NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 241

COURSE TITLE: INTRODUCTION TO REAL ANALYSIS

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Course Title INTRODUCTION TO REAL ANALYSIS
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CONTENTS

MODULE 1

PRELIMINARIES

In this initialmodule we will present the background needed for the study of real analysis. Unit 1.1 consists of a brief survey of set operations and functions, two vitaltools for all of mathematics. In it we establish the notationand state the basic definitions and properties thatwill be used throughout the book. We will regard the word "set" as synonymous with thewords "class", "collection", and "family", and we willnot define these terms or give a listof axioms for set theory. This approach, often referred to as "naive" set theory, is quite adequate for working with sets in the context of real analysis.

Unit1.2 is concerned witha special method of proof calledMathematical Induction. It is related to the fundamental properties of the natural number system and, though it is restricted to proving particular types of statements, it is important and used frequently.' Aninformal discussion of the different types of proofs that are used in mathematics,such as contrapositives and proofs by contradiction, can be found in Appendix A.

In Unit 1.3 we apply some of thetools presented in the first two units of this unit to a discussion of what it means for a set to be finite or infinite. Careful definitions are given and some basic consequences of these definitions are derived. The important resultthattheset of rational numbers is countably infinite is established.

In additionto introducingbasicconceptsand establishingterminology and notation, thismodulealso provides the reader with some initialexperience in workingwithprecise definitions and writingproofs. The careful studyof real analysisunavoidablyentailsthe readingand writing of proofs, and likeany skill, it is necessary to practice.Thismodule is a starting point.

Unit 1 Sets and Functions

1.0 INTRODUCTION

To the reader: In thisunitwe give a brief reviewof the terminology and notation thatwillbe used in this text.We suggest thatyou look through quickly and come back laterwhen you need to recall the meaning of a term or a symbol.

If an element *x* is in a set A, we write

$$x \in A$$

and say that x is a **member** of A, or that*x* belongs to A. If *x* is *not* in A, we write

$$x \notin A.$$

If every element of a set A also belongs to a set *B*, we say that A is a subset of *B* and write

$$A \subseteq B \qquad \text{or} \qquad B \supseteq A$$

We say that a set A is a **proper subset** of a set B if A $\subseteq$B, but there is at least one element of *B* that is not in A. In this case we sometimes write

$$A \subset B.$$

2.0 OBJECTIVES

At theend of the Unit, readers should be able to

(i) Understand the notation and state the basic definition and properties that will be used throughout the study.

(ii) Understand the background needed in Real Analysis

(iii) Understand different axioms use in set theory.

3.0 MAIN CONTENT

1.1.1 Definition Two sets A and B are saidto be **equal,** and we write A = B, if they containthesame elements.

Thus, to prove that the sets A and B are equal, we must show that

$$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

A set is normally denned by either listing its elements explicitly, or by specifying a property that determines the elements of the set. If P denotes a property that is meaningful and unambiguous for elements of a set S, then we write

$$(x \in S : P(x)\}$$

for theset of all elements x in S for which the property P is tine. If the set S is understood from the context, thenit is often omitted in thisnotation.

Several specialsets are used throughout thisbook, and they are denoted by standard symbols. (We willuse the symbol := to mean that the symbol on the left is being *defined* by the symbol on the right.)

- **The** set of **natural numbers** $\mathbb{N}$ **:= {1, 2, 3, …),**
- The set of **integers**$\mathbb{Z}$:= {0, 1, -1, 2, -2,…),
- **The** set of **rational numbers** $\mathbb{Q}$ **:=** $(m/n : m, n \in \mathbb{Z}$ and $n \neq 0)$,
- The set of **real numbers**$\mathbb{R}$.

  The set$\mathbb{R}$ of real numbers is of fundamental importance for us and willbe discussed at length in Module 2.

1.1.2 Examples (a) The set

$$\{x \in \mathbb{N} : .x^2 - 3x + 2 = 0\}$$

consists of those natural numbers satisfying the stated equation. Since the only solutions of this quadratic equation are $x = 1$ and $x = 2$, we can denote this set more simply by {1, 2}.

(b) A natural number n is **even** if it has the form $n = 2k$ for some $k \in \mathbb{N}$. The set of even natural numbers can be written

$$\{2k: k \in \mathbb{N}).$$

which is less cumbersome than $\{n \in \mathbb{N} : n = 2k, k \in \mathbb{N}\}$. Similarly, the set of **odd** natural numbers can be written

$$\{2k - 1 : k \in \mathbb{N}). \qquad \square$$

Set Operations

We now define the methods of obtaining new sets from given ones. Note that these set operations are based on the meaning of the words "or", "and", and "not". For the union, it is important to be aware of the fact that the word "or" is used in the *inclusive sense,* allowing the possibilitythat x may belong to both sets. In legal terminology, this inclusive sense is sometimes indicatedby "and/or".

1.1.3 Definition (a) The **union** of sets A and B is the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

(b) The **intersection** of the sets A and B is the set

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

(c) The **complement** of B **relative** to A is the set

$$A \backslash B := \{x : x \in A \text{ and } x \notin B\}$$

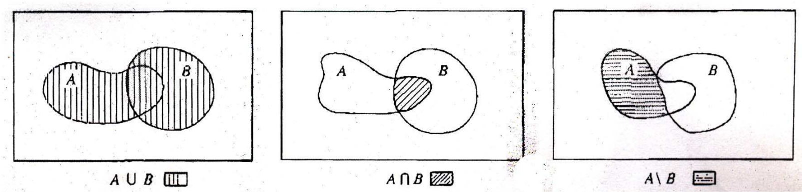


Figure 1.1.1 (a) $A \cup B$ (b) $A \cap B$ (c) $A \backslash B$

The set that has no elements is called the **empty** set and is denoted by the symbol $\emptyset$. Two sets A and B are said to be **disjoint** if they have no elements in common; this can be expressed by writing $A \cap B = \emptyset$

To illustrate the method of proving set equalities, we will next establish one of the *DeMorgan laws* for three sets. The proof of the other one is left as an exercise.

1.1.4 Theorem If A, B, C are sets, then

(a) $A\backslash (B \cup C) = (A\backslash B) \cap (A\backslash C)$,

(b) $A\backslash (B \cap C) = (A\backslash B) \cup (A\backslash C)$.

Proof: To prove (a), we will show that every element in $A\backslash (B \cup C)$ is contained in both $(A\backslash B)$ and $(A\backslash C)$, *and conversely*.

If x is in $A\backslash(B \cup C)$, *then* x is in A, but x is not in $B \cup C$. Hence x is in A, but x is neither in B nor in C. Therefore, x is in A but not B, and x is in A but not C. Thus, $x \in A\backslash B$ and $x \in A\backslash C$, which shows that $x \in (A\backslash B) \cap (A\backslash C)$.

Conversely, if $x \in (A\backslash B) \cap (A\backslash C)$, then $x \in (A\backslash B)$ *and* $x \in (A\backslash C)$. *Hence* $x \in A$ and both $x \notin B$ *and* $x \notin C$. Therefore, $x \in A$ and $x \notin B \cup C)$, so that $x \in (A\backslash (B \cup C)$.

Since the sets $(A\backslash B) \cap (A\backslash C)$ and $A\backslash(B \cup C)$ contain the same elements, they are equal by Definition 1.1.1. **Q.E.D**

There are times when it is desirable to form unions and intersections of more than two sets. For a finite collection of sets $\{A_1, A_2, \ldots, A_n\}$, their unions is the set A consisting of all elements that belong to at least one of the sets A_k, and their intersection consists of all elements that belong to all of the sets A_k.

This is extended to an infinite collection of sets $\{A_1, A_2, \ldots, A_n, \ldots\}$ as follows. Their **union** is the set of elements that belong to *at least one of the sets* A_n. In this case we write

$$\bigcup_{n=1}^{\infty} A_n := \{ x : x \in A_n \text{ for some } n \in \mathcal{N}\}.$$

Similarly, their **intersection** is the set of elements that belong to all of these sets A_n. In this case we write

$$\bigcap_{n=1}^{\infty} A_n := \{ x : x \in A_n \text{ for all } n \in \mathcal{N} \}.$$

Cartesian Products

In order to discuss functions, we define the Cartesian product of two sets.

1.1.5 Definition If *A* and *B* are nonempty sets, then the **Cartesian product** *A* x *B* of *A* and *B* is the set of all ordered pairs (a, b) with a $\in$ A and b $\in$ *B*. That is,

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

Thus if A = {1, 2, 3} and B = (1,5), then the set *A* x *B* is the set whose elements are the ordered pairs.

(1, 1), (1, 5), (2, 1), (2, 5), (3, 1), (3, 5).

We may visualize the set *A* x *B* as the set of six points in the plane with the coordinates that we have just listed.

We often draw a diagram (such as Figure 1.1.2) to indicate the Cartesian product of two sets *A and B*. However, it should be realized that this diagram may be a simplification. For example, if $A := \{x \in \mathbb{R} : 1 \le x \le 2\}$ and $B := \{ y \in \mathbb{R} : 0 \le y \le 1 \text{ or } 2 \le y \le 3\}$, then instead of a rectangle, we should have a drawing such as Figure 1.1.3.

We will now discuss the fundamental notion of a *function or a mapping*.

To the mathematician of the early nineteenth century, the word "function" meant a definite formula, such as $f(x) := x^2 + 3x - 5$, which associates to each real number *x* another number *f(x)*. (Here, $f(0) = -5$, $f(1) = -1$, $f(5) = 35$). This understanding excluded the case of different formulas on different intervals, so that functions could not be defined "in pieces".

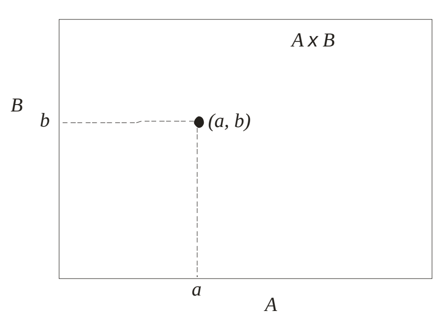


Figure 1.1.2

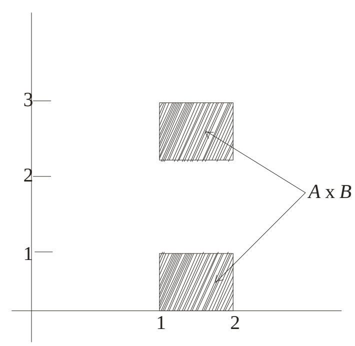


Figure 1.1.3

As mathematics developed, it became clear that a more general definition of "function" would be useful. It also became evident that it is important to make a clear distinction between the function itselfand the values of the function. A reviseddefinition might be:

A function *f* from a set *A* intoa set *B* is a rule of correspondence that assigns to each element *x* in *A* a uniquely determined element *f (x)* in *B*.

But however suggestive this revised definition might be, there, is the difficulty of interpreting the phrase "rule of correspondence". In order to clarify this, we will express the definition entirely in terms of sets; in effect, we will define a function to be its **graph.** While this has the disadvantage of being somewhat artificial, it has the advantage of being unambiguous and clearer.

1.1.6 Definition Let A and B be sets. Then a **function** from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. (In other words, if $(a,b) \in f$ and $(a, b') \in f$, then $b = b'$.)

The set A of first elements of a function f is called the **domain** of f and is often denoted by $D(f)$. The set of all second elements in f is called the **range** of f and is often denoted by $R(f)$. Note that, although $D(f) = A$, we only have $R(f) \subseteq B$. (See Figure 1.1,4,)

The essential condition that:

$$(a, b) \in f \quad \text{and} \quad (a, b') \in f \text{ implies that} \quad b = b'$$

is sometimes called the *vertical line test.* In geometrical terms it says every vertical line $x = a$ with $a \in A$ intersects the graph of f exactly once.

The notation

$$f : A \to B$$

is often used to indicate that f is a function from A into B. We will also say that f is a ***mapping*** of A into B, or that f **maps** A into B. If (a, b) is an element in f, it is customary to write

$$b = f(a) \qquad \text{or sometimes} \qquad a \to b.$$

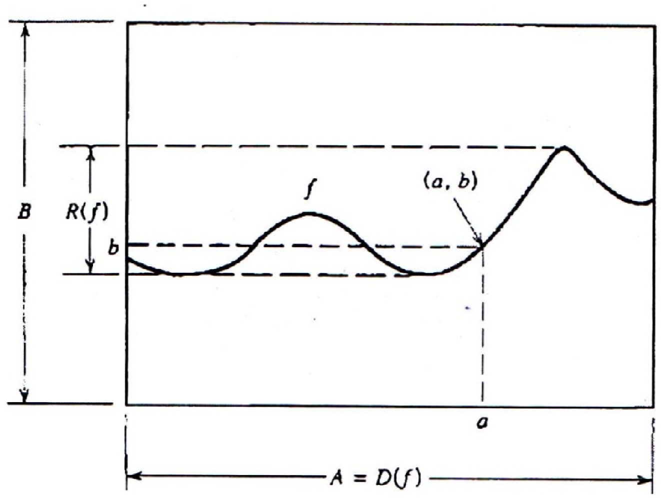


Figure 1.1.4 A function as a graph

If $b = f(a)$, we often refer to b as the **value**of f at a, or as the **image** of a under f.

Transformations and Machines

Aside from using graphs, we can visualize a function as a *transformation* of the set D(f) = A into the set R (f) $\subseteq B$. In this phraseology, when $(a, b) \in f$, we think of f as taking the element a from A and "transforming" or "mapping" it into an element $b = f(a)$ in R(f) $\subseteq B$. We often draw a diagram, such as Figure 1.1.5, even when the sets A and B are not subsets of the plane.

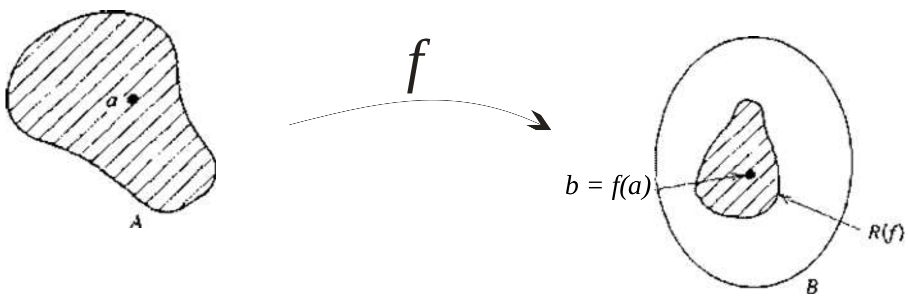


Figure 1.1.5 A function as a transformation

There is another way of visualizing a function: namely, as a *machine* that accepts elements of *D (f) — A* as *inputs* and produces corresponding elements of *R(f)* C *B* as *outputs.* If we take an element* € *D(f)* and put it into /.thenoutcomes the corresponding value /(x).If we put a different element *y e D(f)* into /, then outcomes /(v)which may or may not differ from *f(x).* If we try to insert something that does not belong to *D(f)* into/, we find that it is not accepted, for / can operate only on elements from D(/).(See Figure 1.1.6.)

This last visualization makes clear the distinction between / and *f(x):* the first is the machine itself,and the second is the output of the machine / when x is the input. Whereas no one is likely to confuse a meat grinder with ground meat, enough people have confused functions with their values that it is worth distinguishing between them notationally.

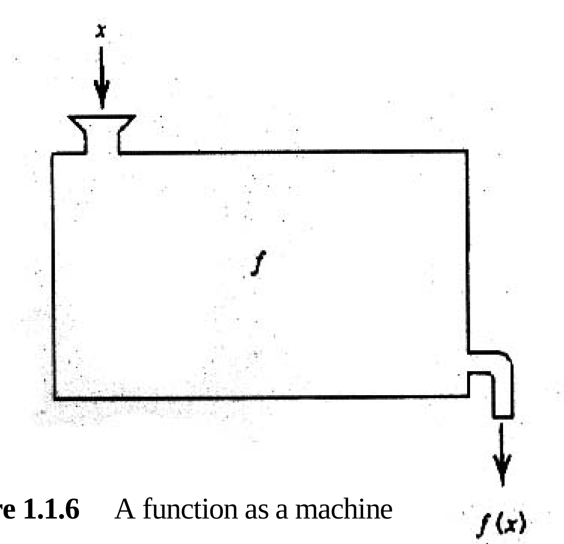


Figure 1.1.6 A function as a machine

Direct and Inverse Images

Let $f : A \to B$ be a function with domain $D(f) = A$ and range $R(f) \subseteq B$.

1.1.7 Definition If E is a subset of A, then the **direct image** of E under f is the subset $f(E)$ of B given by

$$f(E) := \{f(x) : x \in E\}.$$

If H is a subset of B, then the **inverse image** of H under f is the subset $f^{-1}(H)$ of A given by

$$f^{-1}(H) := \{x \in A : f(x) \in H\}.$$

Remark The notation $f^{-1}(H)$ used in this connection has its disadvantages. However, we will use it since it is the standard notation.

Thus, if we are given a set $E \subseteq A$, then a point $y_1 \in B$ is in the direct image $f(E)$ if and only if there exists at least one point $x_1 \in E$ such that $y_1 = f(x_1)$. Similarly, given a set $H \subseteq B$, then a point x_2 is in the inverse image $f^{-1}(H)$ if and only if $y_2 := f(x_2)$ belongs to H. (See Figure 1.1.7.)

1.1.8 Examples (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$. Then the direct image of the set $E := \{x : 0 \le x \le 2\}$ is the set $f(E) = \{y : 0 \le y \le 4\}$.

If $G := \{y : 0 \le y \le 4\}$, then the inverse image of G is the set $f^{-1}(G) = \{x : -2 \le x \le 2\}$. Thus, in this case, we see that $f^{-1}(f(E)) \ne E$.

On the other hand, we have $f(f^{-1}(G)) = G$. But if $H := \{y : -1 \le y \le 1\}$, thenwe have $f(f^{-1}(H)) = \{y : 0 \le y \le 1\} \ne H$.

A sketchof the graphof f mayhelp to visualizethese sets.

(b) Let $f : A \to B$, and let G, H be subsetsof B. Wewill show that

$$f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H).$$

For, if $x \in f^{-1}(G \cap H)$, then $f(x) \in G \cap H$, so that $f(x) \in G$ and $f(x) \in H$. But this implies that x E $f^{-1}(G)$ and $x \in f^{-1}(H)$,*whence*$x \in f^{-1}(G) \cap f^{-1}(H)$.Thusthestated implication is proved. [The opposite inclusion is also true, so that we actually have set equality between these sets; see Exercise 13.]

Further facts about direct and inverse images are given in the exercises.

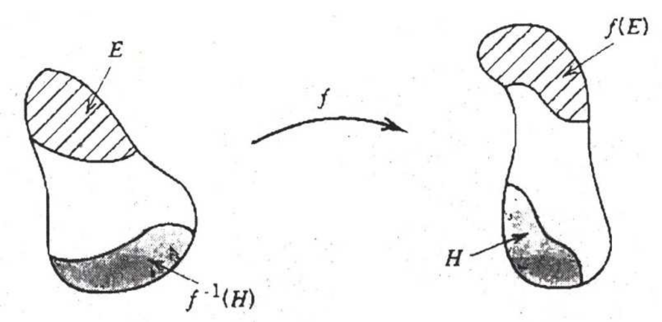


Figure 1.1.7 Direct and inverse images

4.0 CONCLUSION

SpecialTypesofFunctions

Thefollowingdefinitionsidentifysome veryimportanttypesoffunctions.

1.1.9 Definition Let$f$:$A \longrightarrow B$ beafunctionfrom $A$to$B$.

(a) The function$f$issaid tobe**injective**(or tobe**one-one**) ifwhenever$x_1 \neq x_2$,,then$f(x_1) \neq f(x_2)$.If $f$isaninjective function,wealsosaythat f isan**injection**.

(b) Thefunction$f$issaid tobe**surjective**(ortomap Aonto B)if $f(A)=B$; thatis,if therange $R(f)=B$.If $f$isasurjectivefunction,wealsosaythat f isa**surjection.**

(c) If$f$isbothinjective andsurjective,then f issaid tobe**bijective**.If f isbijective,we alsosay that$f$isa**bijection.**

Inorder toprove thatafunction $f$isinjective.wemustestablishthat:

forallx_1, $x_2$in$A$,if $f(x_1) =f(x_2)$, then$x_1=x_2$.

Todothisweassumethat$f(x_1)=f(x_2)$ andshow that$x_1=x_2$

[In other words, thegraph of$f$satisfiesthe*firsthorizontal linetest:*Every horizontal line $y =b$with$b \in B$intersectsthegraph $f$in*atmost*onepoint.]

Toprovethatafunction f issurjective,wemustshow thatforany$b \in B$thereexistsat least one $x \in A$such that $f(x)=b$.

[Inother words.thegraph *of* $f$satisfies the*secondhorizontallinetest:*Every horizontal line $y=b$with$b \in B$intersects thegraph $f$in*atleast*onepoint.]

1.1.10 Example LetA:={$x_2 \in \mathbb{R}$: $x \neq 1$}and define $f(x):=2x/(x-1)$for all $x \in A$. Toshow that f isinjective,wetake$x_1$and$x_2$in Aandassume that $f(x_1)=f(x_2)$.Thuswe have

$$\frac{2x_1}{x_1-1} = \frac{2x_2}{x_2-1}$$

which implies that $x_1(x_2 - 1) = x_2(x_1 - 1)$, and hence $x_1 = x_2$. Therefore f is injective.

To determine the range of f, we solve the equation $y =2x/(x - 1)$ for x in terms 0: y. We obtain $x =y/(y- 2)$, which is meaningful for $y \neq 2$. Thus the range of $f$is the se $B := \{y \in \mathbb{R}: y \neq 2\}$. Thus, $f$is a bijection of A onto B. □

InverseFunctions

If $f$isafunctionfrom Ainto$B$,then$f$isaspecial subset of$A$x B(namely, one passingthe *verticallinetest.*)The set of orderedpairs in$B$xAobtainedbyinterchanging th e membersoforderedpairsin$f$isnotgenerallyafunction.(Thatis,theset$f$maynotpass *both* of the *horizontal line tests.)* However, if *f* is a bijection, then this interchange does lead to a function, called the "inverse function of f.

1.1.11 Definition If $f : A \to B$ is a bijection of A onto B, then

$$g := \{(b, a) \in B \times A : (a, b) \in f\}$$

is a function on B into A. This function is called the **inverse function** of f and is denote by f^{-1}. The function f^{-1} is also called the **inverse** of f.

We can also express the connection between f and its inverse f^{-1} by noting that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$ and that

$$b = f(a) \text{ if and only if } a = f^{-1}(b)$$

For example, we saw in Example 1.1.10 that the function

$$f(x) := \frac{2x}{x-1}$$

is a bijection of $A := \{x \in \mathbb{R} : x \neq 1\}$ onto the set $B := \{y \in \mathbb{R} : y \neq 2\}$. The function inverse to f is given by

$$f^{-1}(y) := \frac{y}{y-2} \quad \text{for } y \in B.$$

Remark We introduced the notation $f^{-1}(H)$ in Definition 1.1.7. It makes sense even if f does not have an inverse function. However, if the inverse function f^{-1} does exist, then $f^{-1}(H)$ is the direct image of the set $H \subseteq B$ under f^{-1}

Composition of Functions

It often happens that we want to "compose" two functions f, g by first finding $f(x)$ and then applying g to get $g(f(x))$; however, this is possible only when $f(x)$ belongs to the domain of g, In order to be able to do this for *all* $f(x)$, we must assume that the range of f is contained in the domain of g. (See Figure 1.1.8).

1.1.12 Definition If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ (note the order!) is the function from A into C defined by

$$(g \circ f)(x) := g(f(x)) \quad \text{for all } x \in A.$$

1.1.13 Examples (a) The order of the composition must be carefully noted. For, let f and g be the functions whose values at $x \in \mathbb{R}$ are given by

$$f(x) := 2x \qquad \text{and} \qquad g(x) := 3x^2 - 1.$$

Since $D(g) = \mathbb{R}$ and $R(f) \subseteq \mathbb{R} = D(g)$, then the domain $D(g \circ f)$ is also equal to $\mathbb{R}$, and the composite function $g \circ f$ is given by

$$(g \circ f)(x) = 3(2x)^2 - 1 = 12x^2 - 1.$$

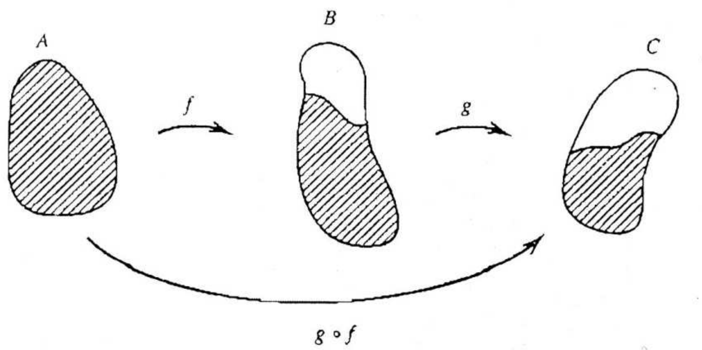


Figure 1.1.8 The composition of f and g

On the other hand, the domain of the composite function $f \circ g$ *is,* also ℝ, but

$$(f \circ g)(x) = 2(3x^2 - 1) = 6x^2 - 2.$$

Thus, in this case, we have $g \circ f \neq f \circ g$.

(b) In considering $g \circ f$, some care must be exercised to be sure that the range of f is contained in the domain of g. For example, if

$$f(x) := 1 - x^2 \quad \text{and} \quad g(x) := \sqrt{x}$$

then, since $D(g) = \{x : x \geq 0)$, the composite function $g \circ f$ is given by the formula

$$(g \circ f)(x) = \sqrt{x - x^2}$$

only for $x \in D(f)$ that satisfy $f(x) \geq 0$; that is, for x satisfying $-1 \leq x \leq 1$.

We note that if we reverse the order, then the composition $f \circ g$ is given by the formula

$$(f \circ g)(x) = 1 - x,$$

but only for those x in the domain $D(g) = \{x : x \geq 0\}$.

We now give the relationship between composite functions and inverse images. The proof is left as an instructive exercise.

1.1.14 Theorem Let $f : A \to B$ and $g : B \to C$ be functions and Jet *H be a subset of C . Then we have*

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$$

Note the *reversal* in the order of the functions.

5.0 SUMMARY

Restrictions of Functions

If $f : A \to B$ is a function and if $A_1 \subset A$, we can define a function $f_1, : A_1 \to B$ by

$$f_1(x) := f(x) \qquad \text{for } x \in A_1$$

The function f_1 is called the restriction of f to A_1. Sometimes it is denoted by $f_1 = f / A_1$
It may seem strange to the reader that one would ever choose to throw away a part of a function, but there are some good reasons for doing so. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the squaring function:

$$f(x) := x^2 \qquad \text{for } x \in \mathbb{R},$$

then f is not injective, so it cannot have an inverse function. However, if we restrict f to the set $A_1 := (x : x \geq 0)$, then the restriction $f|A_1$ is a bijection of A_1 onto A_1. Therefore, this restriction has an inverse function, which is the positive square root function. (Sketch a graph.)

Similarly, the trigonometric functions $S(x) := \sin x$ and $C(x) := \cos x$ are not injective on all of $\mathbb{R}$. However, by making suitable restrictions of these functions, one can obtain the **inverse sine** and the **inverse cosine** functions that the reader has undoubtedly already encountered.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit1.1

1 If A and B are sets, show that $A \subseteq B$ if and only if $A \cap B = A$.

2. Prove the second De Morgan Law [Theorem 1.1. 4(b)].

3. Prove the Distributive Laws:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4. The symmetric difference of two sets A and B is the set D of all elements that belong to either A or B but not both. Represent D with a diagram.
   (a) Show that $D = (A \setminus B) \cup (B \setminus A)$.
   (b) Show that D is also given by $D = (A \cup B) \setminus (A \cap B)$.

5. For each $n \in \mathbb{N}$, let $A_n = \{(n + l)k : k \in \mathbb{N})$.
   (a) What is $A_1 \cap A_2$?
   (b) Determine the sets $\bigcup \{A_n : n \in \mathbb{N})$ and $\bigcap \{A_n : n \in \mathbb{N})$.

6. Draw diagrams in the plane of the Cartesian products A x B for the given sets A and B.
   (a) $A = \{x \in \mathbb{R} : 1 \leq x \leq 2$ or $3 \leq x \leq 4]$, $B = \{x \in \mathbb{R} : x = 1$ or $x = 2\}$.
   (b) $A = \{1,2,3\}$, $B = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

7. Let $A := B := \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ and consider the subset $C := \{(x, y) : x^2 + y^2 = 1\}$ of A x B. Is this set a function? Explain.

8. Let $f(x) := 1/x^2, x \neq 0, x \in \mathbb{R}$.
   (a) Determine the direct image $f(E)$ where $E := (x \in \mathbb{R} : 1 \leq x \leq 2)$.
   (b) Determine the inverse image $f^{-1}(G)$ where $G := \{x \in \mathbb{R} : 1 \leq x \leq 4\}$.

