NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 210

COURSE TITLE: INTRODUCTION TO COMPLEX ANALYSIS

Course Code: MTH 210
Course Title: Introduction to Complex Analysis

Course Developer/Writer: Ishola, Christie Yemisi
School of Science and Technology
National Open University of Nigeria
Victoria Island, Lagos.
Nigeria.

Course Editor: Prof. S.A. Okunuga
Department of Mathematics
University of Lagos
Akoka-Lagos
Nigeria.

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
No. 5 Dares Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja
Nigeria

E-mail: centralinfo@nou.edu.eg
URL: www.nou.edu.ng

TABLE OF CONTENTS

CONTENTS

COURSE GUIDE

CONTENTS

1.0 INTRODUCTION

You are expected to read this study guide carefully at the start of this semester. It contains important information about this course. If you need more clarifications, please consult one of the academic staff on the course. This course material will provide you with in-depth knowledge you will need in order to complete the course.

Code for this course is MTH 210 and the Course title is **Introduction to Complex Analysis**, a three (3) credit unit course for students studying towards acquiring a Bachelor of Science in Mathematics, Computer Science and other related disciplines.

The course is divided into modules and study units. It will first take a brief introduction to Complex Analysis. This course will then go ahead to introduce mathematical operations with complex numbers as well as De Moivre's Theorem and its application as well as obtaining the Limits and continuities of complex functions

The course guide therefore gives you an overview of what the course MTH 210 is all about, the textbooks and other materials to be referenced, what you expect to know in each unit, and how to work through the course material.

You are welcome to the course, introduction to Complex Analysis where you will perform different mathematical operations with complex numbers

2.0 AIM AND OBJECTIVES

Aim

This course aims to introduce you to the basic concepts and features of complex numbers. The study will also introduce you to analytic functions.

Objectives

It is important to note that each unit has specific objectives. You should study them carefully before proceeding to subsequent units. Therefore, it may be useful to refer to these objectives in the course of your study of the unit to assess your progress. You should always look at the unit objectives after completing a unit. In this way, you can be sure that you have done what is required of you by the end of the unit. It is hoped that by the time you complete this course, you will be able to:

- ∇ Define complex numbers
- ∇ Perform mathematical operations with complex numbers
- ∇ Express complex numbers in polar form
- ∇ carry out multiplication and division of complex numbers
- ∇ recall the De Moivre’s theorem and apply it appropriately
- ∇ find roots and work with fractional powers of complex numbers in polar form
- ∇ Define and Identify real and imaginary parts of a complex function
- ∇ Define and discuss the Limit of a complex function
- ∇ Define the Continuity of a complex function
- ∇ Define and discuss the differentiability of a complex function using the concept of limit
- ∇ State the rules of differentiation of complex numbers

3.0 MAIN CONTENT

3.1 A Guide through the Course

3.1.1 Structure of the Course

This Course is divided into three Modules. The first Module has three Units and deals with complex analysis. The second has three Units that deals with limits and continuity of complex functions. The third has two units that deals with analytic functions. The Modules are thus:

3.1.2 Modules and Units

Module 1:

Unit 1: Complex Numbers
Unit 2: Polar Operations with Complex Numbers
Unit 3: De Moivre’s Theorem and Application

Module 2:
Unit 1: Limits of functions of complex variables
Unit 2: Continuity of functions of complex variables
Unit 3: Differentiation of complex functions
Module 3:
Unit 1: Analytic functions I
Unit 2: Analytic functions II

3.1.3 Summary of the Contents

Module 1 introduces you to complex variables and some of its application. Module 2 presents how you obtain limits and continuity of complex variables. Module 3 introduces you to analytic functions. It is recommended that you draw up a schedule on how to accomplish the goals of the course.

3.2 How to Get the Most from this Course

3.2.1 What you will be learning in this Course

The overall aim of this Course is to equip you with the introductory studies of the basics of complex analysis

3.2.2 Working through the Course

For you to complete this Course successfully, you are required to study all the units, the recommended text books, and other relevant materials. Each unit contains some self-assessment exercises and tutor marked assignments, and at some point in this course, you are required to submit the tutor marked assignments. There is also a final examination at the end of this course. Stated below are the components of this course and what you have to do.

3.2.2.1 Course Material

The major components of the course are:

1. Course Guide

2. Modules

3. Study Units

4. Text Books

5. Assignment File

6. Presentation Schedule

In order to complete the learning successfully, you should:

- Apply yourself to undergoing this course.
- Not regard any aspect of this course as simplistic, difficult or complicated.
- Discard any previous biases about other related course(s) when you read this course.
- Regard the present course as an opportunity to engage in lifelong learning as well as enhance your skills.

NOTE: The Course will take you about some weeks to complete.

3.2.2.2 Assignment File

The assignment file will be given to you in due course. In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain for these assignments will count towards the final mark for the course. Altogether, there are 15 tutor marked assignments for this course.

3.2.2.3 Presentation Schedule

The presentation schedule included in this course guide provides you with important dates for completion of each tutor marked assignment. You should therefore endeavour to meet the deadlines.

3.2.2.4 Assessment

There are two aspects to the assessment of this course. First, there are tutor marked assignments; and second, e-examination. Therefore, you are expected to take note of the facts, information and problem solving gathered during the course. The tutor marked assignments must be submitted to your tutor for formal assessment, in accordance to the deadline given. The work submitted will count for 30% of your total course mark. At the end of the course, you will need to sit for a final e-examination which will account for 70% of your total score.

3.2.2.5 Tutor Marked Assignments (TMAs)

There are four tutor marked assignments (TMAs) called TMA 1, TMA 2, TMA 3 and TMA 4 as a (CBT) Computer Based Test in this course. They will be available on your potter as at when due and so you should be conversant with your potter. They must be submitted to your tutor for formal assessment, in accordance to the deadline given. Each TMA carried 10 marks and the best three TMA scores out of the four submitted will be selected and count for you meaning that all TMAs submitted will count for 30% of your total course mark. When you have completed each assignment, send them to your tutor as soon as possible and make certain that it gets to your tutor on or before the stipulated deadline. If for any reason you cannot complete your assignment on time, contact your tutor before the assignment is due to discuss the possibility of extension. Extension will not be granted after the deadline, unless on extraordinary cases.

3.2.2.6 Final Examination and Grading

The final examination for MTH210 will last for a period of 3 hours and have a value of 70% of the total course grade. The examination will consist of questions which reflect the self-assessment exercise and tutor marked assignments that you have previously encountered. Furthermore, all areas of the course will be examined. It would be better to use the time between finishing the last unit and sitting for the examination, to revise the entire course. You might find it useful to review your TMAs and comment on them before the examination. The final examination covers information from all parts of the course.

3.2.2.7 Course marking Scheme

The following table includes the course marking scheme

Table 3.1: Course Marking Scheme

Assessment	Marks
Tutor Marked Assignments (TMAs)	4 Assignments with 10 questions each, 30% for the best 3 TMAs Total = 10% X 3 = 30%
Final Examination	70% of overall course marks
Total	100% of Course Marks

3.2.2.8 Course Overview

This table indicates the units, the number of weeks required to complete them and the assignments.

Table 3.2: Course Organizer

Unit	Title of the work	Weeks Activity	Assessment (End of Unit)
	Course Guide		
Module 1	Complex Variables		
Unit 1	Complex Numbers	Weeks 1 |& 2	Assessment 1
Unit 2	Polar Operations with Complex Numbers	Weeks 3 & 4	Assessment 2
Unit 3	De Moivre's Theorem and Application	Week 5	Assessment 3
Module 2			
Unit 1	Limits of functions of complex variables	Week 6	Assessment 4
Unit 2	Continuity of functions of complex variables	Week 7	Assessment 5
Unit 3	Differentiation of complex functions	Week 8	Assessment 6
Module 3			
Unit 1	Analytic Functions I	Weeks 9 & 10	Assessment 7
Unit 2	Analytic functions II	Week 11	Assessment 8

3.2.2.9 Self-Assessment Exercise

Make a list of what you should and should not do to make a success of this Study Material.

4.0 CONCLUSION

This course guide has given you useful guidelines on how to study and use this course material in order to gain the information and learn the skills that would aid you in studying, using ODL self-instructional materials.

What you have learnt here will help you make the necessary preparations for working through the rest of the course.

5.0 SUMMARY

This Study guide has served as a window to you on what to find in the course material and how best to make use of the information in the course material. You learnt the aims and objectives of the course material, the content of the modules and course units that make up the course material as well as how best to work through the course material.

I wish you an enjoyable learning experience, best of luck.

6.0 REFERENCES AND FURTHER READING

These texts and especially the internet resource links will be of enormous benefit to you in learning this course:

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Prince-ton.
- Ravi P,,Kanishka P.,SandraPinelas (2010), An Introduction to complex Analysis Springer New York Dordrecht Heidelberg London
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics

INTRODUCTION TO COMPLEX ANALYSIS

TABLE OF CONTENTS

MODULE 1 COMPLEX VARIABLES

UNIT 1 COMPLEX NUMBERS

CONTENTS

1.0 Introduction

Perhaps because of their name, it is sometimes thought that complex numbers are in some ways more mysterious than real numbers, that a number such as $i = \sqrt{-1}$ is not as "real" as a number like 2 or -351.127 or even π. However, all of these numbers are equally meaningful; they are all useful mathematical abstractions. Although complex numbers are a relatively recent invention of mathematics, dating back just over 200 years in their current form, it is also the case that negative numbers, which were once called fictitious numbers to indicate that they were less "real" than positive numbers, have only been accepted for about the same period of time, and we have only started to understand the nature of real numbers during the past 150 years or so. In fact, if you think about their underlying meaning, π is a far more "complex" number than i. Although complex numbers originate with attempts to solve certain algebraic equations, such as

$$x^2 + 1 = 0,$$

we shall give a geometric definition which identifies complex numbers with points in the plane. This definition not only gives complex numbers a concrete geometrical meaning, but also provides us with a powerful algebraic tool for working with points in the plane.
The equation $x^2 + 1 = 0$ has no real solutions, because for any real number x, the number x^2 is nonnegative, and so $x^2 + 1$ can never be zero. In spite of this it turns out to be very useful to assume that there is a number i for which one has (1) $i^2 = -1$.

In the early 1800's geometric representation of complex numbers was developed which finally made complex numbers acceptable to all Mathematicians. Since then complex numbers have scoped into all branches of Mathematics, in fact it has even been necessary for developing several areas in modern Physics and Engineering.
This unit aims at familiarising the students with complex numbers and the different ways of representing them. The basic algebraic operations on complex numbers shall be extensively discussed.
Complex analysis is a branch of mathematics that involves functions of complex numbers. It provides an extremely powerful tool with an unexpectedly large number of applications in number theory, applied mathematics, physics, hydrodynamics, thermodynamics, and electrical engineering. Rapid growth in the theory of complex analysis and in its applications has resulted in continued interest in its study by students in many disciplines. This has given complex analysis a distinct place in mathematics curricula all over the world, and it is now being taught at various levels in almost every institution.
Finally, we like to reiterate that whatever Mathematics course you study, you will need the knowledge of the subject matter covered in this unit, hence go through carefully and ensure that you have achieved the following objectives.
Throughout this course, the following well-known notations will be used:

N = *{1, 2, · · ·},* the set of all *natural numbers*;
Z = *{· · ·, −2, −1, 0, 1, 2, · · ·},* the set of all *integers*;
Q = *{m/n*: *m, n* ∈ **Z**, *n* ≠0*},* the set of all *rational numbers*;
R = the set of all *real numbers.*
C = the set of complex numbers

Note that $\boldsymbol{N} \subset \boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R} \subset \boldsymbol{C}$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define complex numbers
- perform mathematical operations with complex numbers
- find modulus and argument of complex numbers
- solve exercises on complex numbers.

3.0 MAIN CONTENT

3.1 Complex Numbers

Consider the linear equation 3x + 5 = 0, which has a solution $x = -\frac{5}{3}$, one can always find a real solution any for linear equation ax+ b =0 where $a.b \in \boldsymbol{R}$. Now, what happens if we try to look for the real solutions of any quadratic equation over **R**? Consider one such equation, namely $x^2 + 1 = 0$ that is $x^2 = -1$. This equation has no solution in **R** since the square of any real number cannot be negative.

From about 250 A.D. onwards, mathematicians have been coming across quadratic equation, arising from real life situations which did not have any real solutions. It was in the 16th century that the Italian mathematicians, Cardano and Bombelli started a serious discussion on extending the number system to include square roots of negative numbers. In the next two hundred years, more and more instances were discovered in which the use of square roots of negative numbers helped in finding the solutions of real problems

Definition: Any complex number is then an expression of the form $x + iy$, where x and y are old- fashioned real numbers. The number x, is called the real part of $x + iy$, and y is called its imaginary part.

In other words, a complex is a number of the form $x + yi$, where x and y are real numbers and $i^2 = -1$. We write $x = Re(x + yi)$ and $y = Im(x + yi)$

Caution: i) i is not a real number ii) $Im(x + yi)$is the real number y and not iy

For example, $3 + i2 = 3 + 2i$, the real part of $3 + 2i$ is 3 and the imaginary part of $3 + 2i$ is 2. There are some shortcut notations. For example, the complex number $3 + (-2)i$ is written as $3 - 2i$. Also, every real number is a complex number; for example, $7 = 7 + i(0)$.

The set of all real numbers is denoted by R, and the set of all the complex numbers is denoted by C. Traditionally the letters z and w are used to stand for complex numbers. We denote the set of all complex numbers by C so, $C = \{x + iy/x, y \in R\}$.

Definition: Consider a complex number $z = x + iy$

If $y = 0$, we say z is purely real and If $x = 0$, we say z is purely imaginary.

We usually write the purely real number $x + 0$ as x and the purely imaginary number $0 + iy$ as iy only

Invention of Complex Numbers

In mathematics, we do a lot of solving of polynomial equations, which amounts to finding a root of a polynomial. For example, the solutions to the equations $x^2 = 1$ are the same as the solutions of $x^2 - 1 = 0$, that is, they are the roots of the polynomial $x^2 - 1$. These roots are $x = 1$ and $x = -1$.
However, some polynomial equations have no real number solutions: for example, the equation

$$x^2 + 1 = 0$$

has no real number solutions, because $x^2 + 1 \geq 1$ if x is a real number.
The complex numbers were invented to provide solutions to polynomial equations. For example if we substitute i for x in the equation above, we get a solution

$$i^2 + 1 = -1 + 1 = 0 .$$

In the early days, the boldness of simply defining a new number i as a solution was considered suspicious (hence the term "imaginary part"), just as the existence of √5 was a matter of religious controversy for the ancient Greeks. Today, the complex number system is so deeply rooted in physical theory (e.g. quantum mechanics) that one could argue the complex number system is a more "real" description of the world than the real number system. (The famous physicist Roger Penrose wrote an essay to this effect,

"Nature is complex".) At any rate, students of today are expected to transcend in a blink the worries of past geniuses.
It is pretty easy (from the quadratic formula) to see that with complex numbers, we can find roots for any quadratic polynomial. For example, the two roots of
$z^2 + 3z + 10$ are

$$z = \frac{-3 \pm \sqrt{3^2 - 4(10)}}{2} = \frac{-3 \pm \sqrt{-31}}{2} = \frac{-3}{2} \pm i\frac{\sqrt{31}}{2}$$

.
However, since C is built from R basically by adding in just that one extra element i, and then just taking combinations $a + bi$, it is a rather amazing fact that ANY non-constant polynomial with real coefficients (or even with complex coefficients) has a root which is a complex number. This fact (which we won't prove) is called the Fundamental Theorem of Algebra, stated next.

Theorem: (Fundamental Theorem of Algebra).
Every non-constant polynomial with coefficients in C (or R) has a root in C.
There is an important corollary to the Fundamental Theorem of Algebra.

Theorem: (Factorization Theorem)
Suppose $p(z)$ is a polynomial of degree n at least 1, with complex coefficients, say $p(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ with $c_n \neq 0$. Then p can be factored as a product of linear terms

$$p(z) = c_n(z - z_1)(z - z_2) \ldots .(z - z_n)$$

where the numbers $z_1, z_{2,,} \ldots, z_n$ are the roots of $p(z)$. (Possibly some roots appear more than once.)

Explanation:

Suppose $p(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ is a polynomial, of degree n at least 1, with complex coefficients (which, again, as a special case could be just real numbers). Polynomial long division works just as well with complex coefficients as with real coefficients. So, given a particular complex number z_0, we could use the polynomial long division to find

$$\frac{p(z)}{z - z_0} = q(z) + \frac{w}{z - z_0}$$

for some polynomial $q(z)$ and some complex number w. By multiplying both sides by $(z - z_0)$, this equation becomes

$$p(z) = q(z)(z - z_0) + w .$$

By substituting z_0 for z in this last line, we obtain $w = p(z_0)$. For example,

$$\frac{z^4 - 1}{z - 2} = z^3 + 2z^2 + 4z + 8 + \frac{15}{z - 2}$$

and this produces

$$z^4 - 1 = (z^3 + 2z^2 + 4z + 8)(z - 2) + 15$$

In the example above, where $p(2) = 15$, the number 2 is not a root of $p(z)$ and we got the proper remainder 15 in the last line. If instead of $z - 2$ we use a $z - z_0$ where

z_0 is a root of $p(z)$, then we will have zero remainder, and $z - z_0$will be a factor of $p(z)$. The point: if $p(w) = 0$, then $(z - w)$ is a factor of $p(z)$.
For example, using $p(z) = z^4 - 1$, we see that the complex number i is a root. Doing the polynomial long division, we could find

$$\frac{z^4-1}{z-i} = z^3 + iz^2 - z + i \text{ and } z^4 - 1 = (z^3 + iz^2 - z + i)(z - i)$$

Applying the same procedure to the polynomial $z^3 + iz^2 - z + i$ and one of its roots, we could factor out another linear term, and then another, to end up with

$$z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i),$$

the factorization which corresponds to the four roots $1, -1, i, -i$ of the polynomial $z^4 - 1$. This approach works on any polynomial to produce a factorization into linear terms, as stated in the Factorization Theorem.

Examples:

i) $z^3 + 2z^2 + z = z(z+1)^2$ roots are 0, -1, -1
ii) $z^2 + 2z + 5$ =(z - [-1 + 2i]) (z - [-1 - 2i]) roots are-$-1 \pm 2i$

Geometrical Representation

As already known, the real number can be geometrically represented on the number line. In fact, there is a one-one corresponding between real numbers and points on the number line. You have also seen that a complex number is determined by two real numbers, namely its real and imaginary parts. This observation led the mathematicians Wessel and Gauss to think of representing complex numbers as points in a plane. This geometric representation was given in the early 1800's, it is called an Argand diagram, after the Swiss Mathematician J. R.Argand, who propagated this idea.

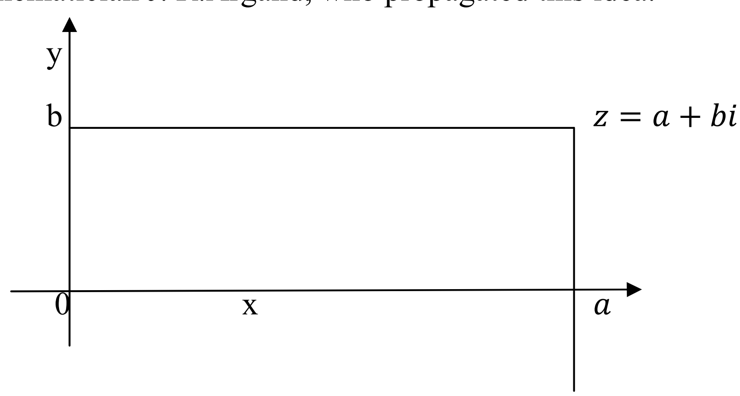


Fig.1.1 Geometric representation of a complex number using Argand diagram

Given real numbers a and b, we picture the complex number $a + ib$ as a point in the plane, with (x, y) coordinate as point (a, b) (see Figure 1.1). The complex plane is just the usual, two dimensional plane, with the interpretation that a point (a, b) in the plane corresponds to the complex number $a + ib$.

Notice that the horizontal axis of the complex plane (the "real axis") corresponds to the set of real numbers. The vertical axis is called the "imaginary axis". The plane in which one plot these complex numbers is called the Complex plane, or Argand plane.

Example

Plot the complex numbers $2 + 3i, -3 + 2i, -3 - 2i,\ 2 - 5i,\ 6i$ on an Argand diagram.

Solution

The figure below shows the Argand diagram. Note that purely real numbers lie on the real axis. Purely imaginary numbers lie on the imaginary axis. Note that complex conjugate pairs such as $-3 \pm 2j$ lie symmetrically on opposite sides of the real axis

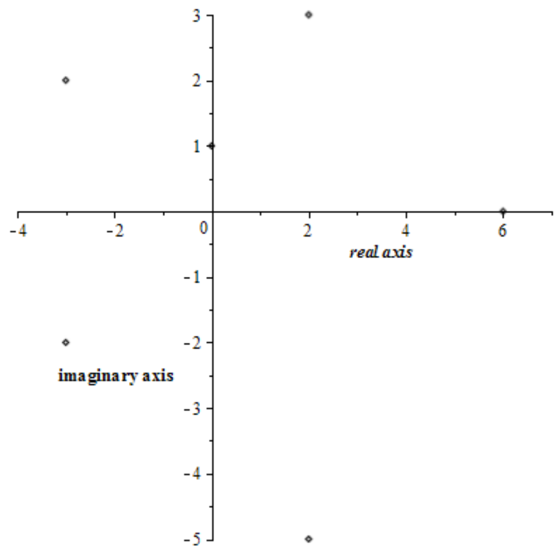


Fig.1.2

3.2 Operations with Complex Numbers

In this section, we shall consider some mathematical operations on complex numbers.

The definitions of addition and multiplication of real numbers are extended to the complex numbers in the only reasonable way.

Addition: Two complex numbers are added simply by adding together their real parts and imaginary parts: we define (a + ib) + (c + id) = (a + c) + i(b + d)

For example, (3 + 2i) + (4 − 6i) = (3 + 4) + i (2 - 6) = (7 − 4i).

Remark: the addition of two complex numbers (x_1, y_1) and (x_2, y_2) are equal and their imaginary parts are equal

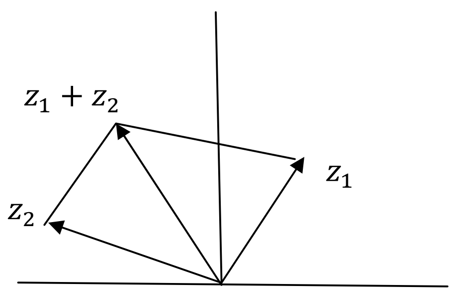


Figure 1.3: Addition of complex numbers.

