}{8} + 2\frac{2}{3}}{\frac{1}{4} \times 9\frac{1}{2}}$
 - $\frac{9}{4} \div (\frac{10}{3} + \frac{11}{4})$
- A quadrilateral has vertices at A(1, 2), B(1, 3), C(2, 3) and D(3, 1). Find the vertices of the image of the quadrilateral after a reflection in the following lines:
 - $y = x + 1$
 - $y + x = 4$
 - $y = 5$

10. Given that $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find two values of λ for which the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is zero, leaving your answers in surd form.
11. ABC is an isosceles triangle with $AB = AC$ and its perimeter is 64 cm. The altitude from A to BC is 24 cm. Find the lengths of AC and BC.
12. Two variables x and y are connected by an equation of the form $y = Ax^n$, where A and n are constants. If $y = 80$ when $x = 2$ and $y = 52$ when $x = 3$, find the values of A and n to 2 s.f.
13. Find the distance in nautical miles between point A(4°S , 50°E) and B(4°S , 80°E).
14. Draw a line $AB = 4.5$ cm. Construct the locus of a point P such that $\angle APB = 40^\circ$.
15. In an experiment, a student obtained the following corresponding values of two variables F and E :

F	1	2	3	4	5
E	2	10	25	46	100

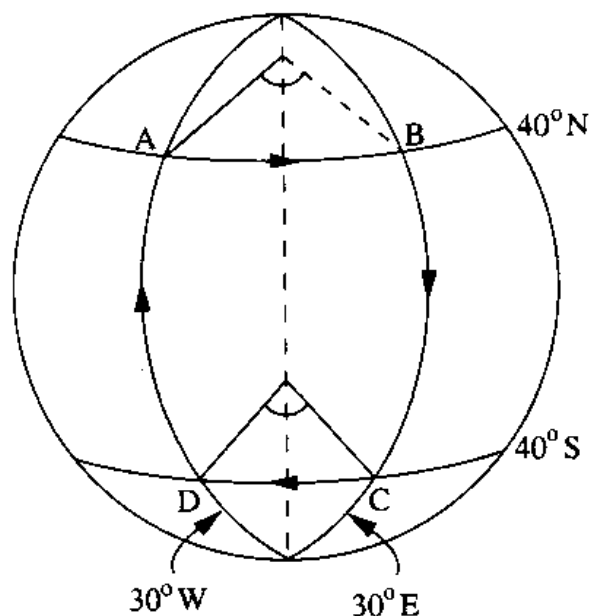
The relation between E and F is suspected to be of the form $E = AF^n$, where A and n are constants. Verify graphically that the results confirm the suspicion. Use your graph to estimate the values of A and n .

Revision Exercise 2

1. The cost of six shirts and two pairs of trousers is sh. 1 320 while that of three shirts and four pairs of trousers is sh. 1 290. How much will George pay for two shirts and two pairs of trousers?
2. Given that the ratio of $x : y : z$ is $5 : 2 : 1$, find the ratios of:
 (a) $2x : y$ (b) $2x : 3z$ (c) $5y : 9z$
3. Vector \mathbf{a} passes through the points $(5, 10)$ and $(3, 5)$ and vector \mathbf{b} passes through $(x, 6)$ and $(-5, -4)$. If \mathbf{a} and \mathbf{b} are parallel, find the value of x .
4. Triangle PQR has vertices at $P(2, 2)$, $Q(6, 2)$ and $R(4, 6)$. Triangle $P'Q'R'$ is the image of triangle PQR under the transformation given by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Find the co-ordinates of the vertices of triangle $P'Q'R'$. Describe the transformation that would map triangle PQR onto triangle $P'Q'R'$.
5. Use tables to find the squares of the following numbers:
 (a) 7.38 (b) 0.0045 (c) 2.331
 (d) 8.281 (e) 0.00724 (f) 0.002416

6. The displacement S in metres of a particle is given by $S = t^3 - \frac{t^2}{4} + 15t$, where t is time in seconds. Find:
- the velocity and acceleration after 3 seconds.
 - the time when acceleration is zero.
7. Solve the equation $3^{(2x+1)} - 4 \times 3^{x+1} + 9 = 0$
8. Sketch the curve $y = \frac{1}{2} \sin(x + 60)^\circ$ for the interval $0^\circ \leq x \leq 360^\circ$ and state:
- the amplitude,
 - the period of the curve.
9. A square garden is enclosed by a path of width 1 metre. If the area of the path is 64 m^2 , determine the perimeter of the outer boundary of the path.
10. A fair coin is tossed four times. Find the probability of getting:
- at least 3 heads?
 - exactly 2 tails?
 - exactly 3 tails?
11. What range of values of x satisfy each of the following pairs of inequalities?
- $3x - 1 < 4$
 - $5 - \frac{x}{2} < 3$
 - $\frac{x+4}{4} > 4$
 - $2x - \frac{1}{5} < 1$
- $2x + 1 < 1$
 - $\frac{1}{2}x + 2 > x - 1$
 - $3 < 10 - \frac{x}{2}$
 - $-\frac{1}{5} - 2x > 1$
12. The line $y = x$ is reflected in the line $x = 2$.
- What is the equation of its image?
 - Describe another single transformation which is equivalent to the reflection.
13. In the expansion of each of the following in ascending powers of x , find the term indicated:
- $(x - y)^7$, 5th term.
 - $(2x - 3y)^8$, 4th term.
 - $(2x - \frac{1}{3}y)^9$, 5th term.
 - $(2 - \frac{1}{3}x)^9$, 4th term.
 - $(1 - \frac{1}{5}x)^6$, 3rd term.
14. Two points on the line $x + 2y = 6$ are each $\sqrt{17}$ units from the origin. Find the distance to 1 d.p. between the points.

15. The figure below shows a globe of radius 0.42 m. A, B, C and D are points on its surface with A(40°N, 30°W), B(40°N, 30°E), C(40°S, 30°E) and D(40°S, 30°W):



A fly walks from A to B along latitude 40°N, B to C along longitude 30°E, C to D along latitude 40°S and D to A along longitude 30°W. Calculate to 4 s.f. the total distance it covers.

Revision Exercise 3

1. Draw the rhombus KLMN in which KL = 8 cm and the diagonal KM = 10 cm. Measure LN and angle LMN.
2. Evaluate, giving your answer to 4 s.f.:

$$\frac{3.14^2 + 6.25^{\frac{1}{2}}}{356 - 9.2^2}$$

3. I have two red beads and one green bead in my pocket. If I pick out two beads, find probability that I will obtain:
 - (a) two red beads.
 - (b) one bead of each colour.
4. Solve the simultaneous equations:

$$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad 3\mathbf{a} + 5\mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$
5. Draw the locus of points that satisfy the following inequalities:

$$x \leq 5, y \geq 0, x \geq 0, y \leq 2x - 3$$

6. A three-digit number is such that the ones digit is the sum of the tens and hundreds digits. When the digits are reversed, the value of the number is increased by 495. If the one's digit of the number is 6, find the number.
7. A circle passes through the points A, B and C, with AB as a diameter of the circle. If the diameter is 20 cm longer than the chord BC and the chord AC is $20\sqrt{3}$ cm long, calculate $\angle BAC$.
8. Expand the expression $(1+x)^9$ up to the fourth term and use your expansion to find the value of $(1.02)^9 + (0.99)^9$, correct to 2 d.p.
9. The number of deaths caused by motor accidents in a certain country for a duration of one year were as shown below:

Age (years)	0-4	5-9	10-14	15-19	20-24	25-29	30-34	35-39
Number of deaths	6	9	22	18	54	64	50	80

Age (years)	40-44	45-49	50-54	55-59	60-64	65-70	71 and over
Number of deaths	75	70	46	32	11	3	0

- (a) Draw a cumulative frequency curve and use it to estimate:
- the median.
 - the interquartile range.
- (b) Find the probability of dying due to motor accident over the age of 45 during that one year.
10. Find the area enclosed by the curve $y = x^2 + 2$ and the line $y = x + 8$.
11. The edges of a rectangular block of wood are 6 cm, 10 cm and 16 cm long. Find the length of its longest diagonal and the angle it makes with the largest face.
12. (a) Draw the graph of $y = 2x^3 - 5x - 3$ for $-2 \leq x \leq 2$.
- (b) By adding suitable lines to your graph, solve the following equation to 1 d.p.:
- $2x^3 - 5x - 3 = 0$
 - $2x^3 - 5x - 4 = 0$
 - $2x^3 - 5x + 3 = 0$
 - $2x^3 - 5x = 0$

13. Complete the following table for the function $y = 2 \cos x + \sin 2x$ and use it to draw the graph of the function for $0^\circ \leq x \leq 360^\circ$:

x°	0	30	60	90	120	150	180	210	240	270	300	330	360
$\cos x$	1	0.87	0.5	0	-0.5	-	-	-0.87	-	-	0.5	-	1
$2\cos x$	2	1.74	1	0	-	-	-	-	-	-	1	-	2
$2x$	0	60	120	180	-	-	-	-	480	540	600	660	720
$\sin 2x$	0	0.87	-	-	-	-	-	-	-	-	0.87	-	0
$y = 2\cos x + \sin 2x$	2	-	-	-	-	-	-	-	-	-	-	-	2

14. Plot the graph of $y = 3x^2 + 2x - 1$ for values of x between -5 and 5 . From your graph, find the turning point. What is the nature of this turning point? Use your graph to obtain solutions to the following equations:
- $3x^2 + 2x - 1 = 0$
 - $3x^2 + 2x - 1 = x$
 - $3x^2 + 2x - 41 = 0$

15. Solve the following equations:

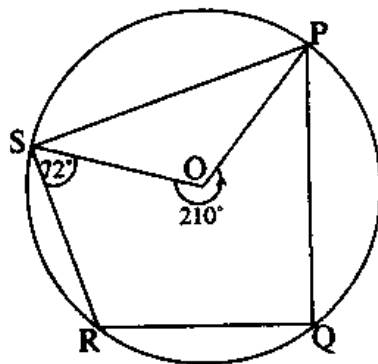
$$x - y + 4z = 1$$

$$2x + 3y + 2z = 0$$

$$\frac{1}{2}x + 5z = 2$$

Revision Exercise 4

- If $\begin{pmatrix} 2 & -3 & 0 \\ 3 & -4 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$, find a , b and c .
- In the figure below, O is the centre of the circle $PQRS$. If angle $OSR = 72^\circ$ and the reflex angle $SOP = 210^\circ$, find angle PQR :

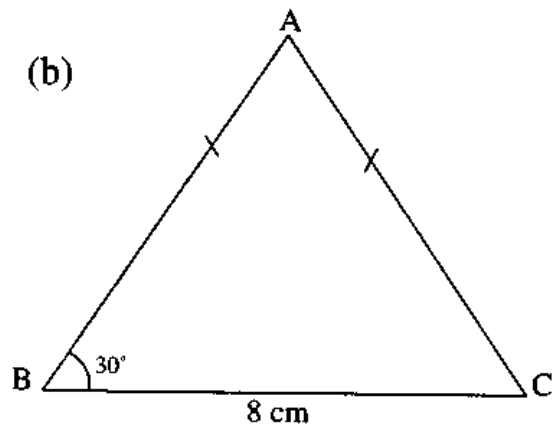
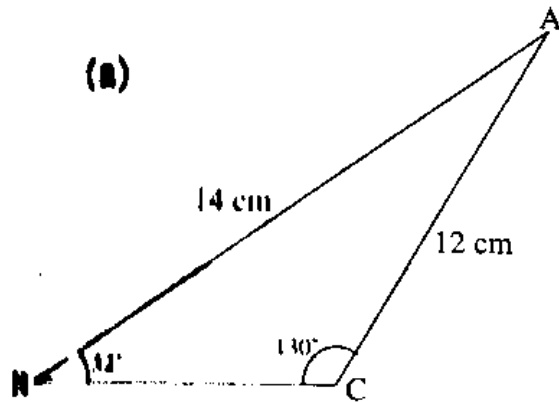


- Points $(3, 6)$ and $(-1, 4)$ are reflected in the line $y = x$. Find:
 - their images.
 - the matrix of transformation in (a).

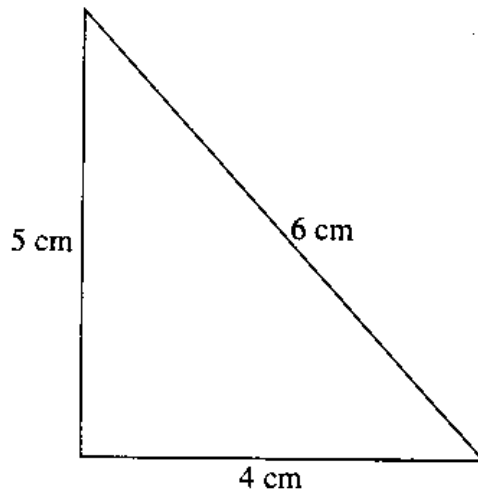
4. In a triangle ABC , $AB = c$, $BC = a$ and points M and N are on BC and AC respectively. If $BM : BC = 2 : 3$ and $AN : NC = 1 : 2$, express AM and BN in terms of a and c . If AM and BN meet at Y , find the ratio $AY : YM$.

5. Simplify $3 \log_{10} x + \frac{1}{3} \log_{10} y - 2 \log_{10} z$.

6. Find the area of each of the following triangles:

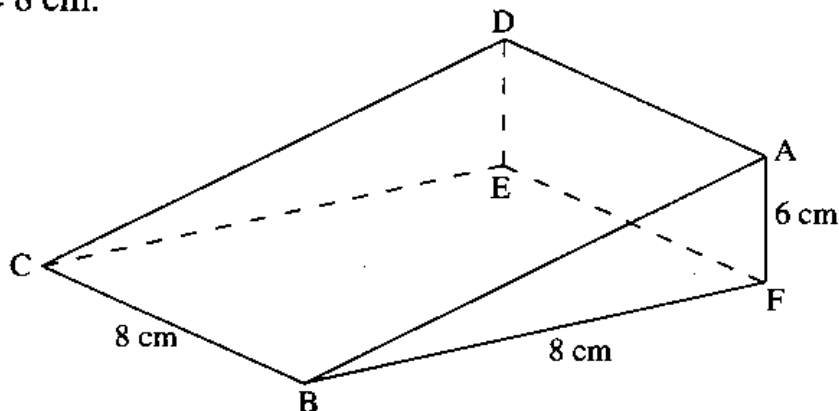


(c)



7. A number is chosen at random from the numbers 2, 3, 4, ..., 30. Find the probability that it is either a multiple of 3 or a factor of 20.
8. Find the area bounded by the curve $y = x^2$ and the lines $y = 4$ and $y = 9$.
9. Find the height and the radius of a closed cylinder of volume $500\pi \text{ cm}^3$ which has the least surface area.
10. Use binomial expansion to find the first four terms of each of the following:
 (a) $(a + b)^n$ (b) $(a - b)^7$ (c) $(a + 2b)^6$
11. Draw a semicircle of radius 3 cm. Use the trapezium rule to estimate its area (use 12 intervals, each of width 0.5 cm).
12. The projection of a line XY on a plane is $X'Y'$. If XY is 20 cm long and $X'Y'$ is 12 cm, find the angle between the line and the plane.