9. Let $g(x) := x^2$ and $f(x) := x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h := g \circ f$.
   (a) Find the direct image $h(E)$ of $E := (x \in \mathbb{R} : 0 \leq x \leq 1)$.
   (b) Find the inverse image $h^{-1}(G)$ of $G := (x \in \mathbb{R} : 0 \leq x \leq 4)$.

10. Let $f(x) := x^2$ for $x \in \mathbb{R}$, and let $E := \{x \in \mathbb{R} : -1 \leq x \leq 0)$ and $F := (x$

$\in \mathbb{R}: 0 \le x \le 1\}$. Show that $E \cap F = \{0\}$ and $f(E \cap F) = (0)$, while $f(E) = f(F) = [y \in \mathbb{R}: 0 \le y \le 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. What happens if 0 is deleted from the sets E and F?

11. Let f and E, F be as in Exercise 10. Find the sets $E \setminus F$ and $f(E) \setminus f(F)$ and show that it is *not* true that $f(E \setminus F) \subseteq f(E) \setminus f(F)$.

12. Show that if $f: A \to B$ and E, F are subsets of A, then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$.

13. Show that if $f: A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

14. Show that the function f defined by $f(x) := x/\sqrt{x^2+1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $(y: -1 \le y \le 1\}$.

15. For $a, b \in \mathbb{R}$ with $a \le b$, find an explicit bijection of $A := \{x : a < x < b\}$ onto $B := \{y : 0 < y < 1\}$.

16. Give an example of two functions f, g on $\mathbb{R}$ to $\mathbb{R}$ such that $f \neq g$, but such that $f \circ g = g \circ f$.

17. (a) Show that if $f: A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an exampleto show that equality need not hold if f is not injective.
    (b) Show that if $f: A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.

18. (a) Suppose that / is an injection. Show that $f^{-1} \circ f(x) = x$ for all $x \in D(f)$ and that $f \circ f^{-1}(y) = y$ for all $y \in R(f)$.
    (b) If f is a bijection of A onto B, show that f^{-1} is a bijection of B onto A.

19. Prove that if $f: A \to B$ is bijective and $g: B \to C$ is bijective, then the composite $g \circ f$ is a bijectivemap of A onto C.

20. Let $f: A \to B$ and $g: B \to C$ be functions.
    (a) Show that *if* $g \circ f$ is injective, then f is injective.
    (b) Show that if $g \circ f$ is subjective, then g is subjective.

21 Prove Theorem 1.1.14.

22. Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

Unit 2 Mathematical Induction

1.0 INTRODUCTION

Mathematical Induction is a powerful method of proof that is frequently used to establish thevalidityof statements that are given in terms of the natural numbers. Although its utility is restricted to this rather special context, Mathematical Induction is an indispensable tool in all branches of mathematics. Since many inductionproofs follow the same formal lines of argument, we will often state only that a result follows from Mathematical Induction and leave it to the reader to provide the necessary details. In this section, we will state the principle and give several examples to illustratehow inductive proofs proceed.

We shallassume familiarity withthe set of natural numbers:

$$\mathcal{N} := \{1, 2, 3, \cdots\}.$$

withthe usual arithmetic operations of addition and multiplication, and with the meaning of a natural number being less than another one. We will also assume thefollowing fundamental property of $\mathcal{N}$.

2.0 OBJECTIVES

At the end of the unit, readers should be able to

(i) understand special method of proof called "Mathematical Induction"

(ii) understand that the course is related to the fundamental properties of the natural number system

(iii) Understand different types of proof that are used in mathematics such as contrapositives and proof by contradiction.

3.0 MAIN CONTENT

1.2.1 Well-Ordering Property of $\mathcal{N}$ *Every nonempty subset of* $\mathbb{N}$ *has a least element.*

A more detailedstatement of thisproperty is as follows: If S is a subset of $\mathcal{N}$ and if $S \neq 0$, then there exists $m \in S$ such that $m \leq k$ for all $k \in . S$.

On the basis of the Well-Ordering Property, we shall derive a version of the Principle of Mathematical Induction that is expressed in terms of subsets of $\mathcal{N}$.

1.2.2 Principle of Mathematical Induction *Let S* be a subset *of* $\mathcal{N}$ *that* possesses *the* two properties:

(1) The number $1 \in S$.

(2) For every $k \in \mathbb{N}$, if $k \in S$, then $k + 1 \in S$.

Then we *have* $S = \mathbb{N}$.

Proof. Suppose to the contrary that $S \neq \mathcal{N}$. Then the set $\mathcal{N} \setminus S$ is not empty, so by the Well-Ordering Principle it has a least element m. Since $1 \in S$ by hypothesis (1), we know that $m > 1$. But this implies that $m > 1$ is also a natural number. Since $m - 1 < m$ and since m is the least element in $\mathbb{N}$ such that $m \notin S$, we conclude that $m - 1 \in S$.

We now apply hypothesis (2) to the element $k := m - 1$ in S, to infer that $k + 1 = (m - 1) + 1 = m$ belongs to S. But this statement contradicts the fact that $m \notin S$. Since m was obtained from the assumption that $S \setminus \mathcal{N}$ is not empty, we have obtained a contradiction.

Therefore we must have $S = \mathcal{N}$. Q.E.D.

The Principle of Mathematical Induction is often set forth in the framework of properties or statements about natural numbers. If $P(n)$ is a meaningful statement about $n \in \mathcal{N}$, then $P(n)$ may be true for some values of n and false for others. For example, if $P_1(n)$ is the statement: "$n^2 = n$" then $P^1(1)$ is true while $P_1(n)$ is false for all $n > 1$, $n \in \mathcal{N}$. On the other hand, if $P_2(n)$ is the statement: "$n^2 > 1$", then $P_2(1)$ is false, while $P_2(n)$ is true for all $n > 1$, $n \in \mathcal{N}$.

In this context, the Principle of Mathematical Induction can be formulated as follows. For each $n \in \mathbb{N}$, *let* $P(n)$ *be a* statement *about n. Suppose that:*

(1) $P(1)$ *is true,*

(2) For every $k \in \mathcal{N}$, *if* $P(k)$ *is* true, *then* $P(k+1)$ *is true.*

Then $P(n)$ *is true for all* $n \in \mathbb{N}$.

The connection with the preceding version of Mathematical Induction, given in 1.2.2, is made by letting $S := (n \in \mathbb{N}: P(n)$ is true). Then the conditions (1) and (2) of 1.2.2 correspond exactly to the conditions (1') and (2'), respectively. The conclusion that $S = \mathbb{N}$ in 1.2.2 corresponds to the conclusion that $P(n)$ is true for all $n \in \mathcal{N}$.

In (2') the assumption "if $P(k)$ is true" is called the **induction hypothesis**. In establishing (2'), we are not concerned with the actual truth or falsity of P(k'), but only with the validity of the implication "if $P(k)$, then $P(k+1)$". For example, if we consider the statements *P(n): "n = n + 5"*, then (2') is logically correct, for we can simply add 1 to both sides of $P(k)$ to obtain $P(k+1)$. However, since the statement P(l): "1 — 6" is false, we cannot use Mathematical Induction to conclude that $n = n + 5$ for all $n \in \mathcal{N}$.

It may happen that statements $P(n)$ are false for certain natural numbers but then are true for all $n \geq n_o$ for some particular n_0. The Principle of Mathematical Induction can be modified to deal with this situation. We will formulate the modified principle, but leave its verification as an exercise. (See Exercise 12.)

1.2.3 Principle of Mathematical Induction (second version) *Let* $n_0 \in \mathbb{N}$ and let $P(n)$ *be a statement for each natural number* $n \geq n_0$. Suppose *that:*

(1) The statement $P(n_0)$ *is* true.

(2) For *all* $k \geq n_0$, *the truth of* $P(k)$ *implies the truth of* $P(k+1)$.

Then $P(n)$ *is true for all* $n \geq n_0$

Sometimes the number n_0 in (1) is called the **base**, since it serves as the starting point, and the implication in (2), which can be written $P(k) \Rightarrow P(k+1)$, is called the **bridge**, since it connects the case k to the case $k + 1$.

The following examples illustrate how Mathematical Induction is used to prove assertions about natural numbers.

1.2.4 Examples (a) For each $n \in \mathbb{N}$, the sum of the first n natural numbers is given by

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1).$$

To prove this formula, we let S be the set of all $n \in \mathbb{N}$ for which the formula is true. We must verify that conditions (1) and (2) of 1.2.2 are satisfied. If $n = 1$, then we have $1 = \frac{1}{2} \cdot 1 \cdot (1 + 1)$ so that $1 \in S$, and (1) is satisfied. Next, we *assume* that $k \in S$ and wish to infer from this assumption that $k + 1 \in S$. Indeed, if $k \in S$, then

$$1 + 2 + \cdots + k = \tfrac{1}{2}k(k+1).$$

If we add $k + 1$ to both sides of the assumed equality, we obtain

$$1 + 2 + \cdots + k + (k + 1) = \tfrac{1}{2}k(k + 1) + (k + 1)$$
$$= \tfrac{1}{2}(k + 1)(k + 2).$$

Since this is the stated formula for $n = k + 1$, we conclude that $k + 1 \in S$. Therefore, condition (2) of 1.2.2 is satisfied. Consequently, by the Principle of Mathematical Induction, we infer that $S = \mathbb{N}$, so the formula holds for all $n \in \mathcal{N}$.

(b) For each $n \in \mathcal{N}$, the sum of the squares of the first n natural numbers is given by

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$

To establish this formula, we note that it is true for $n = 1$, since $1^2 = \tfrac{1}{6}1.\ 2.\ 3$. If we assume it is true for k, then adding $(k+1)^2$ to both sides of the assumed formula gives

$$1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = \tfrac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$$
$$= \tfrac{1}{6}(k + 1)(2k^2 + k + 6k + 6)$$
$$= \tfrac{1}{6}(k + 1)(k + 2)(2k + 3).$$

Consequently, the formula is valid for all $n \in \mathcal{N}$.

(c) Given two real numbers a and b, we will prove that $a - b$ is a factor of $a^n - b^n$ for all $n \in \mathcal{N}$.

First we see that the statement is clearly true for $n = 1$. If we now assume that $a - b$ is a factor of $a^k - b^k$, then

$$a^{k+1} - b^{k+1} = a^{k+1} - ab^k + ab^k - b^{k+1}$$
$$= a(a^k - b^k) + b^k(a - b).$$

By the induction hypothesis, $a - b$ is a factor of $a(a^k - b^k)$ and it is plainly a factor of $b^k(a - b)$. Therefore, $a - b$ is a factor of $a^{k+1} - b^{k+1}$, and it follows from Mathematical Induction that $a - b$ is a factor of $a^n - b^n$ for all $n \in \mathcal{N}$.

A variety of divisibility results can be derived from this fact. For example, since $11 - 7 = 4$, we see that $11^n - 7^n$ is divisible by 4 for all $n \in \mathcal{N}$.

(d) The inequality $2^n > 2^k + 1$ is false for $n = 1, 2$, but it is true for $n = 3$. If we assume that $2^k > 2k + 1$, then multiplication by 2 gives, when $2k + 2 > 3$, the inequality

$$2^{k+1} > 2(2k + 1) = 4^k + 2 = 2^k + (2k + 2) > 2k + 3 = 2(k + 1) + 1.$$

Since $2k + 2 > 3$ for all $k \geq 1$, the bridge is valid for all $k \geq 1$ (even though the statement is false for $k = 1, 2$). Hence, with the base $n_0 = 3$, we can apply Mathematical Induction to conclude that the inequality holds for all $n \geq 3$.

(e) The inequality $2^n \leq (n + 1)!$ can be established by Mathematical Induction.

We first observe that it is true for $n = 1$, since $2^1 = 2 = 1 + 1$. If we assume that $2^k \leq (k + 1)!$, it follows from the fact that $2 \leq k + 2$ that

$$2^{k+1} = 2 \cdot 2^k \leq 2(k + 1)! \leq (k + 2)(k + 1)! = (k + 2)!.$$

Thus, if the inequality holds for k, then it also holds for $k + 1$. Therefore, Mathematical Induction implies that the inequality is true for all $n \in \mathcal{N}$.

(f) If $r \in \mathcal{R}$, $r \neq 1$, and $n \in \mathcal{N}$., then

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

This is the formula for the sum of the terms in a "geometric progression". It can be established using Mathematical Induction as follows. First, if $n = 1$, then $1 + r = (1 - r^2)/(1 - r)$. If we assume the truth of the formula for $n = k$ and add the term r^{k+1} to both sides, we get (after a littlealgebra)

$$1 + r + r^k + \cdots + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} = \frac{1 - r^{k+2}}{1 - r}$$

which is the formula for $n = k + 1$. Therefore, Mathematical Inductionimplies the validity of the formula for all $n \in \mathrm{N}$.

[This result can also be proved without using Mathematical Induction. If we let $S_n := 1 + r + r^2 + \cdots + r^n$, then $rs_n = r + r^2 + \cdots + r^{n+1}$, so that

$$(I - r)\, s_n = s_n - rs_n = 1 - r^{n+1}$$

If we divide by $1 — r$. we obtain the statedformula.]

(g) Careless use of the Principle of Mathematical Inductioncanlead to obviously absurd conclusions.The reader is invited to find the error in the "proof of the following assertion.

4.0 CONCLUSION

Claim: If $n \in \mathrm{N}$and if the maximum of thenatural numbers p and q is n, then $p = q$.

"Proof." Let S be the subsetof $\mathcal{N}$ for which the claimis true. Evidently, $1 \in S$ since if $p, q \in \mathcal{N}$ and their maximum is 1 , then both equal 1 and so $p = q$, Now assume that k e S and thatthemaximum of p and q is $k + 1$. Then the maximum of $p — 1$ and $q - 1$ is k. But since$k \in S$, then$p - 1 = q - 1$ and therefore $p = q$. Thus, $k + 1 \in S$, and we conclude thattheassertionis true for all$n \in \mathcal{N}$.

(h) There are statementsthatare true for *many* natural numbers but that are not truefor *all*of them.

For example, the formula $p(n) := n^2 - n + 41$ givesa prime number for $n = 1, 2, \cdots 40$. However, $p(41)$is obviously divisible by 41, so it is not a prime number.

5.0 SUMMARY

Another version of the Principle of Mathematical Inductionis sometimes quite useful. It is calledthe" Principleof Strong Induction", even though it is in fact equivalent to 1,2.2.

1.2.5 Principle of Strong Induction *Let S be a subset of* N *such that*

(1") $1 \in S$

(2") *For every* $k \in \mathrm{N}$, if $\{1, 2, \cdots, k\} \subseteq S$, *then* $k + 1 \in S$.

Then $S = \mathcal{N}$.

We will leave it to the reader to establish the equivalence of 1.2.2 and 1 .2.5.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 1.2

1. Prove that $1/1.2 + 1/2.3 + \cdots + 1/n(n+1) = n/(n+1)$ for all $n \in \mathcal{N}$.

2. Prove that $1^3 + 2^3 + \cdots + n^3 = [1/2\, n(n+1]^2$ for all $n \in \mathcal{N}$.

3. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathcal{N}$.

4. Prove that $1^2 + 3^2 + \cdots + (2n-1)^2 = (4n^3 - n)/3$ for all $n \in \mathcal{N}$.

5. Prove that $1^2 - 2^2 + 3^2 + \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} n(n+1)/2$ for all $n \in \mathcal{N}$.

6. Prove chat $n^3 + 5n$ is divisibleby 6 for all $n \in \mathcal{N}$.

7. Prove that $S^{2n} - 1$ is divisible by 8 for all $n \in \mathcal{N}$.

8. Prove that $5^n - 4n - 1$ is divisibleby 16 for all $n \in \mathcal{N}$..

9. Prove that $n^3 + (n+1)^3 + (n+2\}^3$ is divisible by 9 for all $n \in \mathcal{N}$.

10. Conjecture a formula for the sum $1/1.3 + 1/3.5 + \cdots + 1/(2n-1)(2n+1)$, and prove your conjecture by using Mathematical Induction.

11. Conjecture a formula for the sum of the first n odd natural numbers $1 + 3 + \cdots + (2n - 1)$, and prove your formula by using Mathematical Induction.

12. Prove the Principle of Mathematical Induction 1.2.3 (second version).
13. Prove that $n < 2^n$ for all $n \in \mathcal{N}$.

14. Prove that $2^n < n!$ for all $n \geq 4$, $n \in \mathcal{N}$.

15. Prove that $2n - 3 \leq 2^{n-2}$ for all $n \geq 5$, $n \in \mathcal{N}$.

16. Find allnatural numbers n such that $n^2 < 2^n$. Prove your assertion.

17. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathcal{N}$. Prove your assertion.

18. Prove that $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$ for all $n \in \mathcal{N}$.

19. Let S be a subset of $\mathbb{N}$ such that (a) $2^k \in S$ for all $k \in \mathcal{N}$ and (b) *if* $k \in S$ and $k \geq 2$, then $k-1 \in S$. Prove that $S = \mathcal{N}$.

20. Let the numbers x_n be defined as follows: $x_1 := 1$, $x_2, := 2$, and $x_{n+2} := ½ (x_{n+1} + x_n)$ for all $n \in \mathcal{N}$. Use the Principle of Strong Induction (1.2.5) to show that $1 \leq x_n \leq 2$ for all $n \in \mathcal{N}$.

7.0 BIBLIOGRAPHY/REFERENCES

Unit 3 Finite and Infinite Sets

1.0 INTRODUCTION

When we count the elements in a set, we say "one, two, three,⋯", stopping when we have exhausted the set. From a mathematical perspective, what we are doing is defining a bijective mapping between sheset and a portion of the set of natural numbers. If the set is such that the counting does not terminate, such as the set of natural numbers itself, then we describe the set as being infinite.

The notions of "finite" and "infinite" are extremely primitive, and it is very likely that the reader has never examined these notions very carefully. In this section we will define these terms precisely and establish a few basic results and state some other important results that seem obvious but whose proofs are a bittricky.

2.0 OBJECTIVES

At the end of the Unit, readers should be able to:

(i) Understand the differences ans similarities of a finite and infinite set

(ii) Understand carefully some definitions and some basic consequences of these definition and their desired

(iii) Understand that the set of rational numbers is countably infinite. It is established.

3.0 MAIN CONTENT

1.3.1 Definition (a) The empty set 0 is said to have 0 **elements.**

(b) If $n \in \mathcal{N}$, a set S is said to have n **elements** if there exists a bijection from the set $\mathcal{N}_n := [1, 2, \cdots, n)$ onto 5.

(c) A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathcal{N}$

(d) A set 5 is said to be **infinite** if it is not finite.

Since the inverse of a bijection is a bijection, it is easy to see that a set S has n elements if and only if there is a bijection from S onto the set $\{1, 2, \cdots n\}$. Also, since the composition of two bijections is a bijection, we see that a set S_1, has n elements if and only if there is a bijection *from* S_1 onto another set S_2 that has n elements. Further, a set T_1, is finite if and only if there is a bijection from T_1, onto another set T_2, that is finite.

It is now necessary to establish some basic properties of finite sets to be sure that the definitionsdo not lead to conclusionsthat conflict withour experience of counting. From the definitions, it is not entirelyclear that a finite set might not have n elements for *more than one* value of n. Also it is conceivably possiblethat the set $\mathcal{N} := \{1, 2, 3, \cdots$ might be a finite set according to this definition. The reader will be relieved that these possibilities do not occur, as the next two theorems state. The proofs of these assertions, which use the fundamental properties of $\mathbb{N}$ described in Unit 1.2, are given in Appendix B.

1.3.2 Uniqueness Theorem *If S is a finite set, then the number of elements in S is a. uniquenumber in* $\mathcal{N}$.

1.3.3 Theorem *The set* $\mathbb{N}$ of natural *numbers is an infinite set.*

The next result gives some elementary properties of finite and infinite sets.

1.3.4 Theorem (a) *If A is a set with m elements and B is a set with n elements and if* $A \cap B = \emptyset$, *then* $A \cup B$ *has m + n elements.*
(b) *If A is* a *set with* $m \in \mathbb{N}$ *elements and* $C \subseteq A$ *is a set with* 1 element, *then* $A \setminus C$ *is a* set*with* $m - 1$ *elements.*
(c) *If C is* an *infinite set and B is* a *finite set, then* $C \setminus B$ *is an infinite set.*

Proof. (a) Let f be a bijection of $\mathbb{N}_m$ onto *A*, and let *g* be a bijection of $\mathcal{N}_n$ onto *B*. We define *h* on $\mathcal{N}_{m+n}$ by $h(i) := f(i)$ for I =1, ⋯, *m* and $h(i) := g(i-m)$ for i = *m* + 1, ⋯, $m + n$. We leave it as an exercise to show that*h* is a bijection from $\mathbb{N}_{m+n}$ onto $A \cup B$.
The proofs of parts (b) and (c) are left to the reader, see Exercise 2. Q.E.D.

It may seem "obvious" that a subset of a finite set is also finite, but the assertion must be deduced from the definitions. This and the corresponding statement for infinite sets are established next.

1.3.5 Theorem *Suppose that S and T are sets andthat* $T \subseteq S$.
(a) *If Sis a finite set, then T is a finite set.*
(b) *IfT is an infinite set, then S is an infinite set.*

Proof. (a) If $T = \emptyset$, we already know that *T* is a finite set. Thus we may suppose that $T \neq \emptyset$. The proof is by induction on the number of elements in *S*.

If *S* has 1 element, then the only nonempty subset *T* of *S* must coincide with*S*, so *T* is a finite set.

Suppose that every nonempty subset of a set with *k* elements is finite. Now let *S* be a set having $k + 1$ elements (so there exists a bijection f of $\mathbb{N}_{k+1}$ onto *S*), and let $T \subseteq S$. If $f(k+1) \in T$, we can consider T to be a subset of $S_1 := S \setminus \{f(k+1)\}$, which has *k* elements by Theorem 1.3.4(b). Hence, by the induction hypothesis, *T* is a finite set.

On the other hand, if $f(k+1) \in T$, then $T_1 := T \setminus \{f(k+1)\}$ is a subset of S_1. Since S_1 has *k* elements, the induction hypothesis implies that T_1 is a finite set. But this implies that $T = T_1 \cup \{f(k+1)\}$ is also a finite set.
(b) This assertion is the contrapositive of the assertion in (a). (See Appendix A for adiscussion of the contrapositive.) Q.E.D.

Countable Sets

We now introduce an important type of infinite set.

1.3.6 Definition (a) A set *S* is said to be **denumerable** (or **countably infinite)** if there exists a bijection of $\mathbb{N}$ onto *S*.
(b) A set *S* is said to be **countable** if it is either finite or denumerable.
(c) A set S is said to be **uncountable** if it is not countable.

From the properties of bijections, it is clear that S is denumerable if and only if there exists a bijection of S onto $\mathcal{N}$. Also a set S_1 is denumerable if and only if there exists a bijection from S_1onto a set S_2that is denumerable. Further, a set T_1 is countable if and only if there exists a bijection from T_1 onto a set T_2 that is countable. Finally, an infinite countable set is denumerable.

1.3.7Examples (a) The set $E := [2n : n \in \mathcal{N}\}$ of even natural numbers is denumerable,since the mapping $f : \mathbb{N} \to E$ defined by $f(n) := 2n$ for $n \in \mathcal{N}$, is a bijection of $\mathcal{N}$ onto E.

Similarly, the set $0 := \{2n — 1 : n \in \mathcal{N}\}$ of *odd* natural numbers is denumerable.

(b) The set $\mathcal{Z}$ of *all* integers is denumerable.

To construct a bijection of $\mathbb{N}$ onto $\mathcal{Z}_1$, we map 1 onto 0, we map the set of even natural numbers onto the set $\mathbb{N}$ of positive integers, and we map the set of odd natural numbers onto the negative integers. This mapping can be displayed by the enumeration:

$$\mathcal{N} = (0, 1, -1, 2. -2, 3, -3, \cdots\} .$$

(c) The union of two disjoint denumerable sets is denumerable.

Indeed, if $A = \{ a_1, a_2, a_3, \cdots$and $B = \{b_1, b_2, b_3, \cdots\}$, we can enumerate the elements of $A \cup B$ as:

$$a_1, b_1, a_2, b_2, a_3, b_3, \cdots$$

1.3.8 Theorem *The set* $\mathcal{N} \times \mathcal{N}$ *is denumerable.*

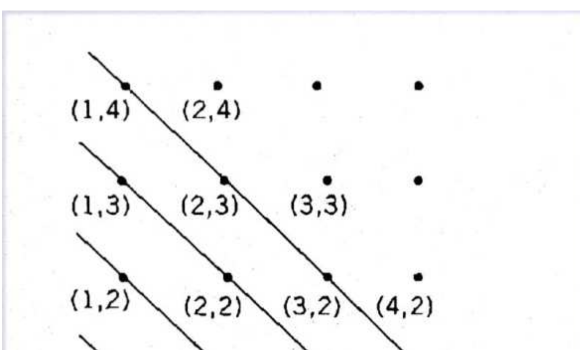
Informal Proof. Recall that $\mathbb{N} \times \mathbb{N}$consists of all ordered pairs (m, n), where $m, n \in \mathcal{N}$. We can enumerate these pairs as:

$$(1,1),\ (1,2),\ (2,1),\ (1,3),\ (2,2),\ (3,1),\ (1,4), \cdots,$$

according to increasing sum $m + n$, and increasing m. (See Figure 1.3.1.) Q.E.D.

The enumeration just described is an instance of a "diagonal procedure", since we move along diagonals that each contain finitely many terms as illustratedin Figure i.3.1. While this argument is satisfying in that it shows exactly what the bijection of $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$should do, it is not a "formal proof, since it doesn't define this bijection precisely. (See Appendix B for a more formal proof.)

As we have remarked, the construction of an explicit bijection between sets is often complicated. The next two results are useful in establishing the countability of sets, since they do not involve showing that certain mappings are bijections. The first result may seem intuitively clear, but its proof is rather technical; it will be given in Appendix B.



1.3.9 Theorem *Suppose that Sand T are sets and that T $\subseteq$S.*

(a) *If S is a countable, set. then T is a countable set.*

(b) *If T is an uncountable set, then S is an uncountable set.*

1.3.10 Theorem *The following statements are equivalent:*

(a) *Sis a countable set.*

(b) There *exists a surjection* of $\mathcal{N}$ onto *S.*

(c) There exists *an injection of S into* $\mathcal{N}$.

Proof. (a)$\Rightarrow$(b)If *S* is finite, there exists a bijection *h* of some set N_n onto *S* and we define *H* on $\mathcal{N}$ by

$$H(k) := \begin{cases} h(k) & \text{for } k = 1, \cdots, n, \\ h(n) & \text{for } k > n. \end{cases}$$

Then *H* is a surjection of $\mathcal{N}$onto*S*.

If *S* is denumerable, there exists a bijection *H* of N onto *S*, which is also a surjection of $\mathcal{N}$onto*S*

(b)$\Rightarrow$ (c) If *H* is a surjectionof NontoS, we define $H_1 : S \Rightarrow N$ by letting/$H_1(S)$ be the least element in the set $H^{-1}(s) := \{n \in \mathcal{N} : H(n] = s\}$. To see that H_1 is an injection of *S*into $\mathcal{N}$,note that *if* $s, t \in S$ and $n_{st} := H_1(s) = H_1(t)$, then $s = H(n_{st}) = t$.

(c) $\Rightarrow$ (a) If H_1 is an injection of *S* into N, then it is a bijection of *S* onto $H_1(S) \subseteq \mathcal{N}$.

By Theorem 1.3.9(a), $H_1(S)$ is countable, whence the set *S* is countable. Q.E.D.

1.3.11 Theorem The set Q of *all rational numbers is denumerable.*

Proof. The idea of the proof is to observe that the set $\mathcal{Q}^+$ of positiverational numbers iscontained in the enumeration:

$$1/1,\ 1/2,\ 2/1,\ 1/3,\ 2/2,\ 3/1,\ 1/4,\ \cdots,$$

which is another "diagonal mapping" (see Figure 1.3.2). However, this mapping is not an injection, since the different fractions $1/2$and $2/4$ represent the same rational number.

To proceed more formally, note that sinceNx Nis countable (by Theorem 1.3.8), it follows from Theorem 1.3.10(b) that there exists a surjection / of Nonto $\mathcal{N}$x $\mathcal{N}$. If

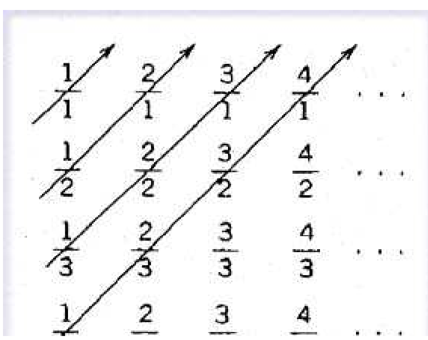


Figure 1.3.2 The set $\mathbb{Q}^+$

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ is the mapping that sends the ordered pair (m, n) into the rational number having a representation m/n_1 then g is a surjection onto $\mathbb{Q}^+$. Therefore, the composition $g \circ f$ is a surjection of $\mathbb{N}$ onto $\mathbb{Q}^+$, and Theorem 1.3.10 implies that $\mathbb{Q}^+$ is a countable set.