As in real number system, $0 = 0 + 0i$ is a complex number such that $z + 0 = z$. There is obviously a unique complex number 0 that possesses this property.

Subtraction: The difference between two complex numbers are obtained simply by finding the difference between their real parts and as well as the difference between their imaginary parts: we define $(a + ib) - (c + id) = (a - c) + i\,(b - d)$.
For example, $(3 + 2i) - (4 + 6\,i) = (3 - 4) + i\,(2 - 6) = (-1 - 4\,i)$.

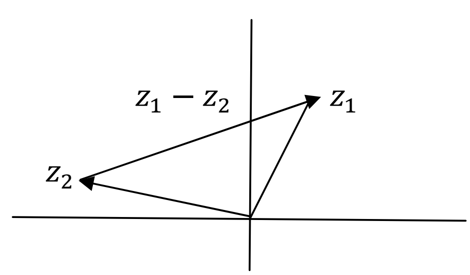


Figure 1.4: Geometry behind the “distance" between two complex numbers.

Multiplication:
$$\begin{aligned} zw &= (a + bi)\,(c + di) \\ &= a(c + d\,i) + b\,i\,(c + di) \\ &= ac + ad\,i + bc\,i + bd\,i^2 \\ &= (ac - bd) + (ad + bc)\,i \end{aligned}$$
where we have used the property $i^2 = -1$ to replace i^2.
$$\begin{aligned} (2 + 3\,i)(4 + 5\,i) &= 2(4 + 5\,i) + 3\,i\,(4 + 5\,i) \\ &= 8 + 10i + 12i + 15i^2 \\ &= 8 + 10i + 12i - 15 \\ &= -7 + 22i \end{aligned}$$
For complex numbers z_1, z_2, z_3 we have the following easily verifiable properties:
(I). Commutativity of addition: $z_1 + z_2 = z_2 + z_1$
(II). Commutativity of multiplication: $z_1 z_2 = z_2 z_1$
(III). Associativity of addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
(IV). Associativity of multiplication: $z_1(z_2 z_3) = (z_1 z_2) z_3$
(V). Distributive law: $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

As an illustration, we shall show only (I).

Let $z_1 = a_1 + b_1 i, z_2 = a_2 + b_2 i$

Then$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i = (a_2 + a_1) + (b_2 + b_1)i$

$$= (a_2 + b_2 i) + (a_1 + b_1 i)$$

$$= (z_2 + z_1)$$

Self Assessment Exercise

Verify properties II) –V)

Clearly, **C** with addition and multiplication forms a field.

We also note that, for any integer *k,*

$$i^{4k} = 1$$
$$i^{4k+1} = i$$
$$i^{4k+2} = -1$$
$$i^{4k+3} = -i$$

Division: To divide two complex numbers one always uses the following trick.

$$\frac{\text{a + bi}}{c+di} = \frac{a+bi}{c+di} \cdot \frac{a-bi}{c-di}$$

$$= \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

Now

$$(c + di)(c - di) = c^2 - (di)^2 = c^2 - d^2 i^2 = c^2 - d^2$$

So $$\frac{\text{a + bi}}{c+di} = \frac{(ac-bd)+(bc-ad)i}{c^2+d^2}$$

$$= \frac{ac-bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2} i, \quad where c^2 + d^2 \neq 0$$

Obviously you do not want to memorize this formula: instead you remember the trick, i.e. to divide $c + di$ into $a + bi$ you multiply numerator and denominator with $c - di$.for any complex number $z = a + bi$, the number $\bar{z} = a - bi$ is called its **complex conjugate**.

Geometrically, conjugating z means reflecting the vector corresponding to z with respect to the real axis.

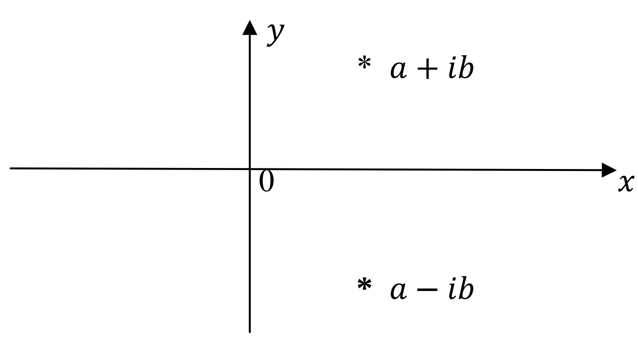


Fig1.5

Notation:

$z = a + bi$, then $\bar{z} = \overline{a + bi}$ or $\bar{z} = a - bi$

A frequently used property of the complex conjugate is the following formula

$$w.\bar{w} = (c + di)(c - di)$$
$$= c^2 - d^2$$
$$= c^2 + d^2$$

Example Find the quotient $\frac{(6+2i)+(1+3i)}{-1+i-2}$

$$\frac{(6+2i)+(1+3i)}{-1+i-2} = \frac{5-i}{-3+i} = \frac{(5-i)}{(-3+i)} \cdot \frac{(-3-i)}{(-3-i)}$$
$$= \frac{-15-1-5i+3i}{9+1}$$
$$= -\frac{8}{5} - \frac{1}{5}i$$

The following notation is used for the real and imaginary parts of a complex number z.

If $z = a + bi$ then a = the Real part of $z = Re\,(z)$, b = the Imaginary Part of $z = Im\,(z)$.

Note that both Rez and Imz are real numbers. A common mistake is to say that $Imz = bi$. the "i" should not be there.

With the definitions above of addition and multiplication, C enjoys all the good arithmetic properties of R (addition and multiplication are commutative and associative; the distributive property holds etc.).

Definition: Two complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$ are equal if $x_1 = x_2$ and $y_1 = y_2$

Thus, the elements of **C** are equal if their real parts are equal and their imaginary parts are equal.

Geometric Properties

From very basic geometric properties of triangles, we get the inequalities

$$-|z| \le Re\,z \le |z| \text{ and } -|z| \le Im\,z \le |z|$$

The square of the absolute value has the property:

$$|x + iy|^2 = x^2 + y^2 = (x + iy)(x - iy)$$

Lemma 1: for any $z, z_1, z_2 \in C$

a) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
b) $|z| \ge 0$,and$|z| = 0$ if and only if $z = 0$
c) $z = -\bar{z}$, if and only if $z \in R$
d) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}},\ \ z_2 \ne 0$
f) $\bar{\bar{z}} = z$
g) $|\bar{z}| = |z|$
h) $|z|^2 = z\bar{z}$

i) $Re\, z = \frac{1}{2}(z + \bar{z})$
j) $Im\, z = \frac{1}{2i}(z - \bar{z})$
k) $\overline{e^{\imath\theta}} = e^{-i\theta}$

A neat formula for the inverse of a non-zero complex number can be generated from part (f) above

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

As an illustration, we shall show the relation d. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, then

$$\overline{z_1 \cdot z_2} = \overline{(a_1 + b_1\imath)(a_2 + b_2\imath)}$$

$$= \overline{(a_1a_2 - b_1b_2) + \imath(a_1b_2 + b_1a_2)}$$

$$= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2) \qquad \text{by c)}$$

$$= (a_1 - b_1 i)(a_2 - b_2 i)$$

$$= \overline{z_1} \cdot \overline{z_2}$$

A famous geometric inequality (which holds for vectors in $\boldsymbol{R^n}$) is the triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2|$$

By drawing a picture in the complex plane, you should be able to come up with a geometric proof of this inequality. To prove it algebraically, we make extensive use of Lemma 1:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) && \text{by h)} \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) && \text{by d)} \\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\ &= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\ &= |z_1|^2 + 2Re\,(z_1\overline{z_2}) + |z_2|^2 && \text{by i)} \end{aligned}$$

Finally by 1)

$$\begin{aligned} |z_1 + z_2|^2 &\le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

which is equivalent to our claim.
For future reference we list several variants of the triangle inequality:

Lemma 2 For $z_1, z_2, \ldots\ldots \in$ **C**, we have the following identities:
(a) The triangle inequality: $|\pm z_1 \pm z_2| \le |z_1| + |z_2|$
(b) The reverse triangle inequality: $|\pm z_1 \pm z_2| \ge |z_1| - |z_2|$.
(c) The triangle inequality for sums: $\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$
The first inequality is just a rewrite of the original triangle inequality, using the fact that $|\pm z| = |z|$ and the last follows by induction.

SELF-ASSESSMENT EXERCISE

1) Find the conjugate of the following expressions:
(i) 3-3i (ii) 4i (iii) -3+4i (iv) 2 - 3i

2) Complete the following table

z	Re z	Im z
i		
$\frac{-2-\sqrt{3}}{5}$		
	0	$0i$
$\frac{(1+i)(3+i)}{2-i}$		
(1 + i) 3		
	31	2i
15		

3) Evaluate the following if w = 3 − 4i and z = −2 + $z = -2 + 7i$.
(a) w + z (b) w – z (c) 3w − 2z (d) $\overline{w}$ (e) zw (f) $\frac{z}{w}$
(g) |z| (h) $\frac{z^2-w}{z+w}$

3.3 Modulus and Argument of Complex Numbers

For any given complex number z = a + bi one defines the absolute value or modulus to be $r = |z| = \sqrt{a^2 + b^2}$, so |z| is the distance from the origin to the point z in the complex plane (see figure 1).

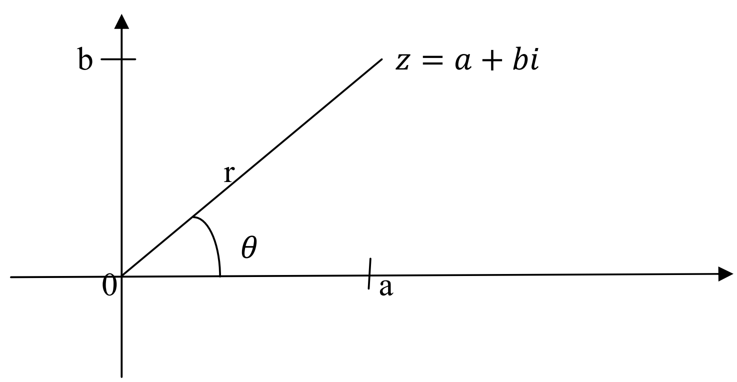


Figure1.6 A complex number $a + ib$

If you picture $z = a + ib$ in the complex plane as in Figure 1, then from the Pythagorean Theorem you can see that |z| is the distance from z to the origin. (Put another way, |z| is the length of the line segment between 0 and z.)

Similarly, if z_1 and z_2 are two complex numbers, then (just as with real numbers) the distance from z_1 to z_2 is $|z_2 - z_1|$. To see this, write z_1 and z_2 in the forms z_1 = a + ib and $z_2 = (a + c) + i(b + d)$ (see Figure 2). Then $z_2 - z_1 = c + id$, and $|z_2 - z_1| = \sqrt{c_2} + d_2$, the distance from z_1 to z_2.

The angle θ is called the argument of the complex number z. Notation: arg z = θ
The argument is defined in an ambiguous way: it is only defined up to a multiple of $2\pi\theta$. E.g. the argument of −1 could be π, or$-\pi$, 2 π, or 3 π, or, etc. In general one says arg(−1) = π + 2k π, where k may be any integer.
From trigonometry one sees that for any complex number z = a + bi one has

$$a = |z|cos\,\theta \text{ and } b = |z|sin\,\theta,$$

so that

$$|z| = |z|cos\,\theta + i|z|sin\,\theta = |z|(cos\,\theta + i\,sin\,\theta)$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

Example: Find argument and absolute value of

i) $z = 2 + i$ ii) $z = \frac{(1+i)(2+i)}{3-i}$

Solution:

i) $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$. z lies in the first quadrant so its argument θ is an angle between 0 and $\frac{\pi}{2}$. From $\tan\tan\theta = \frac{1}{2}$
we then conclude that arg $(2 + i) = \theta = \tan^{-1}(\frac{1}{2}) = 26.5605$

ii) $z = \frac{(1+i)(2+i)}{3-i} = \frac{2+i+2i-1}{3-i} = \frac{1+3i}{3-i}$

this implies that

$$z = \frac{(1+3i)(3+i)}{(3-i)(3+i)} = \frac{3+i+9i-3}{10} = \frac{10i}{10} = i$$

Hence $|z| = 1$ and arg $z = \frac{\pi}{2}$

Example: If $x + iy = r + \frac{s(1+it)}{1-it}$ where r and s are real constant and *x*, *y, t* are real variables. Show that the locus of the point (*x*, *y*) as t varies is a circle.

Solution:

$$x + iy = r + \frac{s(1+it)}{1-it}$$
$$= r + \frac{s(1+it)}{(1-it)} \cdot \frac{(1+it)}{(1+it)}$$
$$= r + \frac{s(1-t^2)}{1+t^2} + \frac{2sit}{1+t^2}$$

Equating the real parts and the imaginary parts in each side of the equation, we have:

$$x = r + \frac{s(1-t^2)}{1+t^2}, \quad y = \frac{2st}{1+t^2}$$

Thus,

$$(x - r)^2 + y^2 = s^2$$

Hence, the locus of the point (*x*, *y*) is a circle centre (a, 0) and radius b.

Another way to describe multiplication of complex numbers is to consider two complex numbers, $x_1 + iy_1$with absolute value r_1 and argument θ_1, and $x_2 + iy_2$ with absolute value r_2 and argument θ_2. This means, we can write

$x_1 + iy_1 = (r_1 \cos\theta_1) + i(r_1 \sin\theta_1)$ and $x_2 + iy_2 = (r_2 \cos\theta_2) + i(r_2 \sin\theta_2)$

To compute the product, we make use of some classic trigonometric identities:

$$x_1 + iy_1 = [(r_1 \cos\theta_1) + i(r_1 \sin\theta_1)][(r_2 \cos\theta_2) + i(r_2 \sin\theta_2)]$$
$$= (r_1 r_2 \cos\theta_1 \cos\theta_2 - r_1 r_2 \sin\theta_1 \sin\theta_2) + i(r_1 r_2 \cos\theta_1 \sin\theta_2 + r_1 r_2 \sin\theta_1 \cos\theta_2)$$
$$= r_1 r_2[(\cos\theta_1 cos\theta_2 - \sin\theta_1 sin\theta_2) + i(\cos\theta_1 sin\theta_2 + \sin\theta_1 cos\theta_2)]$$
$$= r_1 r_2[cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

So the absolute value of the product is $r_1 r_2$ and (one of) its argument is $\theta_1 + \theta_2$. Geometrically, we are multiplying the lengths of the two vectors representing our two complex numbers, and adding their angles measured with respect to the positive x-axis.

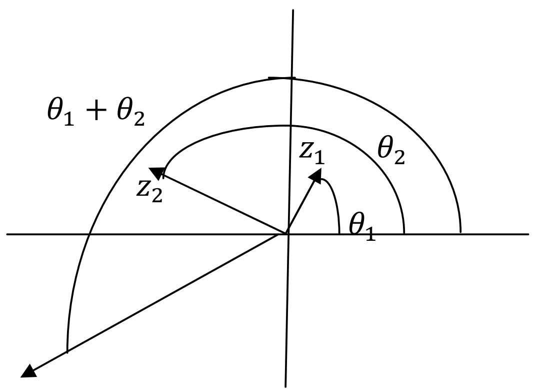


Figure 1.7 Multiplication of complex numbers.

In view of the above calculation, it should come as no surprise that we will have to deal with quantities of the form $\cos\theta + i\ \sin\theta$ (where θ is some real number) quite a bit. We introduce a shortcut notation and define $e^{i\theta} = \cos\theta + i\ \sin\theta$

At this point, this exponential notation is indeed purely a notation, that it has an intimate connection to the complex exponential function. For now, we motivate this ‘maybe’ strange-seeming definition by collecting some of its properties. The reader is encouraged to prove them.

Lemma 2: for any $\theta, \theta_1\theta_2 \in R$,

a) $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

b) $\frac{1}{e^{i\theta}} = e^{-i\theta}$

c) $e^{i(\theta+2\pi)} = e^{i\theta}$

d) $|e^{i\theta}| = 1$

e) $\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$

With this notation, the sentence “The complex number $x + iy$ has absolute value r and argument θ" now becomes the identity $x + iy = ie^{i\theta}$

The left-hand side is often called the rectangular form, the right-hand side the polar form of this complex number.
We now have five different ways of thinking about a complex number: the formal definition, in rectangular form, in polar form, and geometrically using Cartesian coordinates or polar coordinates. Each of these five ways is useful in different situations, and translating between them is an essential ingredient in complex analysis. The five ways and their corresponding notation are listed in Figure 1.4.

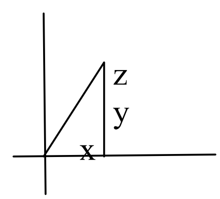
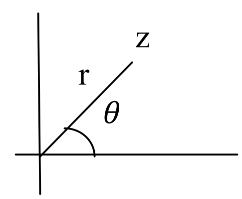
Formal (x, y)	Algebraic	Rectangular $x + iy$	Exponential $re^{i\theta}$
	Geometric	Cartesian 	Polar 

Figure 1.8: Five ways of thinking about a complex number $z \in \boldsymbol{C}$.

4.0 Conclusion

The knowledge acquired in this unit shall be useful in every aspect of our Mathematics Courses to be studied in this university. You are therefore advised to master the materials very well.

5.0 Summary

In the unit, you have studied the following points

1) The definition of a complex number i.e. a complex number is a number of the form $x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. Equivalently, it is a pair $(x, y) \in R\ X\ R$
2) x is the real and y is the imaginary parts of $x + iy$
3) $x_1 + iy_1 = x_2 + iy_2$, iff $x_1 = x_2$ and $y_1 = y_2$
4) The conjugate of $z = x + iy$ is $\bar{z} = x - iy$
5) The geometric representation of complex number is the Argand diagram

6.0 Tutor Marked Assignment

1. Evaluate the following if $w = 3 - 4i$ and $z = -2 + 7i$.

(a) $w + z$ (b) $w - z$

(c) 3w − 2z (d) $\overline{w}$

(e) zw (f) $\frac{z}{w}$

(g) |z| (h) $\frac{z^2-w}{z+w}$

(i) $Re(z - w)$ (j) $Im(3z + w)$

2. Find the real and imaginary parts of each of the following.

$(a)\,\frac{1}{i}$ $(b)\,\frac{3}{1+2i}$

$(c)\,\frac{3-4i}{-2+3i}$ $(d)\,(1+i)^3$

3. For each of the following, write the given z in rectangular coordinates and plot it in the complex plane.

$(a)\ |z| = 3,\ Arg(z) = \frac{\pi}{2}$ $(b)|z| = 5,,\ Arg(z) = \frac{2\pi}{3}$

$(c)\ |z| = 0.5,\ Arg(z) = -\frac{3\pi}{4}$ $(d)\ |z| = 2,\ Arg(z) = \pi$

4. For each of the following, find |z| and $Arg(z)$ and plot z in the complex plane.

$(a)\ z = -i$ $(b)\ z = -5$

$(c)\ z = 1 + i$ $(d)\ z = -1 - i$

$(e)\ z = 2 + 2\sqrt{3i}$ $(f)\ z = \sqrt{3} - i$

5. Suppose w and z are complex numbers with $|w| = 3, Arg(w) = \frac{\pi}{6}$, $|z| = 2$, and $(z) = -\frac{\pi}{3}$. Find both polar and rectangular coordinates for each of the following.

(a) w^2 (b) z^3 (c) wz (d) $\frac{w}{z}$ $(e)\ \frac{z}{w^2}$ $(f)\ w^5$

6. Find all the roots of the polynomial $P(z) = z^6 - 1$ and plot them in the complex plane.

7. Let $v = a_1 + b_1 i, w = a_2 + h_2 i$, and $z = a_3 + b_3 i$ be complex numbers. Verify each of the following. $(a)\ v + w = w + v$ $(b)\ vw = wv$

$(c)\ v(w + z) = vw + vz$ $(d)\ (v + w) + z = v + (w + z)$

$(e)\ v(wz) = (vw)z$ $(f)\ (w + z)^2 = w^2 + 2wz + z^2$

8. Suppose z is a complex number with $|z| = r$ and arg $(z) = \theta$

(a) Let w be a complex number with $|w| = \sqrt{r}$ and $arg(w) = \frac{\theta}{2}$. Show that $w^2 = z$.

(b) Let v be a complex number with $|w| = \sqrt{r}$ and arg(v) = $\frac{\theta}{2} + \pi$. Show that $v^2 = z$.

(c) From (a) and (b) we see that every nonzero complex number has two distinct square roots. Find the square roots, in rectangular form, of $1 + \sqrt{3i}$ and -9.

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex
- Ravi P,,Kanishka; P. SandraPinelas (2010), An Introduction to complex Analysis Springer New York Dordrecht Heidelberg London

UNIT 2 POLAR OPERATIONS WITH COMPLEX NUMBERS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall examine complex numbers in polar forms. The polar form of complex numbers gives interesting results which will be examined in this unit.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- express complex numbers in polar form
- carry out multiplication and division of complex numbers
- recall the De Moivre's theorem and apply it appropriately
- find roots and work with fractional powers of complex numbers in polar form
- solve correctly the exercises that follow after the unit.

3.0 Main Content

3.1 Polar notation

When we write a complex number z in the form $z = x + yi$, we refer to x and y as the rectangular or Cartesian coordinates of z. We now consider another method of representing complex numbers. Let us begin with a complex number $z = x + yi$ written in rectangular form. Assume for the moment that x and y are not both zeroes. If we let θ be the angle between the real axis and the line segment from $(0, 0)$ to (x, y), measured in the counter clockwise direction, then z is completely determined by the two numbers |z| and θ . We call θ the argument of z and denote it by arg (z).