13. The figure below shows a prism ABCDEF with $BC = 8$ cm, $AC = 6$ cm and $BF = 8$ cm:



Calculate:

- the length of the diagonal BD.
 - the angle BD makes with plane BCEF.
14. (a) If A and B are supplementary angles, express $\cos A$ in terms of $\cos B$.
- (b) A cyclic quadrilateral PQRS is such that $PQ = 2$ cm, $QR = 3$ cm, $RS = 5$ cm and $PS = 4$ cm. Write expressions for QS^2 and hence calculate:
- angle QPS
 - QS
15. (a) Draw the graph of $y = x^3$ for the range $-4 \leq x \leq 4$.
- (b) Use your graph to estimate, to one decimal point:
- $\sqrt[3]{40}$
 - $\sqrt[3]{60}$
 - 2.5^3
 - -3.5^3
 - $\sqrt[3]{-45}$

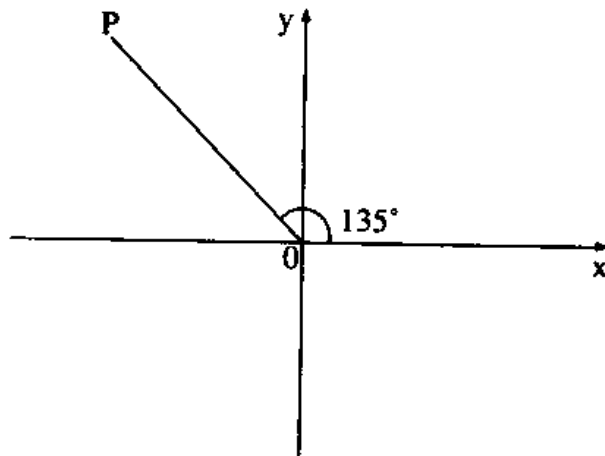
Revision Exercise 5

1. Solve the equation:

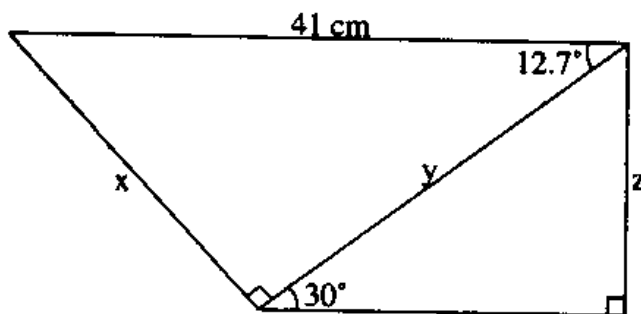
$$\frac{2x+1}{4} - 2 = 5 - \frac{x+2}{4}$$

2. Points P(1, 3), Q(3, 3), R(3, 2) and S(1, 2) are the co-ordinates of the vertices of a rectangle.
- Find the image of the rectangle under reflection in the line $x = 1$ followed by that in $y = 0$.
 - Describe a single transformation which has the same effect.
3. Find the probability of getting at least one diamond if two cards are drawn at random from an ordinary pack of cards.
4. In each of the following, draw the regions that satisfy the inequalities:
- | | | |
|-----------------|-----------------|--------------------|
| (a) $y > 2$ | (b) $y > 0$ | (c) $y \geq x - 3$ |
| $x < 3$ | $3y \leq x + 3$ | $y \leq 2$ |
| $3x + y \geq 5$ | $y \leq 3x - 3$ | $y - 1 \leq x$ |
| | $4y + x \leq 3$ | |

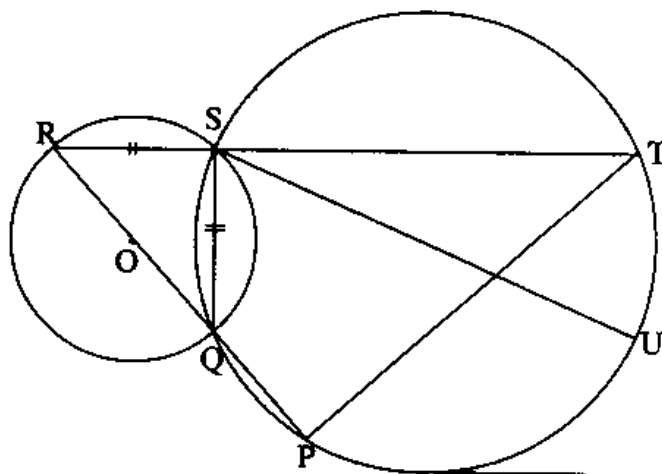
5. A stone is projected vertically upwards and its height S metres after t seconds is given by $S = 16t - 4t^2$. Find:
- the maximum height attained by the stone.
 - the velocity when $t = 1$.
 - the acceleration when $t = 1$.
6. The product of three consecutive positive even numbers is 960 more than the product of the largest and square of the smallest. Find the numbers.
7. Solve for x in the equation;
 $\log_{10} 5 - 2 + \log_{10}(2x + 10) = \log_{10}(x - 4)$.
8. A line parallel to line OP shown in figure below cuts the y -axis at $(0, -1)$. Find the equation of the line:



9. If points $A(1, 3)$, $B(5, -2)$ and $C(-11, y)$ are collinear, calculate the value of y .
10. The volume of a sphere is given by the formula $V = \frac{4\pi r^3}{3}$. Use the formula to calculate the radius of a sphere whose volume is $1\,712\text{ cm}^3$ to three decimal places. (Take $\pi = 3.142$).
11. A man deposits sh. 50 000 in an investment account which pays 12% interest per annum compounded semi-annually. Find the amount in the account after three years.
12. Find the values of x , y and z in the figure below:



13. In the figure below, O is the centre of the circle RQS which is isosceles, with $SR = SQ$. If RQP and RST are straight lines, calculate angle PUS:



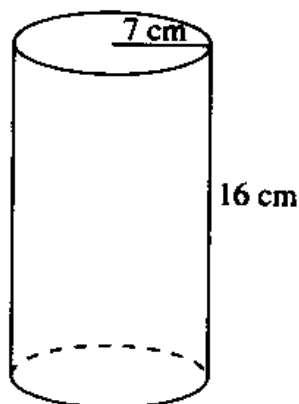
14. Solve the following equations for $0 \leq \theta \leq 360$:

(a) $\cos \frac{2}{3}\theta = -0.7660$

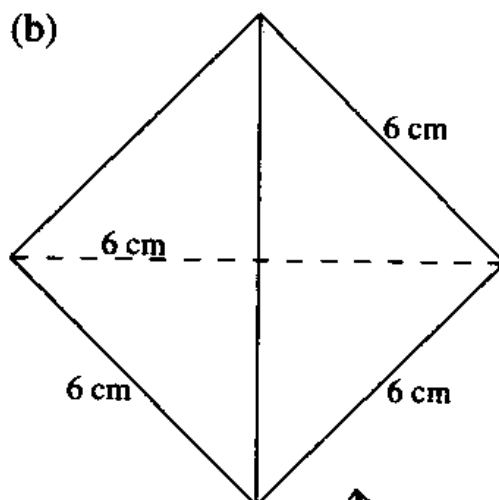
(b) $\sin \frac{2}{3}\theta = 0.6430$

15. Draw the nets of the following solids and find their surface areas:

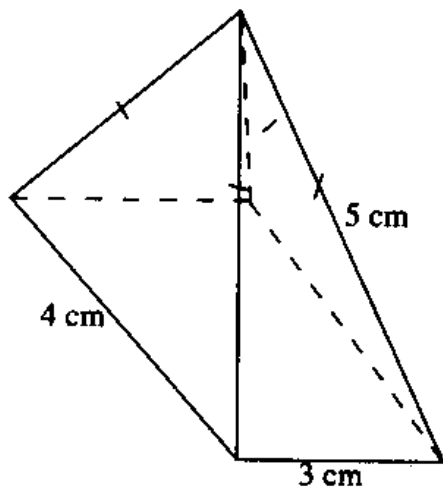
(a)



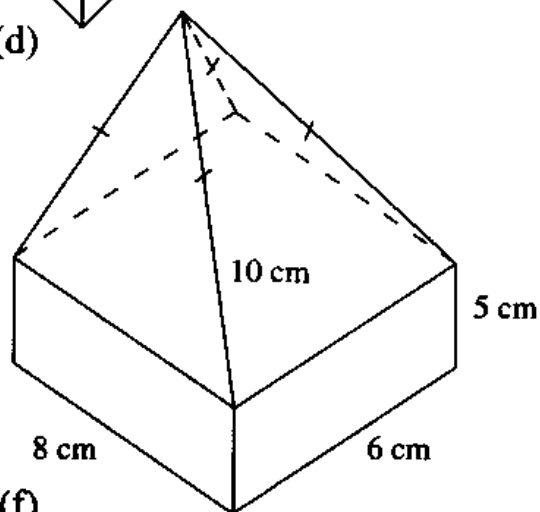
(b)



(c)

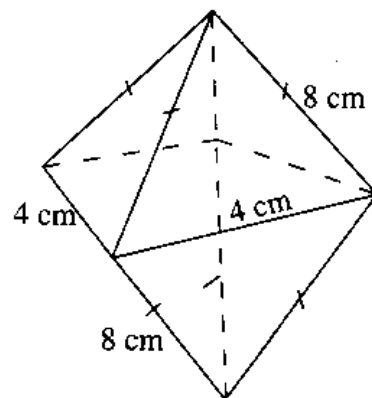
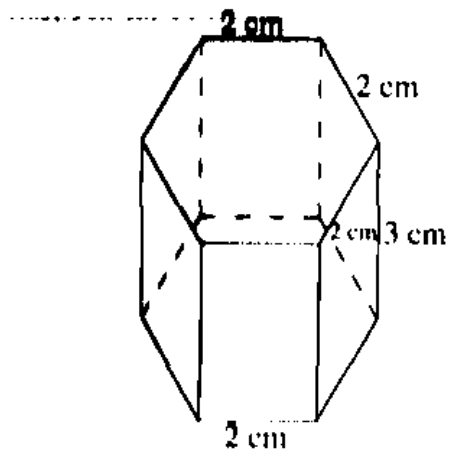


(d)



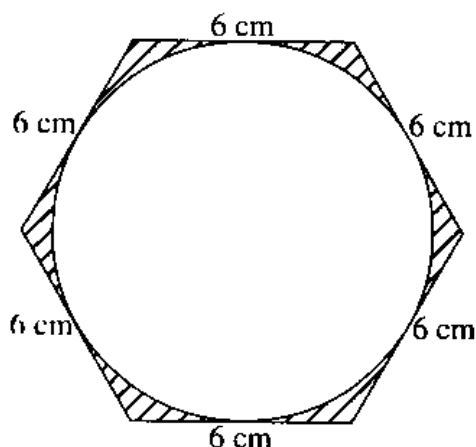
(e)

(f)



Revision Exercise 6

- Round off each of the following to 2, 3 and 4 significant figures:
 (a) 0.06376 (b) 3.00204 (c) 0.256781 (d) 0.000081
- The volume of a cylinder is 196 cm^3 . If the cylinder has a base radius of 1.4 cm, find its height. (Take $\pi = \frac{22}{7}$)
- Find the equation of a tangent and the normal to curve $y = x^3 - 2x^2 + 3x - 1$ at $x = 2$.
- Use the matrix method to solve the following simultaneous equations:
 $3x + 2y = 23$
 $2x - y = 6$
- A line parallel to the line $y = x$ has its y intercept at -5 . If it meets the x and y axes at A and B respectively, determine the position vector of the midpoint of AB .
- The figure below shows a circle drawn inside a regular hexagon whose sides measure 6 cm each:



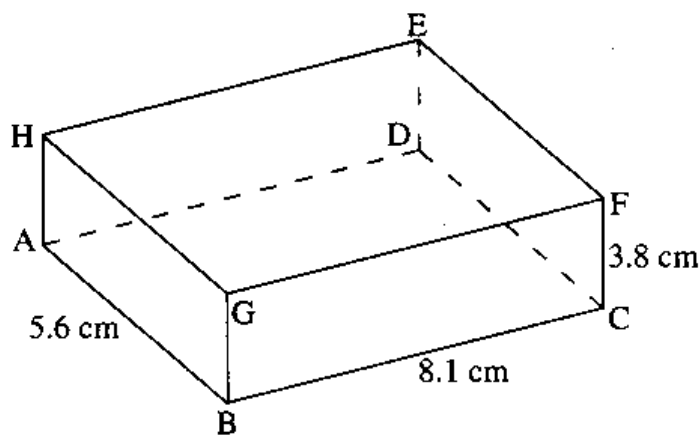
- Find the area of the shaded region.

- (b) If a particle is to fall onto the hexagon, what is the probability that it will fall on the shaded region?

7. For the series;

$43 + 74 + 105 + \dots + 415$, find:

- (a) the number of terms.
 (b) the 10th term.
 (c) the sum.
8. The figure below is a cuboid 8.1 cm long, 5.6 cm wide and 3.8 cm high. Calculate the angle between:
- (a) planes ABFE and CDEF.
 (b) planes ABFE and ABCD.
 (c) planes ACFH and CDEF.