Similarly, the set $\mathbb{Q}$ of all negative rational numbers is countable. It follows as in Example 1.3.7(b) that the set $\mathbb{Q} = \mathbb{Q} \cup \{0\} \cup \mathbb{Q}^+$ is countable. Since $\mathbb{Q}$ contains $\mathbb{N}$, it must be a denumerable set. Q.E.D.

The next result is concerned with unions of sets, In view of Theorem 1.3.10, we need not be womed about possible overlapping of the sets. Also, we do not have to construct a bijection.

1.3.12 Theorem *If A_m is a countable set for each $m \in \mathbb{N}$, then the union $A := \bigcup_{m=1}^{\infty} A_m$ is countable.*

Proof. For each $m \in \mathbb{N}$, let φ_m be a surjection of $\mathbb{N}$ onto $A_{m(}$. We define $\psi : \mathbb{N} \times \mathbb{N} \to A$ by

$$\psi(m, n) := \varphi_m(n).$$

We claim that ^ is a surjection. Indeed, if $a \in A$, then there exists a least $m \in \mathbb{N}$ such that $a \in A_m$, whence there exists a least $n \in \mathbb{N}$ such that $a = \varphi_m(n)$. Therefore, $a = \psi(m, n)$.

Since $\mathbb{N} \times \mathbb{N}$ is countable, it follows from Theorem 1.3. 10 that there exists a surjection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ whence $\psi \circ f$ is a surjection of $\mathbb{N}$ onto A. Now apply Theorem 1.3.10again to conclude that A is countable. Q.E.D.

4.0 CONCLUSION

Remark A less formal (but more intuitive) way to see the truth of Theorem 1.3.12 is to enumerate the elements of $A_m, m \in \mathbb{N}$, as:

$$A_1 = \{a_{11}, a_{12}, a_{13}, \cdots\},$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \cdots\},$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \cdots\}.$$
$$\cdots \quad \cdots \quad \cdots$$

We then enumerate this array using the "diagonal procedure"

$$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, \cdots,$$

as was displayed in Figure 1.3.1.

The argument that the set $\mathbb{Q}$ of rational numbers is countable was first given in 1874 by Georg Cantor (1845-1918). He was the first mathematician to examine the concept of infinite set in rigorous detail. In contrast to the countability of $\mathbb{Q}$, he also proved the set $\mathbb{R}$ of real numbers is an uncountable set. (This result willbe established in Section 2.5.)

In a series of important papers, Cantor developed an extensive theory of infinite sets and transfinite arithmetic. Some of hisresults were quite surprising and generated considerable controversy among mathematicians of that era. In a 1877 letter to his colleague Richard Dedekind, he wrote, after proving an unexpected theorem, "I see it,but I do not believe it",

We close this section with one of Cantor's more remarkable theorems.

5.0 SUMMARY

1.3.13 Cantor's Theorem *If A is any set, then there is no surjection of A onto the* set *P (A) of allsubsets of A.*

Proof. Suppose that $\varphi : A \to P(A)$ is a surjection. Since $\varphi(a)$ is a subset of A, either a belongs to $<p(a)$ or it does not belong to this set. We let

$$D := \{a \in A : a \notin \varphi(a)\}.$$

Since D is a subset of A, if $ is a surjection, then $D = \varphi(a_o)$for some $a_O \in A$.

We must have either $a_0 \in D$ or $a_o \notin$ D. If $a_o \in$ D, then since $D = \varphi(a_0)$, we must have $a_o \in \varphi(a_0)$,contrary to the definition of D. Similarly, if $a_0 \notin D$, then $a_o \notin \varphi(a_o)$ so that $a_o \in D$, which is also a contradiction.

Therefore, φcannot be a surjection. Q.E.D.

Cantor's Theorem implies that there is an unending progression of larger and larger sets. In particular, it implies that the collection $P(\mathbb{N})$ of all subsets of the natural numbers $\mathbb{N}$ is uncountable.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 1.3

1 Prove that a nonempty set T_1, is finite if and only if there is a bijection from T_1onto a finiteset T_2

2 Prove parts (b) and (c) of Theorem 1.3.4.

3. Let $S:=\{l,2\}$ and $T:=\{a.b.c\}$.
   (a) Determine the number of different injections from S into T.
   (b) Determine the number of different surjections from T onto S.

4. Exhibit a bijection between $\mathbb{N}$ and the set of all odd integers greater than 13.

5. Give an explicit definition of the bijection f from $\mathbb{N}$ onto $\mathbb{Z}$ described in Example 1.3.7(b).

6. Exhibit a bijection. between $\mathbb{N}$and a proper subset of itself,

7. Prove that a set $T_1$is denumerable if and *only* if there is a bijection from T_1 onto a denumerable set T_2

8. Give an example of a countable collection of finite sets whose union is not finite.

9. Prove in detail that if S and T are denumerable, then $S \cup T$ is denumerable.

10. Determine the number of elements in $P(S)$, the collection of all subsets of S, for each of the following sets:
    (a) $S:=(1,2)$,
    (b) $S:=\{1.2,3)$,
    (c) S:-{1,2,3. 4).

Be sure to include the empty set and the set S itself in $P(S)$.

11 Use Mathematical Induction to prove that if the set S has n elements, then $P(S)$ has 2^n elements.

12 Prove that the collection $f(\mathbb{N})$ of *all finite* subsets of $\mathbb{N}$ is countable.

7.0 BIBLOGRAPHY/REFERENCES

MODULE 2

THE REAL NUMBERS

In this chapter we will discuss the essential properties of the real number system $\mathcal{R}$. Although it is possible to give a formal construction of this systemon the basis of a more primitive set (such as the set $\mathcal{N}$ of natural numbers or the set $\mathcal{Q}$ of rational numbers), we have chosen not to do so. Instead, we exhibit a list of fundamental properties associated with the real numbers and show how further properties can be deduced from them. This kind of activityis much more useful in learning the tools of analysis than examining the logicaldifficulties of constructing a model for $\mathcal{R}$.

The real number system can be described as a "complete ordered field", and we will discuss that description in considerable detail. In Unit2.1,we first introduce the "algebraic" properties—often called the "field" properties in abstract algebra—that are based on the two operations of addition and multiplication.We continue the section with theintroductionof the "order" properties of $\mathcal{R}$ and we derive some consequences of these properties and illustrate their use in working with inequalities. The notion of absolute value, which is based on the order properties, is discussed in Unit 2.2.

In Unit 2.3, we make the final step by adding the crucial "completeness" property to the algebraic and order properties of $\mathcal{R}$. It is thisproperty, which was not fully understood untilthe late nineteenth century that underlies the theory of limits and continuity and essentially all that follows in this book. The rigorous development of real analysis would not be possible without thisessential property.

In Unit2.4, we apply the Completeness Property to derive several fundamental resultsconcerning $\mathcal{R}$, includingthe Archimedean Property, the existence of square roots, and the density of rational numbers in $\mathcal{R}$. We establish, in Unit 2.5, the Nested Interval Property and use it to prove the uncountability of K. We also discuss its relation to binary and decimal representations of real numbers.

Part of the purpose of Unit 2.1 and 2.2 is to provide examples of proofs of elementary theorems from explicitly stated assumptions. Students can thus gain experience in writing formal proofs before encountering the more subtle and complicated arguments relatedto the Completeness Property and its consequences, However, students who have previously studied theaxiomatic method and the technique of proofs (perhaps in a course on abstract algebra) can move to Unit 2.3 after a cursory look at the earlier sections. A brief discussion of logic and types of proofs can be found in Appendix A at the back of the book.

Unit1 The Algebraic and Order Properties of $\mathcal{R}$

1.0 INTRODUCTION

We begin with a brief discussion of the "algebraic structure" of the real number system. We will give a short list of basic properties of addition and multiplication from which all other algebraic properties can be derived as theorems. In the terminology of abstract algebra, the system of real numbers is a "field" with respect to addition and multiplication. The basic properties listed in 2. 1.1 are known as *{he field axioms.* A *binary operation* associates with each pair *(a, b)* a unique element *B(a, b"),* but we will use the conventional notations of *a* + *b* and *a* . *b* when discussing the properties of addition and multiplication.

2.0 OBJECTIVES

At the end of the Unit, reader should be able to

(i) familiar with the essential properties of real numbers system $\mathbb{R}$

(ii) exhibt a list of fundamental properties associated with the real numbers

(iii) show how further properties can be deducted from them.

3.0 MAIN CONTENT

2.1.1 Algebraic Properties of $\mathbb{R}$ On the set $\mathcal{R}$ of real numbers there are two binary operations, denoted by + and and called **addition and multiplication**, respectively. These operations satisfy the following properties:

(**Al**) $a + b = b + a$ for all a, b in $\mathcal{R}$ *(commutative property of addition);*

(**A2**) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathcal{R}$ *(associative property of addition);*

(**A3**) there exists an element 0 in $\mathcal{R}$ such that $0 + a = a$ and $a + 0 = a$ for all a in $\mathbb{R}$ *(existence of a zero element);*

(**A4**) for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathcal{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ *(existence of negative elements);*

(**Ml**) $a . b = b . a$ for all a, b in $\mathcal{R}$ *(commutative property of multiplication);*

(**M2**) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b,$ c) in $\mathbb{R}$ *(associative property of multiplication);* (**M3**) there exists an element 1 in $\mathcal{R}$ *distinct from* 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$ *(existence of a unit element);*

(**M4**) for each $a \neq 0$ in $\mathcal{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) . a = 1$ *(existence of reciprocals);*

(**D**) $a . (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathcal{R}$ *(distributive property of multiplication over addition).*

These properties should be familiar to the reader. The first four are concerned with addition,the next four with multiplication,and the lastone connects the two operations. The point of the list is that all the familiar techniques of algebra can be derived from these nine properties, in much the same spirit that the theorems of Euclidean geometry can be deduced from the five basic axioms statedby Euclid in his *Elements.* Sincethis taskmore properly belongs to a course in abstract algebra, we will not carry it out here. However, to exhibit the spirit of the endeavor, we will sample a few results and their proofs.

We first establishthe basic fact that the elements 0 and 1, whose existence were asserted in (A3) and (M3), are in fact unique.We also show thatmultiplicationby 0 always results in 0,

2.1.2 Theorem (a) *If z and a* are elements *in* $\mathcal{R}$ with $2 + a = a$, *then* $z = 0$.

(b) *If u* and $b \neq 0$ are *elements* in $\mathbb{R}$ *with* $u \cdot b = b$, *then* $u = 1$.
(c) If $a \in \mathbb{R}$, *then* $a \cdot 0 = 0$.

Proof *(a)* Using (A3), (A4), (A2), the hypothesis $z + a = a$, and (A4), we get

$$z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0.$$

(b) Using (M3), (M4), (M2), the assumed equality $u \cdot b = b$, and (M4) again, we get

$$u = u \,.\, 1 = u \,.\, (b \,.\, (1/b)) = (u \cdot b) \cdot (1/b) = b \cdot (1/b) = 1.$$

(c) We have (why?)

$$a + a.\,0 = a.\,0 = a.\,(1+0) = a\,.1 = a$$

Therefore, we conclude from (a) that $a\,.0 = 0$. Q.E.D.

We next establish two important properties of multiplication: the uniqueness of reciprocals and the fact that a product of two numbers is zero only when one of the factors is zero.

2.1.3 Theorem (a) *If* $a \neq 0$ *and b in* $\mathbb{R}$ are such *that* $a \cdot b = 1$, *then* $b = 1/a$.
(b) *If* $a\,.b = 0$, *then either* $a = 0$ *or* $b = 0$.

Proof. (a) Using (M3), (M4), (M2), the hypothesis $a \cdot b = 1$, and (M3), we have

$$b = 1\,.\,b = ((1/a)\,.\,a)\,.\,b = (1/a)\,.\,(a\,.\,b) = (1/a).\,1 = 1/a.$$

(b) It suffices to assume $a \neq 0$ and prove that $b = 0$. (Why?) We multiply $a \cdot b$ by $1/a$ and apply (M2), (M4) and (M3) to get

$$(1/a)\,.\,(a\,.b) = ((1/a)\,.\,a)\,.\,b = 1\,.b = b.$$

Since $a \cdot b = 0$, by 2.1.2(c) this also equals

$$(1/a)\,.\,(a.b) = (1/a)\,.\,0 = 0.$$

Thus we have $b = 0$. Q.E.D.

These theorems represent a small sample of the algebraic properties of the real number system. Some additional consequences of the field properties are given in the exercises.

The operation of **subtraction** is defined by $a - b := a + (-b)$ for a, b in $\mathbb{R}$. Similarly, division is defined for a, b in $\mathbb{R}$ with $b \neq 0$ by $a/b := a \cdot (1/b)$. In the following, we will use this customary notation for subtraction and division, and we will use all the familiar properties of these operations. We will ordinarily drop the use of the dot to indicate multiplication and write ab for $a \cdot b$. Similarly, we will use the usual notation for exponents and write a^2 for aa, a^3 for $(a^2)a$; and, in general, we define $a^{n+1} := (a^n)a$ for $n \in \mathbb{N}$. We agree to adopt the convention that $a^1 = a$. Further, if $a \neq 0$, we write $a^0 = 1$ and a^{-1} for $1/a$, and if $n \in \mathbb{N}$, we will write a^{-n} for $(1/a)^n$ when it is convenient to do so. In general, we will freely apply all the usual techniques of algebra without further elaboration.

Rational and Irrational Numbers

We regard the set $\mathbb{N}$ of natural numbers as a subset of $\mathbb{R}$, by identifying the natural number $n \in \mathbb{N}$ with the n-fold sum of the unit element $1 \in \mathbb{R}$. Similarly, we identify $0 \in \mathbb{Z}$ with the zero element of $0 \in \mathbb{R}$, and we identify the n-fold sum of -1 with the integer $-n$. Thus, we consider $\mathbb{N}$ and $\mathbb{Z}$ to be subsets of $\mathbb{R}$.

Elements of $\mathbb{R}$ that can be written in the form b/a where $a, b \in \mathbb{Z}$ and $a \neq 0$ are called rational numbers. The set of all rational numbers in $\mathbb{R}$ will be denoted by the standard notation $\mathbb{Q}$. The sum and product of two rational numbers is again a rational number (prove this), and moreover, the field properties listed at the beginning of this section can be shown to hold for $\mathbb{Q}$.

The fact that there are elements in $\mathbb{R}$ that are not in $\mathbb{Q}$ is not immediately apparent. In the sixth century B.C. the ancient Greek society of Pythagoreans discovered that the diagonal of a square with unit sides could not be expressed as a ratio of integers. In view of the Pythagorean Theorem for right triangles, this implies that the square of no rational number can equal 2. This discovery had a profound impact on the development of Greek mathematics. One consequence is that elements of $\mathbb{R}$ that are not in $\mathbb{Q}$ became known as irrational numbers, meaning that they are not ratios of integers. Although the word "irrational" in modern English usage has a quite different meaning, we shall adopt the standard mathematical usage of this term.

We willnow prove that there does not exist a rational number whose square is 2. In the proof we use the notions of even and odd numbers. Recall that a natural number is even if it has the form $2n$ for some $n \in \mathbb{N}$, and it is **odd** if it has the form $2n - 1$ for some $n \in \mathbb{N}$. Every natural number is either even or odd, and no natural number is both even and odd.

2.1.4 Theorem *There does not exist a rational number r such that* $r^2 = 2$.

Proof. Suppose, on the contrary, that p and q are integers such that $(p/q)^2 = 2$. We may assume that p and q are positive and have no common integer factors other than 1. (Why?) Since $p^2 = 2q^2$, we see that p^2 is even. This implies that p is also even (because if $p = 2n - 1$ is odd, then its square $p^2 = 2(2n^2 - 2n + 1) - 1$ is also odd). Therefore, since p and q do not have 2 as a common factor, then q must be an odd natural number.

Since p is even, then $p = 2m$ for some $m \in \mathbb{N}$, and hence $4m^2 = 2q^2$, so that $2m^2 = q^2$. Therefore, q^2 is even, and it follows from the argument in the preceding paragraph that q is an even natural number.

Since the hypothesis that $(p/q)^2 = 2$ leads to the contradictory conclusion that q is both even and odd, it must be false. Q.E.D.

The Order Properties of $\mathbb{R}$

The "order properties" of $\mathbb{R}$ refer to the notions of positivity and inequalities between real numbers. As with the algebraic structure of the system of real numbers, we proceed by isolating three basic properties from which all other order properties and calculations with inequalities can be deduced. The simplest way to do this is to identify a special subset of $\mathbb{R}$ by using the notion of "positivity".

2.1.5 The Order Properties of $\mathbb{R}$ There is a nonempty subset $\mathcal{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:

(i) If a, b belong to $\mathcal{P}$, then $a + b$ belongs to $\mathcal{P}$.

(ii) If a, b belong to $\mathcal{P}$, then ab belongs to $\mathcal{P}$.

(iii) If a belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in \mathcal{P}, \qquad a = 0, \qquad -a \in \mathcal{P}.$$

The first two conditions ensure the compatibility of order with the operations of addition and multiplication, respectively. Condition 2.1.5(iii) is usually called the **Trichotomy Property**, since it divides $\mathbb{R}$ into three distinct types of elements. It states that the set $\{-a : a \in \mathcal{P}\}$ of **negative** real numbers has no elements in common with the set $\mathcal{P}$ of positive real numbers, and, moreover, the set $\mathbb{R}$ is the union of three disjoint sets.

If $a \in \mathcal{P}$, we write $a > 0$ and say that a is a **positive** (ora **strictly positive**) real number. *If* $a \in \mathcal{P} \cup (0)$, we write $a \geq 0$ and say that a is a n**onnegative** real number. Similarly, if —$a \in \mathcal{P}$ we write $a < 0$ and say that a is a **negative** (or a **strictly negative**) real number. *If* $-a \in \mathcal{P} \cup (0)$, we write $a \leq 0$ and say that a is a **nonpositive** real number.

The notion of inequality between two real numbers will now be defined in terms of the set $\mathcal{P}$ of positive elements.

2.1.6 Definition Let a, b be elements of. $\mathcal{R}$.

(a) If $a - b \in \mathcal{P}$, then we write $a > b$ or $b < a$.

(b) If $a - b \in \mathcal{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

The Trichotomy Property 2.1.5(iii)impliesthat for a, $b \in \mathbb{R}$ exactly one of the followingwillhold:

$$a > b, \qquad a = b, \qquad a < b.$$

Therefore, if both $a \leq b$ and $b \leq a$, then $a = b$.

For notational convenience, we will write

$$a < b < c$$

to mean that both $a < b$ and $b < c$ are satisfied. The other "double" inequalities $a \leq b < c$, $a \leq b \leq c$, and $a < b \leq c$ are defined in a similarmanner.

To illustratehow the baste Order Properties are used to derive the "rules of inequalities", we willnow establishseveral resultsthat the reader has used in earlier mathematics courses.

2.1.7 Theorem *Let a, b, c be any elements of* $\mathcal{R}$.

(a) *If $a > b$ and $b > c$. then $a > c$.*

(b) *If $a > b$, then $a + c > b + c$.*

(c) *If $a > b$ and $c > 0$, then $ca > cb$.*
If $a > b$ and $c < 0$. then $ca < cb$.

Proof. (a) If $a - b \in \mathcal{F}$ and $b - c \in \mathcal{P}$, then 2.1.5(1) implies that $(a - b)+(b - c) = a — c$ belongs to $\mathcal{P}$. Hence $a > c$.

(b)*If* $a - b \in \mathrm{P}$, then $(a + c) - (b + c)= a - b$ is in $\mathcal{P}$. Thus $a + c > b + c$.

(c) If $a - b \in \mathcal{P}$ and $c \in \mathcal{P}$, then $ca - cb = c(a - b)$ is in $\mathcal{P}$ by 2.1.5(ii). Thus $ca > cb$ when $c > 0$.

On the other hand, if $c < 0$, then $-c \in \mathcal{P}$, so that $cb - ca = (-c)(a - b)$ is in P. Thus $cb > ca$ when $c < 0$. Q.E.D.

It is natural to expect that thenatural numbers are positivereal numbers. This property is derived from thebasic properties of order. The key observation is that the square of any nonzero real number is positive.

2.1.8 Theorem (a) *If $a \in \mathcal{R}$ and $a \neq 0$, then $a^2 > 0$.*

(b) $1 > 0$,

(c) *If $n \in \mathcal{N}$, then $n > 0$.*

Proof. (a)By the Trichotomy Property, if $a \neq 0$, then either $a \in \mathcal{P}$ or $-a \in \mathcal{P}$. If $a \in \mathcal{P}$, thenby 2.1.5(ii),$a^2 = a \cdot a \in \mathcal{P}$. Also, if $-a \in \mathcal{P}$, then $a^2 = (-a)(-a) \in \mathcal{P}$. We conclude that if $a \neq 0$, then $a^2 > 0$.

(b) Since $1 = 1^2$, it follows from (a) that $1 > 0$.

(c) We use Mathematical Induction. The assertion for $n = 1$ is true by (b). If we suppose the assertion is true for the natural number k, then $k \in \mathcal{P}$, and since $1 \in \mathcal{P}$, we have $k + 1 \in \mathcal{P}$ by 2.1.5(i). Therefore, the assertion is true for all natural numbers. Q.E.D.

It is worth noting that *no smallest positive real number can exist*. This follows by observing that if $a > 0$, then since $½ > 0$ (why?), we have that

$$0 < ½\, a < a.$$

Thus if it is claimed that a is the smallest positive real number, we can exhibit a smaller positivenumber $^1/_2 a$.

This observation leads to the next result, which will be used frequently as a method of proof. For instance, to prove that a number $a \geq 0$ is actually equal to zero, we see that it suffices to show that a is smaller than an arbitrary positive number.

2.1.9 Theorem *If $a \in \mathcal{R}$ is such* that $0 < a < \varepsilon$ for every $\varepsilon > 0$, *then* $a = 0$.

Proof. Suppose to the contrary that $a > 0$. Then if we take $\varepsilon_0 := ½a$, we have $0 < \varepsilon_0 < a$. Therefore, it is falsethat $a < \varepsilon$ for every $\varepsilon > 0$ and we conclude that $a = 0$.
Q.E.D.

Remark It is an exercise to show that if $a \in \mathcal{R}$ is such that $0 \leq a \leq \varepsilon$ for every $\varepsilon > 0$, then $a = 0$,

The product of two positivenumbers is positive. However, the positivity of a productof two numbers does not imply that each factor is positive. The correct conclusion is givenin the next theorem. It is an important tool in working with inequalities.

2.1.10 Theorem *If $ab > 0$, then either*

(i) $a > 0$ and $b > 0$, or

(ii) $a < 0$ and $b < 0$.

Proof. First we note that $ab > 0$ implies that $a \neq 0$ and $b \neq 0$. (Why?) From the Trichotomy Property, either $a > 0$ or $a < 0$. If $a > 0$, then $1/a > 0$ (why?), and therefore $b = (1/a)(ab) > 0$. Similarly, if $a < 0$, then $1/a < 0$, so that $b = (1/a)(ab) < 0$.
Q.E.D.

2.1.11 Corollary *If $ab < 0$, then either*

(i) $a < 0$ and $b > 0$, *or*

(ii) $a > 0$ *and* $b < 0$.

Inequalities

We now show how the Order Properties presented in this section can be used to "solve" certain inequalities. The reader should justify each of the steps.

2.1.12 Examples (a) Determine the set A of all real numbers x such that $2x + 3 \leq 6$. We notethat we have[+]

$$x \in A \Leftrightarrow 2x + 3 \leq 6 \Leftrightarrow 2x \leq 3 \Leftrightarrow x \leq {}^3/_2 .$$

Therefore $A = \{x \in \mathcal{R}: x^2 < {}^3/_2\}$.

(b) Determine the set $B := \{x \in \mathcal{R}: x^2 + x > 2\}$.

We rewrite the inequality so that Theorem 2.1.10 can be applied. Note that Therefore, we either have (i) $x - 1 > 0$ and $x + 2 > 0$, or we have (ii) $x - 1 < 0$ and $x + 2 < 0$. In case (i) we must have both $x > 1$ and $x > -2$, which is satisfied if and only

The symbol $\Leftrightarrow$ shouldbe read "ifandonlyif".

if $x > 1$. In case (ii) we must have both $x < 1$ and $x < -2$, which is satisfied if and only if $x < -2$.

We conclude that $B = \{x \in \mathcal{R}: x > 1\} \cup \{x \in \mathcal{R}: x < -2\}$.

(c) Determine the set

$$C := \left\{x \in \mathcal{R}: \frac{2x + 1}{x + 2} < 1\right\}$$

We note that

$$x \in C \Leftrightarrow \frac{2x + 1}{x + 2} - 1 < 0 \Leftrightarrow \frac{x - 1}{x + 2} < 0.$$

Therefore we have either (i) $x - 1 < 0$ and $x + 2 > 0$, or (ii) $x - 1 > 0$ and $x + 2 < 0$. (Why?) In case (i) we must have both $x < 1$ and $x > -2$, which is satisfied if and only if $-2 < x < 1$. In case (ii), we must have both $x > 1$ and $x < -2$, which is never satisfied.

We conclude that $C = \{x \in \mathcal{R}: -2 < x < 1\}$.

4.0 CONCLUSION

The following examples illustrate the use of the Order Properties of $\mathcal{R}$ in establishing certain inequalities. The reader should verify the steps in the arguments by identifying the properties that are employed.

It should be noted that the existence of square roots of positivenumbers has not yet been established; however, we assume the existence of these roots for the purpose of these examples. (The existence of square roots willbe discussed in Unit 2.4.)

2.1.13 Examples (a) Let $a \geq 0$ and $b \geq 0$. Then

(1) $$a < b \Leftrightarrow a^2 < b^2 \Leftrightarrow \sqrt{a} < \sqrt{b}$$

We consider the case where $a > 0$ and $b > 0$, leaving the case $a = 0$ to the reader. It follows from 2.1.5(i)that $a + b > 0$. Since $b^2 - a^2 = (b - a)(b + a)$, it follows from 2.1. 7(c) that $b - a > 0$ implies that $b^2 - a^2 > 0$. Also, it follows from 2.1.10 that $b^2 - a^2 > 0$ implies that $b - a > 0$.

If $a > 0$ and $b > 0$, then $\sqrt{a} > 0$ and $\sqrt{b} > 0$. Since $a = (\sqrt{a})^2$ and $b = (\sqrt{b})^2$, the second implication is a consequence of the first one when a and b are replaced by $\sqrt{a}$ and $\sqrt{b}$, respectively.

We also leave it to the reader to show that if $a \geq 0$ and $b \geq 0$, then

(1) $$a \leq b \Leftrightarrow a^2 \leq b^2 \Leftrightarrow \sqrt{a} \leq \sqrt{b}$$

(b) If a and b are positive real numbers, then their **arithmetic mean** is ½ $(a + b)$ and their **geometric mean** is $\sqrt{ab}$. The **Arithmetic- Geometric Mean Inequality** for a, b is

(2) $$\sqrt{ab} \le \tfrac{1}{2}(a + b)$$

with equality occurring if and only if $a = b$.

To prove this, note that if $a > 0$, $b > 0$, and $a \neq b$, then $\sqrt{a}>0$, $\sqrt{b}>0$ and $\sqrt{a}\neq\sqrt{b}$. (Why?) Therefore it follows from 2.1.8(a) that $(\sqrt{a} - \sqrt{b})^2 > 0$. Expanding this square, we obtain

$$a - 2\sqrt{ab} + b > 0,$$

whence it follows that

$$\sqrt{ab} < \tfrac{1}{2}(a + b).$$

Therefore (2) holds (with strict inequality) when $a \neq b$. Moreover, if $a = b$ (>0), then both sides of (2) equal a, so (2) becomes an equality. This proves that (2) holds for $a > 0$, $b > 0$.

On the other hand, suppose that $a > 0$, $b > 0$ and that $\sqrt{ab} = \tfrac{1}{2}(a + b\}$. Then, squaring both sides and multiplying by 4, we obtain

$$4ab = (a + b)^2 = a^1 + 2ab + b^2,$$

whence it follows that

$$0 = a^2 - 2ab + b^2 = (a - b)^2$$

But this equality implies that $a = b$. (Why?) Thus, equality m (2) implies that $a = b$.