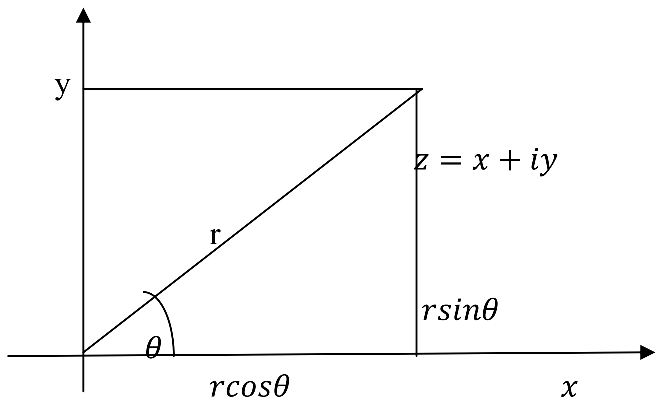


Figure 1 Polar coordinates for a complex number

Geometrically, if we are given |z| and θ, we can locate z in the complex plane by taking the line segment of length |z| lying on the positive real axis, with a fixed endpoint at the origin, and rotating it counter clockwise through an angle θ; the final resting point of the rotating endpoint is the location of z. Algebraically, if $z = x + yi$ is a complex number with r = |z| and $\theta = \arg z$, then

$$x = r\cos\theta \qquad (1)$$

and

$$y = r\sin\theta. \qquad (2)$$

Together, r and θ are called the polar coordinates of z. (see figure 1)

Example If |z| = 2 and arg (z) = $\frac{\pi}{6}$,

Then

$$z = 2cos\frac{\pi}{6} + 2sin\frac{\pi}{6}i$$
$$= \sqrt{3} + i$$

Example If z = 1 – i

then

$$|z| = \sqrt{2} \text{ and arg (z)} = -\frac{\pi}{4}.$$

Note that in the last example we could have taken arg (z) = $\frac{7\pi}{4}$, or, in fact,

$$\arg(z) = -\frac{\pi}{4} + 2n\pi$$

for any integer n. In particular, there are an infinite number of possible values for arg (z) and we will let arg (z) stand for any one of these values. At the same time, it is often important to choose arg (z) in a consistent fashion; to this end, we call the value of arg (z) which lies in the interval $(-\pi, \pi)$ the principal value of arg (z) and denote it by arg(z). For our example, arg (z) = $-\arg z = -\frac{\pi}{4}$.

In general, if we are given a complex number in rectangular coordinates, say $z = x + yi$, then, as we can see from Figure 1., the polar coordinates r = |z| and θ = arg(z) are determined by

$$r = \sqrt{x^2 + y^2} \qquad (3)$$

and

$$\tan\theta = \frac{y}{x} \qquad (4)$$

where the latter holds only if $x \neq 0$. If $x = 0$ and $y \neq 0$, then z is purely imaginary and hence lies on the imaginary axis of the complex plane. In that case, $\theta = \frac{\pi}{2}$ if y > 0 and $\theta = -\frac{\pi}{2}$ if y < 0. If both x = 0 and y = 0, then z is completely specified by the condition r = 0 and θ may take on any value.

Note that, since the range of the arc tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$ the condition

$$\tan\theta = \frac{y}{x}$$

only implies that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

if x > 0, that is, if θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Example Suppose z = −1 - $i\sqrt{3}$. Then

$$|z| = \sqrt{1+3} = 2 = 2$$

and if

$$\theta = \arg(z),$$

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}.$$

Since z lies in the third quadrant, we have

$$\arg(z) = \arg z = -\frac{2\pi}{3}$$

3.2 Multiplication and Division of Complex Numbers in polar form

Suppose z_1 and z_2 are two non zero complex numbers with $||z_1| = r_1$, $|z_2| = r_2$,

$arg(z_1) = \theta_1$ and $arg(z_2) = \theta_2$.

Then

$$z_1 = r_1 \cos\theta_1 + r_1 \sin\theta_1\, i$$
$$= r_1(\cos\theta_1 + \sin\theta_1\, i)$$

and

$$z_2 = r_2 \cos\theta_2 + r_2 \sin\theta_2\, i$$
$$= r_2(\cos\theta_2 + \sin\theta_2\, i)$$

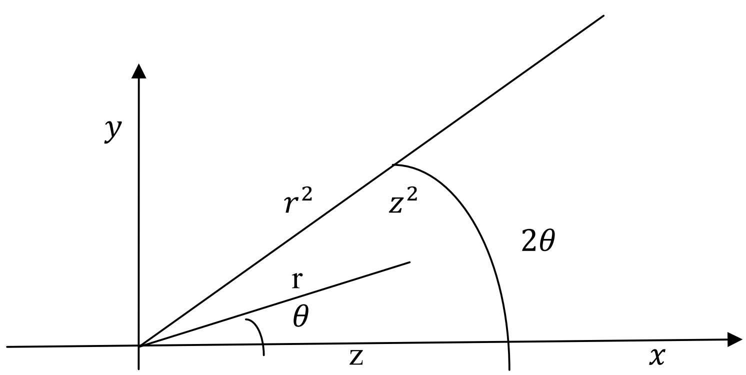


Figure 2 Geometry of z and z^2 in the complex plane

Hence

$$z_1 z_2 = (r_1 \cos\theta_1 + r_1 \sin\theta_1 i)(r_2 \cos\theta_2 + r_2 \sin\theta_2 i)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 i + \sin\theta_1\cos\theta_2 i - \sin\theta_1\sin\theta_2)$$
$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 i + \cos\theta_1\sin\theta_2 i)]$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i]$$

It follows that

$$= |z_1 z_2| = r_1 r_2$$
$$= |z_1||z_2| \qquad (5)$$

and

$$arg(z_1 z_2) = \theta_1 + \theta_2$$
$$= arg(z_1) + arg(z_2) \qquad (6)$$

In other words, the magnitude of the product of two complex numbers is the product of their respective magnitudes and the argument of the product of two complex numbers is the sum of their respective arguments.
In particular, for any complex number z, $|z^2| = |z|^2$ and arg(z_2) = 2 arg(z).
More generally, for any positive integer n,

$$|z^n| = |z|^n \qquad (7)$$

and

$$\arg z^n = n \arg z \qquad (8)$$

See Figure 2.
If z is a complex number with |z| = r and $\arg z = \theta$, then

$$z = r(\cos\theta + \sin\theta\, i)$$

and

$$\bar{z} = r(\cos\theta - \sin\theta)$$
$$r(\cos\theta - \sin\theta i) = r(\cos(-\theta) + \sin(-\theta)i) \qquad (9)$$

hence

$$|\bar{z}| = |z| \qquad (10)$$

and

$$\arg(\bar{z}) = -\arg(z), \qquad (11)$$

in agreement with our previous observation that $\bar{z}$ is obtained from z by reflection about the real axis.

Division

If z_1 and z_2 are two nonzero complex numbers with $|z_1| = r_1$, $|z_2| = r_2$, arg(z_1) = θ_1, and arg(z_2) = θ_2, then

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_1}}{z_2\overline{z_2}}$$
$$= \frac{r_1\, r_2\, (\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\mathrm{i})}{(r_2)^2}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)i]$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (12)$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2. \qquad (13)$$

In other words, the magnitude of the quotient of two complex numbers is the quotient of their respective magnitudes and the argument of the quotient of two complex numbers is the difference of their respective arguments.

Example Let z = 2(cos $\left(\frac{\pi}{12}\right)$+ sin$\left(\frac{\pi}{12}\right) i$) and w = 3(cos$\left(\frac{\pi}{6}\right)$+ sin$\left(\frac{\pi}{6}\right) i$). Then

$$zw = 6\left[cos\left(\frac{\tau}{12}+\frac{\pi}{6}\right)+sin\left(\frac{\tau}{12}+\frac{\pi}{6}\right)i\right]$$
$$= 6\left[cos\left(\frac{\tau}{4}\right)+sin\left(\frac{\tau}{4}\right)i\right]$$
$$= 6\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$
$$=3\sqrt{2}+3\sqrt{2}\,i$$

Also,

$$\frac{z}{w} = \frac{2}{3}\left[cos\left(\frac{\tau}{12}-\frac{\pi}{6}\right)+sin\left(\frac{\tau}{12}-\frac{\pi}{6}\right)i\right]$$
$$= \frac{2}{3}\left[cos\left(-\frac{\tau}{12}\right)+sin\left(-\frac{\tau}{12}\right)i\right]$$
$$= \frac{2}{3}\left[cos\left(\frac{\tau}{12}\right)-sin\left(\frac{\tau}{12}\right)i\right]$$
$$= 0.6440 - 0.1725i$$

where we have rounded the real and imaginary parts to four decimal places .

Example: Write the quotient $\frac{1+i}{\sqrt{3}-i}$ in polar form.

The polar forms of $1 + i = \sqrt{2}\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)$

And

$$\sqrt{3}-i = 2\left(cos\left(-\frac{\pi}{6}\right)+iisin\left(-\frac{\pi}{6}\right)\right)$$

Therefore,

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2}\left\{cos\left[\frac{\pi}{4}-\left(-\frac{\pi}{6}\right)\right]+isin\left[\frac{\pi}{4}-\left(-\frac{\pi}{6}\right)\right]\right\}$$
$$\frac{\sqrt{2}}{2}\left\{cos\left(\frac{5\pi}{12}\right)+isin\left(\frac{5\pi}{12}\right)\right\}$$

Example:

Let $z = cos\left(\frac{\tau}{4}\right)+sin\left(\frac{\tau}{4}\right)i = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

Since $|z| = 1$ and arg $z = \frac{\tau}{4}$

z is a point on the unit circle centred at the origin,

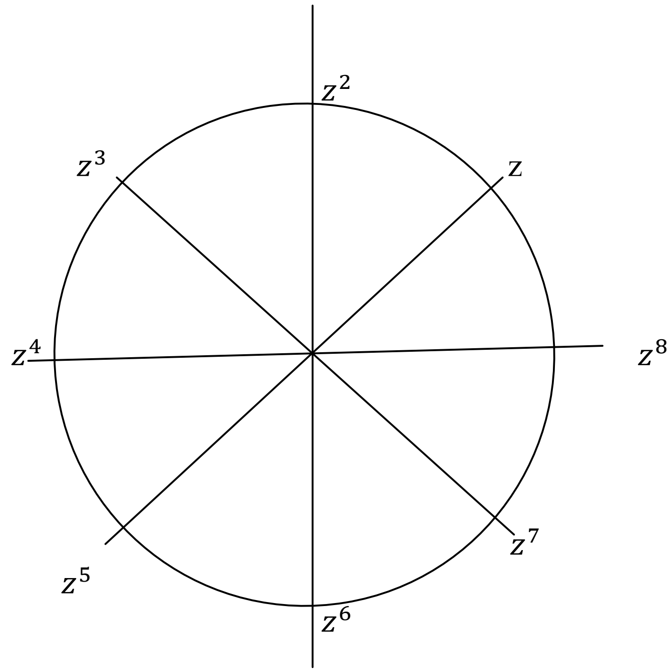


Figure 3 Powers of $z = \cos\frac{\tau}{4} + \sin\frac{\tau}{4} i$

one -eighth of the way around the circle from (1, 0) (see Figure 3). Then

$$z^2 = cos\left(2 \cdot \frac{\tau}{4}\right) + sin\left(2 \cdot \frac{\tau}{4}\right) i = cos\left(\frac{\tau}{2}\right) + sin\left(\frac{\tau}{2}\right) i = i$$
$$z^3 = cos\left(3 \cdot \frac{\tau}{4}\right) + sin\left(3 \cdot \frac{\tau}{4}\right) i = cos\left(\frac{3\tau}{4}\right) + sin\left(\frac{3\tau}{4}\right) i = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$
$$z^4 = cos\left(4 \cdot \frac{\tau}{4}\right) + sin\left(4 \cdot \frac{\tau}{4}\right) i = cos(\pi) + sin(\pi) i = -1$$
$$z^5 = cos\left(5 \cdot \frac{\tau}{4}\right) + sin\left(5 \cdot \frac{\tau}{4}\right) i = cos\left(\frac{5\tau}{4}\right) + sin\left(\frac{5\tau}{4}\right) i = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$
$$z^6 = cos\left(6 \cdot \frac{\tau}{4}\right) + sin\left(6 \cdot \frac{\tau}{4}\right) i = cos\left(\frac{3\tau}{2}\right) + sin\left(\frac{3\tau}{2}\right) i = -i$$
$$z^7 = cos\left(7 \cdot \frac{\tau}{4}\right) + sin\left(7 \cdot \frac{\tau}{4}\right) i = cos\left(\frac{7\tau}{4}\right) + sin\left(\frac{7\tau}{4}\right) i = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$
$$z^8 = cos\left(8 \cdot \frac{\tau}{4}\right) + sin\left(8 \cdot \frac{\tau}{4}\right) i = cos(2\pi) + sin(2\pi) i = 1$$
$$z^9 = zz^8\ (z)(1) = z$$

Hence each successive power of z is obtained by rotating the previous power through an angle of $\frac{\pi}{4}$ on the unit circle centred at the origin; after eight rotations, the point has returned to where it started. See Figure 3. Notice in particular that z is a root of the polynomial

$$P(w) = w^8 - 1.$$

In fact, z^n is a solution of $w^8 - 1 = 0$ for any positive integer n since
$(z^n)^8 - 1 = (z^8)^n - 1 = 1^n - 1 = 1 - 1 = 0$.
Thus there are eight distinct roots of P (w), namely, z, z^2, z^3, z^4, z^5, z^6, z^7, and z^8, only two of which, $z^4 = -1$ and $z^8 = 1$, are real numbers.

3.3 The Complex Exponential Function

In this session, we shall first show that every complex number can be written in exponential form and then use this form to raise a rational power to a given complex number. We shall also extract roots of a complex number. Finally, we shall prove that complex numbers cannot be ordered.

If $z = x + iy$, then e^z is defined to be the complex number

$e^z = e^x(\cos y + i \sin y)$. (3.1)

This number e^z satisfies the usual algebraic properties of the exponential function. For example,

$e^{z_1}e^{z_2} = e^{z_1+z_2}$ and $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

In fact, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then, in view of the previous section, we have

$$e^{z_1}e^{z_2} = e^{z_1}(cosy_1 + isiny_1)e^{z_2}(cosy_2 + isiny_2)$$

$= e^{x_1+x_2}\left(cos(y_1 + y_2) + isin(y_1 + y_2)\right)$

$= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}$

In particular, for $z = iy$, the definition above gives one of the most important formulas of Euler

$$e^{iy} = \cos y + isiny \qquad (3.)$$

which immediately leads to the following identities:

$cosy = Re(e^{iy}) = \frac{e^{iy}+e^{-iy}}{2}, siny = Im(e^{iy}) = \frac{e^{iy}-e^{-iy}}{2i}$

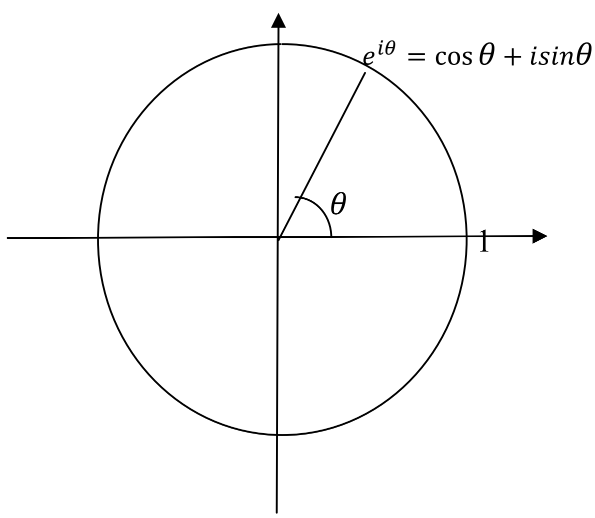


Figure 4. Euler's definition of $e^{i\theta}$

When $y = \pi$, formula (3.2) reduces to the amazing equality

$$e^{\pi t} = \cos\pi + i\sin\pi$$
$$= -1 .$$

This leads to famous Euler's formula

$$e^{\pi i} + 1 = 0,$$

which combines the five most basic quantities in mathematics: $e, \pi, i, 1$, and 0. In this relation, the transcendental number *e* comes from calculus, the transcendental number π

comes from geometry, and i comes from algebra, and the combination $e^{\pi i}$gives -1, the basic unit for generating the arithmetic system for counting numbers.This seems a good definition because e^{it} can be defined and anything can be done with it.
We now substitute it in the Taylor series thus:

$$e^x{:}e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$
$$= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots$$
$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots)$$
$$= \cos t + i \sin t$$

This is not a proof, because before we had only proved the convergence of the Taylor series for e^x if x was a real number, and here we have pretended that the series is also good if you substitute $x = it$.
As a function of t the definition above gives us the correct derivative namely, using the chain rule (i.e. pretending it still applies for complex functions) we would get

$$\frac{de^{it}}{dt} = e^{it}$$

Indeed, this is correct. To see this proceed from our definition

$$\frac{de^{it}}{dt} = \frac{d(cos\ t + i \sin t)}{dt}$$
$$= \frac{d(cos\ t)}{dt} + i\frac{d(sin\ t)}{dt}$$
$$= -\sin t + i\ cos\ t$$
$$= i(\cos t + i\ sin\ t)$$

the formula $e^x . e^x = e^{x+y}$still holds.
Rather, we have $e^{it+is} = e^{it}e^{is}$.
To check this, replace the exponentials by their definition:

$$e^{it}e^{is} = (\cos t + i \sin\ t)(\cos t + i\ sin\ t)$$
$$= \cos(t+s) + i \sin(t+s) = e^{i(t+s)}$$

Requiring $e^x . e^y = e^{x+y}$
to be true for all complex numbers helps us decide what e^{a+bi}shoud be for arbitrary complex numbers a + bi.

Definition

For any complex number a + bi we set

$$e^{a+bi} = e^a e^{bi}$$
$$= e^a(\cos b + i \sin b)$$

One verifies as above that this gives us the right behavior under differentiation. Thus, for any complex number r = a + bi the function

$$y(t) = e^{rt} = e^{at}(\cos bt + i\ sin\ bt).$$

satisfies

$$y'(t) = \frac{de^{rt}}{dt} re^{rt}$$

Example: Evaluate i) $e^{i\pi}$ ii) $e^{-1+\frac{i\pi}{2}}$
Solution

i) Using Euler's equation,

$$e^{i\pi} = \cos\pi + i\sin\pi \; = -1 + i(0)$$
$$= -1$$

ii) $e^{-1+\frac{i\pi}{2}} = e^{-1}e^{\frac{i\pi}{2}}$

$$= e^{-1}\left(\cos\frac{\pi}{2} + sin\frac{\pi}{2}\right)$$
$$= \frac{1}{e}[0 + i(0)]$$
$$= \frac{i}{e}$$

Using Euler's formula, we can express a complex number $z = r(\cos\,\theta + i\,\sin\,\theta)$ in exponential form; i.e., $z \;=\; r(cos\,\theta \;+ i\,sin\,\theta) = re^{i\theta}$
The rules for multiplying and dividing complex numbers in exponential form are given by

$$z_1 z \;\; = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = (r_1 r_2)e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right)e^{i(\theta_1+\theta_2)}$$

Also, the complex conjugate of the complex number $z = re^{i\theta}$ is given by$\bar{z} = re^{-i\theta}$

Example: Compute 1**)** $\frac{1+i}{\sqrt{3}-i}$ 2) $(1+i)^{24}$ 3)$\left(\sqrt{3} - i\right)^{12}$

Solution: $1 + i = \sqrt{2}e^{i\tau/4}$ and $\sqrt{3} - i = 2e^{-i\pi/6}$

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\tau/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{5\pi i/12}$$

$$(1+i)^{24} = \left(\sqrt{2}e^{i\tau/4}\right)^{24} = 2^{12}e^{6\pi i}$$
$$= 2^{12}$$

3) $\left(\sqrt{3} - i\right)^{12} = \left(2e^{-i\pi/6}\right)^{12} = 2^{12}e^{-2\pi i}$

Note that the Euler's equation provides us with an easier method of proving De Moivre's Theorem.

4.0 Conclusion

In this unit, we have studied some theorems and determine the roots of equation using complex variables. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 Summary

In the unit, you have studied the following points

- Multiplication and Division of Complex Numbers in polar form
- The Complex Exponential Function

6.0 Tutor Marked Assignment

1) Write the number in the form $a + bi$

i) $e^{\frac{i\pi}{2}}$ ii) $e^{\frac{i\pi}{3}}$ iii) $e^{\frac{i\pi}{4}}$

iv) $e^{2\pi i}$ v) $e^{-i\pi}$ vi) $e^{2+\pi i}$ vii) $e^{\pi+i}$

2) For each of the following, write the given z in rectangular coordinates and plot it in the complex plane.

(a) $|z| = 3, arg\ (z) = \frac{\tau}{2}$
(b) $|z| = 5, arg\ (z) = \frac{2\tau}{3}$
(c) $|z| = 0.5, arg(z) = -\frac{3\tau}{4}$
(d) $|z| = 2, arg\ (z) = \pi$

3). For each of the following, find |z| and arg(z) and plot z in the complex plane.

(a) $z = -i$ (b) $z = -5$ $(c)\ z = 1 + i$
(d) $z = -1 - i$ (e) $z = 2 + 2\sqrt{3}\,i$ (f) $z = \sqrt{3} - i$

4) Suppose w and z are complex numbers with $|w| = 3, arg(w) = \frac{\tau}{6}$, $|z| = 2$, and $\arg z = -\frac{\tau}{3}$. Find both polar and rectangular coordinates for each of the following.

(a) w^{2} (b) z^2 $(c)\ wz$ (d) $\frac{w}{z}$ e) $\frac{z}{w^2}$ f) w^5

5) Let $v = a_1 + b_1 i$, $w = a_2 + b_2 i$, and $z = a_3 + b_3 i$ be complex numbers. Verify each of the following.

$(a)\ v + w = w + v$ $(b)\ vw = wv$ $(c)\ v(w + z) = vw + vz$
$(d)\ (v + w) + z = v + (w + z)$ $(e)\ v(wz) = (vw)z$
(f) $(w + z)^2 = w^2 + 2wz + z^2$

6) Suppose z is a complex number with $|z| = r$ and $(z) = \theta$.

(a) Let w be a complex number with $|w| = \sqrt{r}$ and $arg(w) = \frac{\theta}{2}$.
Show that $w^2 = z$.

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*

- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex
- Ravi P. Kanishka; P.SandraPinelas (2010), An Introduction to complex Analysis Springer New York Dordrecht Heidelberg London

UNIT 3 De Moivre's Theorem and Application

CONTENTS

1.0 INTRODUCTION

In this unit we introduce De Moivre's theorem and examine some of its consequences. We shall see that one of its uses is in obtaining relationships between trigonometric functions of multiple angles (like sin $3x$, cos $7x$ etc) and powers of trigonometric functions (like sin^2x , cos^4x etc).
Another important aspect of De Moivre's theorem lies in its use in obtaining complex roots of polynomial equations. In this application we re-examine our definition of the argument arg(z) of a complex number.