9. Points $A'(-3, 0)$, $B'(3, 0)$ and $C'(-3, 5)$ are vertices of a triangle which is an image of triangle ABC with vertices at $A(5, 8)$, $B(5, 2)$ and $C(0, 8)$ respectively under reflection. Determine:
- (a) the equation of the line of symmetry.
 (b) the equation of the line perpendicular to the line of symmetry and passing through point $(2, 3)$.
10. A certain alloy is prepared using 6 kg of copper, 2 kg of zinc and 0.5 kg of tin.
- (a) What percentage of the compound is tin?
 (b) Find the ratio of copper to tin.
11. Given that $\log_{10} 3 = x$ and $\log_{10} 7 = y$, express $\frac{\log_{10} 63}{\log_{10} 147}$ in terms of x and y .
12. A rectangular plot measures $(x + 3y)$ m by $(2x + y)$ m. If its area is 77 m^2 and its perimeter 36 m, calculate its dimensions.

13. ... **Shade the region** represented by the following inequalities.

(a) $2x + 3y \geq 15$

$x + y \leq 10$

$x \geq 0$

$y \geq 0$

(b) $4y + 3x \leq 250$

$x + y \geq 60$

$x \geq 15$

$y \geq 30$

14. **Find a 2×2 matrix** of each of the following transformations:

(a) rotation through -90° about the origin.

(b) shearing, $y = x$ invariant and $P(1, 4) \rightarrow P'(3, 6)$.

(c) a stretch perpendicular to the x -axis, $R(1, 1) \rightarrow R'(1, 4)$.

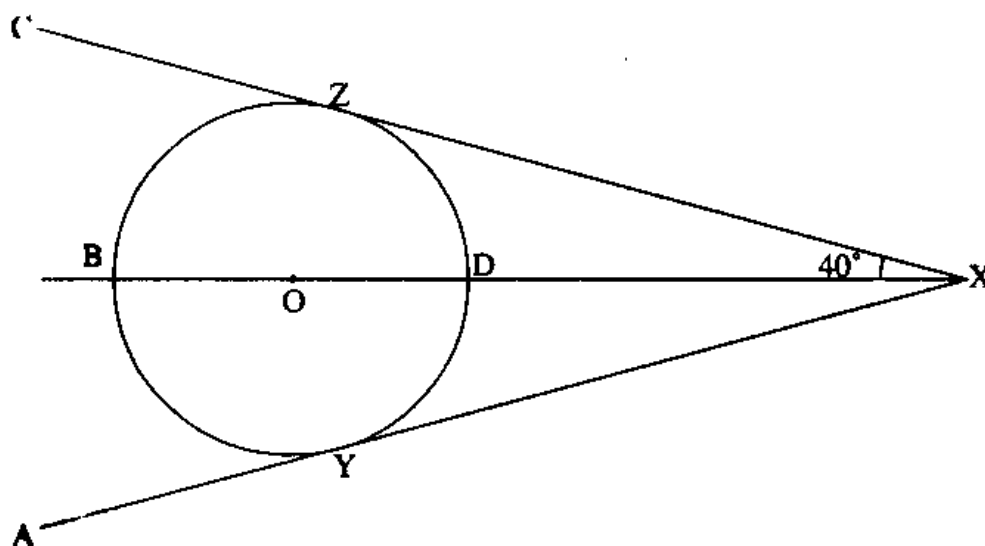
(d) reflection in the line $y = 3x$.

15. A house appreciates at the rate of 20% p.a. If it was valued at sh. 800 000 in January 1990, calculate its value in January 1995.

Revision Exercise 7

1. The co-ordinates of points P and Q are (4, 5) and (10, 2) respectively. Find the co-ordinates of a point R such that $PR : PQ = 1 : 2$.

2. In the figure below, XZC and XYA are tangents to the circle, centre O. The diameter BD is extended to the point X:



If angle $CXB = 40^\circ$, find:

(a) angle YBZ.

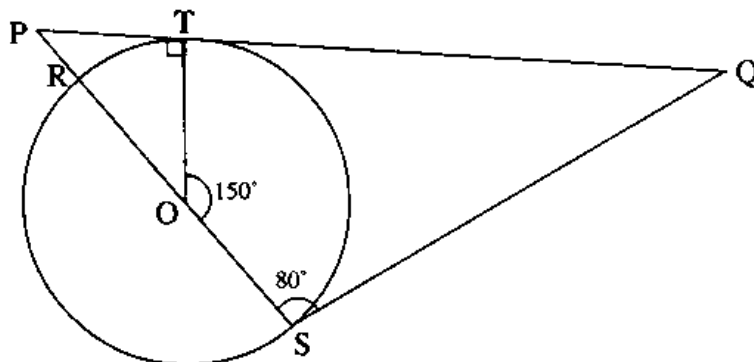
(b) angle YDZ.

(c) angle CZB.

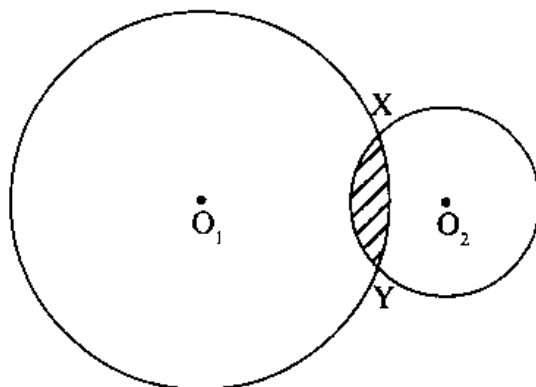
3. A fair die is tossed twice. What is the probability of obtaining a number greater than 4 on both tosses?

4. A quarter of the square of a number is decreased by 9. If the result is -5 , what is the number?
5. Use the trapezium rule to estimate the area enclosed by the curve $y = x^3 + 3x^2$ and the line $y = 0$, using seven ordinates.
6. The base radius of a cylinder is increased by 10% while its height is decreased by 10%. Calculate the percentage increase in volume.
7. Draw a triangle XYZ such that $XY = 8$ cm, $YZ = 6$ cm and angle $XYZ = 40^\circ$. Construct the locus of points equidistant from X and Y to meet the locus of points equidistant from Y and Z at R . Measure RZ .
8. Sketch the graph of $y = x^2$ between $x = 0$ and $x = 6$. Find the area enclosed by the curve and $y = 25$.
9. A man has to repay an overdraft of Ksh. 10 000 by five equal monthly instalments. He also pays an interest of 2 % per month on the outstanding balance. Calculate the total amount of money paid as interest.
10. A line AB of length 13 cm is inclined at 50° to the horizontal. If a line AQ is inclined at 82° and has the same projection in the horizontal as AB , find the length of AQ .
11. Maximise the following functions subject to the given conditions:

(a) $c = 6y + 4x$	(b) $c = 2p + 5q$
$x + y \leq 5$	$p + q \geq 10$
$x \geq 0$	$2 \leq p \leq 10$
$y \geq 0$	$5 \leq q \leq 20$
12. In the figure below, PQ is the tangent to the circle with centre O , SR is the diameter of the circle, angle $PSQ = 80^\circ$ and angle $TOS = 150^\circ$. Find angle SPQ , angle SQP and angle RTO .

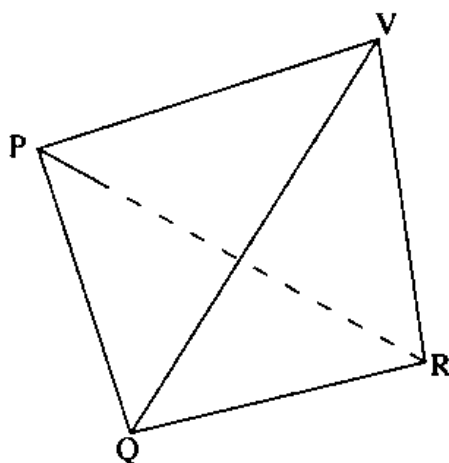


13. The figure below shows two circles intersecting at X and Y . The radius of the larger circle is 6 cm and that of the smaller is 4 cm.



If angle XO_2Y is 72° , calculate:

- (a) the area of the shaded region.
 - (b) the total length of the arcs enclosing the shaded region.
14. The figure below shows a pyramid whose base PQR is an equilateral triangle:



If the vertical height of the pyramid is 10 cm and $PR = 8$ cm, calculate:

- (a) the length VP.
 - (b) the angle which line VR makes with the base.
 - (c) the angle between the plane VPQ and the base.
15. A man standing at the foot of a cliff throws a stone vertically upwards, with a speed of 20 ms^{-1} . Given that the distance h metres covered by the stone after time t seconds is given by the formula $h = 4t(9 - 2t) + 1.5$, complete the following table;

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
h	1.5	—	29.5	37.5	—	—	—	—	17.5	—	-18.5

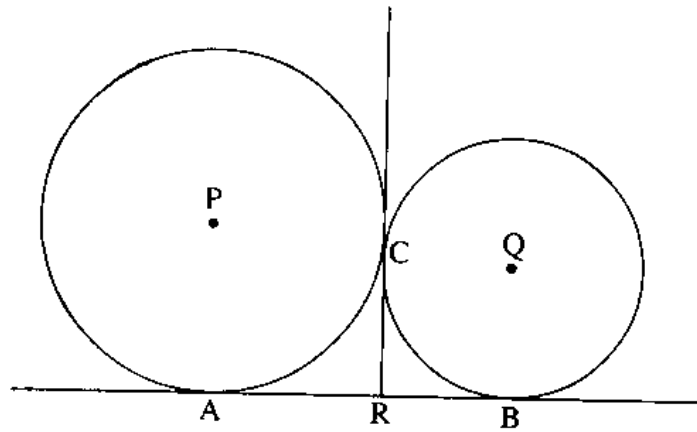
Draw the graph of the function $h = 4t(9 - 2t) + 1.5$ for values of t from 0

to 5 using a scale of 2 cm as the unit for t and 1 mm as the unit for h .
Use your graph to answer the following questions:

- (a) How high above the cliff is the stone after:
 - (i) 0.4 seconds?
 - (ii) 1.2 seconds?
 - (iii) 3.7 seconds?
- (b) After how many seconds is the height of the stone 20 m above the cliff?
- (c) What is the greatest height that the stone reaches and after how many seconds is this attained?
- (d) At what height from the top of the cliff is the stone initially thrown?
- (e) After how long does the stone return to the level at which it was thrown?
- (f) For how many seconds is the stone more than 15 metres above the top of the cliff?

Revision Exercise 8

1. A rotation about a point S is described by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The images of points $P(2, 3)$ and $Q(1, 1)$ when rotated about S are $P'(-3, 2)$ and $Q'(-1, 1)$ respectively. Find:
 - (a) the co-ordinates of S .
 - (b) the angle of rotation.
 - (c) the values of a , b , c , and d .
2. The positions of points A and B are $(25^\circ\text{N}, 45^\circ\text{E})$ and $(25^\circ\text{N}, 63^\circ\text{E})$ respectively. Calculate their distance apart along the latitude.
(Take $R = 6\,370$ km)
3. Juma has money in two denominations, twenty-shilling coins and one hundred-shilling notes. He has four times as many twenty-shilling coins as one hundred-shilling notes. If altogether he has sh. 2 160, how many one hundred-shilling notes does he have?
4. Three craftsmen take $10\frac{1}{2}$ hours to wire a building. How many craftsmen working at the same rate will complete the work in $5\frac{1}{4}$ hours?
5. In the figure below, AB and RC are common tangents to circles with centres at points P and Q . Show that R is a circumcentre of triangle ABC and find the value of angle ACB .



6. Evaluate without using tables:
- $$\frac{0.38 \times 0.23 \times 2.7}{0.114 \times 0.0575}$$
7. Find the sums of the following series up to the terms indicated:
- (a) $1 + \frac{1}{2} + \frac{1}{4} + \dots$, 6 terms
- (b) $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$, 10 terms
- (c) $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \dots$, 8 terms
8. A metallic cuboid measuring 11 cm by 9 cm by 7.2 cm has a right cylindrical hole of diameter 5 cm drilled through the faces measuring 9 cm \times 7.2 cm. Calculate:
- (a) the surface area.
- (b) the volume of the remaining solid.
9. Evaluate without using tables:
- (a) $\frac{1 + \log 4 + \log 9}{1 + \log 36}$
- (b) $\frac{\log 32 + \log 128 - \log 729}{\log 32 + \log 2 - \log 27}$
10. Differentiate with respect to x:
- (a) $y = x^4 - 3x^2 + 5$
- (b) $y = x^2(x + 1)^2$
- (c) $y = \frac{x^3 + 7x^2 + x}{x}$
- (d) $y = (x + 1)^3 - (x - 1)^2$

$$(e) y = \frac{x^4 - 4}{x^2 + 2}$$

11. Integrate with respect to x :

$$(a) -4x^3 + 3x^2$$

$$(b) x^2 - x^{-4} + c$$

$$(c) \frac{x^2(x^2 - 1)}{x + 1}$$

$$(d) \frac{(x^2 + 1)(x + 1)}{x^5}$$

12. Solve the equations:

$$(a) 9^{(x - \frac{1}{2})} = 27^{(\frac{3}{4} - x)}$$

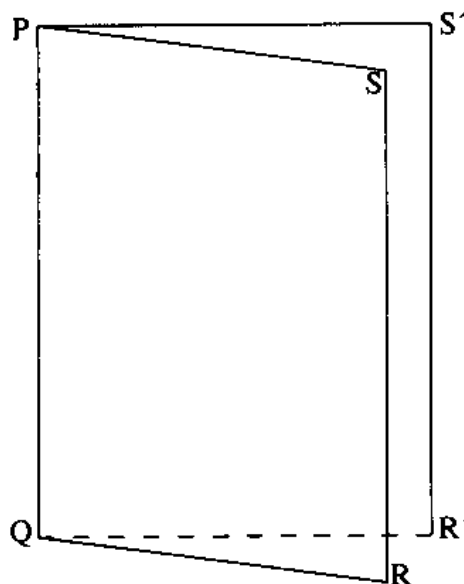
$$(b) \frac{3^x}{9} = \frac{1}{3^{1-2x}}$$

13. The figure below shows a rectangular door with $PQ = 1.9$ m and $QR = 0.9$ m opened through 30° about the vertical line of hinges PQ to the position $PQR'S'$. Calculate:

(a) SQ

(b) SS'

(c) $\angle SQR'$



14. $OABC$ is a parallelogram in which $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$. Y divides \mathbf{OC} in the ratio $3 : 4$ and \mathbf{AY} meets \mathbf{OB} at X . Find \mathbf{OX} in terms of \mathbf{a} and \mathbf{c} .