5.0 SUMMARY

Remark The general Arithmetic-Geometric Mean Inequality for the positive real numbers $a_1, a_2, \cdots, a_n$ is

(3) $$(a_1, a_2, \cdots, a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality occurring if and only if $a_l = a, = \cdots = a_n$. It is possible to prove this more general statement using Mathematical Induction, but the proof is somewhat intricate. A more elegant proof that uses properties of the exponential function is indicated in Exercise 8.3.9 in Module 8.

(c) **Bernoulli's Inequality.** If $x > -1$, then

(4) $$(1 + x)^n \ge 1 + nx \qquad \text{for all } n \in \mathcal{N}$$

The proof uses Mathematical Induction. The case $n = 1$ yields equality, so the assertion is valid in this case. Next, we assume the validity of the inequality (4) for $k \in \mathcal{N}$ and will deduce it for $k + 1$. Indeed, the assumptions that $(1 + x)^k \ge 1 + kx$ and that $1 + x > 0$ imply (why?) that

$$(1 + x)^{k+1} = (1 + x)$$
$$\ge) 1 + kx) . (1 + x) = 1 + (k + 1)x + kx^2.$$
$$\ge 1 + (k + 1)x.$$

Thus, inequality (4) holds for $n = k + 1$. Therefore, (4) holds for all $n \in \mathcal{N}$.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 2.1

1. If $a, b \in \mathbb{N}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$, (b) $-(-a) = a$,
   (c) $(-l)a = -a$, (d) $(-1)(-1) = 1$.

2. Prove that if $a, b \in \mathcal{R}$, then
   (a) $-(a + b) = (-a) + (-b)$, (b) $(-a).(-b) = a.b$
   (c) $1/(-a) = -(l/a)$. (d) $-(a/b) = (-a)/b$ if $b \neq 0$.

3. Solve the following equations, justifying each step by referring to an appropriate property or theorem.
   (a) $2x + 5 = 8$, (b) $x^2 = 2x$,
   (c) $x^2 - l = 3$, (d) $(x-l)(x+2)=0$.

4. If $a \in \mathcal{R}$ satisfies $a.a = a$, prove that either $a = 0$ or $a = 1$.

5. If $a \neq 0$ and $b \neq 0$, show that $l/(ab) = (1/a)(l/b)$.
6 Use the argument in the proof of Theorem 2.1.4 to show that there does not exist a rational number s such that $s^2 = 6$.

7. Modify the proof of Theorem 2.1.4 to show that there does not exist a rational number t such that $t^3 = 3$.

8. (a) Show that if x, y are rational numbers, then $x + y$ and xy are rational numbers.
   (b) Prove chat if x is a rational number and y is an irrational number, then $x + y$ is an irrational number. If, in addition, $x \neq 0$, then show that xy is an irrational number.

9. Let $K := \{s + \sqrt{2}: s, t \in \mathbb{Q}\}$. Show that K satisfies the following:
   (a) If $x_1, x_2 \in. K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.
   (b) If $x \neq 0$ and $x \in K_1$ then $1/x \in K$.
   (Thus the set K is a *subfield* of $\mathbb{R}$. With the order inherited from $\mathbb{R}$, the set K is an ordered fieldthat lies between $\mathbb{Q}$ and $\mathbb{R}$).

10. (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
    (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.

11. (a) Show that if $a > 0$, then $1/a > 0$ and $l/(l/a) = a$.
    (b) Show that if $a < b$, then $a < ½ (a + b) < b$.

12. Let a, b, c, d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.

13. If $a, b \in R$, show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$.

14. If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.
15. If $0 < a < b$, show that (a) $a < \sqrt{ab} < b$, and (b) $1/b < 1/a$.

16. Find all real numbers x that satisfy the following inequalities,
    (a) $x^2 > 3x + 4$, (b) $1 < x^2 < 4$,

(c) $l/x<x$ (d) $l/x < .x^2$.

17. Prove the following form of Theorem 2.1.9: If $a \in \mathbb{R}$ is such that $0 \leq a \leq \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

18. Let $a, b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$ we have $a \leq b + \varepsilon$. Show that $a \leq b$.

19. Prove that $[\frac{1}{2}(a+b)]^2 \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathcal{R}$. Show that equality holds if and only if $a = b$.

20. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.
    (b) If $1 < c$, show that $1 < c < c^2$.

21. (a) Prove there is no $n \in \mathcal{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathcal{N}$.)
    (b) Prove that no natural number can be both even and odd.

22. (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.
    (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n >$].

23. If $a > 0$, $b >> 0$ and $n \in \mathcal{N}$, show that $a < b$ if and only if $a^n < b^n$. [Hint: Use Mathematical Induction].

24. (a) If $c > 1$ and $m, n \in \mathcal{N}$, show that $c^m > c^n$ if and only if $m > n$.
    (b) If $0 < c < 1$ and $m, n \in \mathcal{N}$, show that $c^m < c^n$ if and only if $m > n$.

25. Assuming the existence of roots, show that if $c > l$, then $c^{1/m} < c^{1/n}$ if and only if $m > n$.

26. Use Mathematical Induction to show that if $a \in \mathcal{R}$ and $m, n, \in \mathcal{N}$, then $a^{m+n} = a^m a^n$ and $(a^m)^n = a^{mn}$.

7.0 REFERENCES / TEXTBOOK

Unit 2 Absolute Value and the Real Line

1.0 INTRODUCTION

From the Trichotomy Property 2.1.5(iii), we are assured that if $a \in \mathcal{R}$ and $a \neq 0$, then exactlyone of the numbers a and $-a$ is positive.The absolute value of $a \neq 0$ is defined to be the positiveone of these two numbers. The absolute value of 0 is defined to be 0.

2.0 OBJECTIVES

At the end of the Unit, the readers should be able to:

(i) understand and define the concept of “Absolute Value”
(ii) understand the concept of “Field” properties
(iii) understand and farmiliar with the “Order” properties in Real Analysis

3.0 MAIN CONTENT

2.2.1 Definition The absolute value of a real number a, denoted by $|a|$, is defined by

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|5| = 5$ and $|-8| = 8$. We see from the definition that $|a| \geq 0$ for all $a \in \mathcal{R}$, and that $|a| = 0$ if and only if $a = 0$. Also $|-a| = |a|$ for all $a \in \mathcal{R}$. Some additional properties are as follows.

2.2.2 Theorem (a) $|ab| = |a||b|$ *for all* $a, b \in \mathcal{R}$.

(b) $|a|^2 = a^2$ *for all* $a \in \mathcal{R}$.

(c) *If* $c \geq 0$, *then* $|a| \leq c$ *if and only if* $-c \leq a \leq c$,

(d) $-|a| \leq a \leq |a|$ *for all* $a \in \mathcal{R}$.

Proof. (a) If either a or b is 0, then both sides are equal to 0. There are four other cases to consider. If $a > 0$, $b > 0$, then $ab > 0$, so that $|ab| = ab = |a||b|$. If $a > 0$, $b < 0$, then $ab < 0$, so that $|ab| = -ab = a(-b) = |a||b|$. The remaining cases are treated similarly.
(b) Since $a^2 > 0$, we have $a^2 = |a^2| = |aa| = |a||a| = |a|^2$.
(c) If $|a| < c$, then we have both $a < c$ and $-a \leq c$ (why?), which is equivalent to $-c \leq a \leq c$. Conversely, if $-c \leq a \leq c$, then we have both $a < c$ and $-a \leq c$ (why?), so that $|a| \leq c$.
(d) Take $c = |a|$ in part (c). Q.E.D.

The following important inequality willbe used frequently.

2.2.3 Triangle Inequality ***If a, b*** $\in \mathcal{R}$, then $|a + b| \leq |a| + |b|$.

Proof. From 2.2.2(d), we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. On adding these inequalities, we obtain

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Hence, by 2.2.2(c) we have $|a + b| \leq |a| + |b|$. Q.E.D.

It can be shown that equality occurs in the Triangle Inequality if and only if $ab > 0$, which is equivalent to saying that a and b have the same sign. (See Exercise 2.) There are many useful variations of the Triangle Inequality. Here are two.

2.2.4 Corollary ***If a, b*** $\in \mathcal{R}$, *then*

(a) $||a| - |b|| \leq |a - b|$,

(b) $|a - b| \leq |a| + |b|$.

Proof, (a) We write $a = a - b + b$ and then apply the Triangle Inequality to get $|a| = |(a - b) + b| \leq |a - b| + |b|$. Now subtract $|b|$ to get $|a| - |b| \leq |a - b|$. Similarly, from $|b| = |b - a + a| \leq |b - a| + |a|$, we obtain $-|a - b| = -|b - a| \leq |a| - |b|$. If we combinethese two inequalities, using 2.2.2(c), we get the inequality in (a).
(b) Replaced in the Triangle Inequality by $-b$ to get $|a - b| \leq |a| + |-b|$. Since $|-b| = |b|$ we obtain the inequality in (b). Q.E.D.

A straightforward application of Mathematical Induction extends the Triangle Inequality to any finite number of elements of $\mathbb{R}$.

2.2.5 Corollary If $a_1, a_2 \cdots, a_n$ are any *real numbers, then*

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

The following examples illustrate how the propenies of absolute value can be used.

2.2.6 Examples (a) Determine the set A of $x \in \mathcal{R}$ such that $|2x + 3| < 7$.

From a modification of 2.2.2(c)for thecase of strict inequality,we see that $x \in A$ ifand only if $-7 < 2x + 3 < 7$, which is satisfied if and only if $-10 < 2x < 4$. Dividing by 2, we conclude that $A = \{x \in \mathcal{R}: -5 < x < 2\}$.

(b) Determine the set $B := \{x \in \mathcal{R}: |x - 1| < |x|\}$.

One method is to consider cases so that the absolute value symbols can be removed. Here we take the cases

$$\text{(i) } x \geq 1, \qquad \text{(ii) } 0 \leq x < 1 \qquad \text{(iii) } x < 0$$

(Why did we choose these three cases?) In case (i) the inequality becomes $x - 1 < x$, which is satisfied without further restnction. Therefore all x such that $x \geq 1$ belong to the set B. In case(ii), the inequality becomes $-(x - 1) < x$, which requires that $x > ½$. Thus, this case contributes all x such that $½ < x < 1$ to the set B. In case (iii), the inequality becomes $-(x - 1) < -x$, which is equivalent to $1 < 0$. Since this statement is false, no value of x from case (iii)satisfies the inequality. Forming the union of the three cases, we conclude that $B = \{x \in \mathcal{R}: x > 1/2\}$.

There is a second method of determining the set B based on the fact that $a < b$ if and only if $a^2 < b^2$ when both $a \geq 0$ and $b \geq 0$. (See 2.1.13(a).) Thus, the inequality $|x - 1| < |x|$ is equivalent to theinequality $|x-1|^2 < |x|^2$. Since $|a|^2 = a^2$ for any a by 2.2.2(b), we can expand the square to obtain $x^2 - 2x + 1 < x^2$, which simplifies to $x > ½$. Thus, we again find that

$B = \{x \in \mathcal{R}: > 1/2\}$ This method of squaring can sometimes be used to advantage, but often a case analysis cannot be avoided when dealing with absolute values.

(c) Let the function f be defined by $f(x) := (2x^2 + 3x + 1)/(2x - 1)$ for $2 \leq x \leq 3$.

Find a constant M such that $|f(x)| \leq M$ for all x satisfying $2 \leq x \leq 3$. We consider separately the numerator and denominator of

$$|f(x)| = \frac{|2x^2 + 3x + 1|}{|2x - 1|}$$

From the Triangle Inequality, we obtain

$$|2x^2 + 3x + 1| \leq 2|x|^2 + 3|x| + 1 \leq 2.3^2 + 3.3 + 1 = 28$$

since $|x| \leq 3$ for the x under consideration. Also, $|2x - 1| \geq 2|x| - 1 \geq 2.2 - 1 = 3$
since $|x| \leq 2$ for the x under consideration. Thus, $1/|2x - 1| \leq 1/3$ for $x \geq 2$. (Why?)
Therefore, for $2 \leq x \leq 3$ we have $|f(x)| \leq 28/3$. Hence we can take $M = 28/3$. (Notethat we have found one such constant M; evidently any number $H > 28/3$ will also satisfy $|f(x)| \leq H$. It is also possible that 28/3 is not the smallest possible choice for M.) □

4.0 CONCLUSION

The Real Line

A convenient and familiar geometric interpretation of thereal number system is the real line. In this interpretation, the absolute value $|a|$ of an element a in $\mathcal{R}$ is regarded as the distance from a to the origin 0. More generally, the **distance** between elements a and b in $\mathcal{R}$ is $|a-b|$. (See Figure 2.2.1.)

We will later need precise language to discuss the notion of one real number being "close to" another. If a is a given real number, then saying that a real number* is "close to" a should mean that the distance $|x - a|$ between them is "small". A context in which this idea can be discussed is provided by the terminology of neighborhoods, which we now define.

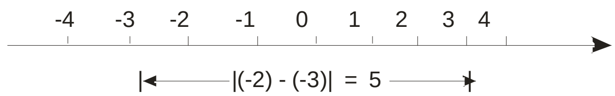


Figure 2.2.1 The distance between a = -2 and b = 3

2.2.7 Definition Let $a \in \mathcal{R}$ and $\varepsilon > 0$. Then the ***ε*- neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathcal{R} : |x - a| < \varepsilon\}$.

For $a \in \mathcal{R}$, the statement that x belongs to $V_\varepsilon(a)$ is equivalent to either of the statements (see Figure 2.2.2)

$$-\varepsilon < x - a < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon.$$

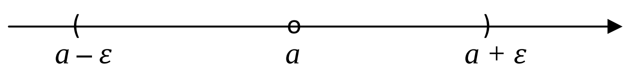


Figure 2.2.2 An ε-neighborhood of a

5.0 SUMMARY

2.2.8 Theorem *Let $a \in \mathcal{R}$. If x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.*

Proof. If a particular x satisfies $|x - a| < \varepsilon$ for every $\varepsilon > 0$, then it follows from 2.1.9 that $|x - a| = 0$, and hence $x = a$. Q.E.D.

2.2.9 Examples (a) Let $U := \{x : 0 < x < 1\}$. If $a \in U$, then let ε be the smaller of the two numbers a and $1 - a$. Then it is an exercise to show that $V_\varepsilon(a)$ is contained in U. Thus each element of U has some ε-neighborhood of it contained in U.

(b) If $I := \{x : 0 \le x \le 1\}$, then for any $\varepsilon > 0$, the ε-neighborhood $V_\varepsilon(0)$ of 0 contains points not in I, and so $V_\varepsilon(0)$ is not contained in 1. For example, the number $x_\varepsilon := -\varepsilon/2$ is in $V_\varepsilon(0)$ but not in 1.

(c) If $|x - a| < \varepsilon$ and $|y - b| < \varepsilon$, then the Triangle Inequality implies that

$$|(x+y) - (a + b)| = |(x - a) + (y - b)|$$
$$\le |x - a| + |y - b| < 2\varepsilon.$$

Thus if x, y belong to the ε-neighborhoods of a, b, respectively, then $x + y$ belongs to the 2ε-neighborhood of $a + b$ (but not necessarily to the ε-neighborhood of $a + b$). □

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 2.2

1 If $a, b \in \mathcal{R}$ and $b \neq 0$, show that:

(a) $|a| = \sqrt{a^2}$, (b) $\backslash a/b\backslash = \backslash a\backslash/\backslash b\backslash$.

2 If a, $b \in \mathcal{R}$, show that $\backslash a + b\backslash = \backslash a\backslash + \backslash b\backslash$ if and only if $ab \geq 0$,

3. If $x, y, z \in \mathcal{R}$ and $x \leq z$, showthat $x \leq y \leq z$ if and onlyif $|x - y| + |y — z\backslash = |x — z|$. Interpret this geometrically,

4. Show that $|x — a\backslash < \varepsilon$ if and only if $a — \varepsilon < x < a + \varepsilon$.

5. If $a<x<b$ and $a<y<b$, show that $|x — y\backslash < b — a$, Interpret this geometrically.

6. Find all $x \in \mathcal{R}$ that satisfy the following inequalities:

(a) $|4x - 5| \leq 13$, (b) $|x^2 - 1| \leq 3$.

7. Find all $xa \in \mathcal{R}$ thatsatisfythe equation $|x + 1| + \backslash x — 2| = 7$.

8. Find all $x \in \mathcal{R}$ thatsatisfythe following inequalities.

(a) $\backslash x - 1\backslash > \backslash x + 1\backslash$, (b) $\backslash x\backslash + \backslash x + 1| < 2$.

9. Sketch the graph of the equation $y = \backslash x| — \backslash x — 1|$.

10. Findall $x \in \mathcal{R}$ that satisfy the inequality $4 < |x + 2|x + |x - 1| < 5$.

11. Findall $x \in \mathcal{R}$ that satisfyboth $\backslash 2x — 3| < 5$ and $\backslash x + 1| > 2$ simultaneously.

12. Determine and sketch the set of pairs (x, y) in $\mathcal{R}$ x $\mathcal{R}$ that satisfy:

(a) $|x| = |y|$ (b) $\backslash x\backslash + \backslash y\backslash = 1$,

(c) $\backslash xy\backslash = 2$. (d) $|x| - |y| = 2$.

13. Determine and sketchtheset of pairs $(x. y)$ in $\mathcal{R}$ x $\mathcal{R}$ that satisfy:

(a) $|x| \leq |y|$, (b) $|x| + |y| = 1$

(c) $|xy| = 2$ (d) $|x| - |y| \geq 2$.

14. Let $\varepsilon > 0$ and $\delta > 0$, and $a \in \mathcal{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.

15. Show that if $a, b \in \mathcal{R}$, and $a \neq b$, then there exist ε-neighborhoods U *of* a and V *of* b such that $U \cap V = \emptyset$.

16. Show that if a, $b \in \mathcal{R}$ then

(a) $max\{a.b\} = ½ (a + b + |a - b\backslash)$ and $\min\{a, b\} = ½ (a + b - \backslash a - b\backslash)$.

(b) $\min\{a,b, c\} = \min\{\min\{a,b\},c)$.

17. Show that if $a, b, c \in \mathcal{R}$, then the "middle number" $\text{mid}\{a, b, c\} = \min\{\max(a, b\}$, $\max(b, c:\}. \max\{c, a)\}$.

7.0 REFERENCES

Unit3 TheCompletenessProperty of R

1.0 INTRODUCTION

Thusfar,wehavediscussedthealgebraicpropertiesandtheorderpropertiesofthereal numbersystem$\mathcal{R}$.Inthissectionweshallpresentonemorepropertyof$\mathcal{R}$thatisoftencalled the"CompletenessProperty".Thesystem $\mathcal{Q}$ ofrationalnumbersalsohasthealgebraicandorderproperties describedintheprecedingsections,butwehaveseen that$\sqrt{2}$ *cannot*be represented asarationalnumber; therefore $\sqrt{2}$doesnot belongto $\mathbb{Q}$.This observation showsthenecessity ofanadditionalpropertytocharacterizetherealnumber system. Thisadditional property, theCompleteness (ortheSupremum)Property, isanessentialproperty of $\mathbb{R}$,and wewillsay that $\mathbb{R}$isa*complete ordered field*.Itis thisspecial property that permits ustodefineanddevelopthevariouslimiting proceduresthatwillbediscussed in thechaptersthatfollow.

Therearesevealdifferent waystodescribetheCompletenessProperty.Wechooseto givewhatisprobably themostefficientapproachby assumingthateachnonemptybounded subset of $\mathbb{R}$hasasupremum.

2.0 OBJECTIVES

At the end of the unit, readers should be able to
(i) the theory of "Completeness", properties to algebraic and order properties of ℝ
(ii) discuss the theory of limits and continuity.

3.0 MAIN CONTENT

SupremaandInfima

Wenowintroducethenotionsofupperboundandlowerbound forasetofrealnumbers. Theseideaswillbeofutmostimportanceinlatersections.

2.3.1 Definition Let$S$beanonemptysubsetof $\mathbb{R}$.

(a) Theset$S$issaidtobe**boundedabove**ifthereexistsanumber $u \in \mathbb{R}$suchthat $s \leq u$foralls$\in S$.Eachsuchnumber$u$iscalledan**upperbound** of$S$.
(b) Theset$S$issaidtobe**boundedbelow**ifthereexistsanumber $\omega \in \mathbb{R}$suchthat $\omega \leq s$foralls$\in S$.Eachsuchnumberωiscalleda**lowerbound** of$S$.
(c) Asetissaid tobe**bounded** ifitisbothboundedaboveandbounded below.Asetis saidtobe**unbounded**ifitisnotbounded.

Forexample, theset$S := \{x \in \mathbb{R}: x < 2\}$isbounded above; thenumber2andany numberlarger than2isanupperboundofS.Thissethasnolower bounds, sothattheset isnotboundedbelow.Thusitisunbounded(eventhoughitisboundedabove).

If asethasone upperbound, thenithasinfinitelymany upper bounds,because if$u$isan upperbound of$S$,then thenumbers $u+1, u+2, \cdots$arealso upper bounds of$S$.(Asimilarobservation isvalidforlowerbounds.)

Inthesetofupper boundsofSandthesetoflower bounds of$S$,wesingleouttheir leastandgreatest elements,respectively,forspecial attention inthefollowing definition. (SeeFigure2.3.1.)

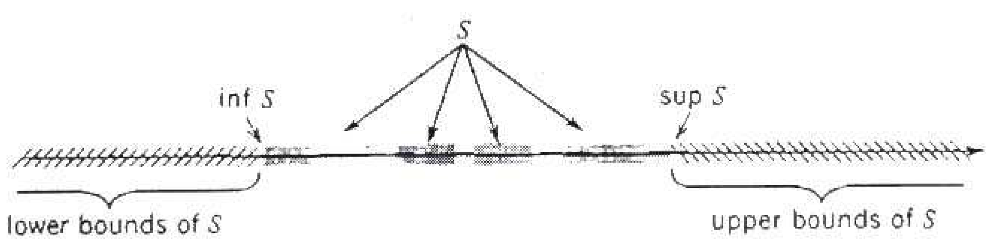


Figure 2.3.1 inf S and sup S

2.3.2 Definition Let S be a nonempty subset of $\mathcal{R}$.

(a) If S is bounded above, then a number u is said to be a supremum (or a **least upper bound**) of S if it satisfies the conditions:

(1) u is an upper bound of S, and

(2) if v is any upper bound of S, then $u \leq v$.

(b) If S is bounded below, then a number ω is said to be an **infimum (or a greatest lower** bound) of S if ii satisfies the conditions:

(1') ω is a lower bound of S, and

(2') if t is any lower bound of S, then $t \leq \omega$.

It is not difficult to see that *there can be only one supremum of a given subset S of* $\mathcal{R}$. (Then we can refer to *the* supremum of a set instead of *a* supremum.) For, suppose that u_1 and u_2 are both suprema of S. If $u_1 < u_2$, then the hypothesis that u_2 is a supremum implies that u_1 cannot be an upper bound of S. similarly, we see that $u_2 < u_1$ is not possible. Therefore, we must have $u_1 = u_2$. A similar argument can be given to show that the infimum of a set is uniquely determined.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup S \quad \text{and} \quad \inf S.$$

We also observe that if u' is an arbitrary upper bound of a nonempty set S, then $\sup S \leq u'$. This is because sup S is the least of the upper bounds of S.

First of all, it needs to be emphasized that in order for a nonempty set S in $\mathcal{R}$ to have a supremum, it must have an upper bound. Thus, not every subset of $\mathcal{R}$ has a supremum; similarly, not every subset of $\mathcal{R}$ has an infimum. Indeed, there are four possibilities for a nonempty subset S of $\mathcal{R}$; it can

(i) have both a supremum and an infimum,

(ii) have a supremum but no infimum,

(iii) have a infimum but no supremum,

(iv) have neither a supremum nor an infimum.

We also wish to stress that in order to show that $u = \sup S$ for some nonempty subset S of $\mathcal{R}$, we need to show that *both* (1) and (2) of Definition 2.3.2(a) hold. It will be instructive to reformulate these statements. First the reader should see that the following two statements about a number u and a set S are equivalent:

(1) u is an upper bound of S,

(1') $s \leq u$ for all $s \in S$.

Also, the following statements about an upper bound u of a set S are equivalent:

(2) if v is any upper bound of S, then $u \leq v$,

(2') if $z < u$, then z is not an upper bound of S,

(2'') if $z < u$, then there exists $s_z \in S$ such that $z < s_z$,

(2''') if $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Therefore, we can state two alternate formulations for the supreme.

2.3.3 Lemma *A number u is the supremum of a nonempty subset S of* $\mathbb{R}$ *if and only if u satisfies the conditions:*

(1) $s \leq u$ for all $s \in S$,

(2) if $v < u$, then there exists $s' \in S$ such that $v < s'$.

We leave it to the reader to write out the details of the proof.

2.3.4 Lemma

An upper bound u of a nonempty set S in $\mathbb{R}$ *is the supremum of S if and only if for every* $\varepsilon > 0$ *there exists an* $s_\varepsilon \in S$ *such that* $u - \varepsilon < s_\varepsilon$.

Proof. If u is an upper bound of S that satisfies the stated condition and if $v < u$, then we put $\varepsilon := u - v$. Then $\varepsilon > 0$, so there exists $s_\varepsilon \in S$ such that $v = u - \varepsilon < s_\varepsilon$. Therefore, v is not an upper bound of S, and we conclude that $u = \sup S$.

Conversely, suppose that $u = \sup S$ and let $\varepsilon > 0$. Since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound of S. Therefore, some element s_ε of S must be greater than $u - \varepsilon$; that is, $u - \varepsilon < s_\varepsilon$ (See Figure 2.3.2.) Q.E.D.

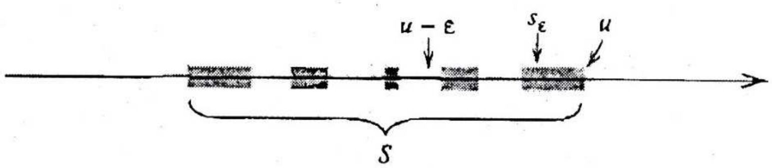


Figure 2.3.2 $u = \sup S$

It is important to realize that the supremum of a set may or may not be an element of the set. Sometimes it is and sometimes it is not, depending on the particular set. We consider a few examples.

2.3.5 Examples (a) If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element ω and a least element ω. Then $u = \sup S_1$ and $\omega = \inf S_1$ and they are both members of S_1 (This is clear if S_1 has only one element, and it can be proved by induction on the number of elements in S_1; see Exercises 11 and 12.)

(b) The set $S_2 := \{x : 0 \leq x \leq 1\}$ clearly has 1 for an upper bound: We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2..

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does *not* contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3.

4.0 CONCLUSION

The Completeness Property of $\mathbb{R}$

It is not possible to prove on the basis of the field and order properties of $\mathbb{R}$ that were discussed in Unit 2.1 that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. However, it is a deep and fundamental property of the real number system that this is indeed the case. We will make frequent and essential use of this property, especially inourdiscussionoflimitingprocesses.Thefollowingstatementconcerningthe existence.ofsuprema isourfinalassumption about $\mathcal{R}$.Thus,wesaythat $\mathbb{R}$ isa*complete orderedfield.*

5.0 SUMMARY

2.3.6 The CompletenessPropertyof$\mathcal{R}$ *Everynonemptysetof*real*numbersthathas anupperboundalsohasasupremumin$\mathcal{R}$.*

Thisproperty isalsocalled theSupremum Property of $\mathcal{R}$.The analogousproperty forinfimacanbededuced fromtheCompletenessProperty asfollows.SupposethatSis anonempty subset of $\mathbb{R}$thatisboundedbelow.Thenthenonempty set$S:=\{-s: s\in S\}$ isbounded above, andtheSupremumProperty implies that$u:=\sup S$exists in $\mathcal{R}$.The readershould verifyindetailthat$-u$istheinfimumof$S$.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 2.3

1. Let $S_1 := (x \in \mathcal{R}: x \geq 0)$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that inf $S_1 = 0$.
2. Let $S_2=\{x \in \mathcal{R}: x \geq 0\}$. Does S_2have lower bounds? Does S_2 have upper bounds? Does inf S_2 exist? Does sup S_2exist? Prove your statements.
3. Let $S_3 = \{1/n: n\in \mathcal{N}\}$.Show that sup S_3 - 1 and $\inf S_3 \geq 0$. (Itwill follow from the Archimedean Property in Unit 2.4 that inf $S_3 = 0$.)
4. Let$S_4 := (1 -(-1)^n/_n : n \in \mathcal{N}\}$. Find inf S_4 and sup S_4.
5. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = — \sup (-s: s\in S\}$.
6. If a set $S\subseteq\mathcal{R}$contains one of its upper bounds, show that thus upper bound is the supremum of S.
7. Let $S \subseteq\mathcal{R}$be nonempty. Show that $u\in\mathbb{R}$ is an upper bound of S if and only if the conditions $t\in\mathbb{R}$and $t > u$ imply that $t \notin S$,
8. Let $S\subseteq\mathcal{R}$be nonempty. Show that if $u = \sup S$, then for every number $n\in\mathcal{N}$the number $u — 1/n$ is not an upper bound of S, but the number $u + 1/n$ is an upper bound of S. (The converse is also true; see Exercise 2.4.3.)
9. Show that if A and B are bounded subsets of $\mathcal{R}$, then $A\cup B$ is a bounded set. Show that $\sup(A\cup B) — \sup\{\sup A, \sup B\}$.
10. Let S be a bounded set in $\mathcal{R}$and let S_0 be a nonempty subset of S. Show that inf $S\leq\inf S_0\leq\sup S_0\leq \sup S$.
11. Let $S\subseteq\mathcal{R}$and suppose that $s^* := \sup S$ belongs to S". If $u\notin S$, show that $\sup(S\cup|u|) = \text{Sup}\{s^*, u\}$.