2.0 Objectives

At the end of this unit, you should be able to:

- employ De Moivre's theorem in a number of applications
- understand more clearly the argument arg(z) of a complex number
- obtain complex roots of complex numbers

3.0 Main Content

3.1 De Moiré's formula.

For any complex number z, the argument of its square z^2 is

$$\begin{aligned}\arg z^2 &= arg(z \cdot z)\\ &= arg\, z + arg\, z\\ &= 2\, arg\, z.\end{aligned}$$

The argument of its cube is

$$\begin{aligned}arg\, z^3 &= \arg z. z^2\\ &= \arg z + arg\, z^2\\ &= arg\, z + 2arg\, z\\ &= 3\, arg\, z.\end{aligned}$$

Continuing like this, one finds that

$$arg\ z^n = n\ arg\ z \text{ for any integer n.}$$

Applying this to $z = cos\theta + i \sin\theta$ you find that z^2 is a number with absolute value $|z^2| = |z|^2 = 1^n = 1$, and argument n, $\arg z = n\theta$. Hence $z^n = cos\ n\theta + i\ sin\ n\theta$

So we have found

$$(\cos\theta + \text{i} \sin\theta)^n = \cos n\theta + i \sin n\theta$$

This is de Moiré's formula named after the French Mathematician Abraham De Moivré's (1667 – 1754)

Theorem: De Moivre's theorem states that if

$$z = r(cos\ \theta + isin\ \theta)$$

and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n\ (cos\ n\theta + i\ sin\ n\theta)$$

The theorem says that to take the n^{th} power of a complex number we take the n^{th} power of the modulus and multiply the argument by n

For instance, for n = 2 this tells us that

$$cos\ 2\theta + i\ sin2\theta = (\cos\theta + i\sin\theta)^2$$
$$= cos^2\theta - sin^2\theta + 2icos\theta sin\theta.$$

Comparing real and imaginary parts on left and right hand sides this gives you the double angle formulas

$$cos2\theta = cos^2\theta - sin^2\theta$$

and

$$sin\ 2\theta = 2 \sin\theta \cos\theta$$

For $n = 3$ you get, using the Binomial Theorem, or Pascal's triangle,

$$(\cos\theta + i\sin\theta)^3 = cos^3\theta + 3\text{i}\ cos^2\theta \sin\theta + 3i^2 \cos\theta sin^2\theta + i^3\ sin^3\theta$$
$$= cos^3\theta - 3\ cos\ \theta sin^2\theta + i\ (3\ cos^2\theta \sin\theta - sin^3\theta)$$

so that

$$cos3\theta = cos^3\theta\ sin\theta - 3\ cos\ \theta sin^2\theta$$

and

$$sin3\theta = cos^3\theta\ sin\theta - sin^3\theta$$

In this way it is fairly easy to write down similar formulas for sin4θ, sin5θ, etc.. .

It can be shown that the theorem is true for all rational values of n. Now suppose n is a negative integer and we let $n = -m$ where m is a positive integer then,

$$(\cos\theta + isin\ \theta)^{-m} = \frac{1}{(\cos\theta + isin\ \theta)^m}$$
$$= \cos(-m\theta) + isin(-m\theta)$$
$$= \cos(n\theta) + isin(n\theta)$$

It can also be proved for fractional angles. Recall that by De'moivre's theorem

$$\left(\cos\frac{p}{q}\theta + isin\ \frac{p}{q}\theta\right)^q = \cos p\theta + isin\ p\theta$$
$$= (\cos\theta + isin\ \theta)^p$$

It follows that $\cos\frac{p}{q}\theta + isin\ \frac{p}{q}\theta$ is a qth root of $(\cos\theta + isin\ \theta)^p$

De Moivre's theorem has been proved for all rational values of n.

There is a need to find other values of $(\cos\theta + isin\ \theta)^{\frac{p}{q}}$

To do this, suppose that

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \rho(\cos\varphi + i\sin\varphi)$$

Then,

$$(\cos\theta + i\sin\theta)^p = \rho^q(\cos\varphi + i\sin\varphi)^q$$

$$\Rightarrow \quad \cos p\theta + i\sin p\theta = \rho^q(\cos q\varphi + i\sin q\varphi)$$

Equating the real and imaginary parts, we have

$$\cos p\theta = p^q \cos q\varphi$$

and

$$\sin p\theta = \sin q\varphi$$

By squaring and adding, we obtain $p^{2q} = 1$ and (since ρ, which is the modulus of a complex number is positive) $\rho = 1$ therefore

$$\cos p\theta = \cos q\varphi\ ; \qquad \sin p\theta = \sin q\varphi\ ,$$

these equations are satisfied by

$$q\varphi = p\theta + 2k\pi; \quad k = 0 \text{ or any integer.}$$

Therefore,

$$\varphi = \frac{p\theta + 2k\pi}{q}$$

Example: Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$

Solution:

$$\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$$

The polar form of $\frac{1}{2} + \frac{1}{2}i$ is $\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

by De Moivre's theorem

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

$$= \mathbf{\frac{1}{32}}\boldsymbol{i}$$

Example: Evaluate $(1 + i)^{20}$ using De Moivre's theorem

Solution: $z = 1 + i, r = \sqrt{2}\ and \tan\theta = \frac{1}{1} = 1$

This implies that $\theta = \frac{\pi}{4}$

$$\therefore\ z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Hence, $(1 + i)^{20} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{20}$

$$= \left(2^{1/2}\right)^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right)$$

$$= 2^{10}(\cos 5\pi + i\sin 5\pi)$$

$$= 2^{10}[-1 + i(0)]$$

$$= -2^{10} = -1024$$

3.2 Roots and Fractional Power of a Complex Number

If n is a positive integer, the nth roots of a complex number are by definition the value of ω which satisfies the equation $\omega^n = z$

If $\omega = \rho(\cos\varphi + isin\,\varphi)$ and $z = r(\cos\theta + isin\,\theta)$ then

$\rho^n(\cos\, n\varphi + isin\, n\varphi) = r(\cos\theta + isin\,\theta)$ where ρ^n= r and $n\varphi = \theta + 2k\pi$, k is an integer or zero. By definition ρ and r are positive, such that $\rho = \sqrt[n]{r}$ also, $\varphi = \frac{\theta+2k\pi}{n}$

Taking in succession the values of $k = 0, 1, 2, 3 \ldots n$, we find that

$\cos\left(\frac{\theta+2k\pi}{n}\right) + isin\left(\frac{\theta+2k\pi}{n}\right)$ has n distinct values.

Hence there are n distinct nth roots of z given by the formula

$\omega_k = \sqrt[n]{r}\left[\cos\frac{\theta+2k\pi}{n} + isin\,\frac{\theta+2k\pi}{n}\right]$, $k = 0,1,2,3, \ldots\ldots$ n-1

In a situation where n is a rational number say $n = \frac{p}{q}, p\ and\ q$ are integers and q is positive, the value of $z^n$are the values of ω which satisfy the equation

$$\omega^q = z^p$$

Hence if $z = r(\cos\theta + isin\,\theta)$ then the q values of $z^{p/q}$ given by the formula

$$\omega_m = \sqrt[q]{r^p}\left[\cos\frac{\theta+2m\pi}{q} + isin\,\frac{\theta+2m\pi}{q}\right],$$

where $\sqrt[q]{r^p}$ is the unique positive qth root of r^p

De Moivre's theorem can also be used to find the roots of complex numbers

Example: Find the fifth roots of -1

Solution: Recall that $-1 = \cos\pi + i\sin\pi$

If $z^5 = -1 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$, $k = 0,1,2,3, \ldots\ldots$ n-1

Therefore,

$$z = \cos\left(\frac{\pi+2k\pi}{5}\right) + i\,sin\left(\frac{\pi+2k\pi}{5}\right)$$

For $k = 0,1,2,3,4$, the solutions are:

$$z = \cos\left(\frac{\pi}{5}\right) + i\,sin\left(\frac{\pi}{5}\right)$$
$$z = \cos\left(\frac{3\pi}{5}\right) + i\,sin\left(\frac{3\pi}{5}\right)$$
$$z = \cos\left(\frac{5\pi}{5}\right) + i\,sin\left(\frac{5\pi}{5}\right) = \cos\pi + i\sin\pi$$
$$z = \cos\left(\frac{7\pi}{5}\right) + i\,sin\left(\frac{7\pi}{5}\right)$$
$$z = \cos\left(\frac{9\pi}{5}\right) + i\,sin\left(\frac{9\pi}{5}\right)$$

Example: Find the sixth roots of $z = -8$ and graph these roots in the complex plane.

Solution: In trigonometric form, $z = 8(\cos\pi + i\sin\pi)$

By $\omega_m = \sqrt[q]{r^p}\left[\cos(\frac{\theta+2m\pi}{q}) + i\,sin\,(\frac{\theta+2m\pi)}{q}\right]$

$= \sqrt[6]{8}\left[\cos(\frac{\pi+2k\pi}{6}) + i\,sin\,(\frac{\pi+2k\pi}{6})\right]$

the sixth roots of - 8 is obtained by taking $k = 0,1,2,3,4,5$

$$\omega_0 = \sqrt[6]{8}\left[\cos(\frac{\pi}{6}) + i\,sin\,(\frac{\pi}{6})\right] = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\omega_1 = \sqrt[6]{8}\left[\cos(\tfrac{\pi}{2}) + i\,sin\,(\tfrac{\pi}{2})\right] = \sqrt{2}\,i$$
$$\omega_2 = \sqrt[6]{8}\left[\cos(\tfrac{5\pi}{6}) + i\,sin\,(\tfrac{5\pi}{6})\right] = \sqrt{2}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right)$$
$$\omega_3 = \sqrt[6]{8}\left[\cos(\tfrac{7\pi}{6}) + i\,sin\,(\tfrac{7\pi}{6})\right] = \sqrt{2}\left(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)$$
$$\omega_4 = \sqrt[6]{8}\left[\cos(\tfrac{3\pi}{2}) + i\,sin\,(\tfrac{3\pi}{2})\right] = -\sqrt{2}\,i$$
$$\omega_5 = \sqrt[6]{8}\left[\cos(\tfrac{11\pi}{6}) + i\,sin\,(\tfrac{11\pi}{6})\right] = \sqrt{2}\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)$$

All these points lie on the circle of radius $\sqrt{2}$ as shown in figure 3.1

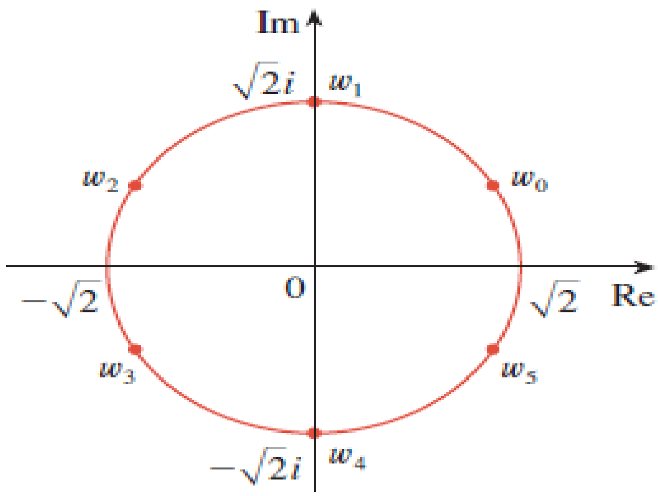


Figure 3.1 Complex numbers as points in the Argand plane

Example: Find all the complex roots of $27i$

Solution: required to find the complex numbers z with the property $z^3 = 27i$.

First write $27i$ in polar form, thus $|27i| = |0 + 27i| = \sqrt{0^2 + (27)^2} = 27$

$$arg(27i) = \frac{\pi}{2}$$

$$27i = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Now suppose $z = r(\cos\theta + i\sin\theta)$ satisfies $z^3 = 27i$, Then, by De Moivre's Theorem, $r^3(\cos 3\theta + i\sin 3\theta) = 27i = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

3.3 The nth Roots of Unity

(Roots of unity) Let n be a positive integer. The complex numbers $e^{2\pi i/n}$ has its nth power equal to 1. Likewise, if k is a non-negative integer in the set $k = 0,1,2, \dots \dots$ n-1, then $e^{2\pi ik/n}$ also has its nth power equal to 1. Such a number is called an nth root of unity. These numbers can be drawn on the unit circle in the complex plane.

We recall that $\cos 0 + i\sin 0 = 1$ this implies that

$$1 = \cos 2\pi k + i\sin 2\pi k,\ k = 0,1,2,3, \dots$$

If ω denotes the root $\cos\left(\frac{2k\pi}{n}\right) + isin\left(\frac{2k\pi}{n}\right)$, $k = 0,1,2,3, \ldots$, then the nth root of unity may be written in the form $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$. We see that they form a geometric progression whose sum $\frac{1-\omega^n}{1-\omega}$ is equal to 0.

Example:
Solve the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ and deduce that

$$\cos(\frac{2\pi}{7}) + cos(\frac{4\pi}{7}) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$$

Solution
We know that

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = \frac{z^7-1}{z-1},$$

Hence we consider the equation

$$z^7 - 1 = 0$$

Note also that

$$1 = \cos 0 + i \sin 0 = \cos 2\pi k + i \sin 2\pi k,$$

Hence

$$z = \cos\left(\frac{2k\pi}{7}\right) + isin\left(\frac{2k\pi}{7}\right), k = 0,1,2,3,4,5,6$$

The equation $z^7 - 1 = 0$ is satisfied by z = 1 and $z = \cos\left(\frac{2k\pi}{7}\right) + i\,sin\left(\frac{2k\pi}{7}\right)$, therefore the given equation is satisfied by $z = \cos\left(\frac{2k\pi}{7}\right) + isin\left(\frac{2k\pi}{7}\right)$, $k = 1,2,3,4,5$
That is,

$$z = \cos\left(\frac{2\pi}{7}\right) + i\,sin\left(\frac{2\pi}{7}\right)$$
$$z = \cos\left(\frac{4\pi}{7}\right) + i\,sin\left(\frac{4\pi}{7}\right)$$
$$z = \cos\left(\frac{6\pi}{7}\right) + i\,sin\left(\frac{6\pi}{7}\right)$$
$$z = \cos\left(\frac{8\pi}{7}\right) + isin\left(\frac{8\pi}{7}\right)$$
$$z = \cos\left(\frac{10\pi}{7}\right) + i\,sin\left(\frac{10\pi}{7}\right)$$

The sum of these roots is $2\left[\cos(\frac{2\pi}{7}) + cos(\frac{4\pi}{7}) + \cos\left(\frac{6\pi}{7}\right)\right]$
But from the given equation the sum of the roots is also -1.
Therefore,

$$\cos(\frac{2\pi}{7}) + cos(\frac{4\pi}{7}) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$$

3.4. Complex roots of a number.

For any given complex number w there is a method of finding all complex solutions of the equation

$$z^n = w \qquad (*)$$

if n = 2, 3, 4, ….. is a given integer.

To find these solutions you write w in polar form, i.e. you find r > 0 and θ such that $w = re^{i\theta}$ then $z = r^{1/n}e^{i\theta/n}$ is a solution to (*). But it isn't the only solution, because the angle θ for which $w = r^{i\theta}$ isn't unique, it is only determined up to a multiple of 2π. Thus if we have found one angle θ for which $w = r^{i\theta}$, then we can also write

$$w = r^{i(\theta+2k\pi)}, k = 0, \pm 1, \pm 2, \ldots$$

The nth roots of w are then

$$z_k = r^{1/n} e^{i\,(\frac{\theta}{n}+2\frac{k}{n}\pi)}$$
$$= \sqrt[n]{r} e^{\frac{i(\theta+2k\pi)}{n}}$$
$$= \sqrt[n]{r}\left(\frac{\cos\theta+2k\pi}{n} + i\frac{\sin\theta+2k\pi}{n}\right), \quad k = 0,1,2,3\ldots., n - 1$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_k = z_0$$
$$z_{n+1} = z_1$$
$$z_{n+2} = z_2$$
$$\vdots$$
$$.$$
$$z_{n+k} = z_k$$

So if you take $k = 0,1,2, \ldots, n-1$ then you have had all the solutions.
The solutions z_k always form a regular polygon with n sides.

Example: Find all sixth roots of w = 1.

Solution: Required to solve $z^6 = 1$.

First write 1 in polar form,

$$1 = 1 \cdot e^{0i}$$
$$= 1.\, z^{2k\pi i}, k = 0, \pm 1, \pm 2, \ldots.$$

Then we take the 6th root and find

$$z_k = 1^{1/6} e^{2k\pi i/6}$$
$$= e^{k\pi i/3}, \; k = 0, \pm 1, \pm 2, \ldots$$

The six roots are

$$z_0 = 1,$$
$$z_1 = e^{\pi i/3} = \frac{1}{2} + \frac{i}{2}\sqrt{3}$$
$$z_2 = e^{2\pi i/3} = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$$
$$z_3 = -1,$$
$$z_4 = e^{\pi i/3} = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$$
$$z_5 = e^{\pi i/3} = \frac{1}{2} - \frac{i}{2}\sqrt{3}$$

4.0 CONCLUSION

In this unit, we have studied some theorems and determine the roots of equations using complex variables. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 SUMMARY

You recall that you learnt about De Moivre's theorem, both for integer quantity and fractional quantity. Also, you learnt about roots of unity among others. You are to study them properly in order to be well equipped for the next course in mathematical methods

6.0 **Tutor Marked Assignment**

1) Use De Moivre's theorem with n = 3 to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

2) Apply De Moivre's formula to prove that

i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

ii) $\sin\theta - 2\sin\theta\cos\theta$

3) Find the indicated power using De Moivre's theorem

i) $(1+i)^6$ ii) $(1-i)^8$ iii) $\left(2\sqrt{3}+2i\right)^5$ iv) $\left(1-i\sqrt{3}\right)^5$

4) Find the indicated roots and sketch the roots in complex plane.

i) The fifth roots of 32

ii) The cube roots of i

iii) The cube roots of $1+i$

5). Find all the roots of the polynomial $P(z) = z^6 - 1$ −1 and plot them in the complex plane.

(b) Let v be a complex number with $|v| = \sqrt{r}$ and $(v) = \frac{\theta}{2} + \pi$.

Show that $v^2 = z$

(c) From (a) and (b) we see that every nonzero complex number has two distinct square roots. Find the square roots, in rectangular form, of 1 +3 i and −9

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14

- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex

MODULE 2

UNIT 1: Limits of functions of complex variables

Content

1.0 Introduction

In one-variable calculus, we study functions $f(x)$ of a real variable x. Likewise, in complex analysis, we study functions $f\ (z)$ of a complex variable $z \in$ C (or in some region of C). Here we expect that $f\ (z)$ will in general take values in C as well. However, it will turn out that some functions are better than others. Basic examples of functions f(z) that we have already seen are:

$f(z) = c$, where c is a constant (allowed to be complex),

$f(z) = z, f(z) = \bar{z}, f(z) = Re\, z,$

$f(z) = Im\, z, f(z) = |z|, f(z) = e^z.$

The "functions" $f(z) = arg\, z, f(z) = \sqrt{z},$ and $f(z) = log\, z$ are also quite interesting, but they are not well-defined (single-valued, in the terminology of complex analysis).

2.0 Objectives

At the end of this unit, you should be able to:

- ✓ Define the function of a complex variable
- ✓ Identify real and imaginary parts of *a complex function*
- ✓ *Define and discuss the Limit of a complex function*

3.0 Main content

3.1 Functions of a complex variable

Definition 1:

Let S be a set of complex numbers in the complex plane. For every point $z = x + iy \in S$, we specific the rule to assign a corresponding complex number $w = u + iv$. This defines a function of the complex variable z, and the function is denoted by $w = f(z)$. The set S is called the domain of definition of the function f and the collection of all values of w is called the range of f.

Definition 2 A (single-valued) function *f* of a complex variable *z* is such that for every *z* in the domain of definition *D* of *f*, there is a unique complex number *w* such that $w = f(z)$.

Definition 3 A complex valued function of a complex variable? If $z = x + iy$, then a function $f(z)$ is simply a function $F(x; y) = u(x; y) + iv(x; y)$ of the two real variables x and y. As such, it is a function (mapping) from R^2 to R^2.

Here are some examples:

1. $f(z) = z$ Corresponds to $F(x, y) = x + iy \quad (u = x, v = y)$
2. $f(z) = \bar{z}$,, with $F(x, y) = x - iy \quad (u = x, v = -y\ )$;
3. $f(z) = Re\, z$, with $F(x, y) = x \quad (u = x, v = 0$, taking values just along the real axis);
4. $f(z) = |z|$, with $F(x, y) = \sqrt{x^2 + y^2}$, $\quad (u = \sqrt{x^2 + y^2}\,, v = 0$ taking values just along the real axis);
5. $f(z) = z^2$, with $F(x, y) = (x^2 - y^2) + i(2xy) \quad (u = x^2 - y^2, v = 2xy)$
6 $f(z) = e^z$, with $F(x, y) = e^z \cos y + i(\,e^z \sin y) \quad (u = e^z \cos y, v = e^z \sin y)$

3.1.1 Real and imaginary parts of *a complex function*

If $f(z) = u + iv$, then the function $u(x; y)$ is called the real part of f and $v(x; y)$ is called the imaginary part of f. Of course, it will not in general be possible to plot the graph of $f(z)$, which will lie in C^2, the set of ordered pairs of complex numbers, but it is the set

$$\{f(z, w) \in C^2 : w = f(z)\}.$$

The graph can also be viewed as the subset of R^4 given by

$$\{(x;\ y;\ s;\ t) : s = u(x;\ y);\ t = v(x;\ y)\}.$$

In particular, it lies in a four-dimensional space.The usual operations on complex numbers extend to complex functions: given a complex function $f(z) = u + iv$, we can define functions $Re\, f(z) = u, Im\, f(z) = v$, $\overline{f(z)} = u - iv$, $|F(z)| = \sqrt{u^2 + v^2}$.