15. Use the graph of $y = 2x^2$ to solve the following equations:

$$(a) 2x^2 + 4x - 3 = 0$$

$$(b) 2x^2 + 4x - 2 = 0$$

$$(c) 2x^2 + 4x = 3x + 2$$

$$(d) 2x^2 + 7 - 2x = 10 - 9x$$

$$(e) x^2 - 3x + 1 = 0$$

Revision Exercise 9

1. A bus left town A for town B at 8.30 a.m. The journey took 6 hours and 5 minutes. At what time did the bus reach its destination?
2. A hollow cylindrical alloy of length 4 cm weighs 352 g. If its internal and external radii are 3 cm and 4 cm respectively, calculate the density of the metal.
3. The table below shows exam marks obtained by 40 pupils:

8	33	45	45	19
11	34	43	29	28
26	16	22	38	39
37	17	25	34	41
22	26	33	27	40
41	21	32	17	22
18	25	27	35	40
23	36	30	44	42

- (a) Make a frequency distribution table using a class interval of 10, with marks starting from 1 - 10.
 - (b) Draw a histogram.
 - (c) Find the mean mark.
4. Solve for x and y in each of the following equations:
 - (a) $\begin{pmatrix} x \\ y + 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$
 - (b) $\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
 - (c) $\begin{pmatrix} 1 & 2 \\ y & 4 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$
 - (d) $\begin{pmatrix} x & 2y \\ 2x & 3y \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$
 5. Describe the locus of a point P in each of the following:
 - (a) A fixed point O and P is such that $OP = 5$ cm.
 - (b) A fixed line XY and the perpendicular distance of P from XY 4 cm.
 6. Three people A, B and C are to share sh. 2 200 among themselves in the ratio $a : b : c$ respectively. If $a = \frac{1}{2}b$ and $c = \frac{1}{3}b$, find how much each person will get.
 7. Use mid-ordinate rule to estimate the area enclosed by the curve $y = \frac{3}{4}x^2 + 2$ between $x = 0$, $x = 4$ and the x -axis.

8. Find and classify the turning points in each of the following functions:

(a) $y = \frac{1}{3}x^3 - x + 7$

(b) $y = \frac{1}{3}x^3 + 2x^2 + 4x + 2$

9. Evaluate the following:

(a) $\int_0^1 (x^2 + 1) dx$

(b) $\int_{-1}^1 (x^3 + 2x^2 - 1) dx$

10. Show that $2 \log a + \log \left(1 - \frac{2b}{a} + \frac{b^2}{a^2} \right) = \log (a - b)^2$.

11. A man makes equal strides of 0.8 m each all round a rectangular plot.

(a) Write an expression for the total number of strides he will make if the length of the plot is x m and its breadth y m.

(b) (i) Find an expression for the number of strides he would make if the plot is 4 m longer than it is wide.

(ii) If the man makes a total of 98 strides, find the perimeter and the area of the plot.

12. In a circle ABCDE, O is the centre and AOE is a straight line. If angle AOD = $(x - y)^\circ$, angle BOE = $2y^\circ$, angle ABC = $(3x + y)^\circ$ and BC = CD, find angle CDE in terms of x and y .

13. Draw a triangle PQR in which PQ = 5.6 cm, QR = 7.2 cm and PR = 4.2 cm. With sides PQ and PR produced, draw an escribed circle to triangle PQR. Draw an inscribed circle to ΔPQR . Measure the distance between the in-centre and ex-centre.

14. The distribution in percentage of a total population of a certain country is as shown below:

Age (years)	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
%	12.4	22.6	16.2	15.4	10.6	8.3	7.5	4.3	1.6	1.1

Calculate the mean age and the standard deviation of the population.

15. Use the graph of $y = x^3$ to solve the following equations:

(a) $x^3 + 3x^2 - 27 = 0$

(b) $x^3 - 4x = 0$

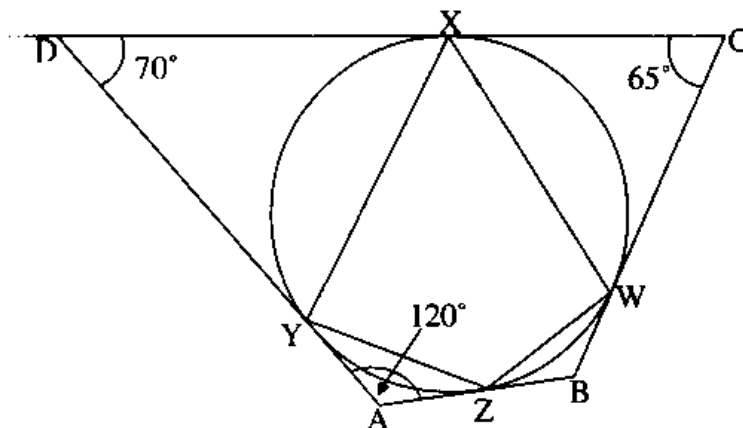
(c) $x^3 - 5x - 8 = 0$

(d) $x^3 - 2x^2 + x - 1 = 0$

Revision Exercise 10

1. Factorise the expression $2x^2 + x - 15$ and hence solve the equation $2x^2 + x - 15 = 0$.

2. John spent $\frac{1}{4}$ of his September salary on rent, $\frac{1}{3}$ of the remainder on food and $\frac{1}{10}$ of what was left on other bills. If he still had sh. 4 500, what was his salary in September?
3. If $3 - \frac{1}{4x} = \frac{1}{8x}$, find the ratio $\frac{1}{2}x + 2 : \frac{1}{3}x + 1$
4. Work out the following:
- (a) $\frac{7}{0.256} + \frac{4}{0.053}$ (b) $\frac{8}{0.375} - \frac{10}{37.5}$ (c) $\frac{2}{0.0049} \times \frac{3}{0.25}$
5. A parallelogram R has three of its vertices at (2, -2), (0, -4) and (4, -4). T is a translation $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ and X is a reflection in the line $y = 0$.
- (a) Draw the parallelogram R.
(b) Locate the position of TX(R).
6. Find the sum of the A.P. having 15 terms, the fourth term being -3.2 and the eighth term 8.4.
7. Find the matrix products **AB** and **BA**, where:
- $$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}$$
8. The ratio of boys to girls in a school is 4 : 5. One day $\frac{1}{3}$ of the boys and $\frac{1}{5}$ of the girls were absent. If 8 less pupils had been absent, $\frac{3}{4}$ of the school would have been present. Calculate the number of pupils in this school on that day.
9. In the figure below, AB, BC, CD and AD are tangents to the circle WXYZ, angle ADC = 70° , angle BCD = 65° and angle BAD = 120° :



- (a) Show that $AB + CD = AD + BC$.
(b) Calculate all angles of the quadrilateral WXYZ.

10. In triangle PQR, M is a point on QR such that $QM : MR = 2 : 3$. A point N divides PQ externally in the ratio 4 : 3. If point $PQ = \mathbf{p}$, $PR = \mathbf{q}$, find QM , QN and NM in term of \mathbf{p} and \mathbf{q} .
11. **E** represents an enlargement, centre origin and scale factor -3 . **M** represents a reflection in $y = 2x$. Find the matrices of **E**, **M** and the composite mapping **EM**. Find also the matrix for the inverse transformation $(\mathbf{EM})^{-1}$.

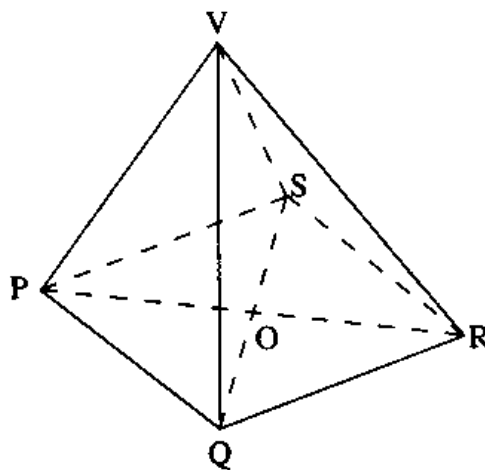
12. Evaluate:

$$(a) \int_0^3 (x^4 - 22x^2) dx \qquad (b) \int_3^5 (x^3 - 7x^2 + 7x + 15) dx$$

13. A building contractor has to move 150 tonnes of cement to a site 30 kilometres away. He has at his disposal a fleet of five lorries. Two of the lorries have a carrying capacity of 12 tonnes each, while each of the remaining can carry 7 tonnes. The cost of operating a 7 tonne lorry is sh. 100 per kilometre and that of operating a 12 tonne lorry sh. 125 per kilometre. The number of trips by the bigger lorries should be more than twice that made by the smaller lorries.

How should the contractor deploy his fleet in order to minimise the cost of moving the cement?

14. The figure below represents a right pyramid on a square base PQRS of side 12 cm. O is the centre of the base and $VO = 14$ cm:



Calculate:

- the length of VP.
- the angle which VP makes with the base.
- the volume of the pyramid.

15. Mr Akung'a is a teacher in a technical institute. He earns a basic monthly salary of sh. 22 145, a house allowance of sh. 12 000 and medical allowance of sh. 2 680. He is entitled to a personal relief of sh. 1 056. He also has an insurance scheme for which he pays a monthly premium of sh. 2 000. He is entitled to relief on the premium at 15 % of the premium paid. Use the taxation schedule below to calculate his net monthly salary:

<i>Income (K£ p.a.)</i>	<i>Rate (%)</i>
1-5 808	10 %
5 809-11 280	15 %
11 281-16 752	20 %
16 753-22 224	25 %
22 225 and above	30 %

Sample Test Papers

Instructions to candidates

1. The paper contains two sections, Section I and Section II.
2. Attempt all questions in Section I and any five questions from Section II.
3. Marks may be given for correct working, even if the answer is wrong.
4. Electronic calculators and mathematical tables may be used.
5. Time $2\frac{1}{2}$ hours.

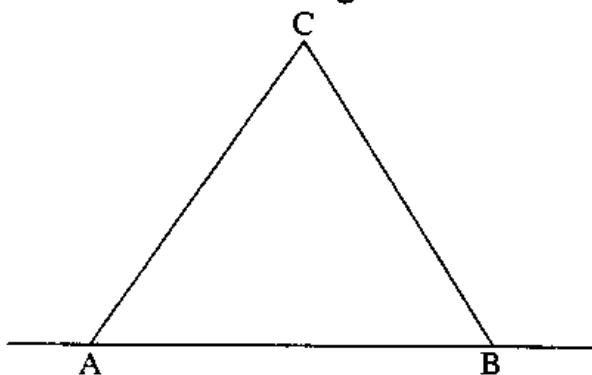
SAMPLE TEST PAPER 1

Section I (50 marks)

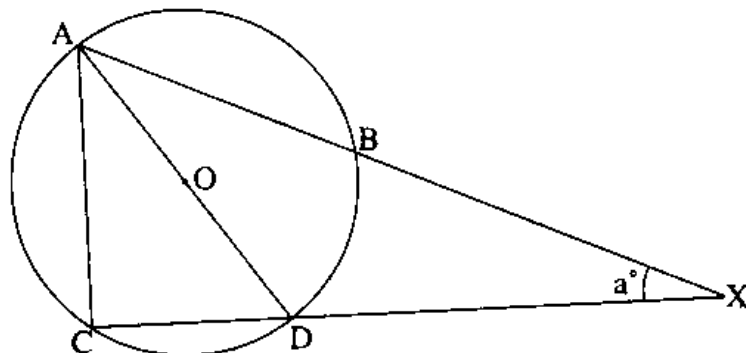
Attempt all questions in this section

1. Solve the equation $\frac{x+1}{2} - \frac{x-3}{3} = 4$ (3 marks)
2. Use mathematical tables to evaluate:
$$\sqrt[3]{\frac{41.07 \times 0.03142}{0.0156 \times 5.8941}}$$
 (4 marks)
3. Make u the subject of the formula in the expression $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$ (3 marks)
4. Given that y varies inversely as x and $y = 3$ when $x = 2$, find the value of x when $y = 4$. (3 marks)
5. Simplify the expression;
 $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$, giving your answer in the form $a + b\sqrt{c}$. (3 marks)
6. Express $\frac{x-3}{x+3} - \frac{x+3}{x-3}$ as a single fraction. (3 marks)
7. Given that $8 \leq y \leq 12$ and $1 \leq x \leq 6$, find the maximum possible value of $\frac{x+y}{y-x}$. (3 marks)

8. Find the equation of a line which passes through $(-1, -4)$ and is perpendicular to the line $y + 2x - 4 = 0$. (3 marks)
9. Given that $\cos A = \frac{5}{13}$ and angle A is acute, find $\tan A$. (2 marks)
10. The figure below shows triangle ABC :



- Construct triangle ABQ equal in area to triangle ABC . (2 marks)
11. Evaluate $\int_1^3 (x^2 + x + 1) dx$ (3 marks)
12. Find the equation of the tangent and the normal to the curve $y = 2x^3 - 3x^2 + 6$ at the point $(2, 10)$. (4 marks)
13. Vector \mathbf{m} passes through the points $(6, 8)$ and $(2, 4)$. Vector \mathbf{n} passes through $(x, -2)$ and $(-5, 0)$. If \mathbf{m} is parallel to \mathbf{n} , determine the value of x . (3 marks)
14. Expand $(a - b)^6$. Hence, find the value of $(0.98)^6$ correct to 4 s.f. (4 marks)
15. In the figure below, O is the centre of the circle. Chords AB and CD intersect at X . If AD is the diameter, prove that $\angle BDC = (90 + a)^\circ$:



(3 marks)

16. Tap A can fill a tank in 10 minutes, tap B can fill the same tank in 20 minutes. Tap C can empty the tank in 30 minutes. The three taps are left open for 5 minutes, after which tap A is closed. How long does it take to fill the tank?