12. Show that a nonempty finite set $S \subseteq E$ contains its supremum. [Hint: Use Mathematical Induction and the preceding exercise.]
13. Show that the assertions (1) and (1') before Lemma 2.3.3 are equivalent.
14. Show that the assertions'(2), (2'), (2''), and (2''') before Lemma 2.3.3 are equivalent.
15. Write out the details of the proof of Lemma 2.3.3.

7.0 REFERENCES/TEXTBOOK

Unit 4 Applications of the Supremum Property

1.0 INTRODUCTION

We will now discuss how to work with suprema and infima. We willalso give some very important applications of these concepts to derive fundamental properties of $\mathbb{R}$. We begin with examples that illustrate useful techniques in applying the ideas of supremum and infimum.

2.0 OBJECTIVES

At the end of the unit, readers should be able to

(i) apply the concept of "Completeness" properties to derive several fundamental results concerning $\mathcal{R}$

(ii) understand the conception of Archimedean Property, the existence of square root and the density of rational numbers in $\mathbb{R}$

3.0 MAIN CONTENT

2.4.1 Example (a) It is an important fact that taking suprema and infima of sets is compatible with the algebraic properties of $\mathbb{R}$. As an example, we present here the compatibility of taking suprema and addition.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := (a + s : s \in S\}$. We will prove that

$$\sup(a+S) = a + \sup S.$$

If we let $u := \sup S$, then $x \leq u$ for all $x \in S$, so that $a + x \leq a + u$. Therefore, $a + u$ is an upper bound for the set $a + S$; consequently, we have $\sup(a+S) \leq a + u$.

Now if u is *any* upper bound of the set $a + S$, then $a + x \leq v$ for all $x \in S$. Consequently $x \leq v — a$ for all $x \in S$, so that $v — a$ is an upper bound of S. Therefore, $u = \sup S \leq v — a$, which gives us $a + u < v$. Since v is any upper bound of $a + S$, we can replace v by $\sup(a + S)$ to get $a + u \leq \sup(a + S)$.

Combining these inequalities,we conclude that

$$\sup(a+ S) = a + u = a + \sup S.$$

For similar relationships between the suprema and infima of sets and the operations of addition and multiplication, see the exercises.

(b) If the suprema or infima of two sets are involved, it is often necessary to establish results in two stages, working with one set at a time. Here is an example, Suppose that A and B are nonempty subsets of ℝ that satisfy the property:

$$a \leq b \quad \text{for all } a \in A \text{ and all } b \in B.$$

We will prove that

$$\sup A \leq \inf B .$$

For, given $b \in B$, we have $a \leq b$ for all $a \in A$.
-This means that b is an upper bound of A, so that $\sup A \leq b$. Next, since the last inequality holds for all $b \in B$, we see that the number sup A is a lower bound for the set B. Therefore, we conclude that $\sup A \leq \inf B$. v□

Functions

The idea of upper bound and lower bound is applied to functions by considering the range of a function. Given a function $f : D \to \mathcal{R}$, we say that f is **bounded above** if the set $f(D) = \{f(x) : x \in D)$ is bounded above in $\mathcal{R}$; that is, there exists $B \in \mathcal{R}$ such that/(;c) < B for all $x \in D$. Similarly, the function / is **bounded below** if the set $f(D)$ is bounded below. We say that f is **bounded** if it is bounded above and below; this is equivalent to saying that there exists $B \in \mathcal{R}$ such that $|f(x)| \leq B$ for all $x \in D$.

The following example illustrates how to work with suprema and infima of functions.

2.4.2 **Example** Suppose that / and g are real-valued functions with common domain D $\subseteq \mathcal{R}$. We assume that f and g are bounded,

(a) If $f(x) \leq g(x)$ for all $x \in D$, then $\sup f(D) \leq \sup g(D)$, which is sometimes written:

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(D)$$

We first note that $f(x) \leq g(x) \leq \sup g(D)$, which implies that the number $\sup g(D)$ is an upper bound for f D). Therefore, $\sup f(D) \leq \sup g(D)$.

(b) We note that the hypothesis $f(x) \leq g(x)$ for all $x \in D$ *in* part (a) does not imply any relation between sup $f(D)$ and inf $g(D)$,

For example, if $f(x) := x^2$ and $g(x) := x$ with $D = \{x : 0 \leq x \leq 1\}$, then $f(x) \leq g(x)$ for all $x \in D$. However, we see that $\sup f(D) =$ l and $\inf g(D) = 0$. Since $\sup g(D) = 1$, the conclusion of (a) holds.

(c) If $f(A) \leq g(y)$ for all $x, y \in D$, then we may conclude that $\sup f(D) \leq \inf g(D)$, which we may write as:

$$\sup_{x \in D} f(x) \leq \text{mf}_{y \in D}\, g(y).$$

(Note that the functions in (b) do not satisfy this hypothesis.)
The proof proceeds in two stages as in Example 2.4. l(b). The reader should write out the details of the argument,

Further relationships between suprema and infima of functions are given in the exercises.

The Archimedean Property

Because of your familiarity with the set $\mathcal{R}$ and the customary picture of the real line, it may seem obvious that the set N of natural numbers is *not* bounded in $\mathcal{R}$. How can we prove this "obvious" fact? In fact, we cannot do so by using only the Algebraic and Order Properties given in Unit2.1. Indeed, we must use the Completeness Property of $\mathcal{R}$ as well as the Inductive Property of N(that is, if $n \in \mathcal{N}$, then $n + 1 \in \mathcal{N}$).

The absence of upper bounds for $\mathcal{N}$ means that given any real number x there exists a naturalnumber n (depending on x) such that $x < n$.

2.4.3 Archimedean Property If $x \in \mathcal{R}$, then there exists $n_x \in \mathcal{N}$ *such that* $x < n_x$

Proof. If the assertion is false, then $n \leq x$ for all $n \in \mathcal{N}$; therefore, x is an upper bound of $\mathcal{N}$. Therefore, by the Completeness Property, the nonempty set $\mathcal{N}$ has a supremum $u \in \mathcal{R}$. Subtracting 1 from u gives a number $u - 1$ which is smaller than the supremum u *of* $\mathcal{N}$. Therefore $u - 1$ is not an upper bound of $\mathcal{N}$, so there exists $m \in \mathcal{N}$ with $u - 1 < m$. Adding 1 gives $u < m + 1$, and since $m + 1 \in \mathcal{N}$, this inequality contradicts the fact that u is an upper bound of $\mathcal{N}$, Q.E.D.

2.4.4 Corollary *If* $S := \{1/n : n \in \mathcal{N}\}$. *then* $\inf S = 0$.

Proof. Since $S \neq 0$ is bounded below by 0, it has an infimum and we let $\omega := \inf S$. It is clear that $\omega \geq 0$. For any $\varepsilon > 0$, the Archimedean Property implies that there exists $n \in \mathcal{N}$ suchthat $1/\varepsilon < n$, which implies $1/n < \varepsilon$. Therefore we have

$$0 \leq \omega \leq 1/n < \varepsilon.$$

But since $\varepsilon > 0$ is arbitrary, it follows from Theorem 2.1.9 that $\omega = 0$. Q.E.D.

2.4.5 Corollary *If* $t > 0$, there *exists* $n_t \in \mathcal{N}$ such *that* $0 < 1/n_t < t$.

Proof. Since $\inf\{1/n : n \in \mathcal{N}\} = 0$ and $t > 0$, then t is not a lower bound for the set $\{1/n : n \in \mathrm{N}\}$. Thus there exists $n_t \in \mathcal{N}$ such that $0 < 1/n_t < t$. Q.E.D.

2.4.6 Corollary *If* $y > 0$, there *exists* $n_y \in \mathcal{N}$ such that $n_y - 1 \leq y < n_y$.

Proof. The Archimedean Property ensures that the subset $E_y := (m \in \mathcal{N} : y < m\}$ *of* $\mathcal{N}$ is not empty. By the Well-Ordering Property 1.2,1, E_y has a leastelement, which we denote by n_y. Then $n_y - 1$ does not belong to E_y, and hence we have $n_y - 1 \leq y < n_y$. Q.E.D.

Collectively, the Corollaries 2.4.4-2.4.6 are sometimes referred to as the Archimedean Property of $\mathcal{R}$.

The Existence of $\sqrt{2}$

The importance of the Supremum Property lies in the fact that it guarantees the existence of real numbers under certain hypotheses. We shall make use of it in this way many times. At the moment, we shall illustrate this useby proving the existence of a positive real number x

such that $x^2 = 2$; that is, the positive square root of 2. It was shown earlier (see Theorem 2,1.4) that such an x cannot be a rational number; thus, we will be deriving the existence of at least one irrational number.

2.4.7 Theorem There *exists a positive real number x such that* $x^2 = 2$.

Proof. Let $S := \{s \in \mathcal{R} : 0 \leq s,\ s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathcal{R}$, and we let $x := \sup S$. Note that $x > 1$.

We will prove that $x^2 = 2$ by ruling out the other two possibilities: $x^2 < 2$ and $x^2 > 2$.

First assume that $x^2 < 2$. We will show that this assumption contradicts the fact that $x = \sup S$ by finding an $n \in \mathcal{N}$ such that $x + 1/n \in S$, thus implying that x is not an upper bound for S. To see how to choose n, note that $1/n^2 \leq 1/n$ so that

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{1}{n}(2x + 1)$$

Hence if we can choose n so that

$$\frac{1}{n}(2x + 1) < 2 - x^2$$

then we get $(x + 1/n)^2 < x^2 + (2 - x^2) = 2$. By assumption we have $2 - x^2 > 0$, so that $(2 - x^2)/(2x + 1) > 0$. Hence the Archimedean Property (Corollary 2.4.5) can be used to obtain $n \in \mathcal{N}$ such that

$$\frac{1}{n} < \frac{2 - x^2}{2x + 1}$$

These steps can be reversed to show that for this choice of n we have $x + 1/n \in S$, which contradicts the fact that x is an upper bound of S. Therefore we cannot have $x^2 < 2$.

Now assume that $x^2 > 2$. We will show that it is then possible to find $m \in \mathcal{N}$ such that $x - 1/m$ is also an upper bound of S. contradicting the fact that x — sup S. To do this, note that

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

Hence if we can choose m so that

$$\frac{2x}{m} < x^2 - 2$$

then $(x - 1/m)^2 > x^2 - (x^2 - 2) = 2$. Now by assumption we have $x^2 - 2 > 0$, so that $(x^2 - 2)/2x > 0$. Hence, by the Archimedean Property, there exists $m \in \mathcal{N}$ such that

$$\frac{1}{m} < \frac{x^2 - 2}{2x}$$

These steps can be reversed to show that for this choice of m we have $(x - 1/m)^2 > 2$. Now if $s \in S$, then $s^2 < 2 < (x - 1/m)^2$, whence it follows from 2.1.13(a) that $s < x - 1/m$. This implies that $x - 1/m$ is an upper bound for S, which contradicts the fact that $x = \sup S$. Therefore we cannot have $x^2 > 2$.

Since the possibilities $x^2 < 2$ and $x^2 > 2$ have been excluded, we must have $x^2 = 2$.

Q.E.D.

By slightly modifying the preceding argument, the reader can show that if $a > 0$, then there is a unique $b > 0$ such that $b^2 = a$. We call b the **positive square root** of a and denote it by $b = \sqrt{a}$ or $b = a^{1/2}$. *A* slightly more complicated argument involving the binomial theorem can be formulated to establish the existence of a unique **positive nth root** of a, denoted by $\sqrt{a}$or $a^{1/n}$, for each $n \in \mathcal{N}$.

4.0 CONCLUSION

Remark If in the proof of Theorem 2.4.7 we replace the set S by the set of rational numbers $T := \{r \in \mathcal{Q}: 0 < r, r^2 < 2\}$, the argument then gives the conclusion that $y := \sup T$ satisfies $y^2 = 2$. Since we have seen in Theorem 2.1.4 that y cannot be a rational number, it follows that the set T that consists of rational numbers does not have a supremum belonging to the set $\mathcal{Q}$. Thus the ordered field $\mathcal{Q}$ of rational numbers does *not* possess the Completeness Property.

Density of Rational Numbers in $\mathcal{R}$

We now know that there exists at least one irrational real number, namely $\sqrt{a2}$. Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable (as shown in Section 1.3), while the set of irrational numbers is uncountable (see Unit 2.5). However, we next show that in spite of this apparent disparity, the set of rational numbers is "dense" in $\mathcal{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

2.4.8 The Density Theorem *If x and y are any real numbers with $x < y$, then there* exists a *rational* number $r \in \mathcal{Q}$ such *that* $x < r < y$.

Proof. It is no loss of generality (why?) to assume that $x > 0$. Since $y - x > 0$, itfollows from Corollary 2.4.5 that there exists $n \in \mathcal{N}$ such that $1/n < y - x$. Therefore,we have $nx + 1 < ny$. If we apply Corollary 2.4.6 to $nx > 0$, we obtain $m \in \mathcal{N}$ with $m - 1 < nx < m$. Therefore, $m \leq nx + 1 < ny$, whence $nx < m < ny$. Thus, the rationalnumber $r := m/n$ satisfies $x < r < y$.

Q.E.D.

5.0 SUMMARY

To round out the discussion of the interlacing of rational and irrational numbers, we have the same "betweenness property" for the set of irrational numbers.

2.4.9 Corollary *If x and y* are *real numbers with $x < y$, then there exists an irrational number z such that* $x < z <$ y.

Proof. If we apply the Density Theorem 2.4.8 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \neq 0$ (why?) such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

Then $z := r\sqrt{2}$ is irrational (why?) and satisfies $x < z < y$. Q.E.D.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 2.4

1. Show that $\sup(1 - 1/n : n \in \mathcal{N}) = 1$.

2. If $S := \{1/n - 1/m : n : m \in \mathcal{N}\}$, find inf S and sup S.

3. Let $S \subseteq \mathcal{R}$ be nonempty. Prove that if a number u in $\mathcal{R}$ has the properties; (i) for every $n \in \mathcal{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathcal{N}$ the number $u - 1/n$ is an upper bound of S, then $u = \sup S$. (This is the converse of Exercise 2.3.8.)

4. Let S be a nonempty bounded set in $\mathcal{R}$.
   (a) Let $a > 0$, and let $aS := (as: s \in S)$. Prove that
   $$\inf (aS) = a \inf S, \qquad \sup(aS) = a \sup S.$$
   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that
   $$\inf (bS) = b \sup S, \qquad \sup(bS) = b \inf S.$$

5. Let X be a nonempty set and let f: X $\to \mathcal{R}$ have bounded range in $\mathcal{R}$. If $a \in \mathcal{R}$, show that Example 2.4.1 (a) implies that
   $$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$$
   Show that we also have
   $$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}$$

6. Let A and B be bounded nonempty subsets of $\mathcal{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

7 Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathcal{R}$. Show that
   $$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$
   and that
   $$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}$$

Give examples to show that each of these inequalities can be either equalities or strict inequalities.

8. Let $X = Y := \{x \in \mathcal{R} : 0 < x < 1\}$. Define $h: X \times Y \to \mathcal{R}$ by $h(x, y) := 2x + y$.
   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; 'then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

9. Perform the computations in (a) and (b) of the preceding exercise for the function $h: X \times Y \to \mathcal{R}$ defined by
   $$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

10. Let X and Y be nonempty sets and let $h : X \times Y \to \mathcal{R}$ have bounded range in $\mathcal{R}$. Let $f : X \to R$ and $g : Y \to \mathcal{R}$ be defined by
   $$f(x) := \sup\{h(x, y) : y \in Y\}, \qquad g(y) := \inf\{h(x, y) : x \in X\}.$$
   Prove that

$$\sup\{g(y): y \in Y\} \leq \inf\{f(x) : x \in X\}$$

We sometimes express this by writing

$$\sup_y \inf_x h(x, y) \leq \inf_x \sup_y h(x, y)$$

Note that Exercises 8 and 9 show that the inequality may be either an equality or a strict inequality.

11. Let X and Y be nonempty sets and let $h: X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F: X \to \mathbb{R}$ and $G: Y \to \mathbb{R}$ be defined by

$$F(x) := \sup\{h(x,y): y \in Y\}, \quad G(y) := \sup\{h(x, y) : x \in X\}.$$

Establish the **Principle of the Iterated Suprema**:

$$\sup\{h(x,y): x \in X, y \in Y\} = \sup\{F(x): x \in X\} = \sup\{G(y): y \in Y\}$$

We sometimes express this in symbols by

$$\sup_{x,y} h(x,y) = \sup_x \sup_y h(x,y) = \sup_y \sup_x h(x,y).$$

12. Given any $x \in \mathbb{R}$, show that there exists a *unique* $n \in \mathbb{Z}$ such that $n - 1 \leq x < n$.

13. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.

14. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number y such that $y^2 = 3$.

15. Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number z such that $z^2 = a$.

16. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number u such that $u^3 = 2$.

17. Complete the proof of the Density Theorem 2.4.8 by removing the assumption that $x > 0$.

18. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru : r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

7.0 REFERENCES

Unit 5 Intervals

1.0 INTRODUCTION

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called "intervals". The notations and terminology for these special sets will be familiar from earlier courses. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the **open interval** determined by a and b is the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

The points a and b are called the **endpoints** of the interval; however, the endpoints are not included in an open interval. If both endpoints are adjoined to this open interval, then we obtain the closed interval determined by a and b; namely, the set

$[a,b]:=(x\in \mathcal{R}: a\leq x\leq b\}$.

The two **half open** (or **half closed**) intervals determined by a and b are $[a,b)$, which includes the endpoint a, and $(a,b]$, which includes the endpoint b.

Each of these four intervals is bounded and has length defined by $b-a$. If $a=b$, the corresponding open interval is the empty set $(a,a)=\emptyset$, whereas the corresponding closed interval is the singleton set $[a,a]=\{a\}$.

There are five types of unbounded intervals for which the symbols ∞ (or $+\infty$) and $-\infty$ are used as notational convenience in place of the endpoints. The **infinite open intervals** are the sets of the form

$$(a,\infty):=\{x\in\mathcal{R}: x>a\} \quad \text{and} \quad (-\infty,b):=\{x\in\mathcal{R}: x<b\}.$$

The first set has no upper bounds and the second one has no lower bounds. Adjoining endpoints gives us the **infinite dosed intervals:**

$$(a,\infty):=\{x\in\mathcal{R}: a\leq x\} \quad \text{and} \quad (-\infty,b):=\{x\in\mathcal{R}: x\leq b\}.$$

It is often convenient to think of the entire set $\mathcal{R}$ as an infinite interval; in this case, we write $(-\infty,\infty):=\mathcal{R}$. No point is an endpoint of $(-\infty,\infty)$.

Warning It must be emphasized that ∞ and $-\infty$ are *not* elements of $\mathcal{R}$, but only convenient symbols.

2.0 OBJECTIVE

At the end of this Unit, readers should be able to

(i) understand the "Nested Interval" property and its uses in proving the unaccountability of $\mathcal{R}$

(ii) understand its relation to binary and decimal representation of real numbers.

3.0 MAIN CONTENT

Characterization of Intervals

An obvious property of intervals is that if two points x,y with $x<y$ belong to an interval I, then any point lying between them also belongs to I. That is, if $x<t<y$, then the point t belongs to the same interval as x and y. In other words, if x and y belong to an interval I, then the interval $[x,y]$ is contained in I. We now show that a subset of $\mathcal{R}$ possessing this property must be an interval.

2.5.1 Characterization Theorem *If S is a subset of $\mathcal{R}$ that contains at least two points and has the property*

(1) *if $x,y\in S$ and $x<y$,* then *$[x,y]\subseteq S$, then S is an interval.*

Proof. There are four cases to consider: (i) S is bounded, (ii) S is bounded above but not below, (iii) S is bounded below but not above, and (iv) S is neither bounded above nor below.

Case (i): Let $a:=\inf S$ and $b:=\sup 5$. Then 5 c $[a,b]$ and we will show that $(a,b)\subseteq S$.

If $a < z < b$, then z is not a lower bound of S, so there exists $a \in S$ with $x < z$. Also, z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Therefore $z \in [x, y]$, so property (1) implies that $z \in S$. Since z is an arbitrary element of (a, b) we conclude that $(a, b) \subseteq S$

Now if $a \in S$ and $b \in S$, then $S = [a, b]$, (Why?) If $a \notin S$ and $b \notin S$, then $S = (a, b)$. The other possibilities lead to either $S = (a, b]$ or $S = [a, b)$.

Case(ii): Let $b := \sup S$. Then $S \subseteq (-\infty, b]$ and we will show that $(-\infty, b) \subseteq S$. For, if $z < b$, then there exist $x, y \in S$ such that $z \in [x, y] \subseteq S$. (Why?) Therefore $(-\infty, b) \subseteq S$. If $b \in S$, then $S = (-\infty, b]$, and if $b \notin S$, then $S = (-\infty, b)$.

Cases (iii) and (iv) are left as exercises. Q.E.D.

Nested Intervals

We say that a sequence of intervals $I_n, n \in \mathcal{N}$, is **nested** if the following chain of inclusions holds (see Figure 2.5. 1):

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

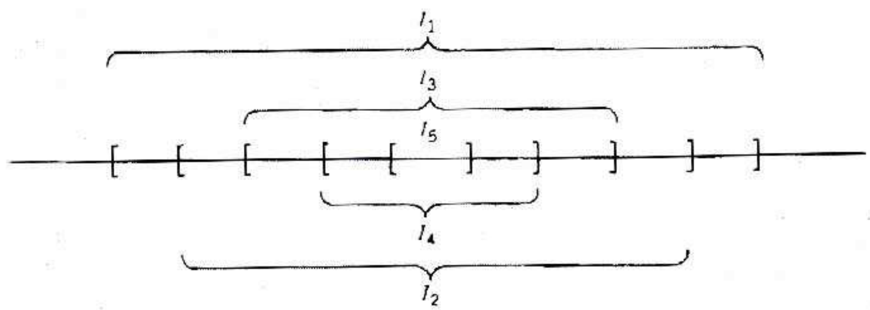


Figure 2.5.1 Nested intervals

For example, if $I_n := [0, 1/n]$ for $n \in \mathcal{N}$, then $I_n \supseteq I_{n+1}$ for each $n \in \mathcal{N}$ so that this sequence of intervals is nested. In this case, the element 0 belongs to all I_n and the Archimedean Property 2.4.5 can be used to show that 0 is the only such common point. (Prove this.) We denote this by writing $\bigcap_{n=1}^{\infty} I_n = \{0\}$,

It is important to realize that, in general, a nested sequence of intervals need *not* have a common point. For example, if $J_n := (0, 1/n)$ for $n \in \mathcal{N}$, then this sequence of intervals is nested, but there is no common point, since for every given $x > 0$, there exists (why?) $m \in \mathcal{N}$ such that $1/m < x$ so that $x \notin J_m$. Similarly, the sequence of intervals $K_n := (n, \infty)$, $n \in \mathcal{N}$, is nested but has no common point. (Why?)

However, it is an important property of $\mathcal{R}$ that every nested sequence of *closed, hounded* intervals does have a common point, as we will now prove. Notice that the completeness of $\mathcal{R}$ plays an essential role in establishing this property.

2.5.2 Nested Intervals Property If $I_n = [a_n . b_n]$, $n \in \mathcal{N}$, is a *nested sequence of closed bounded intervals, then there exists a number* $\xi \in \mathcal{R}$ *such that* $\xi \in I_n$ for *all* $n \in \mathcal{N}$.

Proof. Since the intervals are nested, we have $I_n \subseteq I_1$ for all $n \in \mathbb{N}$, so that $a_n \leq b_1$ for all $n \in \mathcal{N}$. Hence, the nonempty set $\{a_n : n \in \mathcal{N}\}$ is bounded above, and we let ξ be its supremum. Clearly $a_n \leq \xi$ for all $n \in \mathcal{N}$.

We claim also that $\xi \leq b_n$ for all n. This is established by showing that for any particular

n, thenumber b is an upper bound for the set $\{a_k . k \in \mathcal{N}\}$.We consider two cases, (i) If $n \leq k$, then since $I_n \supseteq I_k$, we have $a_k \leq b_k \leq b_n$. (ii) If $k < n$, then since $I_k \supseteq I_n$, we have $a_k \leq a_n \leq b_n$. (See Figure 2.5.2.) Thus, we conclude that $a_k \leq b_n$ for all k, so that b is an upper bound of the set $(a_k . k \in \mathcal{N}\}$. Hence, $\xi \leq b_n$ for each $n \in \mathcal{N}$. Since $a_n \leq \xi \leq b_n$ for all n, we have $\xi \in I_n$ for all $n \in \mathcal{N}$. Q.E.D.

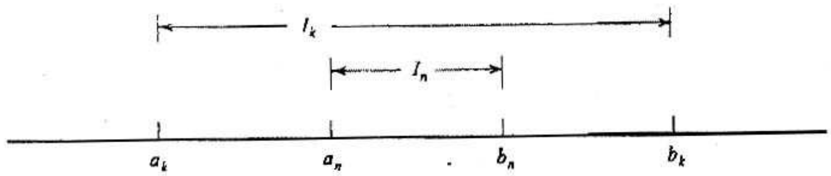


Figure 2.5.2 **If $k < n$, then $I_n \subseteq I_k$**

2.5.3 Theorem If $I_n := [a_n, b_n]$, $n \in \mathcal{N}$, is a *nested sequence ofclosed, bounded intervals such that* the lengths $b_n — a_n$ *of* I_n *satisfy*

$$\inf\{b_n - a_n : n \in \mathcal{N}\} = 0,$$

then the number ξ *contained in* l_n *for all* $n \in \mathcal{N}$ is unique.

Proof. If $\eta := \inf\{b_n : n \in \mathcal{N}\}$,then an argument similar to the proof of 2.5.2 can be used to show that $a_n \leq \eta$ for all n, and hence that $\xi \leq \eta$.In fact, it is an exercise (see Exercise 10) to show that $x. \in I_n$ for all $n \in \mathcal{N}$ if and only if $\xi \leq x \leq \eta$. If we have $\inf(b_n — a_n : n \in \mathcal{N}\} = 0$, then for any $\varepsilon > 0$, there exists an $m \in \mathcal{N}$ such that $0 \leq \eta - \xi \leq b_m — a_m < \varepsilon$. Since thisholds for all $\varepsilon > 0$, it follows from Theorem 2.1.9 that $\eta - \xi = 0$. Therefore, we conclude that $\xi = \eta$ is the only point that belongs to I_n for every $n \in \mathcal{N}$. Q.E.D.

The Uncountability of $\mathcal{R}$

The concept of a countable set was discussed in Unit 1.3 and the countability of the set $\mathcal{Q}$ of rational numbers was established there. We willnow use theNested Interval Property to prove that the set $\mathcal{R}$ is an *uncountable* set. The proof was given by Georg Cantor in 1874 in the first of his papers on infinite sets. He later published a proof that used decimal representations of real numbers, and that proof willbe given later in this section.

2.5.4 Theorem *The set $\mathcal{R}$ of real numbers isnot countable.*

Proof. We will prove that the unitinterval $I := [0, 1]$ is an uncountable set. This implies that theset $\mathcal{R}$ is an uncountable set, for if $\mathcal{R}$ were countable, then the subset / would also be countable, (See Theorem 1.3.9(a).)

The proof is by contradiction.If we assume that I is countable, then we can enumerate the set as $I = \{x_1, x_2, \cdots, x_n, \cdots\}$. We first select a closedsubinterval I_1, of I such that $x_1 \notin I_1$,. then select a closed subinterval I_2 of I_1 such that $x_2 \notin I_2$, and so on. In this waywe obtain nonempty closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that $I_n \subseteq I$ and $x_n \notin .1_n$ for all n. The Nested Intervals Property 2.5.2 implies thatthere exists a point $\xi \in I$ such that $\xi \in I_n$ for all n. Therefore $\xi \neq x_n$ for alt $n \in \mathcal{N}$, so the enumeration of I is not a complete listing of the elements of I, as claimed. Hence, I is an uncountable set. Q.E.D.