Likewise, if $g(z)$ is another complex function, we can define $f(z)g(z)$ *and* $f(z)\,/g(z)$ for those z for which $g(z) \neq 0$.
Some of the most interesting examples come by using the algebraic operations of C. For example, a polynomial is an expression of the form

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

where the a_i are complex numbers, and it defines a function in the usual way. It is easy to see that the real and imaginary parts of a polynomial P(z) are polynomials in x and y. For example,

$$P(z) = (1+i)z^2 - 3iz = (x^2 - y^2 - 2xy + 3y) + (x^2 - y^2 + 2xy - 3x)i$$

and the real and imaginary parts of $P(z)$ are polynomials in x and y. But given two (real) polynomial functions $u(x;\ y)$ and $z(x;\ y)$, it is very rarely the case that there exists a complex polynomial $P(z)$ such that $P(z) = u + iv$.
For example, it is not hard to see that x cannot be of the form $P(z)$, nor can $\bar{z}$. As we shall see later, no polynomial in x and y taking only real values for every z $(i.e.\, v = 0)$ can be of the form $P(z)$.
Of course, since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$**,** every polynomial $F(x, y)$ in x and y is also a polynomial in z and $\bar{z}$, i.e.

$$F(x,y) = Q(z,\bar{z}) = \sum_{i,j\geq 0} c_{ij} z^i \bar{z}^j$$

where c_{ij} are complex coefficients.
Examples:
$f(z) = arg\, z$ is defined everywhere except at $z = 0$, and Arg z can assume all possible real values in the interval (−π, π]. $f\,(z) = z$ is such that $u(x, y) = x$ and $v(x, y) = y$.
Find the real and imaginary parts of $f(z) = \bar{z}$, $f(z) = \frac{1}{\bar{z}}$ is defined for all $z \neq 0$ and is such that

$$u\,(x, y) = \frac{x}{x^2 + y^2} \text{ and } v(x, y) = \frac{y}{x^2 + y^2}$$

3.1.2 Limit of a complex function

A (complex) function f is a mapping from a subset $G \subseteq \boldsymbol{C}$ to **C** (in this situation we will write f : G → **C** and call G the domain of f). This means that each element z ∊ G gets mapped to exactly one complex number called the image of z and usually denoted by f (z). So far there is nothing that makes complex functions any more special than, say, functions from R^m to R^n.
In fact, we can construct many familiar looking functions from the standard calculus repertoire, such as $f\,(z) = z$ (the identity map), $f\,(z) = 2z + i$, $f\,(z) = z^3$, or $f\,(z) = \frac{1}{z}$ The former three could be defined on all of **C**, whereas for the latter we have to exclude the origin z = 0. On the other hand, we could construct some functions which make use of a certain representation of z, for example,
$f\,(x, y) = x - 2iy, f(x, y) = y^2 - ix$, or $f\,(r, \varphi) = 2re^{i(\varphi+\pi)}$.
Maybe the fundamental principle of analysis is that of a limit. The philosophy of the following definition is not restricted to complex functions, but for sake of simplicity we only state it for those functions.

Definition: Suppose f is a complex function with domain G and z_0 is an accumulation point of G. Suppose there is a complex number w_0 such that for every $\varepsilon > 0$, we can find $\delta > 0$ so that for all $z \in G$ satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - w_0 < \varepsilon|$. Then $w_0$is the limit of f as z approaches z_0, in short

$$\lim_{z\to z_0} f(z) = w_0$$

.

This definition is the same as is found in most calculus texts. The reason we require that $z_0$is an accumulation point of the domain is just that we need to be sure that there are points z of the domain which are arbitrarily close to z_0. Just as in the real case, the definition does not require that z0 is in the domain of f and, if z_0 is in the domain of f, the definition explicitly ignores the value of $f(z_0)$. That is why we require

$$0 < |z - z_0|.$$

Just as in the real case the limit w0 is unique if it exists. It is often useful to investigate limits by restricting the way the point z "approaches"z_0. The following is a easy consequence of the definition.

Lemma1: Suppose $\lim_{z\to z_0} f(z)$ exists and has the value w_0, as above. Suppose$G_0 \subseteq G$, and suppose z_0 is an accumulation point of G_0. If f_0 is the restriction of f to G_0 then $\lim_{z\to z_0} f(z)$ exists and has the value w_0.

The definition of limit in the complex domain has to be treated with a little more care than its real companion; this is illustrated by the following example.

Example: $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

To see this, we try to compute this "limit" as $z \to 0$ on the real and on the imaginary axis. In the first case, we can write $z = x \in R$, and hence

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{x\to 0} \frac{\bar{x}}{x} = \lim_{x\to 0} \frac{x}{x} = 1$$

In the second case, we write $z = iy$ where $y \in R$, and then

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{y\to 0} \frac{\overline{iy}}{iy} = \lim_{y\to 0} \frac{-iy}{iy} = -1$$

So we get a different "limit" depending on the direction from which we approach 0. Lemma 1 then implies that $\lim_{z\to 0} \frac{\bar{z}}{z}$does not exist.

On the other hand, the following "usual" limit rules are valid for complex functions; the proofs of these rules are everything but trivial and make for nice exercises.

The absolute value measures the distance between two complex numbers.

Thus, z_1 and z_2 are close when $|z_1 - z_2|$ is small. We can then define the limit of a complex function $f(z)$) as follows: we write

$$\lim_{z\to c} f(z) = L,$$

where c and L are understood to be complex numbers, if the distance from $f(z)$ to L, $|f(z) - L|$ is small whenever $|z - c|$ is small. More precisely, if we want $|f(z) - L|$ to be less than some small specified positive real number δ then there should exist a positive real number ϵ such that, if $|z - c| < \delta$ then $|f(z) - L| < \epsilon$. Note that, as with real functions, it does not matter if $f(c) = L$ or even that $f(z)$ be defined at c. It is easy to see that, if $c = (c_1, c_2), L = a + bi$ and
$f(z) = u + iv$ is written as a real and an imaginary part, then

$$\lim_{z \to c} f(z) = L$$

if and only if

$$\lim_{(x,y)\to(c_1,c_2)} u(x,y) = a \quad \text{and} \quad \lim_{(x,y)\to(c_1,c_2)} v(x,y) = b.$$

Thus the story for limits of functions of a complex variable is the same as the story for limits of real valued functions of the variables x; y. However, a real variable x can approach a real number c only from above or below (or from the left or right, depending on your point of view), whereas there are many ways for a complex variable to approach a complex number c.

Sequences, limits of sequences, convergent series and power series can be defined similarly.

The formal definition of the limit of a function is stated as:

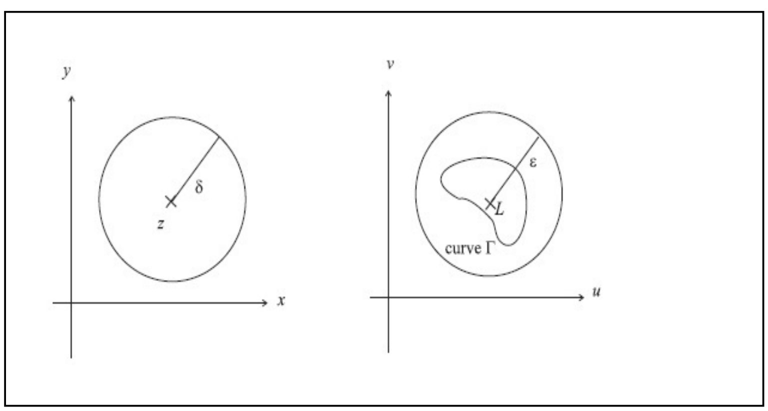
For any $\epsilon > 0$, there exists $\delta > 0$ (usually dependent on ϵ) such that

$|f(z) - L| < \epsilon$ if $0 < |z - z_0| < \delta$

The limit L, if it exists, must be unique. The value of L is independent of the direction along which $z \to z_0$ (see Figure 2.4).

The function $f(z)$ needs not be defined at z_0 in order for the function to have a limit at z0. However, we do require z_0 to be a limit point of S so that it would never occur that $f(z)$ is not defined in some deleted neighbourhood of z_0. For example, let us consider $\lim_{z\to i} \frac{\sin z}{z}$.

The domain of the definition of $\frac{\sin z}{z}$ is $C\backslash\{0\}$. Though $\frac{\sin z}{z}$ is not defined at $z = 0$, this is a limit point of the domain of definition. Hence, the above limit is well defined.



The region $0 < |z - z_0| < \delta$ in the z-plane is mapped onto the region enclosed by the curve in the w-plane. The curve lies completely inside the annulus $0 < |w - L| < \epsilon$

If $L = \alpha + i\beta$, $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ and $z_0 = x_0 + iy_0$
then

$$|u(x,y)-\alpha| \le |f(y)-L| \le |u(x,y)-\alpha| + |v(x,y)-\beta|$$
$$|u(x,y)-\beta| \le |f(y)-L| \le |u(x,y)-\alpha| + |v(x,y)-\beta|$$

From the above inequalities, it is obvious that eq. (2.2.1) is equivalent to the following pair of limits

$$\lim_{(x.y)\to(x_0,y_0)} u(x,y) = \alpha$$
$$\lim_{(x.y)\to(x_0,y_0)} v(x,y) = \beta$$

Therefore, the study of the limiting behavior of $f\,(z)$ is equivalent to that of a pair of real functions $u(x,y)$ and $v(x,y)$. Consequently, theorems concerning the limit of the sum, difference, product and quotient of complex functions hold as to those for real functions.

Suppose that $\lim_{Z\to Z_0} f_1(z) = L_1$ and $\lim_{Z\to Z_0} f_2(z) = L_2$

Then

$$\lim_{Z\to Z_0}\big(f_1(z) \pm f_2(z)\big) = L_1 \pm L_2\ ,$$
$$\lim_{Z\to Z_0}\big(f_1(z) f_2(z)\big) = L_1 L_2\ ,$$
$$\lim_{Z\to Z_0} \frac{f_1(z)}{f_2(z)} = \frac{L_1}{L_2}\ ,\ L_2 \neq 0$$

Example: $\lim_{z\to i} \frac{z^2-1}{z-1} = 2i$

Because the definition of the limit is somewhat elaborate, the following fundamental definition looks almost trivial.

Definition: Suppose f is a complex function. If z_0 is in the domain of the function and either z_0 is an isolated point of the domain or

$$\lim_{z\to z_0} f(z) = f(z_0)$$

then f is continuous at z_0. More generally, f is continuous on $G \subseteq \boldsymbol{C}$ if f is continuous at every $z \in G$.

Just as in the real case, we can "take the limit inside" a continuous function:

Lemma 2: If f is continuous at an accumulation point w_0 and $\lim_{z\to z_0} g(z) = w_0$ then $\lim_{z\to z_0} f(\,g(z)) = f(w_0)$

In other words,

$$\lim_{z\to z_0} f(\,g(z)) = f(\lim_{z\to z_0} g(z))$$

This lemma implies that direct substitution is allowed when f is continuous at the limit point.

In particular, that if f is continuous at w_0 then $\lim_{z\to z_0} g(z) = f(w_0)$.

Example Prove that

$\lim_{z\to\alpha} z^2 = \alpha^2$, α is a fixed complex number using the $\epsilon - \delta$ criterion.

Solution: It suffices to establish that for any given $\epsilon > 0$, there exists a positive number such that

$$|z^2 - \alpha^2| < \epsilon \text{ whenever } 0 < |z - 0 < |z - \alpha| < \delta$$

Observing

$$z^2 - \alpha^2 = (z - \alpha)(z + \alpha) = (z - \alpha)(z - \alpha + 2\alpha)$$

and applying the triangle inequality, we obtain

$$|z^2 - \alpha^2| = |z - \alpha||z - \alpha + 2\alpha| \leq |z - \alpha|(|z - \alpha| + 2|\alpha|)$$

provided that z lies inside the deleted δ neighborhood of α. Here, δ is chosen to be less than $\min(1, \frac{\epsilon}{1+2|\alpha|})$, so

$$|z^2 - \alpha^2| \leq |z - \alpha|(|z - \alpha| + 2|\alpha|) < \frac{\epsilon}{1+2|\alpha|}(1 + 2|\alpha|) = \epsilon$$

Note that the choice of δ depends on ϵ and α.

3.13 Limit at infinity: The definition of limit holds even when z_0 or L is the point at infinity. We can simply replace the corresponding neighbourhood of z_0 or L by the neighbourhood of infinity. The mathematical statement $\lim_{z\to\infty} f(z) = L$ can be understood as:

For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$|f(z) - L| < \epsilon \text{ whenever } |z| > \frac{1}{\delta}$$

Here, z refers to a point in the finite complex plane and $|z| > \frac{1}{\delta}$ is visualized as a deleted neighbourhood of ∞. Also, we must be cautious that the results in the equations above hold for z_0, L_1 and L_2 in the finite complex plane only.

Suppose we define $w = \frac{1}{z}$. Then $z \to \infty$ is equivalent to $w \to 0$. It is then not surprising to have the following properties on limit at infinity

Theorem *If z_0 and w_0 are points in the z-plane and the w-plane respectively, then.*

(a) $\lim_{Z\to Z_0} f(z) = \infty$ if and only if $\lim_{Z\to Z_0} \frac{1}{f(z)} = 0$

(b) $\lim_{z\to\infty} f(z) = w_0$ if and only if $\lim_{Z\to Z_0} f\left(\frac{1}{z}\right) = w_0$

Proof

(a) $\lim_{Z\to Z_0} f(z) = \infty$ implies that for any $\epsilon > 0$, there exists a positive number δ such that

$$|f(z)| > \frac{1}{\epsilon} \text{ whenever } 0 < |Z - Z_0| < \delta$$

The above result may be rewritten as

$$\left|\frac{1}{f(z)} - 0\right| < \epsilon \text{ whenever } 0 < |Z - Z_0| < \delta$$

so we obtain

$$\lim_{Z\to Z_0} \frac{1}{f(z)} = 0$$

(b) $\lim_{z\to\infty} f(z) = w_0$ implies that for any $\epsilon > 0$, there exists a positive number δ such that

$|f(z) - w_0| < \epsilon$ whenever $|z| > \frac{1}{\delta}$, replacing z by $\frac{1}{z}$ we obtain $\left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon$ whenever $0 < |z - 0| < \delta$ so we obtain $\lim_{Z\to 0} f\left(\frac{1}{z}\right) = w_0$

The above results provide the convenient tools to evaluate limits on infinity

For example:

1) $\lim_{z\to 1}\frac{z-1}{z^2+1}=0$ 2) $\lim_{z\to 1}\frac{z^2+1}{z-1}=\infty$

3) $\lim_{z\to 0}\frac{1+4z^2}{5+iz^2}=\frac{1}{5}$ 4) $\lim_{z\to\infty}\frac{z^2+4}{5z^2+i}=\lim_{z\to\infty}\frac{\frac{1}{z^2}+4}{\frac{5}{z^2}+i}=\frac{1}{5}$

5) $\lim_{z\to\infty}\frac{2z+i}{z+1}=2$

4.0 CONCLUSION

In this unit, we have studied limits of complex functions. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 SUMMARY

You recall that you learnt about function of a complex variable as well as limits of complex functions. You are to study them properly in order to be well equipped for the next course in mathematical methods.

6.0 Tutor Marked Assignment

1). Evaluate the following limits or explain why they don't exist.

(a) $\lim_{z\to i}\frac{z^3i-1}{z+1}$

(b) $\lim_{z\to 1-i}[x+i(2x+y)]$

2) Evaluate $\lim_{z\to\infty}\frac{3z+i}{z+2}$

3) Evaluate $\lim_{z\to\infty}\frac{z^4+4z+1}{5z^4+i}$

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex

Unit 2 Continuity of functions of complex variables

Content

1.0 Introduction

Continuity of a complex function is defined in the same manner as for a real function. The Continuity of a complex function is defined using the concept of limits.

2.0 Objectives

At the end of this unit, you should be able to:

- *Define the* Continuity of a complex function
- *Discuss the* Continuity of a complex function at a given point

3.0 Main content

3.1 Continuity of a complex function

As for functions of a real variable, a function f(z) is continuous at c if.

$$\lim_{z\to c} f(z) = f(c)$$

In other words:

1) Limit of the function exists;
2) $f(z)$ is defined at c;
3) Its value at c is the limiting value.

That is, the continuity of a complex function is defined using the concept of limits.

A function $f\ (z)$ is continuous if it is continuous at all points where it is defined. It is easy to see that a function $f\ (z) = u + iv$ is continuous if and only if its real and imaginary parts are continuous, and that the usual functions $z;, \bar{z};\ Re\ z, Im\ z;\ |z|, e^z$ are continuous. (We have to be careful, though, about functions such as $arg\, z$ or log z which are not well-defined.) All polynomials $P(z)$ are continuous, so are all two-variable polynomial functions in x and y. A rational function $R(z) = P(z) = Q(z)$ with $Q(z)$ not identically zero is continuous where it is defined, i.e. at the finitely many points where the denominator $Q(z)$ is not zero. More generally, if $f(z)$) and $g(z)$ are continuous, then so are:

1. $cf(z)$ where c is a constant;
2. $f(z) + g(z)$
3. $f(z). g(z)$
4. $\frac{f(z)}{g(z)}$ is defined (for $g(z) \neq 0$)
5 $(g\ o\ f)(z) = g(f(z)$, the composition of $g(z)$and $f(z)$ are defined.

Example: Consider $f(z) = e^z$; its real and imaginary parts are, respectively, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Since both $u(x, y)$ and $v(x, y)$ are continuous at any point (x_0, y_0) in the finite *x*-*y* plane, we conclude that e^z is continuous at any point $z_0 = x_0 + i\, y_0$ in C

Example: Is the function such that $f(z) = Im \frac{z^2}{|z|^2}$ for $z \neq 0$ and $f(0) = 0$, continuous at $= 0$?

Theorems on real continuous functions can be extended to complex continuous functions. If two complex functions are continuous at a point, then their sum, difference and product are also continuous at that point; and their quotient is continuous at any point where the denominator is non-zero. For example, since $g(z) = z^2$ is continuous everywhere, we conclude by the above remark that both $z^2 \pm e^z$ and z^2/e^z are continuous in C. Examples of complex continuous functions in C are polynomials, exponential functions and trigonometric functions.

Another useful result is that a composition of continuous functions is continuous. If $f(z)$ is continuous at z_0 and $g(z)$ is continuous at ξ, and if $\xi = f(z_0)$, then the composite function $g(f(z))$ is continuous at $z = z_0$. Thus, functions like $\sin z^2$ and $\cos z^2$ are continuous functions in C.

Since continuity of $f(z)$ implies continuity of its real and imaginary parts, the real function

$$|f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}, \qquad f = u + iv \text{ and } z = x + iy$$

is also continuous. By applying the well-known result on boundedness of a continuous real function in a closed and bounded region, we can deduce a related property on boundedness of the modulus of a continuous complex function. We state without proof the following theorem.

Theorem *If $f(z)$ is continuous in a closed and bounded region R, then $|f(z)|$ is bounded in the region, that is,*

$$|f(z)| < M, \qquad \textit{for all } z \in R, \qquad (2.2.7)$$

for some constant M. Also, $|f(z)|$ attains its maximum value at some point z_0 in R.

Example Discuss the continuity of the following complex functions at $z = 0$:

a) $f(z) = \begin{cases} 0, & z = 0 \\ \frac{Re\, z}{|z|}, & z \neq 0 \end{cases}$

b) $f(z) = \frac{Im\, z}{1+\lfloor z \rfloor}$

Solution

a) $z = x + iy\ ,\ z \neq 0$

then,

$$\frac{Re\, z}{\lfloor z \rfloor} = \frac{x}{\sqrt{x^2+y^2}}$$

Suppose z approaches 0 along the half straight line $y = mx$ ($x > 0$).

Then**,**

$$\lim_{z\to 0, y=mx, x>0} \frac{Re\, z}{\lfloor z \rfloor} = \lim_{x\to 0+} \frac{x}{\sqrt{x^2+m^2y^2}}$$
$$= \lim_{x\to 0+} \frac{x}{x\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$

Since the limit depends on m, $\lim_{z\to 0} f(z)$ does not exist. Therefore, $f\,(z)$ cannot be continuous at $z\ =\ 0$.

b) $z = x + iy$, then $\frac{Im\, z}{1+\lfloor z \rfloor} = \frac{y}{1+\sqrt{x^2+y^2}}$

Now, consider the limit,

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{y}{1+\sqrt{x^2+y^2}} = 0 = f(0)$$

Therefore, $f(z)$ is continuous at z = 0.

Example: the function $f(z) = |z|$ is continuous for all z.

For this, let z_0 be given. Then

$\lim_{z\to z_0}|z| = \lim_{z\to z_0} \sqrt{(Re\ z)^2 + (Im\ z)^2} = \sqrt{(Re\ z)^2 + (Im\ z)^2} = |z_0|$

3.2 Uniform continuity

Suppose $f(z)$ is continuous in a region R. Then by definition, at each point z_0 inside R and for any $\epsilon > 0$, we can find $\delta > 0$ such that $|f\ (z) -\ f\ (z_0)| < \epsilon$ whenever $|\ z - z_0| < \delta$. Usually δ depends on ϵ and z_0 together. However, if we can find a single value of δ for each ϵ, independent of $z0$ chosen in R, we say that $f(z)$ is *uniformly continuous* in the region R.

Example Show that

(a) $f_1(z) = z^2$ is uniformly continuous in the region $|z| < R$, where $0 < R < \infty$.

Solution

It suffices to show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|z^2 - {z_0}^2\ | <\ \epsilon$ when $|\ z - z_0| < \delta$, where δ depends on ϵ but not on the particular point $z0$ of the region. If z and $z0$ are any two points inside $|z| < R$, then

$|z^2 - {z_0}^2\ | = |z + z_0||\ z - z_0| \leq \{|\ z| + |\ z_0|\}|\ z - z_0| <\ 2R|\ z - z_0|$

This relation between $|\ f_1(z) - f_1(z_0)|$ and $|\ z - z_0|$ dictates the choice of $\delta\ =\ \epsilon\ 2R$, where δ depends on ϵ but not on z_0 Now, given any $\epsilon > 0$, suppose $|\ z - z_0| <\ \ \delta$. Then by inequality (i), we have

$$|\ f_1(z) - f_1(z_0)| = |z^2 - {z_0}^2| <\ 2R|\ z - z_0|\ <\ 2R\ \delta\ =\ \epsilon$$

Hence, $f_1(z) = z^2$ is uniformly continuous in $|z| < R$.
and $|f_2(z) - f_2(z_0)|$ can be made to be larger than any positive number when $z0$ becomes sufficiently close to 0. It is not possible to find δ that depends on ϵ but not $z0$ such that for any given ϵ, we have

$$|f_2(z) - f_2(z_0)| < \epsilon \text{ for } |z - z_0| < \delta.$$

Hence, $f_2(z) = \frac{1}{z}$ is not uniformly continuous in $0 < |z| < 1$.
Most of the theorems related to the properties of continuity for real functions can be extended to complex functions. However, this is not quite so when we consider differentiation

4.0 CONCLUSION

In this unit, we have studied Continuity of a complex function. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 SUMMARY

You recall that you learnt about Continuity of a complex function You are to study them properly in order to be well equipped for the next course in mathematical methods

6.0 **Tutor Marked Assignment**

Show that $f_2(z) = \frac{1}{z}$ is not uniformly continuous in the region $0 < |z| < 1$.

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex

UNIT 3: Differentiation of complex functions

Content

1.0 Introduction

2.0 Objectives

At the end of this unit, you should be able to:

- Define and discuss the differentiability of a complex function using the concept of limit
- State the rules of differentiation of complex numbers
- Define constant functions
- Obtain Complex velocity and acceleration of functions

3.1 Differentiation of complex functions (or simply Complex derivatives)

The fact that limits such as $\lim_{z\to 0}\frac{\bar{z}}{z}$ do not exist points to something special about complex numbers which has no parallel in the reals—we can express a function in a very compact way in one variable, yet it shows some peculiar behavior "in the limit." We will repeatedly notice this kind of behavior; one reason is that when trying to compute a limit of a function as, say, $z \to 0$, we have to allow z to approach the point 0 in any way. On the real line there are only two directions to approach 0—from the left or from the right (or some combination of those two). In the complex plane, we have an additional dimension to play with. This means that the statement
"A complex function has a limit..." is in many senses stronger than the statement "A real function has a limit..." This difference becomes apparent most baldly when studying derivatives.