(4 marks)

Section II (50 marks)*Attempt any five questions from this section*

17. The frequency distribution of marks of 80 students is given in the table below:

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	1	6	10	20	15	5	14	5	3	1

- (a) Draw a cumulative frequency graph to illustrate the data. (4 marks)
- (b) From your graph estimate:
- the median. (1 mark)
 - semi-interquartile range. (3 marks)
 - the pass mark, if 65% of the students are to pass. (2 marks)
18. Draw the graph of the function $y = -x^2 + 4x - 1$ for $-1 \leq x \leq 5$. (5 marks)
- On the same axes, draw the graph of $y = 2x - 3$. (1 mark)
- Use your graph to solve the following equations:
- $x^2 - 4x + 1 = 0$ (2 marks)
 - $x^2 - 2x - 2 = 0$ (2 marks)
19. (a) Use trapezoidal rule to find the area between the curve $y = x^2 + 4x + 4$, the x-axis and the ordinates $x = -2$ and $x = 1$. Take values of x at intervals of $\frac{1}{2}$ unit. (5 marks)
- (b) Use integration to find the exact area. Hence, find the percentage error in your approximation. (5 marks)
20. The vertices of triangle PQR are P(1, 1), Q(4, 1) and R(5, 4). A transformation represented by a matrix $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ maps ΔPQR onto $\Delta P'Q'R'$. A second transformation represented by $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps $\Delta P'Q'R'$ onto $\Delta P''Q''R''$.
- (a) On the same axes, draw the three triangles PQR, $P'Q'R'$ and $P''Q''R''$. (6 marks)

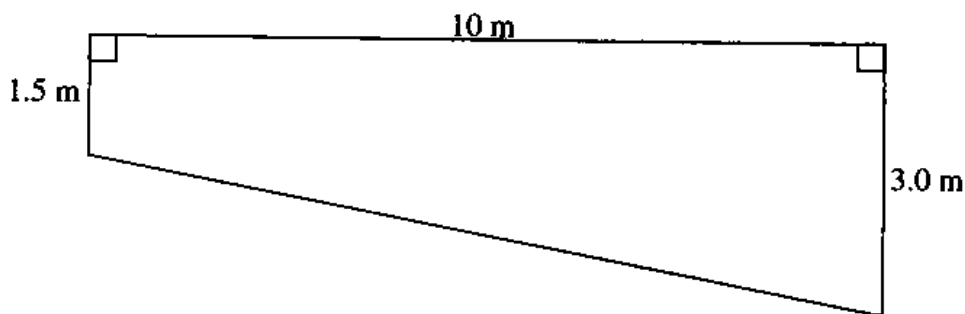
- (b) Describe a single transformation which maps ΔPQR onto $\Delta P''Q''R''$ and find its matrix. (4 marks)

21. (a) Complete the table below:

x	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \sin(x + 30^\circ)$	0.50					0	-0.50						0.50
$y = 2 \cos(x + 30^\circ)$	1.73		0		-1.73							2.00	1.73

(2 marks)

- (b) On the same axes, draw the graphs of $y = \sin(x + 30^\circ)$ and $y = 2\cos(x + 30^\circ)$. (5 marks)
- (c) Use your graphs to solve the equation $2\cos(x + 30^\circ) - \sin(x + 30^\circ) = 0$. (2 marks)
- (d) State the amplitude of each wave. (1 mark)
22. The figure below shows a uniform cross-section of a swimming pool which is 4 m wide. The depth of the pool increases gently from 1.5 m to 3.0 m:



- (a) How much water, in litres, does it hold when full? (3 marks)
- (b) Calculate the total internal surface area of the pool. (5 marks)
- (c) Find the angle at which the bottom of the pool inclines to the horizontal. (2 marks)
23. (a) Using a ruler and compasses only, construct triangle ABC in which $AB = 5$ cm, $BC = 5.9$ cm and $\angle BAC = 45^\circ$. (4 marks)
- (b) Draw the in-circle to the triangle ABC and measure its radius. (3 marks)
- (c) By taking AB as the base, find the area of the triangle. (3 marks)

24. The table below shows the rates of taxation in a certain year.

<i>Income in K£ p.a.</i>	<i>Rate of taxation in sh. per K£</i>
1 - 3 900	2
3 901 - 7 800	3
7 801 - 11 700	4
11 701 - 15 600	5
15 601 - 19 500	7
Above 19 500	9

In that period, Juma was earning a basic salary of Ksh. 21 000 per month. In addition, he was entitled to a house allowance of Ksh. 9 000 p.m. and a personal relief of Ksh. 1056 p.m.

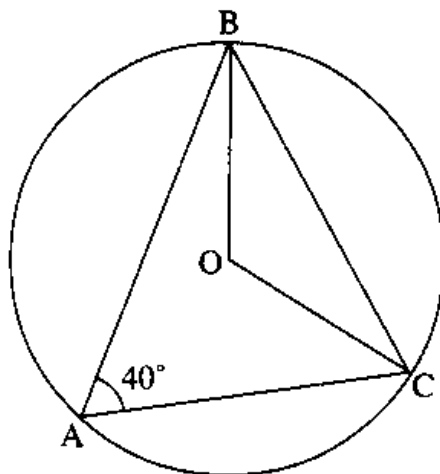
- (a) Calculate how much income tax Juma paid per month. (7 marks)
- (b) Juma's other deductions per month were:
- | | |
|------------------------------------|------------|
| Co-operative society contributions | sh. 2 000. |
| Loan repayment | sh. 2 500. |
- Calculate his net salary per month. (3 marks)

SAMPLE TEST PAPER 2

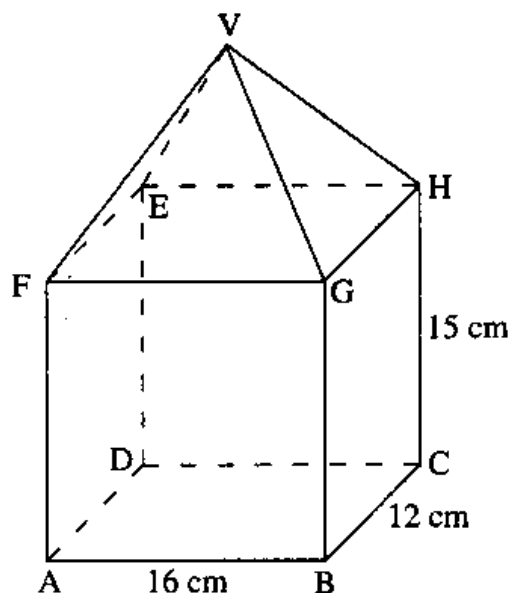
Section I (50 marks)

Attempt all questions in this section

1. Evaluate $\frac{\frac{1}{2} \text{ of } 3\frac{1}{2} + 1\frac{1}{2} (2\frac{1}{2} - \frac{2}{3})}{\frac{3}{4} \text{ of } 2\frac{1}{2} \div \frac{1}{2}}$ (3 marks)
2. Simplify $\frac{12x^2 - 16x}{20 - 11x - 3x^2}$ (3 marks)
3. Determine the equation of a line passing through $(-1, 3)$ and parallel to the line whose equation is $3x - 5y = 10$. (3 marks)
4. Given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 14 & 17 \\ -4 & 2 \end{pmatrix}$ find \mathbf{B} if $\mathbf{A}^2 + \mathbf{B} = \mathbf{C}$. (3 marks)
5. In the figure below, O is the centre of the circle and OB bisects angle ABC . Given that angle $BAC = 40^\circ$, find angle ABO . (3 marks)



6. Find the integral values of x for which;
 $5 \leq 3x + 2$, and,
 $3x - 14 < -2$ (3 marks)
7. A model of the solid shown in the figure below is made by welding together the ends of steel rods. $AB = 16$ cm, $BC = 12$ cm, $CH = 15$ cm and the altitude of V from the base $ABCD$ is 25 cm. Find the total length of the steel rods required to make this model. (3 marks)



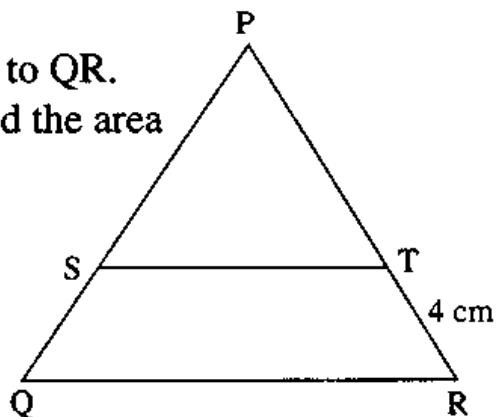
8. A fruit juice dealer sells the juice in packets of 300 ml, 500 ml and 750 ml. Find the size of the smallest container that can fill each of the packets and leave a remainder of 200 ml. (3 marks)

9. Evaluate without using tables;
 $\log(3x + 8) - 3 \log 2 = \log(x - 4)$ (4 marks)

10. A student expands $(x - y)^2$ as $x^2 - y^2$. If the student uses this expansion to evaluate $(12 - 9)^2$, find the percentage error in his calculation. (3 marks)

11. A stone is thrown vertically upwards so that its height h m above the ground after t seconds is given by the equation $h = 3t(5 - t)$. Determine the maximum height attained by the stone. (3 marks)

12. The figure below shows a triangle PQR.
 $PR = 12$ cm, $TR = 4$ cm and ST is parallel to QR .
 If the area of triangle PQR is 336 cm², find the area of the quadrilateral QRTS.



13. A ship covers 60 km on a bearing of 230° . It then changes course and heads due west for 80 km. Determine its direct distance from the starting point. (4 marks)

14. Basket A contains 5 oranges and 3 lemons while basket B contains 4 oranges and 3 lemons. A basket is selected at random and two fruits picked from it, one at a time, without replacement. Find the probability that the fruits so picked are of the same type. (3 marks)
15. A bus leaves town A at 8.30 a.m. and travels at an average speed of 70 km/h. Thirty minutes later, a car leaves the same town and follows the bus at 120 km/h. Find how far apart they will be at 9.30 a.m. (3 marks)
16. Point B is 2 400 nm to the east of the point A(60°N , $35^{\circ} 48' \text{W}$). Find the longitude of B. (3 marks)

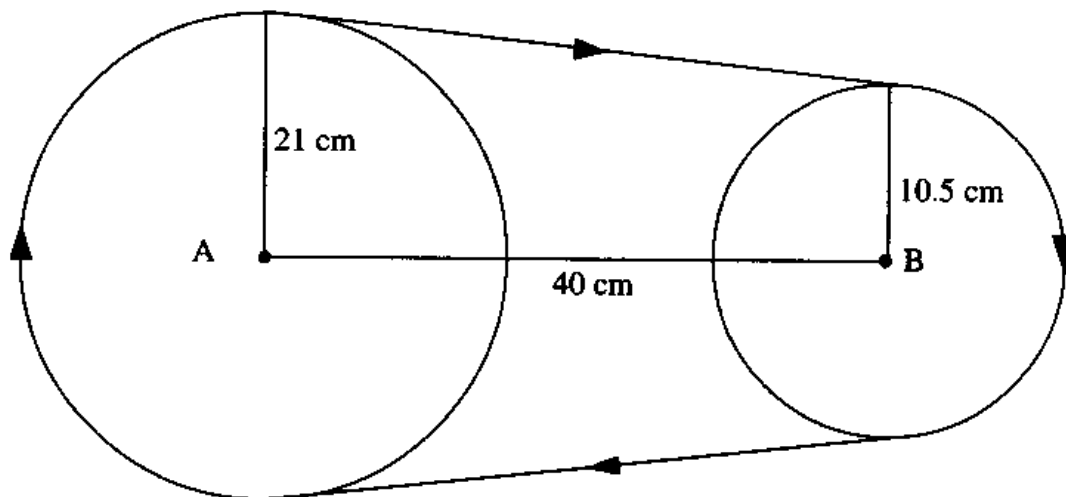
Section II (50 marks)

Attempt any five questions in this section

17. A community water tank is in the shape of a cuboid of base 6 m by 5 m and a height of 4 m. A feeder pipe of diameter 14 cm supplies water to this tank at the rate of $40 \text{ cm}^3/\text{sec}$.
- (a) Calculate:
- the capacity of the tank in litres. (2 marks)
 - the amount of water, in litres, delivered to this tank in one hour. (3 marks)
 - the time taken for the tank to fill. (2 marks)
- (b) The community consumes a full tank a day, with each family consuming an average of 150 litres per day. If each family pays a uniform rate of sh. 350 per month, find the total amount of money due monthly. (3 marks)
18. The cash price of a TV set is sh. 24 000. It can also be bought using either of the two plans below:
- Plan A: A deposit of sh. 6 000 and 15 equal monthly instalments.
- Plan B: 20 equal monthly instalments of sh. 1 680 each.
- (a) If the total payment in plan A is 25% more than the cash price, find:
- the amount of each instalment. (2 marks)
 - the annual rate of interest. (3 marks)
- (b) Find the annual rate of interest in plan B. (3 marks)
- (c) Which plan is cheaper, and by how much? (2 marks)
19. (a) Draw the graph of $y = 8 - 10x - 3x^2$ for $-5 \leq x \leq 3$. (5 marks)
- (b) On the same axes, draw the line $y = 2x + 1$ and hence, find:
- the roots of $8 - 10x - 3x^2 = 2x + 1$ (2 marks)

- (ii) a quadratic equation with the roots in (b) (i) (1 mark)
 (c) By including a straight line, use your graph to solve $3x^2 + 12x - 11 = 0$. (2 marks)

20. Two pulleys of radii 21 cm and 10.5 cm have their centres at A and B, 40 cm apart. A tight belt runs all round the pulleys as shown in the diagram below:



Calculate:

- (a) the length of the belt in contact with the pulley centre B. (4 marks)
 (b) the length of the belt in contact with the pulley centre A. (3 marks)
 (c) the total length of the belt. (3 marks)
21. (a) Complete the table below for the functions $y = 3\cos x$ and $y = \sin 2x$

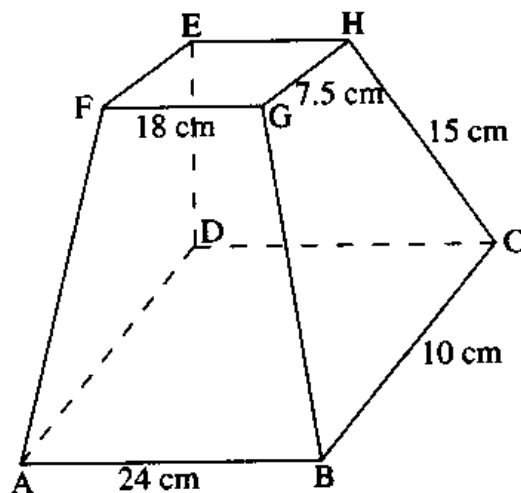
x°	-180	-150	-120	-90	-60	-30	0	30	60	90	120	150	180
$2x^\circ$		-300			-120		0			180			360
$3 \cos x$							3						3
$\sin 2x$							0						0

(2 marks)

- (b) On the same axes, draw the graphs of $y = 3\cos x$ and $y = \sin 2x$ for $-180 \leq x \leq 180^\circ$. (3 marks)
 (c) Use the graph in (b) to find:
 (i) the values of x such that $3\cos x - \sin 2x = 0$. (2 marks)
 (ii) the difference in the values of y when $x = 45^\circ$. (1 mark)
 (iii) the range of values of x such that $3\cos x > 1.5$. (2 marks)
22. The figure below shows a frustum ABCDEFGH of a right pyramid. $AB = 24$ cm, $BC = 10$ cm, $FG = 18$ cm, $GH = 7.5$ cm and $AF = BG = CH = DE = 15$ cm.