The fact that the set $\mathcal{R}$ of real numbers is uncountable can be combined with the fact that the set $\mathcal{Q}$ of rational numbers is countable to conclude that the set $\mathcal{M}\backslash\mathcal{Q}$ of irrational numbers is uncountable. Indeed, since the union of two countable sets is countable (see 1.3.7(c)), if $\mathcal{R}\backslash\mathcal{Q}$ is countable, then since $\mathcal{R} = \mathcal{Q} \cup (\mathcal{R}\backslash\mathcal{Q})$, we conclude that $\mathcal{R}$ is also a countable set, which is a contradiction. Therefore, the set of irrational numbers $\mathcal{R}\backslash\mathcal{Q}$ is an uncountable set.

Binary Representations ———————————————————————

We will digress briefly to discuss informally the binary (and decimal) representations of real numbers. It will suffice to consider real numbers between 0 and 1, since the representations for other real numbers can then be obtained by adding a positive or negative number.

The remainder of this section can be omitted on a first reading.

If $x \in [0, 1]$, we willuse a repeated bisectionprocedure to associate a sequence (a_n) of 0s and 1s as follows. If $x \neq ½$ belongs to the left subinterval $[0, {}^1/_2]$we take $a_1 := 0$, while if x belongs to the right subinterval $[{}^1/_2, l]$ we take $a_1 = 1$. If $x = ½$, then we may take a, to be either 0 or 1. In any case, we have

$$\frac{a_1}{2} \leq x \leq \frac{a_1 + 1}{2}$$

We now bisect the interval $\{{}^1/_2 a_1, {}^1/_2(a_1 + 1)\}$. If x is not the bisection point and belongs to the left subinterval we take $a_2 := 0$, and if x belongs to the right subinterval we take $a_2 := 1$. If $x =¼$ or $x =¾$, we can take a_2, to be either 0 or 1 . In any case, we have

$$\frac{a_1}{2} + \frac{a_2}{2^2} \leq x \leq \frac{a_1}{2} + \frac{a1 + 1}{2^2}$$

We continue thisbisection procedure, assigning at the nth stage the value $a_n := 0$ if x is not the bisectionpoint and lies in the left subinterval, and assigning the value $a_n := 1$if x. lies in the right subinterval. In thisway we obtain a sequence (a_n) of 0s or 1s that correspond to a nested sequence of intervals containing the point x. For each n, we have the inequality

(2)
$$\frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n + 1}{2^n}$$

If x is the bisection point at the nthstage, then $x = m/2^n$with m odd. In this case, we may choose either the left or the right subinterval; however, once this subinterval is chosen, then allsubsequent subintervals in the bisection procedure are determined. [For instance, if we choose the left subinterval so that a_n = 0, then x is the right endpoint of all subsequent subintervals, and hence a_k=1 for all $k \geq n + 1$. On the other hand, if we choose the right subinterval so that a_n = I, then x is the left endpoint of all subsequent subintervals, and hence a_k= 0 for all $k \geq n + 1$. For example, if x = ¾ , then the two possible sequences for x are 1,0,1,1,1, ⋯ and 1,1,0,0,0, ⋯.]

To summarize: *If* $x \in [0, 1]$, *then there exists a sequence* (a_n) of 0s *and* 1s *such that inequality* (2) *holds for all* $n \in \mathcal{N}$. In this case we write

(3) $$x = (.a_1a_2\cdots a_n\cdots)_2$$

andcall (3) a **binary representation** of ***x.*** This representation is unique except when $x = m/2^n$ for m odd, in which case x has the two representations

$$x = (.a_1a_2\cdots a_{n-1}1000\cdots)_2 = (.a_1a_2\cdots a_{n-1}0111\cdots)_2$$

one endingin Os and theother ending in 1s.

Conversely, each sequence of 0s and 1s is the binary representation of a unique real number in [0, 1]. The inequality corresponding to (2) determines a closed interval with length $1/2^n$ and the sequence of these intervals is nested. Therefore, Theorem 2.5.3 implies thattherexists a unique real number x satisfying (2) for every $n \in \mathcal{N}$. Consequently, x has thebinary representation $(.a_1a_2\cdots a_n\cdots)_2$

4.0 CONCLUSION

Remark The concept of binary representation is extremely important in this era of digital computers. A number is entered in a digital computer on "bits", and each bit can be put in one of two states—either it will pass current or it will not. These two states correspond to the values 1 and 0, respectively. Thus, the binary representation of a number can be stored in a digital computer on a string of bits. Of course, in actual practice, since only finitely many bits can be stored, the binary representations must be truncated. If n binary digitsare used for a number $x \in [0, 1]$, then the accuracy is at most $1/2^n$. For example, to assure four-decimal accuracy, it is necessary to use at least 15 binary digits(or 15 bits).

Decimal Representations

Decimal representations of real numbers are similar to binary representations, except that we subdivideintervals into*ten* equal subintervals instead of two.

Thus, given $x \in [0, 1]$, if we subdivide [0, 1] into ten equal subintervals, then x belongs to a subinterval $[b_1/10, (b_1 + 1)/10]$ for some integer b_1, in (0, 1, ⋯. 9}. Proceeding as in thebinary case, we obtain a sequence (b_n) of integers with $0 \leq b_n \leq 9$ for all $n \in \mathcal{N}$ such that x satisfies

(4) $$\frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} \leq x \leq \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n + 1}{10^n}$$

In thiscase we say that x has a **decimal representation** given by

$$x = b_1b_2\cdots b_n\cdots$$

If $x \geq 1$ and if $B \in \mathcal{N}$ is such that $B \leq x < B + 1$, then $x = B.b_1b_2\cdots b_n\cdots$ where the decimal representation of $x — B \in [0, 1]$ is as above. Negative numbers are treated similarly.

The fact that each decimal determines a unique real number follows from Theorem 2.5.3, since each decimal specifies a nested sequence of intervals with lengths $1/10^n$.

The decimal representation of $x \in [0, 1]$ is unique except when x is a subdivision pointat some stage,which can be seen to occur when $x = m/10^n$ for some $m, n \in \mathcal{N}$, $1 \leq m \leq 10"$. (We may also assume that m is not divisible by 10.) When x is a subdivision pointat the «thstage, one choice for $b_n$corresponds to selecting the left subinterval, which causesallsubsequent digitsto be 9, and the other choice corresponds to selectingthe right subinterval, which causes all subsequent digits to be 0. [For example, if x = ½ then x = .4999 ⋯= .5000 ⋯, and if y = 38/100 then y = .37999 ⋯= .38000 ⋯.]

Periodic Decimals

A decimal $B.b_1b_2 \cdots b_n \cdots$ is said to be **periodic** (or to be **repeating**), if there exist $k, n \in \mathcal{N}$such that$b_n = b_{n+m}$for all $n \geq k$. In this case, the block of digits $b_k b_{k+1} \cdots b_{k+m-1}$is repeated once the *kth*digitis reached. The smallest number m with this property is called theperiod of the decimal. For example, 19/88 = .2159090 ⋯90 ⋯has period m = 2with repeating block 90 starting at k = 4. A terminating decimal is a periodic decimal where the repeated block is simply the digit 0.

We will give an informal proof of the assertion: A *positive real number is rational if and only if its decimal representation is periodic*.

For, suppose that $x = p/q$ where $p, q \in \mathcal{N}$ have no common integer factors. For convenience we will also suppose that $0 < p < q$. We note that the process of "long division" of q into p gives the decimal representation of p/q. Each step in the division process produces a remainder that is an integer from 0 to q — 1. Therefore, after at most q steps, some remainder will occur a second time and, at that point, the digitsin the quotient will begin to repeat themselves in cycles. Hence, the decimal representation of such a rational number is periodic.

Conversely, if a decimal is periodic, then it represents a rational number. The idea of the proof is best illustrated by an example. Suppose that x =7.31414 ⋯ 14 ⋯ We multiply by a power of 10 to move the decimal point to the first repeating block; here obtaining $10x$ = 73,1414 ⋯.We now multiply by a power of 10 to move one block to the left of the decimal point; here getting $1000x$ = 7314.1414 ⋯. We now subtract to obtain aninteger; here getting $1000x - 10x$ = 7314 - 73 = 7241, whence$x$ = 7241/990, a rational number.

5.0 SUMMARY

Cantor's Second Proof

We will now give Cantor's second proof of the uncountability of $\mathcal{R}$. This is the elegant "diagonal" argument based on decimal representations of real numbers.

2.5.5 Theorem *The unit interval* $[0, 1] := \{x \in \mathcal{R} : 0 \leq x \leq 1\}$ *is not countable.*

Proof. The proof is by contradiction. We willusethe fact that every real number $x \in [0, 1]$ has a decimal representation $x = 0.b_1b_2b_3 \cdots$, where b_i = 0, 1 ⋯, 9. Suppose that there is an enumeration $x_1, x_2, x_3 \cdots$ of all numbers in [0, 1], which we display as:

$$x_1 = 0.b_{11}b_{12}b_{13} \cdots b_{1n} \cdots,$$
$$x_2 = 0.b_{21}b_{22}b_{23} \cdots b_{2n} \cdots,$$
$$x_3 = 0.b_{31}b_{32}b_{33} \cdots b_{3n} \cdots,$$
$$\cdots \quad \cdots$$
$$x_n = 0.b_{n1}b_{n2}b_{n3} \cdots b_{nn} \cdots.$$
$$\cdots \quad \cdots$$

We now define a real number $y := 0.y_1y_2y_3 \cdots y_n \cdots$ by setting $y_1 := 2$ if $b_{11} \geq 5$ and $y_1 := 7$ if $b_{11}, \leq 4$; in general, we let

$$y_n := \begin{cases} 2 & \text{if } b_{nn} \geq 5 \\ 7 & \text{if } b_{nn} \leq 4 \end{cases}$$

Then $y \in [0, 1]$. Note that the number y is not equal to any of the numbers with twodecimal representations, since. $y_n \neq 0, 9$ for all $n \in \mathcal{N}$. Further, since y and x_n differ inthe nth decimal place, then $y \neq x_n$ for any $n \in \mathcal{N}$. Therefore, y is not included in theenumeration of [0, 1], contradicting the hypothesis. Q.E.D.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 2.5

1. If $I := [a, b)$ and $I' := [a', b'\}$ *are*. closed intervals in $\mathcal{R}$, show that $I \subseteq I'$ if and only if $a' \leq a$ and $b \leq b'$,

2. If $S \subseteq \mathcal{R}$ is nonempty, show that S is bounded if and only if there exists a closed bounded interval I such that $S \subseteq I$.

3. If $S \subseteq \mathcal{R}$ is a nonempty bounded set, and $I_s := [\inf S, \sup S]$, show that $S \subseteq I_s$. Moreover, if J is any closed bounded interval containing S, show that $I_s \subseteq J$.
4. In the proof of Case (ii) of Theorem 2.5.1, explain why x, y exist in S.

5. Write out the details of the proof of case (iv) in Theorem 2.5.1.

6. If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a nested sequence of intervals and if $I_n = [a_n, b_n]$, show that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ and $b_1 \geq b_2 \geq \cdots \geq b_n \cdots$.
7. Let $I_n := [0, 1/n]$ for $n \in \mathcal{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = (0)$.

8. Let $J_n := (0, 1/n)$ for $n \in \mathcal{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \emptyset$.

9. Let $K_n := (n, \infty)$ for $n \in \mathcal{N}$, Prove that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

10. With the notation in the proofs of Theorems 2.5.2 and 2.5.3, show that we have $\eta \in \bigcap_{n=1}^{\infty} I_n$. Also show that $[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$
11 Show that the intervals obtained from the inequalities in (2) form a nested sequence.

12. Give the two binary representations of $^3/_8$ and $^7/_{16}$.

13. (a) Give the first four digits in the binary representation of $^1/_3$.
    (b) Give the complete binary representation of $^1/_3$.

14. Show that if $a_k, b_k \in \{0, 1, \cdots, 9\}$ and if

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_m}{10^m} \neq 0$$

then $n = m$ and $a_k = b_k$ for $k = 1, \cdots, n$.

15 Find the decimal representation of $-^2/_7$.

16. Express $^{1}/_{7}$ and $^{2}/_{19}$ as periodic decimals.

17. What rationals are represented by the periodic decimals 1.25137 ⋯137 ⋯and 35. 14653⋯ 653⋯?

7.0 REFERENCES/BIBLIOGRAPHY

MODULE 3

SEQUENCES AND SERIES

Now thatthefoundationsof thereal number system $\mathcal{R}$ have been laid,we are prepared to pursuequestions of a more analytic nature, and we will begin with a study of the convergence of sequences. Some of the early results may be familiar to the reader from calculus, but the presentation here is intended to be rigorous and will lead to certain more profound theorems than are usually discussed in earlier courses.

We willfirst introduce the meaning of theconvergence of a sequence of real numbers and establishsome basic, but useful,resultsabout convergent sequences. We then present some deeper resultsconcerning the convergence of sequences. These include the Monotone Convergence Theorem, the Bolzano-Weierstrass Theorem, and the Cauchy Criterion for convergence of sequences. It is important for the reader to learn both the theorems and how the theorems apply to special sequences.

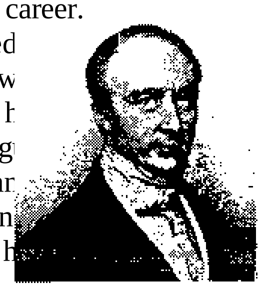
Because of the linear limitations inherent in a book it is necessary to decide where to locate the subject of infinite series. It would be reasonable to follow this chapter with a fulldiscussionof infinite series, but thiswould delay the important topics of continuity, differentiation,and integration.Consequently, we have decided to compromise. A brief introductionto infinite series is given in Unit 3.7 at the end of this Module, and a more extensivetreatment is givenlater in Module 9. Thus readers who want a fullerdiscussion of series at thispointcan move to Module 9 after completing this module.

UNIT 1

1.0 INTRODUCTION

Augustin-Louis Cauchy

Augustin-LouisCauchy (1789-1857) was horn in Paris justafter the start of the French Revolution. His father was a lawyer in the Paris police department, and the family was forced to flee during the Reign of Terror. As a result, Cauchy's early years were difficult and he developed strong anti-revolutionary and pro-royalist feelings. After returning to Pans, Cauchy's father became secretary to the newly-formed Senate, which included the mathematicians Laplace and Lagrange. They were impressed by young Cauchy's mathematical talentand helped him begin his career.



He entered the Ecole Polytechnique in 1805 and soon established exceptional mathematician. In 1815, the year royalty was restored, he w faculty of theEcolePolytechnique,buthisstrong politicalviews and h standards in mathematics often resultedin bad relationswith hiscolleag revolution of 1830, Cauchy refused to sign the new loyaltyoath and left Fran self-imposed exile. In 1838, he accepted a minor teaching post in Paris, an III reinstated him to his former position at the Ecole Polytechnique, where h death.

Cauchy was amazingly versatile and prolific, making substantial contributions to many areas, includingreal and complex analysis, number theory, differential equations, mathematical physics and probability. He published eightbooks and 789 papers, and his collected works fill 26 volumes, He was one of the most important mathematicians in the first half of the nineteenth century.

(i) understand meaning of convergence and sequence of real numbers

(ii) establish some basic, but useful, results about convergent sequences

(iii) understand some deeper results concerning the convergence of sequences Theorem.

3.0 MAIN CONTENT

A sequence in a set S is a function whose domain is the set $\mathcal{N}$ of natural numbers, and whose range is contained in the set S. In this chapter, we will he concerned with sequences in $\mathcal{R}$ and will discuss what we mean by the convergence of these sequences.

3.1.1 Definition A sequence **of real numbers (or a sequence in $\mathcal{R}$)** is a function defined on the set $\mathcal{N}=\{1, 2, \cdots\}$ of natural numbers whose range is contained in the set $\mathcal{R}$ of realnumbers.

In other words, a sequence in $\mathcal{R}$ assigns to each natural number $n = 1, 2, \cdots$ a uniquely determined real number. If $X : \mathcal{N}\to\mathcal{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n rather than using the function notation $X(n)$. The values x_n are also called the **terms** or the **elements** of the sequence. We will denote this sequence by the notations

$$X, \qquad (x_n), \qquad (x_n : n \in \mathcal{N}).$$

Of course, we willoften use other letters, such as $Y = (y_k), Z=(z_i.)$, and so on, to denote sequences.

We purposely use parentheses to emphasize that the ordering induced by the natural order of $\mathcal{N}$ is a matter of importance. Thus, we distinguish notationally between the sequence $(x_n : n \in \mathcal{N})$, whose infinitely many terms have an ordering, and theset of values $\{x_n : n \in \mathcal{N}\}$ in the range of the sequence which are not ordered. For example, thesequence $X := ((-1)^n : n \in \mathcal{N})$ has infinitely many terms that alternate between —1 and 1, whereas the set of values $\{(-1)^n : n \in \mathcal{N}\}$ is equal to theset $\{-1, 1)$, which has only two elements.

Sequences are often defined by givinga formula for the nth term x_n. Frequently, it is convenient to listthe terms of a sequence in order, stopping when the rule of formation seems evident. For example, we may define the sequence of reciprocals of the even numbers by writing

$$X := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right)$$

though a more satisfactory method is to specify the formula for the general term and write

$$X := \left(\frac{1}{2n} : n \right)$$

or more simply $X = (1/2n)$

Another way of defining a sequence is to specify the value of x_1 and give a formula for x_{n+1}. $(n \geq 1)$ in terms of x_n. More generally, we may specify x_1 and give a formula for obtaining x_{n+1} from $x_1, x_2, \cdots, x_n$. Sequences denned in this manner are said to be **inductively** (or **recursively)** defined.

3.1.2 Examples (a) If $b \in \mathcal{R}$, the sequence $B := (b, b^2, b3, \cdots)$, all of whose terms equal b, is called the **constant sequence** b. Thus the constant sequence 1 is the sequence $(1, 1, 1, \cdots)$, and the constant sequence 0 is the sequence $(0, 0, 0, \cdots)$.

(b) If $b \in \mathcal{M}$, then $B := (b^n)$ is the sequence $B = (b, b^2, b^3, \cdots, b^n, \cdots)$. In particular, if $b = {}^1/_2$ then we obtain the sequence

$$\frac{1}{2^n} : n \quad = \quad X := \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \cdots \right\}$$

(c) The sequence of $(2n : n \in \mathcal{N})$ of even natural numbers can be defined inductively by

$$x_1 := 2, \qquad x_{n+1} := x_n + 2,$$

or by the definition

$$y_1 := 2, \qquad y_{n+1} := y_1 + y_2.$$

(d) The celebrated Fibonacci sequence $F := (f_n)$ is given by the inductive definition

$$f_1 := 1. f_2 := 1, \; f_{n+1} := f_{n-1} + f_n \; (n \geq 2).$$

Thus each term pastthe second is the sum of its two immediate predecessors. The first ten terms of F are seen tobe(l,1,2,3,5,8,13,21,34,55,···).

The Limit of a Sequence

There are a number of different limit concepts in real analysis. The notion of limit of a sequence is themost basic, and it willbe thefocus of this module.

3.1.3 Definition A sequence $X = (x_n)$ in $\mathcal{R}$ is said to converge to $x \in \mathcal{R}$, or x is said to be a limit of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, theterms x_n satisfy $|x_n - x| < \varepsilon$.

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit,we say that the sequence is **divergent**.

Note The notation $K(f.)$ is used to emphasize that the choice of K depends on the value of ε. However, it is often convenient to write K insteadof $K(\varepsilon)$, In most cases, a "small" value off will usuallyrequire a "large" value of K to guarantee that the distance $|x_n - x|$ between x_n and x is less than ε for all $n \geq K = K(\varepsilon)$.

When a sequence has limit x, we will use the notation

$$\lim X = x \qquad \text{or} \qquad \lim(x_n) = x.$$

We willsometimes use the symbolism $x_n \to x$, which indicates the intuitive idea that the values x_n "approach" the number x as $n \to \infty$.

3.1.4 Uniqueness of Limits ***A sequence in*** $\mathcal{R}$ **can *have,* at most one *limit.***

Proof. Suppose that x' and x'' are both limits of (x_n). For each ε. >0 there exist K' such that $|x_n - x'| < \varepsilon/2$ for all $n \geq K'$, and there exists K'' such that $|x_n - x''| < \varepsilon/2$ for all $n \geq K''$. We let K be the larger of K' and K''. Then for $n \geq K$ we apply the Triangle Inequality to get

$$\begin{aligned} |x' - x''| &= |x' - x_n + x_n - x''| \\ &\leq |x' - x_n| + |x_n - x''| < \varepsilon|2 + \varepsilon|2 = \varepsilon|2 \end{aligned}$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $x' - x'' - 0$. Q.E.D.

For $x \in \mathbb{R}$ and $\varepsilon > 0$, recall that the ε-neighborhood of x is the set

$$V_\varepsilon(O) := [u \in \mathbb{R} : |u - x\backslash < \varepsilon\}.$$

(See Unit 2.2.) Since $u \in V_\varepsilon(x)$ is equivalent to $\backslash u - x\backslash < \varepsilon$, the definition of convergence of a sequence can be formulated in terms of neighborhoods. We give several different ways of saying that a sequence x_n converges to x in. the following theorem.

3.1.5 Theorem *Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent.*

(a) *X converges to x.*

(b) For every $\varepsilon > 0$, *there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy* $\backslash x_n - x\backslash < \varepsilon$.

(c) *For every $\varepsilon > 0$, there exists* a *natural number K such that for all $n > K$, the terms x_n satisfy* $x - \varepsilon < x_n < x + \varepsilon$.

(d) *For every ε-neighborhood $V_\varepsilon(x)$ of x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\varepsilon(x)$.*

Proof. The equivalence of (a) and (b) is just the definition. The equivalence of (b), (c), and (d) follows from the following implications:

$$|u - x| < \varepsilon \Leftrightarrow -\varepsilon < u - x < \varepsilon \Leftrightarrow \quad x - \varepsilon < u < x + \varepsilon \Leftrightarrow \quad u \in V_\varepsilon(x).$$

With the language of neighborhoods, one can describe the convergence of the sequence $X = (x_n)$ to the number x by saying: *for each ε-neighborhood $V_\varepsilon(x)$ of x, all but a finite number of terms of X belong to $V_\varepsilon(x)$.* The finite number of terms that may not belong to the ε-neighborhood are the terms $x_1, x_2, \cdots, x_{k-1}$

Remark The definition of the limit of a sequence of real numbers is used to verify that a proposed value x is indeed the limit. It does *not* provide a means for initially determining what that value of x might be. Later results will contribute to this end, but quite often it is necessary in practice to arrive at a conjectured value of the limit by direct calculation of a number of terms of the sequence. Computers can be helpful in this respect, but since they can calculate only a finite number of terms of a sequence, such computations do not in any way constitute a proof of the value of the Limit.

The following examples illustrate how the definition is applied to prove that a sequence has a particular limit. In each case, a positive ε *is* given and we are required to find a K, depending on ε, as required by the definition.

3.1.6 Examples (a) $\lim(1/n) = 0$.

If $\varepsilon > 0$ is given, then $1/\varepsilon > 0$. By the Archimedean Property 2.4.5, there is a natural number $K = K(\varepsilon)$ such that $1/K < \varepsilon$. Then, if $n \geq K$, we have $1/n \leq 1/K < \varepsilon$. Consequently, *if* $n \geq K$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, we can assert that the sequence $(1/n)$ converges to 0.

(b) $\lim(1/(n^2+1)) = 0$

Let ε>0 be given. To find K, we first note that if $n \in \mathcal{N}$, then

$$\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}$$

Now choose K such that $1/K < \varepsilon$, as in (a) above. Then $n \geq K$ implies that $1/n < \varepsilon$, and therefore

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n} < \varepsilon$$

Hence, we have shown that the limit of the sequence is *zero.*

Given $s > 0$, we want to obtain the inequality

(c) $\lim \left(\frac{3n+2}{n+1}\right) = 3$

Give ε> 0, we want to obtain the inequality

(1) $$\left|\frac{3n+2}{n+1} - 3\right| < \varepsilon$$

when n is sufficiently large. We first simplify the expression on the left:

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$$

Now if the inequality $1/n < \varepsilon$ is satisfied, then the inequality (1) holds. Thus if $1/K < \varepsilon$, then for any $n \geq K$, *we* also have $1/n < \varepsilon$ and hence (1) holds. Therefore the limit of the sequence is 3.

(d) If $0 < b < 1$, then $\lim(b^n) = 0$.

We will use elementary properties of the natural logarithm function. If ε> 0 is given, we see that

$$b^n < \varepsilon \Leftrightarrow n \ln b < \ln \varepsilon \Leftrightarrow n > \ln \varepsilon / \ln b.$$

(The last inequality is reversed because $\ln b < 0$.) Thus if we choose K to be a number such that $K > \ln \varepsilon / \ln b$, then we will have $0 < b^n < \varepsilon$ for all $n \geq K$. Thus we have $\lim (b^n) = 0$.

For example, if $b = .8$, and if $\varepsilon = .01$ is given, then we would need $K > \ln .01/\ln .8 \approx 20.6377$. Thus K=21 would be an appropriate choice for $\varepsilon = .01$.

4.0 CONCLUSION

Remark **The *K(ε)* Game** In the notion of convergence of a sequence, one way to keep in mind the connection between the ε and the K is to think of it as a game called the *K(ε)* Game. In this game, Player A asserts that a certain number x is the limit of a sequence (x_n). Player B challenges this assertion by giving Player A a specific value for ε> 0. Player A must respond to the challenge by coming up with a value of K such that $|x_n - x| < \varepsilon$ for all $n > K$. If Player A can always find a value of K that works, then he wins, and the sequence is convergent. However, if Player B can give a specific value of ε> 0 for which Player A cannot respond adequately, then Player B wins, and we conclude that the sequence does not converge to x.

In order to show that a sequence $X = (x_n)$ does *no!* converge to the number x, it is enough to produce one number $\varepsilon_0 > 0$ such that no matter what natural number K is chosen,

k 0

one can find a particular n_K satisfying $n_K \geq K$ such that $|x_n - x| \geq \varepsilon$ (This will be discussed in more detail in Unit 3.4.)

3.1.7 Example The sequence (0, 2, 0, 2, ⋯, 0, 2, ⋯) does *not* converge to the number 0.

If Player A asserts that 0 is the limit of the sequence, he will lose the $K(\varepsilon)$ Game when Player B gives him a value of $\varepsilon < 2$. To be definite, let Player B give Player A the value $\varepsilon_0 = 1$. Then no matter what value Player A chooses for K, his response will not be adequate, for Player B will respond by selecting an even number $n > K$. Then the corresponding value is $x_n = 2$ so that $|x_n - 0| = 2 > 1 = \varepsilon_0$. Thus the number 0 is not the limit of the sequence. □

4.0 CONCLUSION

Tails of Sequences

It is important to realize that the convergence (or divergence) of a sequence $K — (x_n)$ depends only on the "ultimate behavior" of the terms. By this we mean that if, for any natural number m, we drop the first m terms of the sequence, then the resulting sequence X_m converges if and only if the original sequence converges, and in this case, the limits are the same. We will state this formally after we introduce the idea of a "tail" of a sequence.

3.1.8 Definition If $X = (x_1, x_2, \cdots, x_n, \cdots)$ is a sequence of real numbers and *if* m is a given natural number, then the 'm-tail of X is the sequence

$$X_m := (x_{m+n} : n \in \mathcal{N}) = (x_{m+n}, x_{m+2}, \cdots)$$

For example, the 3-tail of the sequence $X = (2, 4, 6, 8, 10, \cdots, 2n, \cdots)$. is the sequence $X_3 = (8, 10, 12, \cdots, 2n + 6, \cdots)$.

3.1.9 Theorem *Let $X = (x_n : n \in \mathcal{N})$ be a sequence of real numbers and let $m \in$ N. Then the m-tail $X_m = (x_{m+n} : n \in \mathcal{N})$ of X converges if and only if X converges. In this case,* $\lim X_m = \lim X$.

Proof. We note that for any $p \in \mathcal{N}$, the pth term of X_m is the $(p + m)$th term of X. Similarly, if $q > m$, then the qth term of X is the $(q — m)$th term of X_m.

Assume X converges to x. Then given any $\varepsilon > 0$, if the terms of X for $n > K(\varepsilon)$ satisfy $|x_n - x| < \varepsilon$ then the terms of X_m for $k \geq K(\varepsilon) - m$ satisfy $|x_k - x| < \varepsilon$. Thus we can take $K_m(\varepsilon) = K(\varepsilon) - m$, so that X_m *also converges to* x.

Conversely, if the terms of X_m for $k \geq K_m(\varepsilon)$ satisfy $|x_k - x| < \varepsilon$, then the terms of X for $n \geq K(\varepsilon) + m$ satisfy $|x_n - x| < \varepsilon$. Thus we can take $K(\varepsilon) = K_m(\varepsilon) + m$.