Differentiability Similar to the calculus of real variables, the differentiability of a complex function is defined using the concept of limit. Having discussed some of the basic properties of functions, we ask now what it means for a function to have a complex derivative. Here we will see something quite new: this is very different from asking that its real and imaginary parts have partial derivatives with respect to x and y. We will not worry about the meaning of the derivative in terms of slope, but only ask that the usual difference quotient exists.

Definition A function $f(z)$ is complex differentiable at c if

$$\lim_{z\to c}\frac{f(z)-f(c)}{z-c}$$

exists. In this case, the limit is denoted by $f'(c)$. Making the change of variable $z = c+h,\ f(z)$ is complex differentiable at c if and only if the limit $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists, in which case the limit is again $f'(c)$. A function is complex differentiable if it is complex differentiable at every point where it is defined. For such a function $f(z)$, the derivative defines a new function which we write as

$$f'(z)\ \ or\ \frac{d}{dz}f(z)$$

For example, a constant function f(z) = C is everywhere complex differentiable and its derivative $f'(z) = 0$.The function $f(z) = z$ is also complex differentiable, since in this case

$$\frac{f(z)-f(c)}{z-c} = \frac{z-c}{z-c} = 1$$

Thus, $(z)' = 1$. But many simple functions do not have complex derivatives.
For example, consider $f(z) = Re\, z = x$. We show that the limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

does not exist for any c. Let $c = a + bi$, so that $f(c) = a$. First consider $h = t$ a real number. Then $f(c + t) = a + t$ and so $\frac{f(c+h)-f(c)}{h} = \frac{a+t-a}{t} = 1$

So if the limit exists, it must be 1. On the other hand, we could use $h = it$.
In this case, $f(c + it) = f(c) = a$, and

$$\frac{f(c+h)-f(c)}{h} = \frac{a-a}{it} = 0$$

Thus approaching c along horizontal and vertical directions has given two different answers, and so the limit cannot exist. Other simple functions which can be shown not to have complex derivatives are $Im\, z;\ \bar{z}$, and $|z|$.

3.2 Rules for differentiation

1. If $f(z)$ is complex differentiable, then so is $cf(z)$, where c is a constant, and $(cf(z))' = cf'(z)$;
2. (Sum rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z) + g(z)$, and $(f(z) + g(z))0 = f0(z) + g0(z)$;
3. (Product rule) If f(z) and g(z) are complex differentiable, then so is $f(z).g(z)$ and $(f(z)\,_{-}\,g(z))' = f'(z)g(z) + f(z)g'(z)$;
4. (Quotient rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is

$f(z) = g(z)$, where defined (i.e. where g(z) ≠ 0), and $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z)-f(z)g\prime(z)}{g(z)^2}$

5. (Chain rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(g(z))$ where defined, and $(f(g(z)))' = f'(g(z)).g'(z)$.

6. (Inverse functions) If $f(z)$ is complex differentiable and one-to-one, with nonzero derivative, then the inverse function $f^{-1}(z)$ is also differentiable, and

$$(f^{-1}(z))' = \frac{1}{f'(f^{-1}(z))})$$

Thus for example we have the power rule $(z^n)' = nz^{n-1}$, every polynomial

$\mathrm{P}(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is complex differentiable, with

$P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$;

and every rational function is also complex differentiable. It follows that a function which is not complex differentiable, such as $Re\, z$ or $\bar{z}$ cannot be written as a complex polynomial or rational function.

Assume that *f* is defined in a neighborhood of $z - z_0$. The *derivative* of the function *f* at $z - z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

assuming that this limit exists.

If *f* has a derivative at $z = z_0$, we say that *f* is *differentiable* at $z = z_0$.

Example Show that the functions $\bar{z}$ and $Re\, z$ are nowhere differentiable, while $|z|^2$ is differentiable only at $z = 0$.

Solution According to definition (2.3.1), the derivative of *z* is given by

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\bar{z}+\Delta\bar{z}-\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} e^{-2i\, Arg\Delta z}$$

The value of the limit depends on the path approaching z. Therefore, *z* is nowhere differentiable. Similarly,

$$\frac{d}{dz}Re\, z = \frac{d}{dz}\frac{1}{2}(z+\bar{z})$$

$$= \frac{1}{2}\lim_{\Delta z \to 0} \frac{(\mathrm{z}+\bar{z}+\Delta z+\Delta\,\overline{z)}-(z-\overline{z)}}{\Delta z}$$

$$= \frac{1}{2}\lim_{\Delta z \to 0} \frac{\Delta z+\Delta\bar{z}}{\Delta z}$$

$$= \frac{1}{2} + \frac{1}{2}\lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z}$$

Again, Re z is shown to be nowhere differentiable. Lastly, the derivative of $|z|^2$ is given by

$$\frac{d}{dz}|z|^2 = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\bar{z} + z\frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z}\right]$$

The above limit exists only when $z = 0$, that is, $|z|^2$ is differentiable only at $z = 0$

Examples:

$f(z) = \bar{z}$ is continuous but not differentiable at $z = 0$.

$f(z) = x^3$ is differentiable at any $z \in C$ and $f_z = 3x^2$

Holomorphicity

Definition Suppose $f : G \to \boldsymbol{C}$ is a complex function and z_0 is an interior point of G. The derivative of f at z_0 is defined as $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ provided this limit exists. In this case, f is called differentiable at z_0. If f is differentiable for all points in an open disk centered at z_0 then f is called **holomorphic** at z_0. The function f is holomorphic on the open set G $\subseteq$ **C** if it is differentiable (and hence holomorphic) at every point in G. Functions which are differentiable (and hence holomorphic) in the whole complex plane **C** are called **entire.(**Some sources use the term 'analytic' instead of 'holomorphic'. in our context, these two terms are synonymous. Technically, though, these two terms have different definitions. Since we will be using the above definition, we will stick with using the term 'holomorphic' instead of the term 'analytic**)**

The difference quotient limit which defines $f'(z_0)$ can be rewritten as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

This equivalent definition is sometimes easier to handle. Note that h is not a real number but can rather approach zero from anywhere in the complex plane.

The fact that the notions of differentiability and holomorphicity are actually different is seen in the following examples.

Example the function $f(z) = z^3$ is entire that is, holomorphic in **C**: For any $z_0 \in$ **C**

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{z^3-{z_0}^3}{z-z_0}$$

$$=\lim_{z \to z_0} \frac{(z^2+zz_0+{z_0}^2)(z-z_0)}{z-z_0}$$

$$= \lim_{z \to z_0} z^2 + zz_0 + {z_0}^2$$

$$= 3{z_0}^2$$

Example The function $f\ (z)\ = \bar{z}^2$ is differentiable at 0 and nowhere else (in particular, f is not holomorphic at 0): Let's write $z\ = z_0\ +\ re^{i\theta}$. Then

$$\frac{\bar{z}^2-\bar{z}_0{}^2}{z-z_0} = \frac{\left(\overline{z_0 + re^{i\theta}}\right)^2-\overline{z_0}^2}{z_0 + re^{i\theta}-z_0}$$

$$= \frac{\left(\overline{z_0} + re^{-i\theta}\right)^2-\overline{z_0}^2}{re^{i\theta}}$$

$$= \frac{\overline{z_0}^2+2\overline{z_0}re^{-i\theta}+r^2e^{-2i\theta}-\overline{z_0}^2}{re^{i\theta}} = \frac{2\overline{z_0}re^{-i\theta}+r^2e^{-2i\theta}}{re^{i\theta}}$$

$$= 2\overline{z}_0e^{-2i\theta} + re^{-3i\theta}$$

If $z_0 \neq 0$, then the limit of the right-hand side as $z \to z_0$ does not exist since $r \to 0$ and we obtain different answers for horizontal approach i.e. ($\theta = 0$) and for vertical approach i.e. ($\theta = \frac{\pi}{2}$). (A more entertaining way to see this is to use, for example, $z(t) = z_0 + \frac{1}{t}e^{it}$ which approaches z_0 as $t \to \infty$). On the other hand, if $z_0 = 0$ then the right-hand side equals $re^{-3i\theta} = |z|e^{-3i\theta}$.
Hence,

$$\lim_{z\to 0}\left|\frac{\bar{z}^2}{z}\right| = \lim_{z\to 0}\left||z|e^{-3i\theta}\right|$$

$$= \lim_{z\to 0}|z| = 0$$

Example The function $f\ (z)\ =\ \bar{z}$ is nowhere differentiable:

$$\lim_{z\to z_0}\frac{\bar{z}-\overline{z_0}}{z-z_0} = \lim_{z\to z_0}\frac{\overline{z-z_0}}{z-z_0}$$

$$= \lim_{z\to 0}\frac{\bar{z}}{z}$$

does not exist, as discussed earlier.
The basic properties for derivatives are similar to those we know from real calculus. In fact, one should convince oneself that the following rules follow mostly from properties of the limit. (The 'chain rule' needs a little care to be worked out.)

Lemma 1 Suppose f and g are differentiable at $z \in \boldsymbol{C}$, and that $c \in \boldsymbol{C}, n \in \boldsymbol{Z}$, and h is differentiable at $g(z)$.

(a) $\big(f(z) + cg(z)\big)' = f'(z) + cg'(z)$

(b) $\big(f(z)\ .\ g(z)\big)' = f'(z)g(z) + f(z)g'(z)$

(c) $(f(z)/g(z))' = \frac{f'(z)g(z)+f(z)g'(z)}{g(z)^2}$, whenever $g(z) \neq 0$

(d) $(z^n)' = nz^{n-1}$

(e) $(h(g(z)))' = h'(g(z))g'(z)$

We end this section with yet another differentiation rule, that for inverse functions. As in the real case, this rule is only defined for functions which are bijections. A function $f : G \to H$ is one-to-one if for every image $w \in H$ there is a unique $z \in G$ such that $f(z) = w$. The function is onto if every $w \in H$ has a pre-image $z \in G$ (that is, there exists a z ∈ G such that $f(z) = w$). A bijection is a function which is both one-to-one and onto. If $f : G \to H$ is a bijection then g is the inverse of f if for all $z \in H, f(g(z)) = z$.

Lemma 2 Suppose G and H are open sets in **C**, $f : G \to H$ is a bijection, $g : H \to G$ is the inverse function of , and $z_0 \in H$. If f is differentiable at $g(z_0)$, $f'(g(z_0)) \neq 0$, and g is continuous at z_0 then g is differentiable at z_0 with

$$g'(z_0) = \frac{1}{f'(g(z_0))}$$

Proof. We have, $g'(z_0) = \lim_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0}$

$$= \lim_{z \to z_0} \frac{g(z)-g(z_0)}{f(g(z))-f(g(z_0))}$$

$$= \lim_{z \to z_0} \frac{1}{\frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0}}$$

.

Because $g(z) \to g(z_0)$ as $z \to z_0$, we obtain:

$$g'(z_0) = \lim_{g(z) \to g(z_0)} \frac{1}{\frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0}}.$$

Finally, as the denominator of this last term is continuous at z_0,, by Lemma 2.6 we have:

$$g'(z_0) = \frac{1}{\lim_{g(z) \to g(z_0)} \frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0}} = \frac{1}{f'(g(z_0))}$$

3.3 Constant Functions

As an example application of the definition of the derivative of a complex function, we consider functions which have a derivative of 0. One of the first applications of the Mean-Value Theorem for real-valued functions is to show that if a function has zero derivatives everywhere on an interval then it must be constant.

Lemma 3 If $f : I \to \boldsymbol{R}$ is a real-valued function with $f'(x)$ defined and equal to 0 for all $x \in I$, then there is a constant $c \in \boldsymbol{R}$ such that $f(x) = c$ for all $x \in I$.

Proof:. The proof is easy: The Mean-Value Theorem says that for any $x, y \in I$,

$$f(y) - f(x) = f'(x + a(y - x))(y - x)$$

for some 0 < a < 1. If we know that f' is always zero then we know that

$$f'(x + a(y - x)) = 0,$$

so the above equation yields $f(y) = f(x)$. Since this is true for any $x, y \in I$, f must be constant.

There is a complex version of the Mean-Value Theorem, but we defer its statement to another course. Instead, we will use a different argument to prove that complex functions with derivative that are always 0 must be constant.

Lemma 3 required two key features of the function f, both of which are somewhat obviously necessary. The first is that f be differentiable everywhere in its domain. In fact, if f is not differentiable everywhere, we can construct functions which have zero derivative 'almost' everywhere but which have infinitely many values in their range.

The second key feature is that the interval I is connected. It is certainly important for the domain to be connected in both the real and complex cases. For instance, if we define

$$f(z) = \begin{cases} 1 & if\ Re\, z > 0 \\ -1 & if\ Re\, z < 0 \end{cases}$$

then $f'(z) = 0$ for all z in the domain of f but f is not constant. This may seem like a silly example, but it illustrates a pitfall to proving a function is constant that we must be careful of.

Recall that a region of **C** is an open connected subset.

Theorem If the domain of f is a region $G \subseteq \boldsymbol{C}$ and $f'(z) = 0$ for all z in G then f is a constant.

Proof. We will show that f is constant along horizontal segments and along vertical segments in G. Then, if x and y are two points in G which can be connected by horizontal and vertical segments, we have that $f(x) = f(y)$. But any two points of a region may be connected by finitely many such segments by Theorem 1.16, so f has the same value at any two points of G, proving the theorem.

To see that f is constant along horizontal segments, suppose that H is a horizontal line segment in G. Since H is a horizontal segment, there is some value $y_0 \in \boldsymbol{R}$ so that the imaginary part of any $z \in H$ is $Im(z) = y_0$. Consider the real part $u(z)$ of the function. Since $Im(z)$ is constant on H, we can consider $u(z)$ to be just a function of x, the real part of $z = x + iy_0$. By assumption, $f'(z) = 0$, so for $z \in H$ we have $u_x(z) = Re\big(f'(z)\big) = 0$. Thus, by Lemma 2.13, $u(z)$ is constant on H. We can argue the same way to see that the imaginary part $v(z)$ of $f(z)$ is constant on H, since $v_x(z) = Im(f\ 0(z)) = 0$. Since both the real and imaginary parts of f are constant on H, f itself is constant on H.

This same argument works for vertical segments, interchanging the roles of the real and imaginary parts, so we're done.

3.4 *Complex velocity and acceleration*

A complex number z can be visualized geometrically as a position vector in the complex plane. Suppose $z(t)$ is considered as a position vector with the running parameter t. The terminal point of the position vector traverses a curve C in the complex plane. Similar to the differentiation of a vector function, we define the derivative of $z(t)$ with respect to t to be

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

Suppose we separate $z(t)$ into its real and imaginary parts and write $z(t) = x(t) + iy(t)$. Then the derivative of $z(t)$ can be expressed as

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

The derivative gives the direction of the tangent vector to the curve at t. If the parameter t is considered as the time variable, then $\frac{dz}{dt}$ represents the velocity with which the terminal point moves along the curve. Also, the second-order derivative $\frac{d^2z}{dt^2}$ gives the acceleration of the motion along the curve.

Example Suppose the motion of a particle is described using the polar coordinates (r, θ) and its position in the complex plane is represented by

$$z(t) = r(t)e^{i\theta(t)}$$

By differentiating $z(t)$ with respect to the time variable t, find the velocity and acceleration of the particle, separating them into their radial and tangential components.

Solution Starting with $z = re^{i\theta}$, where z, r and θ are all functions of t, we differentiate z with respect to the time variable t and obtain

$$u = \dot{z}\,\dot{r}e^{i\theta} + ire^{i\theta}\dot{\theta}$$

Here, u is called the *complex velocity* and the dot over a variable denotes differentiation of the variable with respect to t. Also, $e^{i\theta}$ and $ie^{i\theta}$ represent the unit vector in the radial direction and tangential direction, respectively. The radial component of velocity u_r and the tangential component of velocity u_θ are then given by $u_r = \dot{r}$ and $u_\theta = r\dot{\theta}$

The complex velocity may be written as

$$u = (u_r + iu_\theta)e^{i\theta}\ .$$

The *complex acceleration* can be found by differentiating u again with respect to t. We obtain $a = \frac{du}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)e^{i\theta} + (2\dot{r}\,\dot{\theta} + r\ddot{\theta})ie^{i\theta}$

The radial component of acceleration a_r and the tangential component of acceleration a_θ are then given by

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad \text{and} \qquad a_\theta = 2\dot{r}\,\dot{\theta} + r\ddot{\theta}$$

Examples:

Properties involving the sum, difference or product of functions of a complex variable are the same as for functions of a real variable. In particular, the limit of a product (sum) is the product (sum) of the limits.

The *product and quotient rules* for differentiation still apply.

The *chain rule* still applies.

4.0 Conclusion In this unit, we have studied differentiation of complex functions. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 Summary You recall that you learnt about differentiation of complex functions, rules for differentiation as well as Constant Functions and *Complex velocity and acceleration.* You are to study them properly in order to be well equipped for the next course in mathematical methods

6.0 Tutor Marked Assignment

Where are the following functions differentiable? Where are they holomorphic? Determine their derivatives at points where they are differentiable.

(a) $f\ (z)\ =\ e^{-x}e^{-iy}$.
(b) $f\ (z)\ =\ 2x\ +\ ixy^2$.
(c) $f\ (z)\ =\ x^2\ +\ iy^2$.
(d) $f\ (z)\ = e^{x}e^{-iy}$.
(e) $f\ (z) = \ cos\ x\ cosh\ y -\ i\ sin\ x\ sinh\ y$.
(f) $f\ (z)\ =\ Im\ z$.
(g) $f\ (z) = |z|^2 = x^2 + y^2$.
(h) $f\ (z)\ =\ z\ Im\ z$.
(i) $f\ (z) = \frac{xi+1}{y}$
(j) $f\ (z) = \ 4(Re\ z)(Im\ z) -\ i(\bar{z})^2$.
(k) $f\ (z) = \ 2xy -\ i(x+y)^2$.
(l) $f\ (z) = \ z^2 - \bar{z}^2$.

7.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Prince-ton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex

Module 3: Analytic functions

Content

1.0 Introduction

Analytic functions

In mathematics, an **analytic function** is a function that is locally given by a convergent power series. Analytic functions can be thought of as a bridge between polynomials and general functions. There exist both **real analytic functions** and **complex analytic functions**, categories that are similar in some ways, but different in others. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. A function is analytic if and only if it is equal to its Taylor series about x_0 converges to the function in some neighborhood for every x_0 in its domain.

2.0 Objectives

At the end of this unit, you should be able to:

- Define Analytic functions
- State the properties of analytic functions

3.0 Main content

3.1 Definitions & Examples of Analytic functions

Definition:

A function f is ***real analytic*** on an open set D in the real line if for any x_0 in D one can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots.$$

in which the coefficients $a_0 a_1 \ldots .,$ are real numbers and the series is convergent to $f(x)$ for x in a neighborhood of x_0. Alternatively, an analytic function is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

converges to $f(x)$ for x in a neighborhood of x_0. The set of all real analytic functions on a given set D is often denoted by $C^{\omega}(D)$.

A function f defined on some subset of the real line is said to be real analytic at a point x if there is a neighborhood D of x on which f is real analytic. The definition of a *complex analytic function* is obtained by replacing, in the definitions above, "real" with "complex" and "real line" with "complex plane."

Examples

Most special functions are analytic (at least in some range of the complex plane). Typical examples of analytic functions are:

- Any polynomial (real or complex) is an analytic function. This is because if a polynomial has degree n, any terms of degree larger than n in its Taylor series expansion will vanish, and so this series will be trivially convergent. Furthermore, every polynomial is its own Maclaurin series.
- The exponential function is analytic. Any Taylor series for this function converges not only for x close enough to x_0 (as in the definition) but for all values of x (real or complex).

- The trigonometric functions, logarithm, and the power functions are analytic on any open set of their domain.

Typical examples of non analytic functions are:

- The absolute value function when defined on the set of real numbers or complex numbers is not everywhere analytic because it is not differentiable at 0. Piecewise defined functions (functions given by different formulas in different regions) are typically not analytic where the pieces meet.

- The complex conjugate function $z \to z^*$ is not complex analytic, although its restriction to the real line is the identity function and therefore real analytic, and it is real analytic as a function from $\mathbf{R}^2$ to $\mathbf{R}^2$.

The following gives another example of a non-analytic smooth function.

Alternative characterizations

If f is an infinitely differentiable function defined on an open set $D \subset \mathbf{R}$, then the following conditions are equivalent.

1) f is real analytic.
2) There is a complex analytic extension of f to an open set $G \subset \mathbf{C}$ which contains D.
3) For every compact set $K \subset D$ there exists a constant C such that for every $x \in K$ and every non-negative integer k the following bound holds

$$\left|\frac{d^k f}{dx^k}(x)\right| \leq C^{k+1} k!$$

The real analyticity of a function f at a given point x can be characterized using the Fourier–Bros–Iagolnitzer (FBI) transform. Complex analytic functions are exactly equivalent to holomorphic functions, and are thus much more easily characterized.

3.2 Properties of analytic functions

- The sums, products, and compositions of analytic functions are analytic.
- The reciprocal of an analytic function that is nowhere zero is analytic, as is the inverse of an invertible analytic function whose derivative is nowhere zero. (See also the Lagrange inversion theorem.)
- Any analytic function is smooth, that is, infinitely differentiable. The converse is not true; in fact, in a certain sense, the analytic functions are sparse compared to all infinitely differentiable functions.
- For any open set $\Omega \subseteq \boldsymbol{C}$, the set $A(\Omega)$ of all analytic functions $u : \Omega \to \boldsymbol{C}$ is a Fréchet space with respect to the uniform convergence on compact sets. The fact that uniform limits on compact sets of analytic functions are analytic is an easy consequence of Morera's theorem. The set $A_\infty(\Omega)$ of all bounded analytic functions with the supremum norm is a Banach space.

A polynomial cannot be zero at too many points unless it is the zero polynomial (more precisely, the number of zeros is at most the degree of the polynomial). A similar but weaker statement holds for analytic functions. If the set of zeros of an analytic function f has an accumulation point inside its domain, then f is zero everywhere on the connected component containing the accumulation point. In other words, if (r_n) is a sequence of distinct numbers such that $f(r_n) = 0$ for all n and this sequence converges to a point r in the domain of D, then f is identically zero on the connected component of D containing r.