Find:

- (a) the altitude of the pyramid. (2 marks)
- (b) the angle between:
- (i) AH and the base ABCD. (3 marks)
- (ii) the planes ABGF and ABCD. (3 marks)
- (iii) the planes ABHE and ABCD. (2 marks)



23. (a) Complete the table below for the function $y = x^2 - 3x - 4$:

x	-4	-3.5	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1
y											

- (b) Use the table and the trapezoidal rule with 10 strips to estimate the area bounded by the curve $y = x^2 - 3x - 4$, $x = -4$, $x = 1$ and the x -axis. (3 marks)
- (c) Calculate the exact area in (b) above and hence determine the percentage error in the estimate. (5 marks)
24. An electronics dealer wishes to purchase radios and TV sets. He can buy at most 30 of both items. On average, a radio and a TV set cost sh. 4 000 and sh. 12 000 respectively and he has sh. 240 000 to spend. The number of TV sets should be at most twice the number of radios. He must buy more than five TV sets.
- (a) Form all inequalities to represent the above information and graph them. (5 marks)
- (b) If the dealer makes a profit of sh. 600 and sh. 1000 per radio and TV set respectively, find the maximum profit he will make. (2 marks)
- (c) During a sales promotion week, the dealer declared a discount of 10% and 5% on the display prices of each radio and TV set respectively. Determine his new maximum profit. (3 marks)

SAMPLE TEST PAPER 3

Section I (50 marks)

Attempt all questions in this section

1. Use mathematical tables to evaluate:

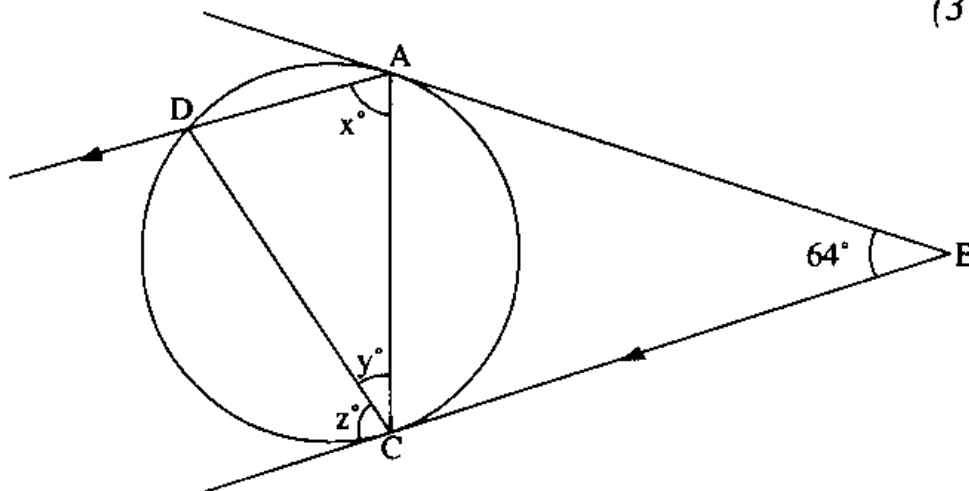
$$\frac{43.25 \times 0.9371}{\sqrt{2.64 + 8.43}}$$

(3 marks)

2. A sum of sh. 6 000 is invested at 8% p.a. compound interest. After how long will this sum amount to sh. 9 250? (Give your answer to the nearest month). (3 marks)

3. Solve for x; $2^{x-3} \times 8^{x^2+2} = 128$ (3 marks)

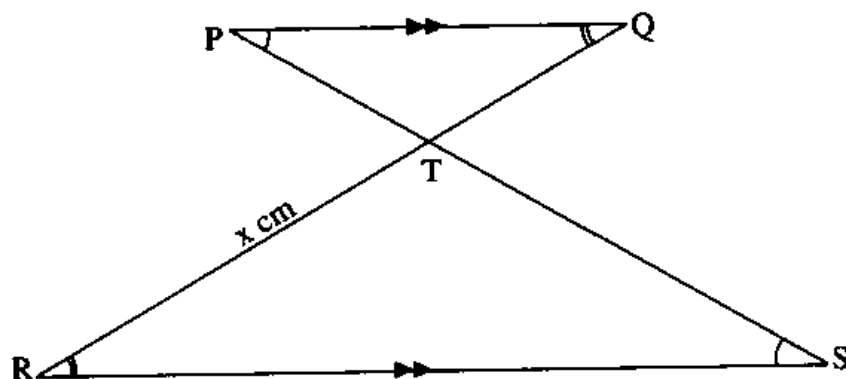
4. In the figure below, AB and BC are tangents to the circle and AD is parallel to BC. Find the angles marked x, y and z: (3 marks)



5. The internal radius of a pipe is 0.35 m. Water flows through the pipe at a rate of 45 cm per second. Calculate the amount of water that passes through the pipe in $2\frac{1}{4}$ hours. (3 marks)

6. Under a transformation whose matrix is $\mathbf{A} = \begin{pmatrix} a-2 & -2 \\ a & a \end{pmatrix}$, a triangle whose area is 12.5 cm^2 is mapped onto a triangle whose area is 50 cm^2 . Find two possible values of a. (3 marks)

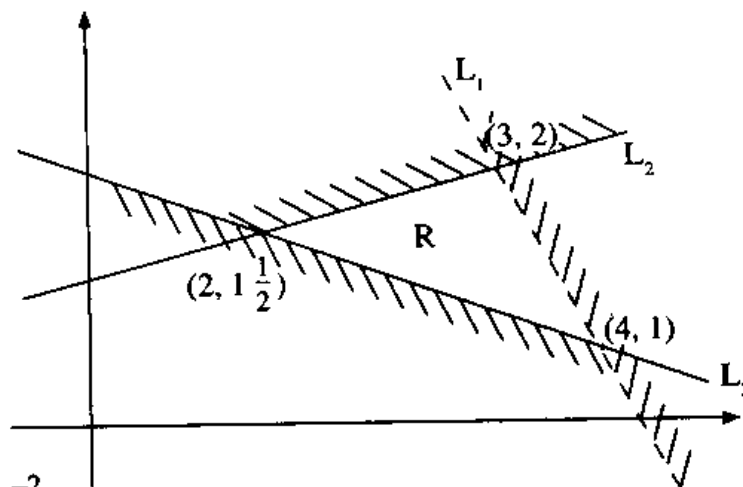
7. In the figure below, PQ is parallel to RS and the lines PS and RQ meet at T. Given that $PT : TS = 2 : 3$ and $RQ = 15 \text{ cm}$, find length RT. (3 marks)



8. Make x the subject of the formula;

$$\frac{p}{q} = \frac{mx - 2}{nx + 4} \quad (3 \text{ marks})$$

9. Five shirts and four pairs of trousers cost a total of sh. 6 160. Three similar shirts and a pair of trousers cost Ksh. 2 800. Find the cost of four shirts and a pair of trousers. (4 marks)
10. A bag contains three white and five blue balls. A second bag contains four white and six blue balls. A ball is chosen at random from each bag. Find the probability that:
- (a) both are white.
- (b) they are of different colours. (4 marks)
11. Write down three inequalities which define region R:



12. Evaluate $\int_{-3}^{-2} (x - 3)(x + 2)(x - 2) dx$ (3 marks)

13. The fifth term of an arithmetic progression is 11 and the twenty fifth term is 51. Calculate the first term and the common difference of the progression.
14. A solid in the shape of a right pyramid on a square base of side 8 cm and height 15 cm is cut at 6 cm height from the base. Find the surface area of the frustum formed. (4 marks)

15. A particle moves so that its displacement S in metres after t seconds is given by $S = 2t^3 + t^2 + 2$. Determine the displacement after the 4th second.
(3 marks)

16. Solve for x , given that matrix $A = \begin{pmatrix} 5 & -3 \\ 2 & x \end{pmatrix}$ is a singular matrix.
(2 marks)

Section II (50 marks)

Attempt any five questions in this section

17. In a $\triangle OAB$, M and N are points on OA and OB respectively, such that $OM : MA = 2 : 3$ and $ON : NB = 2 : 1$. AN and BM intersect at X . Given that $OA = a$ and $OB = b$:
- express in terms of a and b :
 - BM (i) AN (2 marks)
 - taking $BX = tBM$ and $AX = hAN$, where t and h are scalars, find two expressions for OX . (4 marks)
 - find the values of t and h . (4 marks)
18. A square S has vertices at $A(0, 0)$, $B(2, 0)$, $C(2, 2)$ and $D(0, 2)$.
- On a graph paper, draw the square S and its image S' under a transformation whose matrix is;
 $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ (3 marks)
 - S'' is the image of S under a transformation whose matrix is $B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Describe fully the transformation which would map S' to S'' . (4 marks)
 - Draw the image S''' of S under the transformation whose matrix is AB . Hence, describe a single transformation which maps S to S''' . (3 marks)
19. Three warships P , Q and R are at sea such that ship Q is 400 km on a bearing of 030° from ship P . Ship R is 750 km from ship Q and on a bearing of 120° from ship Q . An enemy warship S is sighted 1 000 km due south of ship Q .
- Taking a scale of 1 cm to represent 100 km, locate the position of ships P , Q , R and S . (4 marks)

- (b) Find the compass bearing of:
- ship P from ship S.
 - ship S from ship R. (2 marks)
- (c) Use the scale drawing to determine:
- the distance of S from P.
 - the distance of R from S. (2 marks)
- (d) Find the bearing of:
- Q from R.
 - P from R. (2 marks)

20. Complete the table below for the equation $y = x^2 + 3x - 6$, given $-6 \leq x \leq 4$.

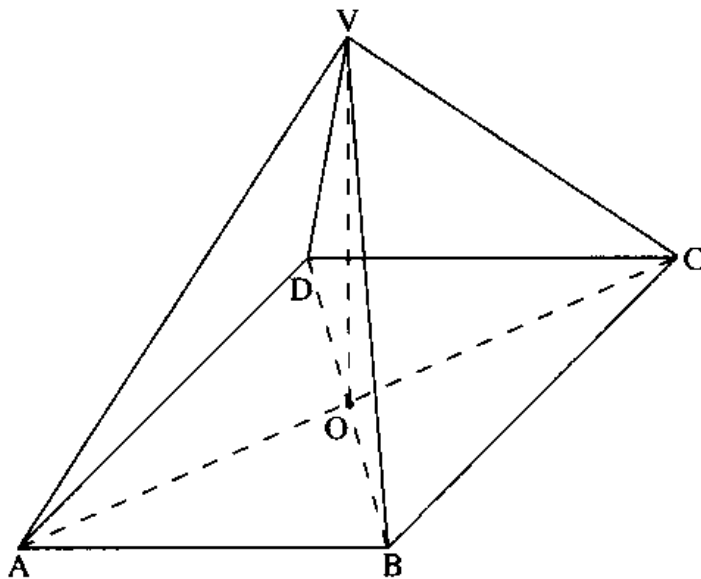
(a)

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4
y	12			-6			-6				22

(2 marks)

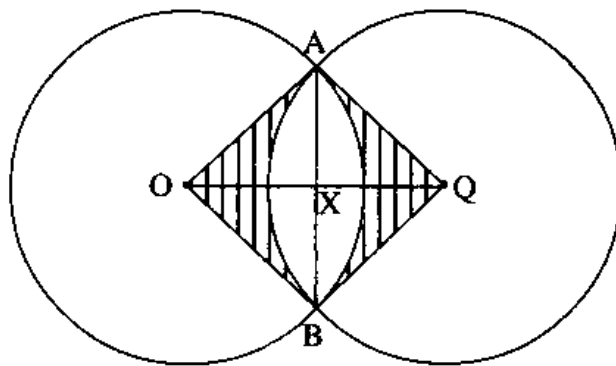
- (b) Using a scale of 1 cm to represent 2 units in both axes, draw the graph of $y = x^2 + 3x - 6$. (3 marks)
- (c) Use your graph to solve the quadratic equations:
- $x^2 + 3x - 6 = 0$ (2 marks)
 - $x^2 + 3x - 2 = 0$ (3 marks)

21. The diagram below shows a right pyramid with a horizontal rectangular base ABCD and vertex V. The area of the base is 60 cm^2 and the volume of the pyramid is 280 cm^3 .



- Calculate the height of the pyramid. (2 marks)
- Given that the ratio of the sides $AB : BC$ is $3 : 5$, find the lengths of:
 - AB
 - BC (4 marks)
- Find the length of the slanting height. (4 marks)

- (d) Calculate the angle between the planes VCB and ABCD. (2 marks)
22. Using a ruler and a pair of compasses only:
- (a) construct a parallelogram PQRS, where $PQ = 10$ cm, $QR = 7$ cm and $\angle PQR = 150^\circ$. Bisect $\angle PQR$ and $\angle SPR$ so that the angle bisectors meet at X. (2 marks)
- (b) construct a perpendicular from X to meet PQ at M. Measure XM. (2 marks)
- (c) calculate the area of ΔPXQ . (2 marks)
23. Two equal circles with centres O and Q and radius 8 cm intersect at points A and B as shown below.