Therefore, X converges to x if and only if X_m converges to x. Q.E.D

We shall sometimes say that a sequence X *ultimately* has a certain property if some tail of X has this property. For example, we say that the sequence (3, 4, 5, 5, 5, ⋯5, ⋯) is "ultimately constant". On the other hand, the sequence (3, 5, 3, 5, ⋯, 3, 5, ⋯) is not ultimately constant. The notion of convergence can be stated using this terminology: A sequence X converges to x if and only if the terms of X are ultimately in every ε-neighborhood of x. Other instances of this "ultimate terminology" will be noted below.

5.0 SUMMARY

Further Examples

In establishing that a number x is the limit of a sequence (x_n), we often try to simplifythe difference $|x_n - x|$ before considering an $\varepsilon>0$ and finding a $K(\varepsilon)$ as required by the definition of limit. This was done in some of the earlier examples. The next result is a more formal statement of this idea, and the examples that follow make use of this approach.

3.1.10 Theorem *Let* (x_n) *be a sequence of real numbers and let* $x \in \mathbb{R}$. If (a_n) *is a sequence of positive real numbers with* $\lim(a_n) = 0$ *and if for some constant* $C > 0$ and some $m \in \mathbb{N}$ we *have*

$$|x_n - x| \leq C a_n \quad \text{for all } n \geq m,$$

then it follows that $\lim(x_n) = x$.

Proof. *If* $\varepsilon>0$ is given, then since $\lim(a_n) = 0$, we know there exists $K = K(\varepsilon/C)$ such that $n \geq K$ implies

$$a_n = |a_n - 0| < \varepsilon/C.$$

Therefore it follows that if both $n \geq K$ and $n \geq m$, then

$$|x_n - x| < C a_n < C(\varepsilon/C) = \varepsilon.$$

Since $\varepsilon > 0$ *is* arbitrary, we conclude that $x = \lim(x_n)$. Q.E.D.

3.1.11 Examples (a) If $a > 0$, then $\lim\left(\frac{1}{1+na}\right) = 0$

Since $a > 0$, then $0 < na < 1 + na$, and therefore $0 < 1/(1+na) < 1/(na)$. Thus we have

$$\left|\frac{1}{1+na} - 0\right| \leq \left(\frac{1}{a}\right)\frac{1}{n} \qquad \text{for all } n \in \mathbb{N}$$

Since $\lim(1/n) = 0$, we may invoke Theorem 3.1.10 with $C = 1/a$ and $m = 1$ to infer that $\lim(1/(1+na)) = 0$.

(b) If $0 < b < 1$, then $\lim(b^n) = 0$.

This limit was obtained earlier in Example 3.1.6(d). We will give a second proof that illustrates the use of Bernoulli's Inequality (see Example 2.1.13(c)).

Since $0 < b < 1$, we can write $b = 1/(1+a)$, where $a := (1/b) - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1+a)^n \geq 1 + na$. Hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$$

Thus from Theorem 3.1.10 we conclude that $\lim(b^n) = 0$.

In particular, if $b = .8$, so that $a = .25$, and if we are given $\varepsilon = .01$, then the preceding inequality gives us $K(\varepsilon) = 4/(.01) = 400$. Comparing with Example 3.1.6(d), where we obtained $K = 25$, we see this method of estimation does not give us the "best" value of K. However, for the purpose of establishing the limit, the size of K is immaterial.

(c) If $c > 0$, then $\lim(c^{1/n}) = 1$.

The case $c = 1$ is trivial, since then $(c^{1/n})$ is the constant sequence $(1, 1, \cdots)$, which evidently converges to 1.

If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$$c = (1 + d_n)^n \geq 1 + nd_n \qquad \text{for} \qquad n \in \mathcal{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n < (c - 1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \ (c-1)^1/_n \qquad \text{for } n \in \text{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1 + h_n)^n} \leq \frac{1}{1 + nh_n} < \frac{1}{nh_n}$$

From which it follows that $0 < h_n < 1/nc$ for $n \in \mathcal{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \frac{h_n}{(1 + h_n)} < h_n < \frac{1}{nc}$$

so that

$$| c^{1/n} - 1| < \left(\frac{1}{c}\right) \frac{1}{n} \qquad \textit{for all } n \in \mathcal{N}$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + {}^1/_2 n(n-1)k^2_n + \cdots \geq 1 + ½^n(n-1)k^2_n$$

whence it follows that

$$n - 1 \ ½\, n\, (n-1)\, k^2_n$$

Hence $k^2_n \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 3.1

1. The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case:
   (a) $x_n := 1 + (-1)^n$, (b) $x^n := (-1)^n/n$,
   (c) $x_n := \dfrac{1}{n(n+1)}$ (d) $x := \dfrac{1}{n^2 + 2}$

2. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) $5, 7, 9, 11, \cdots$, (b) $1/2, -1/4, 1/8, -1/16, \cdots$,
   (c) $1/2, 2/3, 3/4, 4/5, \cdots$, (d) $1, 4, 9, 16, \cdots$.

3. List the first five terms of the following inductively defined sequences,
   (a) $x_n := 1.\ x_{n+1} = 3x_n + 1$,
   (b) $y_1 := 2,\quad y_{n+1} = ½\,(y_n + 2/y_n)$,
   (c) $z_1 := 1,\quad z_2 := 2,\quad z_{n+2} := (z_{n+1} + z_n)/(z_{n+1} - z_n)$,
   (d) $s_1 = 3,\quad s_2 := 5,\quad s_{n+2} := s_n + s_{n+1}$

4. For any $b \in \mathcal{R}$, prove that $\lim(b/n) = 0$.

5. Use the definition of the limit of a sequence to establish the following limits.
   (a) $\lim\left(\frac{n}{n^2+1}\right) = 0$, (b) $\lim\left(\frac{2n}{n+1}\right) = 2$,
   (c) $\lim\left(\frac{3n+1}{2n+5}\right) = 3/2$, (d) $\lim\left(\frac{n^2-1}{2n^2+3}\right) = 1/2$.

6. Show that
   (a) $\lim\left(\frac{1}{\sqrt{n+7}}\right) = 0$, (b) $\lim\left(\frac{2n}{n+2}\right) = 2$,
   (c) $\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0$, (d) $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$.

7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathcal{N}$.
   (a) Use the definition of limit to show that $\lim(x_n) = 0$
   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$

8. Prove that $\lim(x_n) = 0$ if and only $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

9. Show that if $x_n \geq 0$ for all $n \in \mathcal{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x})_n = 0$

10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

11. Show that $\lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$

12. Show that $\lim(1/3_n) = 0$

13. Let $b \in \mathcal{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$. *[Hint:* Use the Binomial Theorem as in Example 3.1.11(d).]

14. Show that $\lim((2n)^{1/n}) = 1$.

15. Show that $\lim(n^2/n!) = 0$

16. Show that $\lim(2n/n!) = 0$. *[Hint:* if $n \geq 3$, then $0 < 2^n/n! \leq 2(^2/_3)^{n-2}$.]

17. if $\lim(xn) = x > 0$, show that there exists a natural number K *such that* if $n \geq K$, *then* $\frac{1}{2}x < x_n < 2x$.

7.0 REFERENCES/FUTHER READINGS

Unit 2 Limit Theorems

1.0 INTRODUCTION

In this unit we will obtain some results that enable us to evaluate the limits of certain sequences of real numbers. These results will expand our collection of convergent sequences rather extensively. We begin by establishing an important property of convergent sequence that will be needed in this and later units.

2.0 OBJECTIVES

At the end of the unit, readers should be able to

(i) understand common Limit theorems and Proofs

(ii) show with example squeeze theorem and divergent sequence

3.0 MAIN CONTENT

3.2.1 Definition A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathcal{N}$.

Thus, the sequence (x_n) is bounded if and only if the set $\{x_n : n \in \mathcal{N}\}$ of its values is a bounded subset of $\mathcal{R}$.

3.2.2 Theorem *A convergent sequence of real numbers is bounded.*

Proof, Suppose that $\lim(x_n) = x$ and let $\varepsilon := 1$. Then there exists a natural number $K = K(1)$ such that $|x_n - x| < 1$ for all $n \geq K$. If we apply the Triangle Inequality with $n \geq K$ we obtain

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

If we set

$$M := [|x_1|, |x_2|, \cdots, |x_{k-1}|, 1 + |x|]$$

then it follows that $|x_n| \leq M$ for all $n \in \mathcal{N}$. Q.E.D.

We will now examine how the limit process interacts with the operations of addition, subtraction, multiplication, and division of sequences. If $X = (x_n)$ and $Y = (y_n)$ are sequences of real numbers, then we define their sum to be the sequence $X + Y := (x_n + y_n)$, their **difference** to be the sequence $X - Y := (x_n - y_n)$. and their **product** to be the sequence $X . Y := (x_n y_n)$. If $c \in \mathcal{R}$, we define the **multiple** of X by ***c*** to be the sequence $cX := (cx_n)$, Finally, *if* $Z = (z_n)$ is a sequence of real numbers with $z_n \neq 0$ for all $n \in \mathcal{N}$, then we define the **quotient** of X and Z to be the sequence $X/Z := (x_n / z_n)$.

For example, if X and Y are the sequences

$$X := (2, 4, 6, \cdots 2n, \cdots), \qquad Y := (^1/_1, ½, ^1/_3, \cdots, ^1/_n, \cdots)$$

then we have

$$X + Y = \left(\frac{3}{1}\frac{9}{2}\frac{19}{3}, \cdots, \frac{2n^2+1}{n}, \cdots\right),$$

$$X - Y = \left(\frac{1}{1}\frac{7}{2}\frac{17}{3}, \cdots, \frac{2n^2-1}{n} \cdots\right),$$

$$X \, . \, Y = (2, 2, 2, \cdots, 2, \cdots),$$

$$3X = (6, 12, 18, \cdots, 6n \cdots),$$

$$X/Y = (2, 8, 18, \cdots, 2n^2, \cdots)$$

We note that if Z is the sequence

$$Z := (0, 2, 0, \cdots, 1 + (-1)^n, \cdots),$$

then we can define $X + Z$, $X - Z$ and $X \, . \, Z$, but X/Z is not defined since some of the terms of Z are zero.

We now show that sequences obtained by applying these operation s to convergent sequences give rise to new sequences whose limits can be predicted.

3.2.3 Theorem (a) Let $X = (x_n)$*and* $Y = (y_n)$ *be sequences of real numbers that converge to x and y, respectively, and let* $c \in \mathcal{R}$. *Then the sequence* $X + Y$, $X - Y$, $X.Y$, *and* cX *converges to* $x + y$, $x - y$, xy, *and* cx, *respectively.*

(b) *If* $X = (x_n)$ *converges to x and* $Z = (z_n)$ *is a sequence of nonzero real numbers that converges to z and if* $z \neq 0$, *then the quotient sequence X/Z converges to x/z.*

Proof (a) To show that $\lim(x_n + y_n)| = x + y$, we need to estimate the magnitude of $|(xn + yn) - (x + y)|$. To do this we use the Triangle Inequality 2.2.3 to obtain

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y|. \end{aligned}$$

By hypothesis, if $\varepsilon > 0$ there exists a natural number K_1such that if $n \geq K_1$, then $|x_n - x| < \varepsilon/2$; also there exists a natural number K_2such that if $n \geq K_2$, then $|y_n - y| < \varepsilon/2$, Hence if $K(\varepsilon) := \sup\{K_1, K_2\}$, it follows that *if* $n \geq K(\varepsilon)$ then

$$\begin{aligned} |(x_n + y_n) - (x + y)| &\leq |x_n - x| + |y_n - y| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $X + Y = (x_n + y_n)$ converges to $x + y$.

Precisely the same argument can be used to show that $X - Y = (x_n - y_n)$ converges to $x - y$.

To show that $X \, . \, Y = (x_n y_n)$ converges to xy, we make the estimate

$$\begin{aligned} |x_n y_n - xy| &= |(x_n y_n - x_n y) + (x_n y - xy)| \\ &\leq |x_n(y_n - y)| + |(x_n - x)y| \\ &= |x_n||y_n - y| + |x_n - x||y|. \end{aligned}$$

According to Theorem 3.2.2 there exists a real number $M_1 > 0$ such that $|x_n| \leq M_1$ for all $n \in \mathcal{N}$ and we set $M := \sup\{M_1, |y|\}$. Hence we have the estimate

$$|x_n y_n - xy| \leq M|y_n - y| + M|x_n - x|.$$

From the convergence of X and Y we conclude that if $\varepsilon > 0$ is given, then there exist natural numbers K_1 and K_2 such that if $n \geq K_1$ then $|x_n - x| < \varepsilon/2M$, and if $n \geq K_2$ then $|y_n - y| < \varepsilon/2M$. Now let $K(\varepsilon) = \sup\{K_1, K_2\}$; then, if $n \geq K(\varepsilon)$ we infer that

$$\begin{aligned} |x_n y_n - xy| &\leq M|y_n - y| + M|x_n - x| \\ &< M(\varepsilon/2M) + M(\varepsilon/2M) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves that the sequence $X.Y = (x_n y_n)$ converges to xy.

The fact that $cX = (cx_n)$ converges to ex can be proved in the same way; it can also be deduced by taking Y to be the constant sequence $(c, c, c, \cdots)$. We leave the details to the reader.

(b) We next show that if $Z = (z_n)$ is a sequence of nonzero numbers that converges to a nonzero limit z, then die sequence $(1/z_n)$ of reciprocals converges to $1/z$. First let $\alpha := \frac{1}{2}|z|$ so that $\alpha > 0$. Since $\lim(z_n) = z$, there exists a natural number K_1 such that if $n > K_1$ then $|z_n - z| < \alpha$. It follows from Corollary 2.2.4(a) of the Triangle Inequality that $-\alpha \leq -|z_n - z| \leq |z_n| - |z|$ for $n \geq K_1$, whence it follows that $\frac{1}{2}|z| = |z| - \alpha \leq |z_n|$ for $n \geq K_1$. Therefore $1/|z_n| \leq 2/|z|$ for $n \geq K_1$ so we have the estimate

$$\begin{aligned} \left| \frac{1}{z} - \frac{1}{z} \right| &= \left| \frac{z - z_n}{z_n z} \right| \frac{1}{=} \frac{}{|z_n z|} |z - z_n| \\ &\leq \frac{2}{|z|^2} |z - z_n| \qquad \text{For all } n \geq K_1 \end{aligned}$$

Now, if $\varepsilon > 0$ is given, there exists a natural number K_2, such that if $n \geq K_2$ then $|z_n - z| < \frac{1}{2}\varepsilon|z|^2$. Therefore, it follows that if $K(\varepsilon) = \sup\{K_1, K_2\}$, then

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| < (\varepsilon) \qquad \text{For all } n > K(\varepsilon)$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim \left(\frac{1}{z_n} \right) = \frac{1}{z}$$

The proof of (b) is now completed by taking Y to be the sequence $(1/z_n)$ and using the fact that $X.Y = (x_n/z_n)$ converges to $x(1/z) = x/z$. Q.E.D.

Some of the results of Theorem 3.2.3 can be extended, by Mathematical Induction, to a finite number of convergent sequences. For example, if $A = (a_n), B = (b_n), \cdots, Z = (z_n)$ are convergent sequences of real numbers, then their sum $A + B + \cdots + Z = (a_n + b_n + \cdots + z)$ is a convergent sequence and

(1) $\lim(a_n + b_n + \cdots + z_n) = \lim(bn) + \cdots + \lim(z_n)$

Also their product $A . B \cdots Z := (a_n b_n \cdots z_n)$ is a convergent sequence and

(2) $\lim(a_n b_n \cdots z_n) = (\lim(a_n))(\lim(b_n)) \cdots (\lim(z_n))$.

Hence, if $k \in \mathscr{N}$ and if $A = (a_n)$ is a convergent sequence, then

(3) $\lim(a^k_n) = (\lim(a_n))^k$

We leavethe proofs of these assertions to the reader.

3.2.4 Theorem *If $X = (x_n)$ is a convergent* sequence *of real numbers and if* $x_n \geq 0$ *for all* $n \in \mathcal{N}$, then $x = \lim(x_n) \geq 0$.

Proof. Suppose the conclusionis not true and that $x < 0$; then $\varepsilon := -x$ is positive. Since X converges to x, there is a natural number K such that

$$x - \varepsilon < x_n < x + \varepsilon \qquad \text{for all} \quad n \geq K$$

In particular, we have $x_K < x + \varepsilon = x + (-x) = 0$. But this contradicts the hypothesis that $x_n \geq 0$ for all $n \in \mathcal{N}$. Therefore, thiscontradiction implies that $x \geq 0$. Q.E.D.

We now give a useful result that is formally stronger than Theorem 3.2.4.

3.2.5 Theorem *If $X = (x_n)$ and $Y = (y_n)$* are *convergent sequences of real numbers and if* $x_n \leq y_n$ *for all* $n \in \mathcal{N}$, then $\lim(x_n) \leq \lim(y_n)$.

Proof. Let $z_n := y_n - x_n$ so that $Z := (z_n) = Y - X$ and $z_n \geq 0$ for all $n \in \mathcal{N}$. It follows from Theorems 3.2.4 and 3,2.3 that

$$0 \leq \lim Z = \lim(y_n) - \lim(x_n),$$

so that $\lim(x_n) \leq \lim(y_n)$. Q.E.D.

The next result asserts that if allthe terms of a convergent sequence satisfy an inequality of the form $a \leq x_n \leq b$, then the limit of the sequence satisfies the same inequality. Thus if the sequence is convergent, one may "pass to the limit" in an inequality of this type.

3.2.6 Theorem *If $X = (x_n)$ is* a *convergent sequence and if $a \leq x_n \leq b$ for all $n \in \mathcal{N}$*, then $a \leq \lim(x_n) \leq b$.

Proof. Let Y be the constant sequence $(b, b, b, \cdots)$•Theorem 3.2.5 implies that $\lim X \leq \lim Y = b$. Similarly one shows that $a \leq \lim X$. Q.E.D.

The next result asserts that if a sequence Y is squeezed between two sequences that converge to the *same limit*, then it must also converge to this limit.

3.2.7 Squeeze Theorem *Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that*

$$< \quad x_n \leq y_n \leq z_n \qquad \text{for all} \quad n \in \mathcal{N}$$

and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is *convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Proof. Let $\omega := \lim(x_n) = \lim(z_n)$. If $\varepsilon > 0$ isgiven, then it followsfrom the convergence of X and Z to ω that there exists a natural number K such that if $n \geq K$ then

$$|x_n - \omega| < \varepsilon \qquad \text{and} \qquad |z_n - \omega| < \varepsilon$$

Since the hypothesis implies that

$$x_n - \omega \le y_n - \omega \le z_n - \omega \qquad \text{for all} \quad n \in \mathcal{N}$$

it follows (why?) that

$$-\varepsilon < y_n - \omega < \varepsilon$$

for all $n \ge K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\lim(y_n) = \omega$. Q.E.D.

4.0 CONCLUSION

Remark Since any tail of a convergent sequence has the same limit, the hypotheses of Theorems 3.2.4, 3.2.5, 3.2.6, and 3.2.7 can be weakened to apply to the tail of a sequence. For example, in Theorem 3.2.4, if $X = (x_n)$ is "ultimately positive" in the sense that there exists $m \in \mathcal{N}$ such that $x_n \ge 0$ for all $n \ge m$, then the same conclusion that $x \ge 0$ will hold. Similar modifications are valid for the other theorems, as the reader should verify.

3.2.8 Examples (a) The sequence (n) is divergent.

It follows from Theorem 3.2.2 that if the sequence $X := (n)$ is convergent, then there exists a real number $M > 0$ such that $n = |n| < M$ for all $n \in \mathcal{N}$. But this violates the Archimedean Property 2.4.3.

(b) The sequence ((-!)") is divergent.

This sequence $X = ((-1)^n)$ is bounded (take $M := 1$), so we cannot invoke Theorem 3.2.2. However, assume that $a := \lim X$ exists. Let $\varepsilon := 1$ so that there exists a natural number K_1 such that

$$|(-1) - a| < 1 \qquad \text{for all} \quad n \ge K_1$$

If n is an odd natural number with $n \ge K_1$, this gives $|-1-a| < 1$, so that $-2 < a < 0$. (Why?) On the other hand, if n is an even natural number with $n \ge K_1$, this inequality gives $|1 - a| < 1$ so that $0 < a < 2$. Since a cannot satisfy both of these inequalities, the hypothesis that X is convergent leads to a contradiction. Therefore the sequence X is divergent.

(c) $\lim \left(\frac{2n+1}{n}\right) = 2$

If we let $X := (2)$ and $Y := (1/n)$, then $((2n+1)/n) = X + Y$. Hence it follows from Theorem 3.2.3(a) that $\lim(X+Y) = \lim X + \lim Y = 2 + 0 = 2$.

(d) $\lim \left(\frac{2n+1}{n+5}\right) = 2$

Since the sequences $(2n + 1)$ and $(n + 5)$ are not convergent (why?), it is not possible to use Theorem 3.2.3(b) directly. However, if we write

$$\frac{2n+1}{n+5} = \frac{2+1/n}{1+5/n}$$

we can obtain the given sequence as one to which Theorem 3,2.3(b) applies when we take $X := (2 + 1/n)$ and $Z := (1 + 5/n)$. (Check that all hypotheses are satisfied.) Since $\lim X = 2$ and $\lim Z = 1 \ne 0$. we deduce that $\lim((2n+1)/(n+5)) = 2/1 = 2$.

(e) $\lim \left(\frac{2n}{n^2+1}\right) = 0$

Theorem 3.2.3(b) does not apply directly. (Why?) We note that

$$\frac{2n}{n^2+1} = \frac{2}{n+1/n}$$

but Theorem 3.2.3(b) does not apply here either, because $(n + 1/n)$ is not a convergemt sequence, (Why not?) However, if we write

$$\frac{2n}{n^2+1} = \frac{2/n}{1+1/n^2}$$

then we can apply Theorem 3.2.3(b), since $\lim(2/n) = 0$ and $\lim(1+1/n^2) = 1 \neq 0$
Therefore $\lim(2n/(n^2+1)) = 0/1 = 0$.

(f) $\lim \left(\frac{\sin n}{n}\right) = 0$

We cannot apply Theorem 3.2.3(b) directly, since the sequence (n) is not convergent [neither is the sequence $(\sin n)$]. It does not appear that a simple algebraic manipulation willenable us to reduce the sequence intoone to which Theorem 3.2.3 will apply. However, if we note that $-1 \leq \sin n \leq 1$. then it follows that

$$-\frac{1}{n} < \frac{\sin n}{n} < \frac{1}{n} \qquad \text{for all}$$

Hence we can apply the Squeeze Theorem 3.2.7 to infer that $\lim(n^{-1} \sin n) = 0$. (We note that Theorem 3.1.10 could also be applied to thissequence.)

(g) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathcal{R}$. Let p be a polynomial; for example, let

$$p(t) := a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0,$$

where $k \in \mathcal{N}$ and $a_j \in \mathcal{R}$ for $j = 0, 1, \cdots, k$. It follows from Theorem 3.2.3 that the sequence $(p(x_n))$ converges to $p(x)$. We leave the details to the reader as an exercise.

(h) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathcal{R}$. Let r be a rational function (that is, $r(t) := p(t)/q(t)$, where p and q are polynomials). Suppose that $q(x_n]$ ^ 0 for all n e N and that $q(x\}$ ^ 0. Then the sequence $(r(x_n))$ converges to $r(x) — p(x\}/q(x)$. We leave the details to the reader as an exercise.

We conclude this section with several results that will be useful in the work that follows.

5.0 SUMMARY

3.2.9 Theorem Let *the sequence* $X = (x_n)$ *converge to* x. *Then the sequence* $(|x_n|)$ *of absolute values converges to* $|x|$. *That is, if* $x = \lim(x_n)$, then $\backslash x \backslash = \lim(|x_n|)$.

Proof. It follows from the Triangle Inequality (see Corollary 2.2.4(a) that

$$\big||x_n| - |x|\big| \leq |x_n - x| \qquad \text{for all} \quad n \in \mathcal{N}$$

The convergence of $(x_n|)$ to $|x|$ is then an immediate consequence of the convergence of (x_n) to x.

3.2.10 Theorem *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and* lim $(\sqrt{x_n}) = \sqrt{x}$.

Proof. It follows from Theorem 3.2.4 that $x = \lim(x_n) \geq 0$ so the assertion makes sense. We now consider the two cases: (i) $x = 0$ and (ii) $x > 0$.

Case (i) If $x = 0$, let $\varepsilon > 0$ be given. Since $x_n \to 0$ there exists a natural number K such that if $n \geq K$ then

$$0 \leq x_n = x_n - 0 < \varepsilon^2.$$

Therefore [see Example 2.1.13(a)], $0 \leq \sqrt{x_n} < \varepsilon$ for $n \geq K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

Case (ii) If $x > 0$, then $\sqrt{x} > 0$ and we note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \left(\frac{1}{\sqrt{x}}\right)|x_n - x|$$

The convergence of $\sqrt{x_n} \to \sqrt{x}$ follows from the fact that $x_n \to x$. Q.E.D

For certain types of sequences, the following result provides a quick and easy "ratio test" for convergence. Related results can be found in the exercises.

3.2.11 Theorem *Let (x_n) b e a sequence of positive real numbers such that $L :=$* $\lim(x_{n+1}/x_n)$ *exists. If $L < 1$, then (x_n) converges and* $\lim(x_n) = 0$.

Proof. By 3.2.4 it follows that $L \geq 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$$

It follows from this (why?) that if $n \geq K$, *then*

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_k r^{n-k+1}$$

If we set $C := x_K / r^K$, we see that $0 < x_{n+1} < C r^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from 3.3.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

so that $\lim(x_{n+1}/x_n) = ½$. Since ½ < 1, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 3.2

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1. For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$

   (a) $x_n := \frac{n}{n+1}$ (b) $x_n := \frac{(-1)^n n}{n+1}$

   (c) $x_n := \frac{n^2}{n+1}$ (d) $x_n := \frac{2n^2+3}{n^2+1}$

2. Give an example of two divergent sequences X and Y such that:

   (a) their sum $X + Y$ converges, (b) their product XY converges.

3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.

4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.

5. Show that the following sequences are not convergent.

   (a) (2^n), (b) $((-1)^n n^2)$.

6. Find the limits of the following sequences:

   (a) $\lim\left((2 + 1/n)^2\right)$ (b) $\lim\left(\frac{(-1)^n}{n+2}\right)$

   (c) $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$ (d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)$

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$. show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot be used.*

8. Explain why the result in equation (3) before Theorem 3.2.4 *cannot* be used to evaluate the limit of the sequence $((1 + 1/n)^n)$.

9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that (y_n) and $(\sqrt{n}\,y_n)$ converge. Find their limits.

10. Determine the following limits,

    (a) $\lim\left((3\sqrt{n})^{1/2n}\right)$ (b) $\lim\left((n+1)^{1/\ln(n+1)}\right)$

11. If $0 < a < b$, determine $\lim \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$

12. If $a>0$, $b>0$, show that $\lim (\sqrt{(n+a)(n+b)} - n) = (a+b)/2$.

13. Use the Squeeze Theorem 3.2.7 to determine the limits of the following.
(a) (n^{1/n^2}) (b) $((n!)^{1/n^2})$

14. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim (z_n) = b$.

15. Apply Theorem 3.2:11 to the following sequences, where a, b satisfy $0 < a < 1$, $b > 1$.

16. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.
(b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)

17. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.

18. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1$, $b > 1$.
(a) $(n^2 a^n)$, (b) (b^n/n^2),
(c) $(bn/n!)$ (d) $(n!/n^n)$.

f
19. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathcal{N}$. Use this to show that $\lim(x_n) = 0$.

20. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
(b) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
(Thus, this property cannot be used as a test for convergence.)

21. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

22. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.16.)

23. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \text{mid}\{x_n, y_n, z_n\}$ is also convergent. (See Exercise 2.2.17.)

7.0 REFERENCES/FURTHER READINGS

Unit 3.3 Monotone Sequences

1.0 INTRODUCTION

Until now, we have obtained several methods of showing that a sequence $X = (x_n)$ of real numbers is convergent:

(i) We can use Definition 3.1.3 or Theorem 3.1.5 directly. This is often (but not always) difficult to do.

(ii) We can dominate $|x_n - x|$ by a multiple of the terms in a sequence (a_n) known to converge to 0, and employ Theorem 3.1.10.

(iii) We can identify X as a sequence obtained from other sequences that are known to be convergent by taking tails, algebraic combinations, absolute values, or square roots, and employ Theorems 3.1.9, 3.2.3, 3.2.9, or 3.2.10.

(iv) We can "squeeze" X between two sequences that converge to the same limit and use Theorem 3.2.7.

(v) We can use the "ratio test" of Theorem 3.2.11.

Except for (iii), all of these methods require that we already know (or at least suspect) the value of the limit, and we then verify that our suspicion is correct.