Also, if all the derivatives of an analytic function at a point are zero, the function is constant on the corresponding connected component.

These statements imply that while analytic functions do have more degrees of freedom than polynomials, they are still quite rigid.

3.3 Analyticity and differentiability

As noted above, any analytic function (real or complex) is infinitely differentiable (also known as smooth or C^{∞}). (Note that this differentiability is in the sense of real variables; compare complex derivatives below.) There exist smooth real functions that are not analytic: see non-analytic smooth function. In fact there are many such functions.

The situation is quite different when one considers complex analytic functions and complex derivatives. It can be proved that any complex function differentiable (in the complex sense) in an open set is analytic. Consequently, in complex analysis, the term *analytic function* is synonymous with *holomorphic function*(See the last unit).

3.4 Real versus complex analytic functions

Real and complex analytic functions have important differences (one could notice that even from their different relationship with differentiability). Analyticity of complex functions is a more restrictive property, as it has more restrictive necessary conditions and complex analytic functions have more structure than their real-line counterparts.

According to Liouville's theorem, any bounded complex analytic function defined on the whole complex plane is constant. The corresponding statement for real analytic functions, with the complex plane replaced by the real line, is clearly false; this is illustrated by

$$f(x) = \frac{1}{x^2+1}$$

Also, if a complex analytic function is defined in an open ball around a point x_0, its power series expansion at x_0 is convergent in the whole ball (analyticity of holomorphic functions). This statement for real analytic functions (with open ball meaning an open interval of the real line rather than an open disk of the complex plane) is not true in general; the function of the example above gives an example for $x_0 = 0$ and a ball of radius exceeding 1, since the power series

$$1 - x^2 + x^4 - x^6 \ldots \text{ diverges for } |x| > 1.$$

Any real analytic function on some open set on the real line can be extended to a complex analytic function on some open set of the complex plane. However, not every real analytic function defined on the whole real line can be extended to a complex function defined on the whole complex plane. The function $f(x)$ defined in the paragraph above is a counterexample, as it is not defined for $x = \pm i$. This explains why the Taylor series of $f(x)$ diverges for $|x| > 1$, i.e., the radius of convergence is 1 because the complexified function has a pole at distance 1 from the evaluation point 0 and no further poles within the open disc of radius 1 around the evaluation point.

4.0 Conclusion In this unit, we have studied differentiation of complex functions. You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment.

5.0 Summary

In the unit, you have studied the following

- Analytic functions
- the properties of analytic functions
- typical examples of analytic and non analytic functions
- important differences between real and complex analytic functions

6.0 References/ Further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Princeton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitetsforlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex

Unit 2 Analytic functions 2

Content

1.0 Introduction

A function of a complex variable $\omega = f(z)$ can be viewed as a mapping of points in the z-plane to points in the ω-plane. If to each value of the independent variable *z* there is one and only one image point ω, *then* the mapping is said to be **single valued**. In contrast, let us examine a multiple-valued function. Consider a small circular path about a point z_0. This circular path can be represented by the equation $z = z_0 + re^{i\theta}$ where *r* > 0 is some small constant value and θ varies in a counter clockwise direction about the point z_0. If we have a function $\omega = f(z)$ such that $\omega = f(z_0 + re^{i\theta})$ takes on different values as θ increases by 2π, then the point z_0 is called a **branch point** of the function and the different values of ω are called **branches** of the function.
By definition, a multiple-valued function occurs if to each value of *z* there is more than one value for the dependent variable ω. The several values of ω are said to be branches of the complex valued function. If it is possible to solve an equation of the form $F(z, \omega) = 0$, connecting the complex variables

$$z = x + iy \quad \text{and} \quad \omega = u + iv,$$

to obtain single-valued functions

$$\omega_1 = f_1(z),\ \omega_2 = f_2(z),\ \omega_3 = f_3(z), \ldots\ldots$$

then these functions are called branches of the function ω. A point z_0 satisfying the property that there is no neighbourhood $|z - z_0| < \varepsilon$ in which the function $\omega = f(z)$ is single-valued, then the point z_0 is called a branch point of $f(z)$.

2.0 Objectives

At the end of the unit, you should be able to

- Define Branch Points and Branch Cuts

- Identify Cauchy-Riemann Equations
- State the necessary and sufficient conditions for the existence of the derivative of a complex function $f(z) = u + iv$
- Identify and define Harmonic functions

3.0 Main content

3.1 Branch Points and Branch Cuts

A branch point is said to be of **order** $n - 1$ whenever a function $\omega = f(z)$ is an n-valued function in the neighbourhood $|z - z_0| < \varepsilon$. A line which connects two and only two branch points is called a **branch cut** or **branch line**.

Roughly speaking, branch points are the points where the various sheets of a multiple valued function come together. The branches of the function are the various sheets of the function. For example, the function $w = z^{1/2}$ has two branches: one where the square root comes in with a plus sign, and the other with a minus sign. A **branch cut** is a curve in the complex plane such that it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve. Branch cuts are usually, but not always, taken between pairs of branch points.

Branch cuts allow one to work with a collection of single-valued functions, "glued" together along the branch cut instead of a multivalued function. For example, to make the function $f(z) = \sqrt{z}\sqrt{1 - z}$ single-valued, one makes a branch cut along the interval [0, 1] on the real axis, connecting the two branch points of the function. The same idea can be applied to the function $\sqrt{z}$; but in that case one has to perceive that the *point at infinity* is the appropriate 'other' branch point to connect to from 0, for example along the whole negative real axis.

The branch cut device may appear arbitrary (and it is); but it is very useful, for example in the theory of special functions. An invariant explanation of the branch phenomenon is developed in Riemann surface theory (of which it is historically the origin), and more generally in the ramification and monodromy theory of algebraic functions and differential equations.

A branch cut is a curve (with ends possibly open, closed, or half-open) in the complex plane across which an analytic multivalued function is discontinuous. For convenience, branch cuts are often taken as lines or line segments. Branch cuts (even those consisting of curves) are also known as cut lines (Arfken 1985, p. 397), slits (Kahan 1987), or branch lines.

For example, consider the function z^2 which maps each complex number z to a well-defined number z^2. Its inverse function $\sqrt{z}$, on the other hand, maps, for example, the value $z = 1$ to $\sqrt{1} = \pm 1$. While a unique principal value can be chosen for such functions (in this case, the principal square root is the positive one), the choices cannot be made continuous over the whole complex plane. Instead, lines of discontinuity must occur. The most common approach for dealing with these discontinuities is the adoption of so-called branch cuts. In general, branch cuts are not unique, but are instead chosen by convention to give simple analytic properties (Kahan 1987). Some functions have a relatively simple branch cut structure, while branch cuts for other functions are extremely complicated.

An alternative to branch cuts for representing multi-valued functions is the use of Riemann surfaces.

In addition to branch cuts, singularities known as branch points also exist. It should be noted, however, that the endpoints of branch cuts are not necessarily branch points.

Branch cuts do not arise for the single-valued trigonometric, hyperbolic, integer power, and exponential functions. However, their multi-valued inverses do require branch cuts. The plots and table below summarize the branch cut structure of inverse trigonometric, inverse hyperbolic, non-integer power, and logarithmic functions adopted in *Mathematica*.

Function name	**Function**	**Branch cut(s)**
inverse cosecant	$\csc^{-1} z$	$(-1, 1)$
inverse cosine	$\cos^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
inverse cotangent	$\cot^{-1} z$	$(-i, i)$
inverse hyperbolic cosecant	csch^{-1}	$(-i, i)$
inverse hyperbolic cosine	$\cosh^{-1}$	$(-\infty, 1)$
inverse hyperbolic cotangent	$\coth^{-1}$	$[-1, 1]$
inverse hyperbolic secant	sech^{-1}	$(-\infty, 0]$ and $(1, \infty)$
inverse hyperbolic sine	$\sinh^{-1}$	$(-i\infty, -i)$ and $(i, i\infty)$
inverse hyperbolic tangent	$\tanh^{-1}$	$(-\infty, -1]$ and $[1, \infty)$
inverse secant	$\sec^{-1} z$	$(-1, 1)$

inverse sine	$\sin^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
inverse tangent	$\tan^{-1} z$	$(-i\infty, -i]$ and $[i, i\infty)$
natural logarithm	$\ln z$	$(-\infty, 0]$
power	$z^n, n \notin \mathbb{Z}$	$(-\infty, 0)$ for $\Re[n] \le 0$; $(-\infty, 0]$ for $\Re[n] > 0$
square root	$\sqrt{z}$	$(-\infty, 0)$

3.1.1Branch point at infinity

Consider the two-valued function

$$\omega_1 = f(z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)\ldots\ldots(z-z_n)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which has singularities at the points $z_1, z_2, \ldots, z_n$ in the finite z-plane. Let z denote a variable point in the z -plane and construct straight lines from the point z to each of the points $z_1, z_2, \ldots, z_n$ and denote the length of these lines by $r_1, r_2, \ldots, r_n$. These straight lines make angles $\theta_1, \theta_2, \ldots \theta_n$ respectively with a horizontal line through each of the points $z_1, z_2, \ldots, z_n$.

The figure 4-1 (a) illustrates these constructions for the points z_k and z_m. The variable point z can then be represented in terms of moduli $r_1, r_2, \ldots, r_n$ and arguments $\theta_1, \theta_2, \ldots \theta_n$ by writing

$$z = z_1 + r_1 e^{i\theta_1},\ z = z_2 + r_2 e^{i\theta_2},\ \ldots\ldots,\ z = z_n + r_n e^{i\theta_n} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

and the equation (4.1) can then be expressed as

$$\omega_1 = \frac{1}{\sqrt{r_1, r_2, \ldots, r_n\, e^{i(\theta_1+\theta_2+\ldots+\theta_n}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

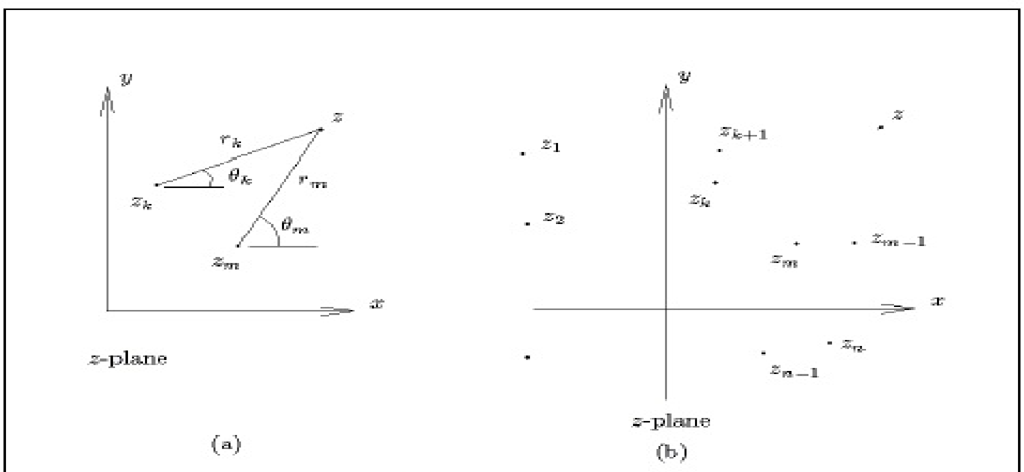


Figure 4-1. Branch points of the function $\omega = f(z) = \frac{1}{\sqrt{(z-z_1)(z-z_2)\ldots(z-z_n)}}$

Afte

r_i, i

$\theta_m + 2\pi$ and the other values of $\theta, i \neq m$ return to their original values. The equation (4.1) becomes $\omega_2 = \omega_1 e^{-i\pi} = -\omega_1$.

Observe, that if we move the point z in a small circle about any one of the points $z_1, z_2, \ldots, z_n$, the same thing happens. We observe that ω_1 changes to $\omega_2 = -\omega_1$.

In order to examine the behaviour of the function ω_1 at the"point" $z = \infty$ we make the substitution $z = \frac{1}{z'}$ and examine the behaviour of $\omega_1$for z' near the origin 0' of the z'$''$-plane.
To make the algebra tractable, replace $z_1, z_2, \ldots, z_n$ by $\frac{1}{{z_1}'}, \frac{1}{{z_2}'}, \ldots, \frac{1}{{z_n}'}$

and write

$$\omega_1 = \frac{1}{\sqrt{\left(\frac{1}{z'}-\frac{1}{{z_1}'}\right)\left(\frac{1}{z'}-\frac{1}{{z_2}'}\right)\ldots\left(\frac{1}{z'}-\frac{1}{{z_n}'}\right)}.} = \frac{\sqrt{{z_1}'{z_2}'\ldots{z_n}'}\,(z')^{n/2}}{\sqrt{({z_1}'-z')({z_2}'-z')\ldots({z_n}'-z')}.}$$

For z' near the origin 0' let $z' = re^{i\theta}$ and show that as *r* tends toward zero one obtains $\omega_1 = \left(re^{i\theta}\right)^{n/2}$. Also as z' moves about the origin 0' the angle θ' changes to $\theta + 2\pi$ and $\omega_1$changes to $\omega_2 = \omega_1 e^{in\pi}$. Therefore, if *n* is even, ω_1 keeps its same value and if *n* is odd, then
ω_1 becomes $\omega_2 = -\omega_1$. This shows that functions of the form

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)}}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4))}}$$

$$\vdots$$

$$\vdots$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_{2m}))}}$$

have respectively 2*,* 4, . . .*,* 2*m* branch points but no branch point at infinity. In contrast, functions having the forms

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2(z-z_3))}}$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5))}}$$

$$\vdots$$

$$\vdots$$

$$\omega = \frac{1}{\sqrt{(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_{2m})(z-z_{2m+1}))}}$$

have respectively, 3*,* 5*,* . . .*,* 2*m* + 1 branch points with each function having a branch point at infinity. In the case of an even number of branch points, the branch points are connected in groups of any two pairs, where the connecting cuts do not cross one another. In the case of an odd number of branch points the cuts are made in groups of any two pairs, where connecting cuts do not cross one another. The remaining point is joined to infinity by a cut line which does not cross the other cut lines.

3.1.2 Riemann surface for *n*-valued functions

To avoid the problem that the same value of *z* corresponds to two or more values of *ω* the *z*-plane is split into *n* parallel *z*-planes called sheets of a Riemann surface, where *n* corresponds to the multiplicity of the function. These *n*-sheets are separated by an infinitesimal distance and connected along a branch cut or along each of the branch cuts if more than one branch cut exists. In this way as *z* moves around the first sheet the image of *ω* is that of the first branch $\omega_1 = f_1(z)$. As *z* moves around the second sheet, the image of *ω* is that of the second branch $\omega_2 = f_2(z)$. In general, the value of *z* on the *i*th-sheet, for *i* = 1, 2, . . .,*n* , produces a single-valued function $\omega_i = f_i(z)$.. As *z* moves around a sheet and crosses a branch cut or branch line, then there occurs a change in the branch of the function. All the sheets are connected along the branch line(s) or branch cut(s) and is to be regarded as a continuous surface called the Riemann surface. The following are some examples to illustrate the above concepts.

Example

Consider the function $\omega^2 = z$.This function has a branch point at *z* = 0 and is two-valued.

It has the two branches $\omega_1 = f_1(z)$. $= +\sqrt{z}$ and $\omega_2 = f_2(z) = -\sqrt{z}$.

Let $z = re^{i(\theta+2k\pi)}$ where $k = 0{,}1$ and write

$$\omega^2 = z \text{ in the form } \omega^2 = z = re^{i(\theta+2k\pi)}$$

and then solve for *ω* to obtain the functions

$$\omega = z^{1/2} = r^{1/2}e^{i(\theta+2k\pi)/2} \quad , \quad k = 0, 1$$

We obtain

for *k* = 0 the first branch of the function $\omega_1 = f_1(z)$. $= +\sqrt{z} = +\sqrt{r}\, e^{i\theta/2}$

for *k* = 1 the second branch of the function $\omega_2 = f_2(z)$. $= -\sqrt{z} = +\sqrt{r}\, e^{i(\frac{\theta}{2}+\pi)}$

Note that when *k* = 3, 5, 7 . . . we are back on the first branch and when *k* = 4, 6, 8, ... We are back on the second branch. We desire to define a domain where these branches of the function are single-valued and analytic at each point of the domain. The derivative of the function

$$\omega_1 = f_1(z) = r^{1/2}\cos\frac{\theta}{2} + ir^{1/2}\sin\frac{\theta}{2} = u(r,\theta) + iv(r,\theta).$$

can be obtained from the derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and the formula of equations (1.78) and (1.79). One can verify the partial derivatives

$$\frac{\partial u}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} \qquad \frac{\partial v}{\partial r} = \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2}$$
$$\frac{\partial u}{\partial \theta} = -\frac{1}{2}r^{1/2}\sin\frac{\theta}{2} \qquad \frac{\partial u}{\partial \theta} = \frac{1}{2}r^{1/2}\sin\frac{\theta}{2}$$

and the derivative

$$.\frac{d\omega_1}{dz} = f'_1(z) = \frac{d}{dz}\sqrt{z} = \frac{1}{2}z^{-1/2} = \frac{1}{2}r^{-1/2}e^{-i\theta/2}$$

In a similar manner one can verify the derivative

$$\frac{d\omega_2}{dz} = f'_2(z) = \frac{d}{dz}\left(-\sqrt{z}\right) = \frac{1}{2}z^{-1/2} = \frac{1}{2}r^{-1/2}e^{-i(\pi-\frac{\theta}{2})}$$

The functions ω_1 and ω_2 fail to be analytic at the points $z = 0$ and $z = \infty$. The points $z = 0$ and $z = \infty$ are singular points associated with the function $\omega = z^{1/2}$. Let us examine the behaviour of the function ω_1 as we move around the singular point $z =$

0. If we hold r constant and let θ vary from θ to $\theta + 2\pi$ we find $\omega_1 = r^{1/2}e^{i\theta/2}$ changes to $r^{1/2}e^{i(\theta+2\pi)/2}$
and similarly if we investigate the behaviour of the function !2 as we move around the singular point $z = 0$, holding r constant, and letting θ change to $\theta + 2\pi$ we find that $\omega_2 = -r^{1/2}e^{i\theta/2}$ changes to $-r^{1/2}e^{i(\theta+2\pi)/2} = -e^{i\pi}\, r^{1/2}e^{i\theta/2} = \omega_1$
This shows that as θ increases by 2π the functions ω_1 and $\omega_2$change into each other. If we construct a branch cut from 0 to 1 along the negative *x*-axis and require that *z* not be allowed to cross the branch cut, then the functions $\omega_1$and $\omega_2$will become single-valued and analytic when defined by the equations

$$\omega_1 = f_1(z) = \sqrt{r}\, e^{i\theta/2}, \qquad r > 0, \quad -\pi < \theta \leq \pi$$

$$\omega_2 = f_2(z) = \sqrt{r}\, e^{i(\theta+2\pi)/2}, \quad r > 0, \quad -\pi < \theta \leq \pi$$

Note that at each point on the branch line or branch cut there occurs a discontinuity in the functions ω_1 and ω_2. The branch cut is a way of preventing these discontinuities to occur and hence keep the square root function single-valued.

3.1.3 Transcendental and logarithmic branch points

Suppose that *g* is a global analytic function defined on a *punctured disc* around z_0. Then *g* has a **transcendental branch point** if z_0 is an *essential singularity* of *g* such that *analytic continuation* of a function element once around some simple closed curve surrounding the point z_0 produces a different function element. An example of a transcendental branch point is the origin for the multi-valued function

$$g(z) = \exp\,(z^{-\frac{1}{k}})$$

for some integer $k > 1$. Here the monodromy around the origin is finite.

By contrast, the point z_0 is called a **logarithmic branch point** if it is impossible to return to the original function element by analytic continuation along a curve with nonzero winding number about z_0. This is so called because the typical example of this phenomenon is the branch point of the complex logarithm at the origin. Going once counterclockwise around a simple closed curve encircling the origin, the complex logarithm is incremented by $2\pi i$. Encircling a loop with winding number w, the logarithm is incremented by $2\pi i\, w$. There is no corresponding notion of ramification for transcendental and logarithmic branch points since the associated covering Riemann surface cannot be analytically continued to a cover of the branch point itself. Such covers are therefore always unramified.

In complex analysis, an **essential singularity** of a function is a "severe" singularity near which the function exhibits extreme behavior.

In complex analysis, a branch of mathematics, **analytic continuation** is a technique to extend the *domain* of a given analytic function. Analytic continuation often succeeds in defining further values of a function, for example in a new region where an infinite series representation in terms of which it is initially defined becomes divergent.

Examples of Branch points

- 0 is a branch point of the *square root* function. Suppose $w = z^{1/2}$, and z starts at 4 and moves along a circle of radius 4 in the complex plane centred at 0. The dependent variable w changes while depending on z in a continuous manner. When z has made one full circle, going from 4 back to 4 again, w will have made one half-circle, going from the positive square root of 4, i.e., from 2, to the negative square root of 4, i.e., −2.

- 0 is also a branch point of the natural logarithm. Since e^0 is the same as $e^{2\pi i}$, both 0 and $2\pi i$ are among the multiple values of Log (1). As z moves along a circle of radius 1 centred at 0, w = Log (z) goes from 0 to $2\pi i$.

- In trigonometry, since tan(π/4) and tan (5π/4) are both equal to 1, the two numbers π/4 and 5π/4 are among the multiple values of arc tan(1). The imaginary units i and $-i$ are branch points of the arctangent function (arc tan (z) = $(1/2i)\log(i - z)/(i + z)$). This may be seen by observing that the derivative (d/dz) arc tan (z) = $1/(1 + z^2)$ has simple poles at those two points, since the denominator is zero at those points.

- If the derivative f' of a function f has a simple pole at a point a, then f has a logarithmic branch point at a. The converse is not true, since the function $f(z) = z^{\alpha}$ for irrational α has a logarithmic branch point, and its derivative is singular without being a pole.