Given that the distance between O and Q is 12 cm and that line AB meets OQ at X, find:

- (a) the length of chord AB. (2 marks)
- (b) the area of the shaded region. (6 marks)
- (c) the reflex angle AOB (2 marks)
24. Two friends Jane and Tom live 40 km apart. One day Jane left her house at 9.00 a.m. and cycled towards Tom's house at an average speed of 15 km/h. Tom left his house at 10.30 a.m. on the same day and cycled towards Jane's at an average speed of 25 km/h.
- (a) Determine:
- (i) the distance from Jane's house, where the two friends met. (4 marks)
- (ii) the time they met. (2 marks)
- (iii) how far Jane was from Tom's house when they met. (2 marks)
- (b) The two friends took 10 minutes at the meeting point and then cycled to Tom's house at an average speed of 12 km/h. Find the time they arrived at Tom's house. (2 marks)

SAMPLE TEST PAPER 4

Section I (50 marks)

Attempt all questions in this section

- Evaluate without using tables or calculator;
$$\frac{0.036 \times 0.0049}{0.07 \times 0.048}$$

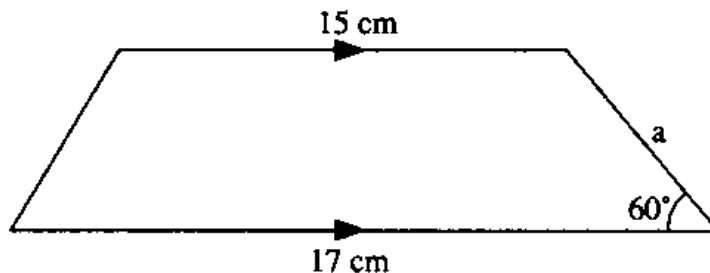
(3 marks)
- A metallic sphere of radius 10.5 cm was melted. The material was then used to make a cube. Find the length of one side of the cube. (3 marks)
- Factorise the expression $6x^2 - 13x + 6$ and hence solve the equation $6x^2 - 13x + 6 = 0$. (3 marks)
- Use reciprocal tables to find the value of $\frac{1}{0.325}$. Hence, evaluate;
$$\frac{\sqrt[3]{0.000125}}{0.325}$$

(3 marks)
- The sum of interior angles of a regular polygon is $1\ 080^\circ$.
 - Find the size of each exterior angle.
 - Give the name of the polygon.(3 marks)
- Wanjiru, Atieno and Jeptoo shared the profit of their business in the ratio $3 : 7 : 9$ respectively. If Atieno received sh. 60 000, how much profit did the business yield?

(3 marks)
- Wanjala broke three beakers and four test tubes while Mghanga broke two beakers and five test tubes during practical lessons in the laboratory. If Wanjala was charged sh. 560 and Mghanga sh. 490 for the breakages, find the cost of each beaker and each test tube. (3 marks)
- The length of an arc of a circle is $\frac{1}{5}$ of its circumference. If the area of the circle is 346.5 cm^2 , find:
 - the angle subtended by the arc at the centre of the circle.
 - the area of the sector enclosed by this arc.

(3 marks)

9. A bucket is 44 cm in diameter at the top and 24 cm in diameter at the bottom. Find its capacity in litres if it is 36 cm deep.
10. Find the value of 'a' in the figure below, if its area is 128 cm^2 :



(3 marks)

11. The thickness of a cylindrical piece of solid is given as 0.8 cm. If ten such pieces of solid are piled, one on top of the other, find the limits within which the height of the pile lies. (3 marks)
12. A man deposits his money in a savings bank on a monthly basis. Each deposit exceeds the previous one by sh. 500. If he started by depositing sh. 1 500, how much will he have deposited in 12 months? (3 marks)
13. The probability of a team losing a game is $\frac{1}{4}$. The team plays the game until it wins. Determine the probability that the team wins in the fifth round. (3 marks)
14. A ship leaves an island (5°N , 45°E) and sails due east for 120 hours to another island. The average speed of the ship is 27 knots. Find the position of the second island. (3 marks)
15. The gradient function of a curve at any point (x, y) is $6x^2$. Given that the curve passes through the point $(1, 5)$, find its equation. (3 marks)
16. A quantity P is partly constant and partly varies as the square of Q . When $Q = 2$, $P = 40$ and when $Q = 3$, $P = 65$. Determine the value of P when $Q = 4$. (4 marks)

Section II (50 marks)*Attempt any five questions in this section*

17. (a) (i) Draw the graph of $y = 2x^2 - 3x - 5$ for $-2 \leq x \leq 3$. (4 marks)
- (ii) Use the graph to solve the equation $2x^2 - 3x - 5 = 0$. (1 mark)
- (b) Using the same axes, draw the graph of $y = -2x - 2$. (2 marks)

- (c) From your graphs, find the values of x which satisfy the simultaneous equations;

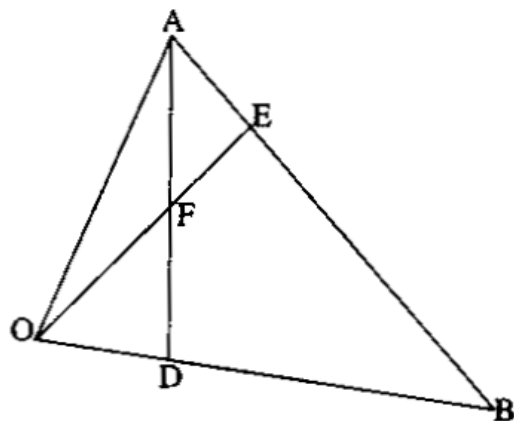
$$y = 2x^2 - 3x - 5$$

$$y = -2x - 2$$

(1 mark)

- (d) Write down the quadratic equation which is satisfied by the values of x where the two graphs intersect. (2 marks)

18. In the figure below, E is the midpoint of AB , $OD : DB = 2 : 3$ and F is the point of intersection of OE and AD :



- (a) Given that $OA = \mathbf{a}$ and $OB = \mathbf{b}$, express OE and AD in terms of \mathbf{a} and \mathbf{b} . (2 marks)
- (b) Given further that $AF = tAD$ and $OF = sOE$, find the values of s and t . (5 marks)
- (c) Show that O , F and E are collinear. (3 marks)
19. A and B are two points on the latitude 50°N . The two points lie on the longitudes 30°E and 150°W respectively.
- (a) Calculate:
- the distance in km from A to B along a parallel of latitude.
(Take $\pi = \frac{22}{7}$ and $R = 6\,370$ km) (5 marks)
 - the shortest distance from A to B along a great circle in nautical miles. (3 marks)
- (b) An aircraft takes 54 hours to fly between the two points A and B along the great circle. Calculate its speed in knots. (2 marks)
20. A triangular plot ABC is such that $AB = 72$ m, $BC = 80$ m and $AC = 84$ m.
- (a) Calculate the:
- area of the plot in square metres. (3 marks)
 - acute angle between the edges AB and BC . (3 marks)
 - the perpendicular height from A to the side BC . (2 marks)
- (b) A water tap is to be installed inside the plot such that the tap is equidistant from each of the vertices A , B and C . Calculate the distance of the tap from the vertex A . (3 marks)

21. The table below shows the distribution of the wages in a week for 50 employees in a certain factory:

Wage (Ksh)	800-899	900-999	1 000-1 099	1 100-1 199	1 200-1 299
No. of workers	3	10	25	9	3

- (a) Using Ksh. 1049.5 per week as the assumed mean wage, calculate:
- the mean for the grouped wages. (3 marks)
 - the standard deviation of the wages. (4 marks)
- (b) Estimate the median wage. (3 marks)
22. In driving to work, Wakesho has to pass through three sets of traffic lights. The probability that she will have to stop at any of the lights is $\frac{3}{5}$.
- (a) Draw a tree diagram to represent the above information. (2 marks)
- (b) Using the diagram, determine the probability that on any one journey, she will have to stop at:
- all the three sets. (2 marks)
 - only one of the sets. (2 marks)
 - only two of the sets. (2 marks)
 - none of the sets. (2 marks)
23. The velocity of a particle after t seconds is given by $v = t^2 - 2t + 4$.
- (a) Use the mid-ordinate rule with six strips to estimate the displacement of the particle between $t = 1$ and $t = 13$. (3 marks)
- (b) Determine:
- the exact displacement of the particle between $t = 1$ and $t = 13$. (3 marks)
 - the acceleration of the particle at $t = 4$. (2 marks)
- (c) Calculate the percentage error arising from the estimate in (a). (2 marks)
24. Four towns P, Q, R, and S are such that town Q is 120 km due east of town P. Town R is 160 km due north of town Q, town S is on a bearing of 330° from P and on a bearing of 300° from R.
- (a) Use a ruler and compasses only to show the positions of towns P, Q, R and S. (Take a scale of 1 cm = 50 km) (5 marks)
- (b) Determine;
- the distance SP. (2 marks)
 - the distance SR. (2 marks)
 - the bearing of town S from town Q. (1 mark)

SAMPLE TEST PAPER 5

Section I (50 marks)

Attempt all questions in this section

1. Evaluate the following and give your answer to 4 s.f.:

$$\frac{(0.28)^3 - (0.214)^{\frac{1}{2}}}{(0.123)^2 + (0.817)^{\frac{1}{3}}}$$

(4 marks)

2. Simplify without using tables or a calculator;

$$\frac{\sin 480^\circ - \cos 765^\circ}{\tan 225^\circ - \cos(-330^\circ)}$$

(3 marks)

3. How long will it take a bus 8 m long and moving with a speed of 80 km/h to go past a truck 15 m long heading the opposite direction with a speed of 60 km/h?

(3 marks)

4. Solve for x in the equation;

$$\frac{1}{2} \log_2 81 + \log_2(x^2 - \frac{x}{3}) = 1$$

(3 marks)

5. Given that $10.5 \leq x \leq 20$ and $1.5 \leq y \leq 3$, find:

(a) the maximum value of $\frac{x}{y}$.

(b) the minimum value of $y - x$.

(4 marks)

6. A shopkeeper mixes sugar costing sh. 40 per kg with another type which costs sh. 60 per kg. Find the ratio in which the two types should be mixed so that if a kilogram of the mixture is sold at sh. 55, a profit of 10% is realised.

(3 marks)

7. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$. Hence, determine the point of intersection of the lines;

$$y + x = 7$$

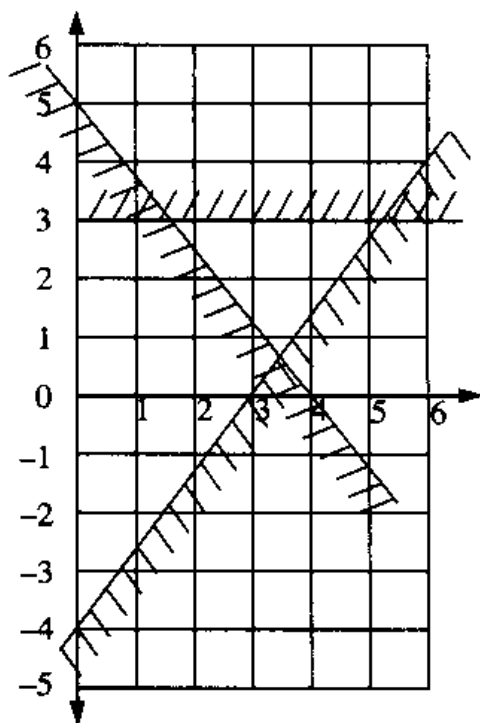
$$3x + y = 15$$

(3 marks)

8. The current price of a vehicle is sh. 500 000. If the vehicle depreciates at a rate of 15% p.a., find the number of years it will take for its value to fall to sh. 180 000.

(3 marks)

9. Find the first five terms of the expansion $(2 - \frac{1}{x})^8$. Hence, evaluate $(1.75)^8$.
(4 marks)
10. Find the centre and the radius of a circle whose equation is $x^2 - 6x + y^2 - 10y + 30 = 0$.
(3 marks)
11. A farmer has 1 200 m of metal railing to form two adjacent sides of a rectangular enclosure, the other sides being a live fence forming a right angle. Find the dimensions which will give the maximum possible area.
(3 marks)
12. A variable y varies as the square of x and inversely as the square root of z . Find the percentage change in y when x is increased by 5% and z reduced by 19%.
(3 marks)
13. Find the least number of terms that must be taken for the sum of the A.P. $5 + 7 + 9 + \dots$, to exceed 1 000 000.
(3 marks)
14. The scale of a map is given as 1 : 20 000. Find the actual area in hectares of a region represented by a triangle of sides 6 cm, 7 cm and 4 cm.
(4 marks)
15. A two-digit number is such that the sum of the ones and the tens digit is ten. If the digits are reversed, the new number formed exceeds the original number by 54. Find the number.
(3 marks)
16. Determine the inequalities which define the unshaded region below:



(3 marks)

Section II (50 marks)

Attempt any five questions in this section.

17. A triangle has vertices $A(-5, -2)$, $B(-3, -2)$ and $C(-5, -5)$. The triangle is rotated through $+90^\circ$ about the origin to obtain the image $A'B'C'$. $\Delta A'B'C'$ is then reflected on the line $y + x = 0$ to get $\Delta A''B''C''$.