3, 2 Cauchy-Riemann Equations

This section discusses the necessary and sufficient conditions for the existence of the derivative of a complex function $f(z) = u + iv$ If the complex derivative $f'(z)$ is to exist, then we should be able to compute it by approaching z along either horizontal or vertical lines (i.e. direction parallel to the y-axis or direction parallel to the x-axis. Thus we must have

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it}$$

where t is a real number. In terms of u and v,

$$\lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{u(x+t,y) + iv(x+t,y) - u(x,y) - v(x,y)}{t}$$
$$= \lim_{t\to 0} \frac{u(x+t,y) - u(x,y)}{t} + i\lim_{t\to 0} \frac{v(x+t,y) - v(x,y)}{t}$$
$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Taking the derivative along a vertical line gives

$$\lim_{t\to 0} \frac{f(z+it) - f(z)}{it} = -i\lim_{t\to 0} \frac{u(x,y+t) + iv(x,y+t) - u(x,y) - v(x,y)}{t}$$
$$= -i\lim_{t\to 0} \frac{u(x,y+t) - u(x,y)}{t} + i\lim_{t\to 0} \frac{v(x,y+t) - v(x,y)}{t}$$
$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we see that If a function $f(z) = u + iv$ is complex differentiable, then its real and imaginary parts satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Moreover, the complex derivative $f'(z)$ is then given by

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

If $f\,(z) = u(x,y) + iv\,(x,y)$ is defined in a neighborhood of $z = x + iy$, and if *f* is differentiable at *z*, then

$$u_x(x,y) = v_y\,(x,y), \quad \text{and} \quad u_y\,(x,y) = -\,v_x\,(x,y). \qquad (1)$$

These are called the Cauchy-Riemann equations.

Conversely, if the partial derivatives of *u* and *v* exist in a neighborhood of $z = x + i\,y$, if they are continuous at *z* and satisfy the Cauchy-Riemann equations at *z*, then

$$f\,(z) = u(x,y) + iv\,(x,y) \text{ is differentiable at } z.$$

The Cauchy-Riemann equations therefore give a criterion for analyticity. Indeed, if a function is analytic at *z*, it must satisfy the Cauchy-Riemann equations in a neighborhood of *z*. In particular, if *f* does not satisfy the Cauchy-Riemann equations, then *f* cannot be analytic.

Conversely, if the partial derivatives of *u* and *v* exist, are continuous, and satisfy the Cauchy-Riemann equations in a neighborhood of $z = x + iy$, then $f\,(z) = u(x,y) + i\,v\,(x,y)$ is analytic at z.

Examples:

3.2.1 Cauchy–Riemann relations in polar coordinates.

Consider the polar coordinates

$$z^2 = (x^2 - y^2) \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

Differentiating *r* and θ with respect to both *x* and *y*, we obtain

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2-y^2} = -\frac{1}{r}\sin\theta\,, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{r}\cos\theta$$

Using the chain rule, the first-order partial derivatives of *u* are given by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta, \qquad \text{(i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \qquad \text{(ii)}$$

Similarly, the first-order partial derivatives of *v* are given by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \qquad \text{(iii)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \qquad \text{(iv)}$$

Using one of the Cauchy–Riemann relations, we combine equations. (i) and (iv) to give

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad \text{(v)}$$

Similarly, using equations (ii) and (iii) and applying the other Cauchy–Riemann relation, we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta + \left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \qquad \text{(vi)}$$

In order that equations. (v) and (vi) are satisfied for all θ, we must have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}$$

These are the Cauchy–Riemann relations expressed in polar coordinates.

Example Discuss the differentiability of the function

$$f(z) = f(x+iy) = \sqrt{|xy|} \text{ at } z = 0$$

Solution: let $f(x+iy) = u(x,y) + iv(x,y)$ so that

$$u(x,y) = \sqrt{|xy|} \text{ and } v(x,y) = 0$$

Since $u(x,0)$ and $u(0,y)$ are identically equal to zero, we have $u_x(0,0) = u_y(0,0) = 0$
Also, since $v(x,y)$ is identically zero, it is obvious that $v_x(0,0) = v_y(0,0) = 0$
Hence, the Cauchy–Riemann relations are satisfied at the point (0, 0).
However, suppose z approaches the origin along the ray $x = \alpha t, y = \beta t, t > 0$, assuming that α and β cannot be zero simultaneously. For $z = \alpha t + i\,\beta t$, we then have

$$\frac{f(z)-f(0)}{z-0} = \frac{f(z)}{z} = \frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}$$

The limit of the above quantity as $z \to 0$ depends on the values of α and β, so the limit is non-unique. Therefore, $f(z)$ is not differentiable at $z = 0$, though the Cauchy–Riemann relations are satisfied at $z = 0$.
Let us check the continuity of v_x at $(0,0)$. Since

$$\frac{\partial u}{\partial x} = \sqrt{|y|}\frac{d}{dx}\sqrt{|xy|}$$

u_x fails to be continuous at $(0,0)$. By virtue of Theorem 2.4.2, it is not surprising that $f(z) = \sqrt{|xy|}$ can fail to be differentiable at $z = 0$ since the Cauchy– Riemann relations are necessary but not sufficient for differentiability.

Example: The function $z^2 = (x^2 - y^2) + 2xyi$ satisfies the Cauchy- Riemann equations, since

$$\frac{\partial}{\partial x}(x^2 - y^2) = 2x = \frac{\partial}{\partial y}(2xy) \text{ and } \frac{\partial}{\partial x}(2xy) = 2y = -\frac{\partial}{\partial y}(x^2 - y^2)$$

Likewise $e^z = e^x \cos y + ie^x \sin y$ satisfies the Cauchy-Riemann equations,
Since $\frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y = \frac{\partial}{\partial y}(e^x \sin y)$ and

$$\frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y = -\frac{\partial}{\partial y}(e^x \cos y)$$

Moreover, e^z is in fact complex differentiable, and its complex derivative is

$$\begin{aligned}\frac{d}{dz}e^z &= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial x}(e^x \sin y)\\ &= e^x \cos y + e^x \sin y\\ &= e^z\end{aligned}$$

The chain rule then implies that, for a complex number α, $\frac{d}{dz}e^{\alpha z} = \alpha e^{\alpha z}$.One can define $cos\ z$ and $sin\ z$ in terms of e^{iz} and e^{-iz}.
From the sum rule and the expressions for $cos\ z$ and $sin\ z$ in terms of e^{iz} and e^{-iz}, it is easy to check that cos z and sin z are analytic and that the usual rules hold :

$$\frac{d}{dz}cos\ z = -\sin z; \quad \frac{d}{dz}\sin z = \cos z$$

On the other hand, _z does not satisfy the Cauchy-Riemann equations, since

$$\frac{\partial}{\partial x}(x) = 1 \neq \frac{\partial}{\partial y}(-y)$$

Likewise, $f(z) = x^2 + iy^2$ does not. Note that the Cauchy-Riemann equations are two equations for the partial derivatives of u and v, and both must be satisfied if the function $f(z)$is to have a complex derivative. We have seen that a function with a complex derivative satisfies the Cauchy-Riemann equations. In fact, the converse is true:

Theorem: Let $f(z) = u + iv$ be a complex function defined in a region (open subset) D of C, and suppose that u and v have continuous first partial derivatives with respect to x and y. If u and v satisfy the Cauchy-Riemann equations, then $f(z)$ has a complex derivative.

The proof of this theorem is not difficult, but involves a more careful understanding of the meaning of the partial derivatives and linear approximation in two variables.

Thus we see that the Cauchy-Riemann equations give a complete criterion for deciding if a function has a complex derivative (that is, the Cauchy-Riemann equations gives a criterion for analyticity) . There is also a geometric interpretation of the Cauchy-Riemann equations. Recall that $\Delta u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and that $\Delta v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$. Then u and v satisfy the Cauchy-Riemann equations if and only if

$$\Delta v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)$$

If this holds, then the level curves: $u = c_1$ and $v = c_2$ are orthogonal where they intersect.

Instead of saying that a function $f(z)$ has a complex derivative, or equivalently satisfies the Cauchy-Riemann equations, we shall call $f(z)$ analytic or holomorphic. Here are some basic properties of analytic functions, which are easy consequences of the Cauchy-Riemann equations:

Theorem: Let $f(z) = u + iv$ be an analytic function.

1. If $f'(z)$ is identically zero, then $f(z)$ is a constant.
2. If either Re $f(z) = u$ or $Im\, f(z) = v$ is constant, then $f(z)$ is constant.

In particular, a non constant analytic function cannot take only real or only pure imaginary values.

3. If $|f(z)|$is constant or arg $f(z)$ is constant, then $f(z)$ is constant.

For example, if $f'(z) = 0$, then

$$0 = f'^{(z)} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0.$$

By the Cauchy-Riemann equations, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$ as well. Hence $f(z)$ is a constant. This proves (1).

To see (2), assume for instance that u is constant. Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, and, as above, the Cauchy-Riemann equations then imply that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Again, $f(z)$ is constant.

Part (3) can be proved along similar but more complicated lines.

3.3 Harmonic functions

Let $f(z) = u + iv$ be an analytic function, and assume that u and v have partial derivatives of order 2 (in fact, this turns out to be automatic). Then, using the Cauchy-Riemann equations and the equality of mixed partials, we have

$$:\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

In other words, u satisfies: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The above equation is a very important second order partial differential equation, and solutions of it are called harmonic functions. Thus, the real part of an analytic function is harmonic. A similar argument shows that v is also harmonic, i.e. the imaginary part of an analytic function is harmonic. Essentially, all harmonic functions arise as the real parts of analytic functions.

Theorem: Let D be a simply connected region in C and let $u(x;\ y)$ be a real-valued, harmonic function in D. Then there exists a real-valued function $v(x;\ y)$ such that $f(z) = u + iv$ is an analytic function.

We will discuss the meaning of the simply connected condition in the exercises in the next handout. The problem is that, if D is not simply connected, then it is possible that u can be completed to an analytic "function" $f(z) = u + iv$ which is not single-valued, even if u is single valued. The basic example is Re log $z = \frac{1}{2}\ln(x^2 + y^2)$ A calculation (left as homework) shows that this function is harmonic. But an analytic function whose real part is the same as that of $log\ z$ must agree with log z up to an imaginary constant, and so cannot be single-valued.

The point to keep in mind is that we can generate lots of harmonic functions, in fact essentially all of them, by taking real or imaginary parts of analytic functions. Harmonic functions are very important in mathematical physics, and one reason for the importance of analytic functions is their connection to harmonic functions.

3.3.1 Harmonic conjugate

Given two harmonic functions φ(x, y) and ψ(x, y) and if they satisfy the Cauchy-Riemann relations throughout a domain D, with $\varphi_x = \psi_x$ and $\varphi_y = -\psi_y$

We call ψ a harmonic conjugate of φ in D.

Note that harmonic conjugacy is not a symmetric relation because of the minus sign in the second Cauchy-Riemann relation. While ψ is a harmonic conjugate of φ, $-\varphi$ is a harmonic conjugate of ψ.

For example, e^xsin y is a harmonic conjugate of e^xcos y while $-e^x$cos y is a harmonic conjugate of e^xsin y.

Theorem

A complex function $f(z) = u(x,y) + iv(x,y), z = x + iy$, is analytic in a domain D if and only if v is a harmonic conjugate to u in D.

Proof

$\Rightarrow$ given that $f = u + iv$ is analytic, then u and v are harmonic and Cauchy-Riemann relations are satisfied. Hence, v is a harmonic conjugate of u.

$\Leftarrow$ Given that v is a harmonic conjugate of u in D, we have the satisfaction of the Cauchy Riemann relations and the continuity of the first order partials of u and v in D.

Hence, $f = u + iv$ is differentiable for all points in D. Since D is an open set, every point in D is an interior point, so f is analytic in D.

Example Find a harmonic conjugate of the harmonic function

$$u(x,y) = e^{-x}\cos y + xy$$

Solution

1. Take$(x_0, y_0) = (0,0)$, $u_y(x,0) = x$ and $u_x(x,y) = -e^{x}\cos y + y$

$$v(x,y) = \int_0^x -x dx + \int_0^y (-e^{-x}\cos y + y)dy$$
$$= -\frac{x^2}{2} - e^{-x}\sin y + \frac{y^2}{2}$$

2. From the first Cauchy-Riemann relation, we have

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} = -e^{-x}\cos y + y$$

Integrating with respect to y, we obtain

$$v(x,y) = -e^{-x}\sin y + \frac{y^2}{2} + \eta(x)$$

where η(x) is an arbitrary function arising from integration.
Using the second Cauchy-Riemann relation, we have

$$\frac{\partial u}{\partial x} = e^{-x}\sin y + \eta'(x) = -\frac{\partial u}{\partial y} = e^{-x}\sin y - x$$

Comparing like terms, we obtain $\eta'(x) = -x$, and subsequently,

$$\eta(x) = -\frac{x^2}{2} + C, \qquad \text{C is an arbitrary constant.}$$

Hence, a harmonic conjugate is found to be (taking C to be zero for convenience)

$$v(x,y) = -e^{-x}\sin y + \frac{y^2 - x^2}{2}$$

The corresponding analytic function, $f = u + iv$, is seen to be

$$f(z) = e^{-z} - \frac{iz^2}{2}, \qquad z = x + iy$$

which is an entire function.

3. It is readily seen that $e^{-x}\cos y = Re\, e^{-z}$ and $xy = \frac{1}{2} Im\, z^2$

A harmonic conjugate of $Re\, e^{-z}$ is $Im\, e^{-z}$, while that of $\frac{1}{2} Im\, z^2$ is $-\frac{1}{2} Re\, z^2$.
Therefore, a harmonic conjugate of $u(x,y)$ can be taken to be

$$v(x,y) = Im\, e^{-z} - \frac{1}{2} Re\, z^2$$
$$= -e^{-x}\sin y + \frac{y^2 - x^2}{2}$$

Example

Show that$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)$. Use the result to find a harmonic conjugate of

$$u(x,y) = e^{-x}(x \sin y - y\cos y)$$

Solution

Observe that $f'(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y)$. Putting $y = 0$, we obtain

$f'(x) = \frac{\partial u}{\partial x}(x,0) - i\frac{\partial u}{\partial y}(x,0)$

Replacing x by z, we obtain $f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)$

Application of the C-R Equations

A consequence of the Cauchy-Riemann equations is that

$$f(z) = ux + ivx = vy - iuy\,.(2)$$

We will use these formulas later to calculate the derivative of some analytic functions.

Another consequence of the Cauchy-Riemann equations is that an entire function with constant absolute value is constant. In fact, a more general result is that *an entire function that is bounded* (including at infinity) is *constant*

Given a harmonic function u, one can use the Cauchy- Riemann equations to find its harmonic conjugate$v$, and vice-versa.

Examples:

Check that $u(x,y) = 2xy$ is harmonic, and find its harmonic conjugate v.

Given a harmonic function $v(x,y)$, how would you find its harmonic conjugate $u(x,y)$?

Theorem

If ψ is a harmonic conjugate of φ, then the two families of curves

$$\varphi(x, y) = \alpha \text{ and } \psi(x, y) = \beta$$

are mutually orthogonal to each other.

Proof

Consider a particular member from the first family

$$\varphi(x,y) = \alpha_1$$

the slope of the tangent to the curve at (x, y) is given by $\frac{dy}{dx}$ where

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 0,$$

Giving

$$\frac{dy}{dx} = -\frac{\partial\varphi}{\partial x}\Big/\frac{\partial\varphi}{\partial y}.$$

Similarly, the slope of the tangent to a member from the second family at (x, y) is given by

$$\frac{dy}{dx} = -\frac{\partial\psi}{\partial x}\Big/\frac{\partial\psi}{\partial y}.$$

The product of the slopes of the two tangents to the two curves at the same point is found to be $\left(-\frac{\partial\varphi}{\partial x}\Big/\frac{\partial\psi}{\partial y}\right)\left(-\frac{\partial\psi}{\partial x}\Big/\frac{\partial\psi}{\partial y}\right) = -1$

by virtue of the Cauchy-Riemann relations: $\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}$ and $\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$

Hence, the two families of curves are mutually orthogonal to each other.

Example

Supposing the isothermal curves of a steady state temperature field are given by the family of parabolas $y^2 = \alpha^2 + 2\alpha x$ is real positive, in the complex plane, find the general solution of the temperature function T(x, y). Also, find the family of flux lines of the temperature field.

Solution

First, we solve for the parameter α in the equation of the isothermal curves. This gives

$$\alpha = -x + \sqrt{x^2 + y^2} ,$$

where the positive sign is chosen since $\alpha > 0$. .

A naive guess may suggest that the temperature function T(x, y) is given by

$$T(x, y) = -x + \sqrt{x^2 + y^2}$$

However, since T(x, y) has to be harmonic, the above function cannot be a feasible solution. We set

$$T(x, y) = f(t)$$

Where $t = \sqrt{x^2 + y^2} - x$ and f is some function to be determined such that $T(x, y)$ is harmonic. To solve for f(t), we first compute

$$\frac{\partial^2 T}{\partial x^2} = f''(t)\left(\frac{\partial t}{\partial x}\right)^2 + f'(t)\frac{\partial^2 t}{\partial x^2}$$

$$= f''(t)\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right)^2 + f'(t)\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 T}{\partial y^2} = f''(t)\frac{y^2}{x^2+y^2} + f'(t)\frac{x^2}{(x^2+y^2)^{3/2}}$$

Since the Laplace equation T(x, y) satisfies

$$2\left(1 - \frac{x}{\sqrt{x^2+y^2}}\right) f''(t) + \frac{1}{\sqrt{x^2+y^2}} f'(t) = 0 \text{ or } \frac{f''(t)}{f'(t)} = \frac{1}{2t}$$

Integrating once gives

$$\ln f'(t) = -\frac{1}{2}\ln t + C \quad \text{or} \quad f'(t) = \frac{C\prime}{\sqrt{t}}$$

Integrating twice gives $f(t) = C_1\sqrt{t} + C_2$

where C_1 and C_2 are arbitrary constants. The temperature function is

$$T(x, y) = f(t) = C_1\sqrt{\sqrt{x^2 + y^2} - x} + C_2$$

When expressed in polar coordinates

$$T(r, \theta) = C_1\sqrt{r(1 - \cos\theta)} + C_2 = C_1\sqrt{2r}\sin\frac{\theta}{2} + C_2$$

Since $T(r, \theta)$can be expressed as $\sqrt{2}C_1 Im\, z^{1/2} + C_2$, the harmonic conjugate of $T(r, \theta)$ is easily seen to be $F(r, \theta) = -\sqrt{2}C_1 Re z^{1/2} + C_3 = -C_1\sqrt{2r}\cos\frac{\theta}{2} + C_3$

Where C_3 is another arbitrary constant.

Note that $\sqrt{2r}\cos\frac{\theta}{2} = \sqrt{r + r\cos\theta} = \sqrt{\sqrt{x^2 + y^2} + x}$

So that $F(x, y) = -C_1\sqrt{\sqrt{x^2 + y^2} + x} + C_3$

The family of curves defined by

$$x + \sqrt{x^2 + y^2} = \beta \text{ or } y^2 = \beta^2 - 2\beta x, \qquad \beta > 0$$

are orthogonal to the isothermal curves $y^2 = \alpha^2 - 2\alpha x,\ \alpha > 0$
Physically, the direction of heat flux is normal to the isothermal lines. Therefore, the family of curves orthogonal to the isothermal lines is called the **flux lines**. These flux lines indicate the flow directions of heat in the steady state temperature field.

The flux function F(r, θ) is a harmonic conjugate of the temperature function. The families of curves: T(r, θ) = α and F(r, θ) = β, α and β being constant, are mutually orthogonal to each other.

4.0 CONCLUSION

In this unit, we have studied some examples of branch Points and branch Cuts, the necessary and sufficient conditions for the existence of the derivative of a complex function You are required to study this unit properly before attempting to answer questions under the tutor-marked assignment

5.0 Summary

In the unit, you have studied the following

- Definitions and Examples of Branch Points and Branch Cuts
- Cauchy-Riemann Equations
- the necessary and sufficient conditions for the existence of the derivative of a complex function $f(z) = u + iv$
- Harmonic functions

6.0 Tutor Marked Assignment

1. Write the function $f(z)$ in the form $u + iv$:
   i. a) $z + i\,z^2$ b) $\frac{1}{z}$ c) $\frac{\bar{z}}{z}$
2. $f(z) = e^z$, describe the images under f (z) of horizontal and vertical lines, i.e. what are the sets $f(a + it)$ and $f(t + ib)$, where a and b are constants and t runs through all real numbers?
3. Is the function $\frac{\bar{z}}{z}$ continuous at 0? Why or why not? Is the function $\frac{\bar{z}}{z}$ analytic where it is defined? Why or why not?
4. Compute the derivatives of the following analytic functions, and be prepared to justify your answers:
   a) $\frac{iz+3}{z^2-(2+i)z+(4-3i)}$
   b) e^{z^2}
   c) $\frac{1}{e^z + e^{-z}}$
5. Let $f(z)$ be a complex function. Is it possible for both $f(z)$ and $\overline{f(z)}$ to be analytic?
6. Use the Cauchy-Riemann equations to show that $\bar{z}$ is not analytic.

7. Use the Cauchy-Riemann equations to show that $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

8. Let $f(z) = u + iv$ be analytic. Recall that the Jacobian is the function given by the following determinant:

   a. $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$

   b. Using the Cauchy-Riemann equations, show that this is the same as $|f'(z)|^2$

9. Define the complex sine and cosine functions as follows:

   i. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$; $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

10. Note that, if t is real, then this definition of cos t and sin t agree with the usual ones, and that (for those who remember hyperbolic functions) $\cos z = -i \cosh iz$ and $\sin z = -i \sinh iz$. Verify that cos z and sin z are analytic and that $(\cos z)' = -\sin z$ and that $(\sin z)' = \cos z$. Write, $z\ as\ u + iv$where u and v are real-valued functions of x and y, and similarly for sin z.

11. Verify that $1/_Z$, $Im\ 1/_Z$, and $Re\ log\ z = \frac{1}{2}\ln(x^2 + y^2)$ are harmonic.

12. Which of the following are harmonic?

    a) $x^3 - y^3$ b) $x^3y - xy^3$ c) $x^2 - 2xy$

13. If $f(z) = u + iv$ is a complex function such that u and v are both harmonic, is $f(z)$ necessarily analytic?

7.0 Reference/further Readings

- K.A Stroud; Engineering Mathematics Palgrave New York(2011)
- Complex Variables (2nd Edition), M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman, Schaum's Outline Series, Mc Graw Hill (USA), ISBN 978-0-07-161569-3
- Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 0-07-912147-0, "In electrical engineering, the letter *j* is used instead of *i*."
- Kasana, H.S. (2005), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 81-203-2641-5, Extract of chapter 1, page 14
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", *Journal of Online Mathematics and its Applications*
- Hamilton, J. D. (1994).Time Series Analysis. Princeton University Press, Prince-ton.
- SydsÊter, K. (1978).Matematisk analyse, Bind II. Universitets forlaget, Oslo.
- SydsÊter, K. and P. Hammond (2002).Essential Mathematics for Economics Analysis. Pearson Education, Essex
- Steven G. Krantz, *Function Theory of Several Complex Variables* (1992)

- Volker Scheidemann, *Introduction to complex analysis in several variables*, Birkhäuser, 2005, ISBN 3-7643-7490-X