- (a) Plot the triangles ABC , $A'B'C'$ and $A''B''C''$ on the grid. (5 marks)
- (b) Find a single transformation that maps ΔABC onto $\Delta A''B''C''$. (3 marks)
- (c) Find the co-ordinates of the image of ABC under a stretch scale factor 2 parallel to the x-axis. (2 marks)

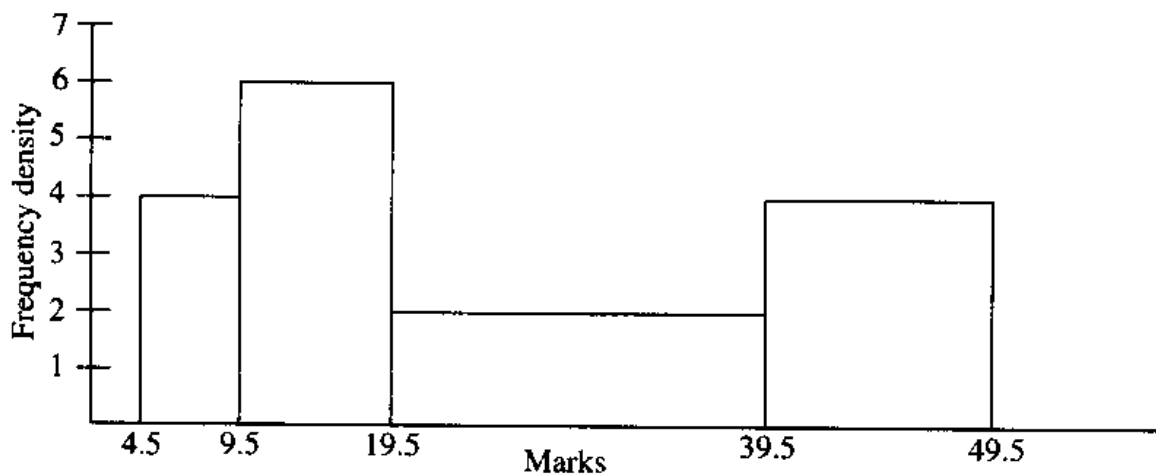
18. The displacement of a particle after t seconds is given by $S = 40t^3 - t^2 - 3t + 3$. Find the:

- (a) velocity of the particle when $t = 2$. (3 marks)
- (b) acceleration of the particle when $t = 3$. (3 marks)
- (c) (i) maximum displacement. (2 marks)
(ii) minimum velocity of the particle. (2 marks)

19. (a) Draw the graph of $y = x^3 + 4x^2 - x - 6$ for $-5 \leq x \leq 3$. (5 marks)

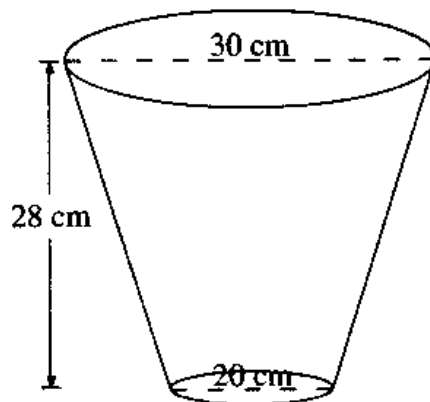
- (b) Use your graph to solve the following equations:
- (i) $x^3 + 4x^2 - x - 6 = 0$ (1 mark)
- (ii) $3x^3 + 12x^2 - 15x - 21 = 0$ (2 marks)
- (iii) $-x^3 - 4x^2 + 2x + 9 = 0$ (2 marks)

20. The diagram below shows a histogram representing marks obtained in a certain test:



- (a) Develop a frequency distribution table for the data. (4 marks)
- (b) Estimate the mean. (3 marks)
- (c) Calculate the standard deviation. (3 marks)

21. (a) Using a ruler and pair of compasses only, construct triangle ABC in which $AB = 9$ cm, $BC = 8.5$ cm and $\angle BAC = 60^\circ$. (3 marks)
- (b) On the same side of AB as C:
- (i) determine the locus of a point P such that $\angle APB = 60^\circ$. (3 marks)
- (ii) construct the locus of R such that $AR > 4$ cm. (2 marks)
- (iii) determine the region T such that $\angle ACT \geq \angle BCT$. (2 marks)
22. The diagram below shows a bucket with top diameter 30 cm and bottom diameter 20 cm. The height of the bucket is 28 cm:



Find:

- (a) the capacity of the bucket in litres. (5 marks)
- (b) the area of the metal sheet required to make 100 such buckets, taking 10% extra for overlapping and wastage. (5 marks)
23. The table below gives values of Q with corresponding values of S.

Q	90.1	222.3	371.2	693.3	4450.1	11 000
S	60	105	147	226	780	1 500

Q and S are connected by a law of the form $Q = kS^n$, where k and n are constants.

- (a) Draw a suitable straight line graph and determine the values of k and n.
- (b) State the value of:
- (i) Q when $S = 113$.
- (ii) S when $Q = 400$. (10 marks)
24. A farmer wishes to grow two crops, potatoes and beans. He has 70 hectares of land available for this purpose. He has 240 man-days of labour available to work out the land and he can spend up to sh. 180 000. The requirements for the crops are as follows:

	<i>Potatoes</i>	<i>Beans</i>
Minimum number of hectares to be sown	10	20
Man-day per hectare	2	4
Cost per hectare in sh.	3 000	2 000
Profit per hectare in sh.	15 000	10 000

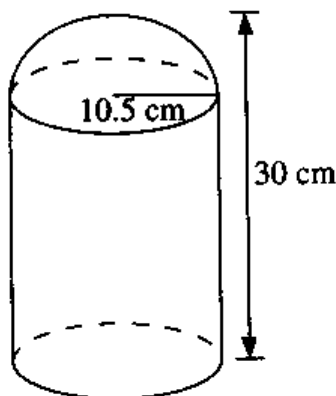
- (a) If x and y represent the number of hectares to be used for potatoes and beans respectively, write down in their simplest form, the 5 inequalities which x and y must satisfy. *(4 marks)*
- (b) Represent the inequalities on the same axes. *(3 marks)*
- (c) Using the graph, find the number of hectares to be used for potatoes and beans so as to give maximum profit. *(2 marks)*
- (d) Find the maximum profit. *(1 mark)*

SAMPLE TEST PAPER 6

Section I (50 marks)

Attempt all questions in this section

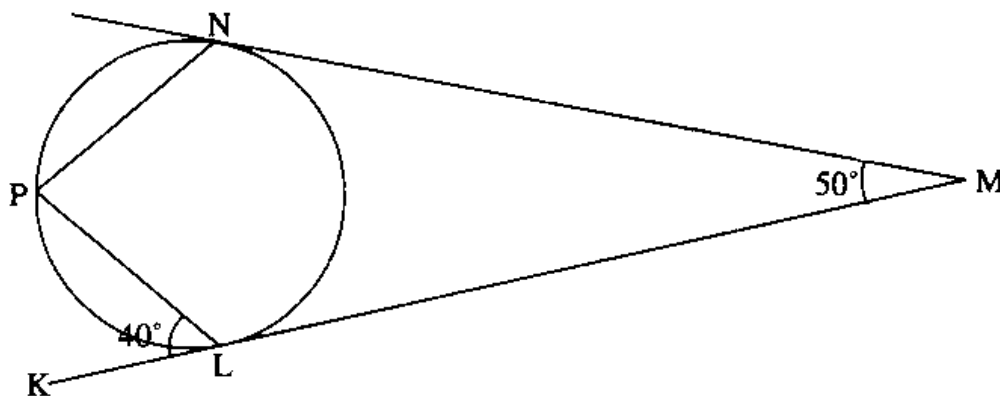
1. Use logarithms to evaluate $\sqrt[3]{\left(\frac{1.23 \times 0.0468}{\log 6}\right)}$ (4 marks)
2. Simplify: $\frac{x-3}{x+3} - \frac{x^2-3x}{x^2-9}$ (3 marks)
3. The sum of the ages of three sisters Rhoda, Sally and Tabitha is 39 years. Sally is twice as old as Tabitha and one and a half times as old as Rhoda. Determine their ages. (3 marks)
4. A perpendicular line is drawn from a point (1, 2) to the line $3y + 2x + 1 = 0$. Find the equation of the perpendicular in the form $ay = bx + c$. (3 marks)
5. Simplify; $\frac{\sqrt{15}}{\sqrt{5} - \sqrt{3}} - \frac{\sqrt{15}}{\sqrt{5} + \sqrt{3}}$ (3 marks)
6. Make m the subject of the formula; $b = \frac{nm}{n-m}$. (3 marks)
7. The solid shown below consists of a cylinder and a hemisphere of equal radius 10.5 cm. If the height of the solid is 30 cm, find its volume.
(Take $\pi = \frac{22}{7}$) (4 marks)



8. Solve $9^{x+1} = 243$ (3 marks)

9. The length and breadth of a rectangular card were measured to the nearest millimetre and found to be 14.5 cm and 10.6 cm respectively. Find the percentage error in the perimeter. (3 marks)
10. Three ladies Agnes, Betty and Cate contributed a total amount of sh. 21 000 to start a salon. The ratio of the contributions of Agnes to Betty was 3 : 5 and that of Betty to Cate was 2 : 1. How much did Betty contribute? (3 marks)
11. Aggrey paid sh. 450 for a trouser after getting a discount of 10%. The retailer still made a profit of 25% on the sale of this trouser. What profit would the retailer have made if no discount was allowed? (3 marks)
12. A steel factory starts producing hinges at the rate of 2 000 per hour. The rate of production decreases by 10% every two hours. Calculate the number of hinges produced in the first 6 hours. (3 marks)
13. Solve the following equation;
 $\sin (2x - 30) = \frac{\sqrt{3}}{2}$ for $0^\circ \leq x \leq 180^\circ$. (3 marks)

14.



In the figure above, line KLM and NM are tangents to the circle at L and N respectively. $\angle LMN = 50^\circ$ and $\angle KLP = 40^\circ$. Find the size of $\angle MNP$. (2 marks)

15. Expand $(a - b)^5$. Use the expansion to find the value of 1.98^5 . (4 marks)
16. A wildlife club has 13 form four students and 11 form three students. The club has three officials. Find the probability that two of the officials are form fours. (3 marks)

Section II (50 marks)*Attempt any five questions in this section*

17. A kite ABCD has vertices at A(1, 1), B(6, 2), C(6, 6) and D(2, 6).

(a) On the same axes:

(i) draw the image $A'B'C'D'$ of ABCD under a rotation of 90° anticlockwise about the origin. (3 marks)(ii) draw the image $A''B''C''D''$ of $A'B'C'D'$ under a reflection in the line $y = x$. (3 marks)(iii) draw the image $A'''B'''C'''D'''$ of $A''B''C''D''$ under a reflection in the line $x = 0$. (2 marks)(b) Describe a single transformation that maps $A'''B'''C'''D'''$ onto ABCD. (2 marks)

18. The table below shows marks scored by 100 form four students in a mathematics examination.

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
No. of students	2	10	13	17	18	14	10	6	6	4

(a) Draw an ogive to represent the above information. (4 marks)

(b) Using your graph, estimate:

(i) the median. (1 mark)

(ii) the quartile deviation. (4 marks)

(c) If the pass mark is 45%, how many students passed? (1 mark)

19. The velocity v m/s of a vehicle at any time t seconds is given by the equation $v = t^2 - 4t + 5$.

(a) Complete the table below:

t	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
v	2				2		5				17				

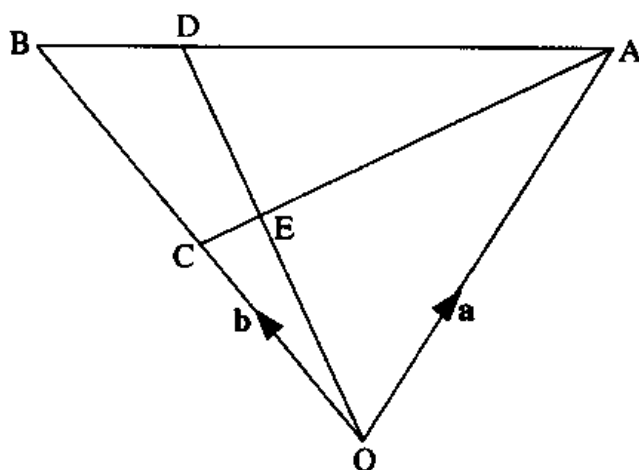
(2 marks)

(b) Use the mid-ordinate rule with seven strips to estimate the area enclosed by the curve $v = t^2 - 4t + 5$, the x -axis and the lines $x = 1$ and $x = 8$. (2 marks)

(c) Find the exact area of the region in (b) above. (3 marks)

(c) Calculate the percentage error introduced by using the mid-ordinate rule in (b) above. (3 marks)

20.



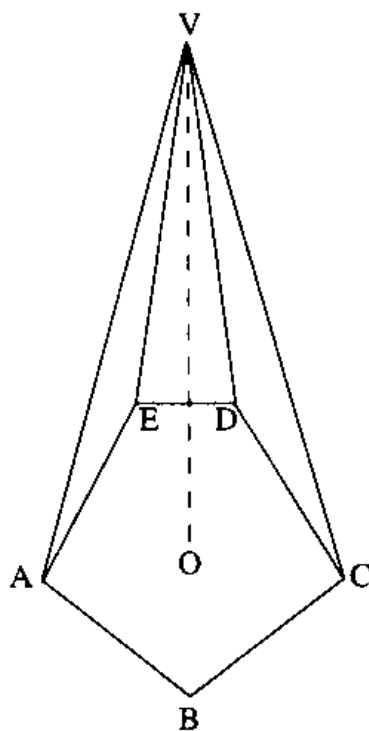
The figure above shows $\triangle OAB$ in which $BD : DA = 1 : 2$, $OE : ED = 3 : 2$ and C is the midpoint of OB .

- (a) Given that $OA = \mathbf{a}$ and $OB = \mathbf{b}$, express the following vectors in terms of \mathbf{a} and \mathbf{b} :
- AB (1 mark)
 - OD (2 marks)
 - AE (2 marks)
- (b) Show that points A, E and C lie on a straight line. Hence, determine the ratio $CE : EA$. (5 marks)
21. The area $A \text{ cm}^2$ of a cylinder depends partly on r and partly on r^2 , where r is the radius of the base. When $r = 1 \text{ cm}$, $A = 7 \text{ cm}^2$ and when $r = 2$, $A = 16 \text{ cm}^2$.
- Find an expression for A in terms of r . (4 marks)
 - Calculate the radius when the area is 115 cm^2 . Give your answer to 1 d.p. (4 marks)
 - Find the value of r for which the two parts are equal. (2 marks)
22. Water flows through a cylindrical pipe of diameter 4.2 cm at a speed of 50 m/min.
- Calculate the volume of water delivered by the pipe per minute in litres. (3 marks)
 - A cylindrical storage tank of depth 3 m is filled by water from this pipe and at the same rate of flow. Water begins flowing into the empty storage tank at 8.30 a.m. and is full at 3.10 p.m. Calculate the area of cross-section of this tank in m^2 . (4 marks)
 - A family consumes the capacity of this tank in one month. The cost of water is sh. 40 per thousand litres plus a fixed basic charge of sh. 1 650. Calculate the cost of this family's water bill for a month. (3 marks)

23. A plane leaves an airport X (41.5°N , 36.4°W) at 9.00 a.m. and flies due north to airport Y on latitude 53.2°N .
- (a) Calculate the distance covered by the plane in km. (3 marks)
- (b) After stopping for 30 minutes to refuel at Y, the plane then flies due east to airport Z, 2 500 km from Y. Find:
- (i) the position of Z.
- (ii) the time the plane lands at Z, if its speed is 500 km/h.
- (Take the value of π as $\frac{22}{7}$ and radius of the earth as 6 370 km)

(7 marks)

24. The diagram below shows a right pyramid whose base is a regular pentagon of side 11 cm. $VA = VB = VC = VD = VE = 21$ cm, and O is the centre of the pyramid:



Calculate:

- (a) the height of the pyramid. (2 marks)
- (b) (i) area of the pentagon. (3 marks)
- (ii) volume of the pyramid. (2 marks)
- (c) surface area of the pyramid. (3 marks)