There are many instances, however, in which there is no obvious candidate for the limit of a sequence, even though a preliminary analysis may suggest that convergence is likely. In this and the next two sections,we shall establish results that can be used to show a sequence is convergent even though the value of the limit is not known. The method we introduce in thissectionis more restricted in scope than the methods we give in the next two, but it is much easier to employ. It applies to sequences that are monotone in the following sense.

2.0 OBJECTIVES

At the end of the unit readers should be able to

(i) understand monotone increasing and decreasing

(ii) state monotone Convergence Theorem

(iii) understand Euler's number with example.

3.0 MAIN CONTENT

3.3.1 Definition Let $X = (x_n)$ be a sequence of real numbers, We say that X is increasing if it satisfies the inequalities

$$x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots$$

We say that X is decreasing if it satisfies the inequalities

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots$$

We saythat X is monotone if it is either increasing or decreasing.

The following sequences are increasing:

$$(1,2,3,4,\cdots,n,\ \cdots), \qquad (1,2,2,3,3,3,\cdots),$$
$$(a,a^2,a^3,\cdots,a^n,\cdots) \quad \text{if} \quad a > 1.$$

The following sequences are decreasing:

$$(1, 1/2, 1/3, \cdots, 1/n, \cdots), \qquad (1, 1/2, 1/2^2, \cdots, 1/2^{n-1}, \cdots),$$
$$(b,\ b^2, b^3, \cdots, b^n,\ \cdots) \quad \text{if} \quad 0 < b < 1.$$

The following sequences are not monotone:

$$(+1, -1, +1, \cdots, (-1)^{n+1}, \cdots), \qquad (-1, +2, -3, \cdots, (-1)^n n \cdots)$$

The following sequences are not monotone, but they are "ultimately" monotone:

$$(7, 6, 2.1, 2.3, 4, \cdots), \qquad (-2, 0.1, 1/2, 1/3, 1/4, \cdots).$$

3.3.2 Monotone Convergence Theorem *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*

(a) *If $X = (x_n)$ is abounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathcal{N}\}.$$

(b) *If $Y = (y_n)$ is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathcal{N}\}.$$

Proof. It was seen in Theorem 3.2.2 that a convergent sequence must be bounded. Conversely, let X be a bounded monotone sequence. Then X is either increasing or decreasing.

(a) We first treat thecase where $X = (x_n)$is a bounded, increasing sequence. Since X is bounded, there exists a real number M such that $x_n \leq M$ for all $n \in \mathcal{N}$. According to the Completeness Property 2.3.6, the supremum $x^* = \sup\{x_n : n \in \mathcal{N}\}$ exists in $\mathcal{R}$; we will show that $x^* = \lim(x_n)$.

If $\varepsilon > 0$ is given, then $x^* — \varepsilon$is not an upper bound of the set $\{x_n : n \in \mathcal{N}\}$, and hence there exists a member of set $x_K$such that $x^* — \varepsilon < x_K$. The fact that X is an increasing sequence implies that $x_K \leq x_n$whenever $n \geq K$, so that

$$x^* — \varepsilon < x_k, \leq x_n \leq x^* < x^* + \varepsilon \quad \text{for all} \quad n \geq K.$$

Therefore we have

$$|x_n - x^*| < \varepsilon \text{for all} \quad n \geq K.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (x_n) converges to x^*.

(b) If $Y = (y_n)$ is a bounded decreasing sequence, then it is clear that $X := — Y = (—y_n)$is a bounded increasing sequence. It was shown in pan (a) that $\lim X =— \sup\{-y_n : n \in \mathcal{N}\}$. Now $\lim X = — \lim Y$ and also, by Exercise 2.4.4(b), we have

$$\text{Sup}\{-y_n : n \in \mathcal{N}\}$$

Therefore $\lim Y = -\lim X = \inf\{y_n : n \in \mathcal{N}\}$

The Monotone Convergence Theorem establishes the existence of the limit of a bounded monotone sequence, It also gives us a way of calculating the limit of the sequence *provided* we can evaluate the supremum in case (a), or the infimum in case (b). Sometimes it is difficult to evaluate this supremum (or infimum), but once we know that it exists, it is often possible to evaluate the limit by other methods.

3.3.3 Examples (a) $\lim(1/\sqrt{n}) = 0$.

It is possible to handle this sequence by using Theorem 3.2.10; however, we shall use the Monotone Convergence Theorem. Clearly 0 is a lower bound for the set $(1/\sqrt{n} : n \in \mathcal{N})$,and it is not difficult to show that 0 is the infimum of the set $\{1/\sqrt{n} : n \in \mathcal{N}\}$; hence $0 = \lim(1/\sqrt{n})$.

On the other hand, once we know that $X := (1/\sqrt{n})$ is bounded and decreasing, we know that it converges to some real number x. Since $X = (1/\sqrt{n})$ converges to x, it follows from Theorem 3.2.3 that $X . X = (1/n)$converges to x^2.Therefore $x^2 = 0$, whence$x = 0$.

(b) Let.$x_n := 1 + 1/2 + 1/3 + \cdots + 1/n$ for $n \in \mathcal{N}$.

Since $x_{n+1} = x_n + 1/(n+1) > x_n$,we see that (x_n) is an increasing sequence. By the Monotone Convergence Theorem 3.3.2, the question of whether the sequence is convergeni or not is reduced to the question of whether the sequence is bounded or not. Attempts to use direct numerical calculations to arrive at a conjecture concerning the possible boundedness of the sequence (x_n) lead to inconclusive frustration.

A computer run will reveal the approximate values $x_n \approx 11.4$ for $n = 50{,}000$, and $x_n \approx 12.1$ for $n = 100{,}000$. Such numerical facts may lead the casual observer to conclude that the sequence is bounded. However, the sequence is in fact divergent, which is established by noting that

$$x_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= 1 + \frac{n}{2}$$

Since (xn) is unbounded, Theorem 3.2.2 implies that it is divergent.

The terms x_n increase extremely slowly. For example, it can be shown that to achieve $x_n > 50$ would entail approximately 5.2×10^{21} additions, and a normal computer performing 400 million additions a second would require more than 400,000 years to perform the calculation (there are 31,536,000 seconds in a year). Even a supercomputer that can perform more than a trillion additions a second, would take more than 164 years to reach that modest goal.

Sequences that are defined inductively must be treated differently. If such a sequence is known to converge, then the value of the limit can sometimes be determined by using the inductive relation.

For example, suppose that convergence has been established for the sequence (x_n) defined by

$$x_1 = 2, \qquad x_{n+1} = 2 + \frac{1}{x}, \qquad n \in \mathcal{N}$$

If we let $x = \lim(x_n)$ then we also have $x = \lim(x_{n+1})$ since the 1-tail (x_{n+1}) converges to the same limit. Further, we see that $x_n \geq 2$, so that $x \neq 0$ and $x_n \neq 0$ for all $n \in \mathcal{N}$. Therefore, we may apply the limit theorems for sequences to obtain

$$x = \lim(x_{n+1}) = 2 + \frac{1}{\lim(x_n)} = 2 + \frac{1}{x}$$

Thus, the limit x is a solution of the quadratic equation $x^2 - 2x — 1 = 0$, and since x must be positive, we find that the limit of the sequence is $x = 1 + \sqrt{2}$.

Of course, the issue of convergence must not be ignored or casually assumed. For example, if we assumed the sequence (y_n) defined by $y_1 := 1$, $y_{n+1} := 2y_n + 1$ is convergent with limit y, then we would obtain y = 2y + I, so that $y = — 1$. Of course, this is absurd.

In the following examples, we employ thus method of evaluating limits, but only after carefully establishing convergence using the Monotone Convergence Theorem. Additional examples of this type will be given in Unit 3.5.

3.3.4 Examples (a) Let $Y = (y_n)$ be defined inductively by $y_1 := 1$, $y_{n+1} := ¼\,(2y_n + 3)$ for $n \geq 1$. We shall show that $\lim Y = 3/2$.

Direct calculation shows that $y_2 = 5/4$. Hence we have y, $< y_2 < 2$. We show, by Induction, that $y_n < 2$ for all $n \in \mathcal{N}$. Indeed, this is true for $n = 1, 2$. If $y_k < 2$ holds for some $k \in \mathcal{N}$, then

$$y_{k+1} = ¼\,(2y_k+3) < ¼\,(4+3) = {}^7/_4 < 2,$$

so that $y_{k+1} < 2$. Therefore $y_n < 2$ for all $n \in \mathcal{N}$.
for some k\ then 2^ + 3 <$2y_{k+l}$ + 3,

We now show, by Induction, that $y_n < y_{n+1}$ for all $n \in \mathcal{N}$. The truth of this assertion has been verified for $n = 1$. Now suppose that $y_k < y_{k+1}$ for some k; then $2y_k + 3 < 2y_k+1 + 3$, whence it follows that

$$y_{k+1} = ¼\,(2y_k+3) < ¼\,(2y_{k+1} + 3) = y_k < 2,$$

Thus $y_k < y_{k+l}$ implies that $y_{k+1} < y_{k+2}$. Therefore $y_n < y_{n+1}$ for all $n \in \mathcal{N}$.

We have shown that the sequence $Y = (y_n)$ is increasing and bounded above by 2. It follows from the Monotone Convergence Theorem that Y converges to a limit that is at most 2, In this case it is not so easy to evaluate $\lim(y_n)$ by calculating $\sup\{y_n : n \in \mathcal{N}\}$. However, there is another way to evaluate its limit. Since $y_{n+1} = ¼\,(2y_n + 3)$ for all $n \in \mathcal{N}$, the nth term in the 1-tail Y_1, of Y has a simple algebraic relation to the nth term of Y. Since, by Theorem 3. .1.9, we have y := lim Y_1 = lim Y, it therefore follows from Theorem 3.2.3 (why?) that

$$y = ¼\,(2y + 3)$$

from which it follows that $y = 3/2$.

(b) Let $Z = (z_n)$ be the sequence of real numbers defined by $z_1 := 1$, $z_{n+1} := \sqrt{2z_n}$ for $n \in \mathcal{N}$. We will show that $\lim(z_n) = 2$.

Note that $z_l = 1$ and $z_2 = \sqrt{2}$; hence $1 < z_l < z_2 < 2$. We claim that the sequence Z is increasing and bounded above by 2. To show this we will show, by Induction, that

$1 \leq z_n < z_{n+1} < 2$ for all $n \in \mathcal{N}$. This fact has been verified for $n = 1$. Suppose that it is true for $n = k$; then $2 \leq 2z_k < 2z_{k+1} < 4$, whence it follows (why?) that

$$1 < \sqrt{2} \leq z_{k+1} = \sqrt{2z_k} < z_{k+2} = \sqrt{2z_{k+1}} < \sqrt{4} = 2$$

[In thus laststep we have used Example 2.1.13(a).] Hence the validity of the inequality $1 \leq z_k < z_{k+1} < 2$ implies the validity of $1 \leq z_{k+1} < z_{k+2} < 2$. Therefore $1 \leq z_n < z_{n+1} < 2$ for all $n \in \mathcal{N}$.

Since $Z = (z_n)$ is a bounded increasing sequence, it follows from the Monotone Convergence Theorem that it converges to a number $z := \sup\{z_n\}$. It may be shown directly that $\sup\{z_n\} = 2$, so that $z = 2$. Alternatively we may use the method employed in part (a). The relation $z_{n+1} = \sqrt{2z_n}$ gives a relation between the nth term of the 1-tail Z_1 of Z and the *nth* term of Z. By Theorem 3.1.9, we have $\lim Z_1 = z = \lim Z$. Moreover, by Theorems 3.2.3 and 3.2.10, it follows that the limit z must satisfy the relation

$$z = \sqrt{2_z}$$

Hence z must satisfy the equation $z^2 = 2z$ which has theroots $z = 0, 2$. Since the terms of $z = (z_n)$ all satisfy $1 \leq z_n \leq 2$, it follows from Theorem 3.2.6 that we must have $1 \leq z \leq 2$. Therefore $z = 2$. □

The Calculation of Square Roots

We now give an application of the Monotone Convergence Theorem to the calculation of square roots of positive numbers.

3.3.5 Example Let $a > 0$; we will construct a sequence (s_n) of real numbers that converges to $\sqrt{a}$.

Let $s_1 > 0$ he arbitrary and define $s_{n+1} := \frac{1}{2}(s_n + a/s_n)$ for $n \in \mathbb{N}$. We now show that the sequence (s_n) converges to $\sqrt{a}$. (This process for calculating square roots was known in Mesopotamia before 1500 B.C.)

We first show that $s_n^2 \geq a$ for $n \geq 2$. Since s_n satisfies the quadratic equation $s_n^2 - 2s_{n+1}s_n + a = 0$, this equation has a real root. Hence the discriminant $4s_{n+1}^2 - 4a$ must be nonnegative; that is, $s_{n+1}^2 \geq a$ for $n \geq 1$.

To see that (s_n) is ultimately decreasing, we note that for $n \geq 2$ we have

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) = \frac{1}{2}\left(\frac{s_n^2 - a}{s_n}\right) = \geq 0$$

Hence, $s_{n+1} \leq s_n$ for all $n \geq 2$. The Monotone Convergence Theorem implies that $s := \lim(s_n)$ exists. Moreover, from Theorem 3.2.3, the limit s must satisfy the relation

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right)$$

whence it follows (why?) that $s = a/s$ or $s^2 = a$. Thus $s = \sqrt{a}$.

For the purposes of calculation, it is often important to have an estimate of *how rapidly* the sequence (s_n) converges to $\sqrt{a}$. As above, we have $\sqrt{a} \leq s_n$ for all $n \geq 2$, whence it follows that $a/s_n \leq \sqrt{a} \leq s_n$. Thus we have

$$0 \leq s_n - \sqrt{a} \leq s_n - a/s_n = (s_n^2 - a)/s_n \qquad \text{for } n \geq 2$$

Using this inequality we can calculate $\sqrt{a}$ to any desired degree of accuracy.

4.0 CONCLUSION

Euler's Number

We conclude this section by introducing a sequence that converges to one of the most important "transcendental" numbers in mathematics, second in importance only to π.

3.3.6 Example Let $e_n := (1 + 1/n)^n$ for $n \in \mathbb{N}$. We will now show that the sequence $E = (e_n)$ is bounded and increasing; hence it is convergent. The limit of this sequence is the famous *Euler number e*, whose approximate value is 2.718281828459045 ⋯, which is taken as the base of the "natural" logarithm.

If we apply the Binomial Theorem, we have

$$e_n = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1}\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}$$

$$+ \cdots + \frac{n(n-1)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

If we divide the powers of n into the terms in the numerators of the binomial coefficients, we get

$$e_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

Similarly we have

$$e_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right)$$
$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right)$$

Note that the expression for e_n contains $n + 1$ terms, while that for e_{n+1} contains $n + 2$ terms, Moreover, each term appearing in e_n is less than or equal to the corresponding term in e_{n+1}, and e_{n+1} has one more positive term. Therefore we have $2 < e_1 < e_2 < \cdots < e_n < e_{n+1} < \cdots$, so that the terms of E are increasing.

To show that the terms of E are bounded above, we note that if $p = 1, 2, \cdots, n$, then $(1 - p/n) < 1$. Moreover $2^{p-1} \le p!$ [see 1.2.4(e)] so that $1/p! \le 1/2^{p-1}$. Therefore, if $n > 1$, then we have

$$2 < e_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

Since it can be verified that [see 1.2.4(f)]

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}} < 1$$

we deduce that $2 < e_n < 3$ for all $n \in \mathcal{N}$. The Monotone Convergence Theorem implies that the sequence E converges to a real number that is between 2 and 3. We define the lumber e to be the limit of this sequence.

By refining our estimates we can find closer rational approximations to e, but we cannot evaluate it *exactly*, since e is an irrational number. However, it is possible to calculate e toas many decimal places as desired. The reader should use a calculator (or a computer) to evaluate e_n for "large" values of n. □

5.0 SUMMARY

Leonhard Euler

Leonhard Euler (1707-1783) was born near Basel, Switzerland. His clergy-man father hoped that his son would follow him into theministry, but when Euler entered the University of Basel at age 14, hjs mathematical talentwas

noted by Johann Bernoulli,who became his mentor. In 1727, Euler went to Russia to join Johann's son, Daniel, at the new St. Petersburg Academy. There he met and married Katharina Gsell, the daughter of a Swiss artist. During their long marriage they had 13 children, but only five survived childhood.

In 1741, Euler accepted an offer from Frederick the Great tojointhe Berlin Academy, where he stayed for 25 years. During this period he wrote landmark books on calculus and a steady stream of papers. In response to a request for instruction in science from the Princess of Anhalt-Dessau. ha wrote a multt-volume work on science that became famous under the title *Letters to a German Princess.*

In 1766, he returned to Russiaat the invitationof Catherine the Great. His eyesight had deteriorated over the years, and soon after his return to Russia he became totally blind. Incredibly, his blindness made little impact on his mathematical output, for he wrote several books and over 400 papers whileblind.He remained busy and active until the day of his death.

Euler's productivity was remarkable: he wrote textbooks on physics, algebra, calculus, real and complex analysis, analytic and differential geometry, and the calculus of variations. He also wrote hundreds of original papers, many of which won prizes. A current edition of his collected works consists of 74 volumes.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathcal{N}$. Show that (x_n) is bounded and monotone. Find the limit.
2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathcal{N}$. Show that (x_n) is bounded and monotone. Find thelimit.
3. Let$x_1 \geq 2$ and$x_{n+1} := 1 + \sqrt{x - 1}_n$ for $n \in \mathcal{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.
4. Let $x_1 := 1$ and $x_{n+1} := 1 + \sqrt{2 + x}_n$ for $n \in \mathcal{N}$. Show that (x_n) converges and find the limit.
5. Let $y_1 := \sqrt{p}$ where $p > 0$, and $y_{n+1} := \sqrt{p + y}_n$ for $n \in \mathcal{N}$. Show that (y_n) converges and find the limit.*[Hint:* One upper bound is $1 + 2\sqrt{p}$]
6. Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{p + y}_n$ for $n \in \mathcal{N}$. Show that (z_n) converges and find the limit,
7. Let$x_1 := a > 0$ and$x_{n+1} := x_n + 1/x_n$ for $n \in \mathcal{N}$. Determine if (x_n) converges or diverges.
8. Let (a_n) be an increasing sequence, (b_n) a decreasing sequence, and assume that $a_n \leq b_n$ for all$n \in \mathcal{N}$. Show that $\lim(a_n) \leq \lim(b_n)$,and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.
9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$, Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathcal{N}$such that $u = \lim(x_n)$.

10. Let (x_n) be a bounded sequence, and for each $n \in \mathcal{N}$ let $s_n := \sup\{x_k : k \geq n\}$ and $t_n := \inf\{x_1 : k \geq n\}$. Prove that (s_n) and (t_n) are monotone and convergent. Also prove that if $\lim(s_n) = \lim(t_n)$, then (x_n) is convergent. [One calls $\lim(s_n)$ the **limit superior** of (x_n), and $\lim(t_n)$ the **limit inferior** of (x_n).]

11. Establish the convergence or the divergence of the sequence (y_n), where

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2^n} \qquad \text{for } n \in \mathcal{N}$$

12. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathcal{N}$. Prove that (x_n) is increasing and bounded, and hence converges. [*Hint:* Note that if $k \geq 2$, then $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$.]

13. Establish the converges and find the limits of the following sequences.
(a) $((1 + 1/n)^{n+1})$, (b) $((1 + 1/n)^{2n})$,
(c) $\left(\left(1 + \frac{1}{n+1}\right)^n\right)$ (d) $((1 - 1/n)^n)$.

14. Use the method in Example 3.3.5 to calculate $\sqrt{2}$, correct to within 4 decimals.

15. Use the method in Example 3.3.5 to calculate $\sqrt{5}$, correct to within 5 decimals.

16. Calculate the number e_n in Example 3.3.6 for $n = 2, 4, 8, 16$.

17. Use a calculator to compute e_n for $n = 50$, $n = 100$, and $n = 1{,}000$.

7.0 REFERENCES/FURTHER READINGS

Unit 4 Subsequences and the Bolzano – Weierstrass Theorem

1.0 INTRODUCTION

Inthis sectionwewillintroducethenotionofasubsequenceofasequenceofrealnumbers. Informally, asubsequenceofasequence isaselection oftermsfrom thegivensequence suchthattheselectedtermsforma newsequence. Usuallytheselectionismadeforadefinite purpose.Forexample, subsequencesareoftenusefulinestablishingtheconvergenceorthe divergence ofthesequence. Wewillalsoprovetheimportant existencetheorem knownas theBolzano-WeierstrassTheorem, whichwillbeusedtoestablish anumber ofsignificant results.

2.0 OBJECTIVES

At the end of the unit, readers should be able to

(i) understand the notion of *a* Subsequence of a sequence of a real numbers.
(ii) prove important existence theorem known as the Bolzano-Weierstrass Theorem.

3.0 MAIN CONTENT

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots)$$

is called a **subsequence** of X.

For example, if $X := (^1/_1, ½, ^1/_3, \cdots)$, then the selection of even indexed terms produces the subsequence

$$X = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2k}, \cdots\right)$$

where $n_1 = 2, n_2 = 4, \cdots, n_k = 2k, \cdots$. Other subsequences of $X = (1/n)$ are the following:

$$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2k-1}, \cdots\right) \quad \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \cdots, \frac{1}{(2k)!}, \cdots\right)$$

The following sequences are *not* subsequences of $X = (1/n)$

$$\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \cdots\right) \quad \left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \cdots\right).$$

A tail of a sequence (see 3,1.8) is a special type of subsequence. In fact, the m-tail corresponds to the sequence of indices

$$n_1 = m + 1, n_2 = m + 2, \cdots, n_k = m + k, \cdots,$$

But, clearly, not every subsequence of a given sequence need be a tail of the sequence.

Subsequences of convergent sequences also converge to the same limit, as we now show.

3.4.2 Theorem *If a sequence $X = (x_n)$ of real numbers converges to* a real number x, *then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Proof: Let $\varepsilon > 0$ be given and let $K(\varepsilon)$ be such that if $n \geq K(\varepsilon)$, then $|x_n - x| < \varepsilon$. Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easilyproved *(by* Induction) that $n_k \geq k$. Hence, if $k \geq K(\varepsilon)$, we also have $n_k \geq k \geq K(\varepsilon)$ sothat $|x_{n_k} - x| < \varepsilon$ Therefore the subsequence (x_{n_k}) also converges to x.

Q.E.D

3.4.3 Example (a) $\lim(b^n) = 0$ if $0 < b < 1$.

We have already seen, in Example 3.1.1l(b), that if $0 < b < 1$ and if $x_n := b^n$, then it follows from Bernoulli's Inequality that $\lim(x_n) = 0$. Alternatively, we see that since $0 < b < 1$, then $x_{n+1} = b^{n+1} < b^n = x_n$ so that the sequence (x_n) is decreasing. It is also clear that $0 \leq x_n \leq 1$, so it follows from the Monotone Convergence Theorem 3.3.2 that the sequence is convergent. Let $x := \lim x_n$. Since (x_{2n}) is a subsequence of (x_n) it follows from Theorem 3.4.2 that $x = \lim(x_{2n})$. Moreover, it follows from the relation $x_{2n} = b^{2n} = (b^n)^2 = x^2_n$ and Theorem 3.2.3 that

$$x = \lim(x_{2n}) = (\lim(x_n))^2 = x^2$$

Therefore we must either have $x = 0$ or $x = 1$. Since the sequence (x_n) is decreasing and bounded above by $b < 1$, we deduce that $x = 0$.

(b) $\lim(c^{1/n}) = 1$ for $c > 1$.

This limit has been obtained in Example 3.1.11(c) for $c > 0$, using a rather ingenious argument. We give here an alternative approach for the case $c > 1$. Note that if $z_n := c^{1/n}$, then $z_n > 1$

and $z_{n+1} < z_n$ for all $n \in \mathcal{N}$. (Why?) Thus by the Monotone ConvergenceTheorem, the limit z := lim(z_n) exists. By Theorem 3.4.2, it follows that $z = \lim(z_{2n})$. In addition, it follows from the relation

$$z_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = z^{1/2}_n$$

and Theorem 3.2.10 that

$$z = \lim(z_{2n}) = (\lim(z_n))^{1/2} = z^{1/2}$$

Therefore we have $z^2 = z$ whence it follows that either z = 0 or z = 1. Since $z_n > 1$ for all $n \in \mathcal{N}$, we deduce that $z = 1$.

We leave it as an exercise to the reader to consider the case $0 < c < 1$. □

The following result is based on a careful negation of the definition of $\lim(x_n) = x$. It leads to a convenient way to establish the divergence of a sequence.

3.4.4 Theorem *Let $X = (x_n)$ be a sequence of real numbers. Then the following are equivalent:*

(i) *The sequence $X = (x_n)$ does not converge to $x \in \mathcal{R}$.*

(ii) *There, exists an $\varepsilon_0 > 0$* such that for *any* $k \in \mathcal{N}$, there exists $n_k \in \mathcal{N}$ such that $n_k \geq k$ *and* $|x_{n_k} - x| \geq \varepsilon_0$.

(iii) There exists *an* $\varepsilon_0 > 0$ and a subsequence $X' \sim (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathcal{N}$

***Proof**:* (i) ⇒ (ii) If (x_n) does not converge to x, then for some $\varepsilon_0 > 0$ it is impossible to find a natural number k such that for all $n \geq k$ the terms x_n satisfy $|x_n - x| \geq \varepsilon_0$. That is,for each $k \in \mathcal{N}$ it is *not true* that for *all* $n \geq k$ the inequality $|x_n - x| < \varepsilon_0$ holds. In other words, for each $k \in \mathcal{N}$ there exists a natural number $n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.

(ii) ⇒ (iii) Let ε_0 be as in (ii) and let $n_1 \in \mathcal{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - x| \geq \varepsilon_0$. Now let $n_2 \in \mathcal{N}$ be such that $n_2 > n_1$ and $|x_{n_2} - x| \geq \varepsilon_0$; let $n_3 \in \mathcal{N}$ be such that $n_3 > n_2$ and $|x_{n_3} - x| \geq \varepsilon_0$. Continue in this way to obtain a subsequence X' = (x_{n_k}) of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathcal{N}$.

(iii) ⇒ (i) Suppose X = (x_n) has a subsequence $X' = (x_{n_k})$ satisfying the condition in (iii). Then X cannot converge to x;for if it did, then, by Theorem 3.4.2, the subsequence X' would also converge to x. But this is impossible, since none of the terms of X' belongs to the ε_0-neighborhood of x Q.E.D.

Since all subsequences of a convergent sequence must converge to the same limit, we have part (i) in the following result. Part (ii) follows from the fact that a convergent sequence is bounded.

3.4.5 DivergenceCriteria If a sequence X = (x_n) *of* real numbers has either of the following properties, then X is *divergent.*

(i) X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ *whose limits* are not equal.

(ii) X is unbounded.

3.4.6 Examples (a) The sequence $X := ((-1^n))$is divergent.

The subsequence $X' := ((—1)^{2n}) = (1, 1, \cdots)$ converges to 1, and the subsequence $X'' := ((—1)^{2n-1}) = (-1, —1, \cdots)$ converges to —1. Therefore, we conclude from Theorem 3.4.5(i) that X is divergent.

(b) The sequence (1, ½ , 3, ¼ , ⋯) is divergent.

This is the sequence $Y = (y_n)$, where $y_n = n$ if n is odd, and $y_n — 1/n$ if n is even. It can easily be seen that Y is not bounded, Hence, by Theorem 3.4.5(ii), the sequence is divergent.

(c) The sequence $S := (\sin n)$ is divergent.

This sequence is not so easy to handle. In discussing it we must, of course, make use of elementary properties of the sine function. We recall that $\sin(\pi/6) = ½ = \sin(5\pi/6)$ and that $\sin x > ½$ for x in the interval $1_1 := \pi/6, 5\pi/6)$. Since the length of 1_1 is $5\pi/6 – \pi/6 = 2\pi/3 > 2$, there are at least two natural numbers lying inside 1_1; we let n_1 be the first such number. Similarly, for each $k \in \mathcal{N}$, $\sin x > ½$ for x in the interval.

$$1_k := (\pi/6 + 2\pi(k – 1), 5\pi/6 + 2\pi(k - 1)).$$

Since the length of l_k is greater than 2, there are at least two natural numbers lying inside 1_k; we let n_k be the first one. The subsequence $S` := (\sin n_k)$ of S obtained in this way has the property that all of its values lie in the interval $[^1/_2, 1]$.

Similarly, if $k \in \mathcal{N}$ and J_k is the interval

$$J_k := (7\pi/6 + 2\pi(k – 1), 11\pi/6 + 2\pi(k – 1).$$

then it is seen that $\sin x < - ½$ for all $x \in J_k$ and the length of J_k is greater than 2. Let m_k be the first natural number lying in J_k. Then the subsequence $S'' := (\sin m_k\}$ of S has the property that all of its values lie in the interval [- 1,- ½].

Given any real number c, it is readily seen that at least one of the subsequences S' and S'' lies entirely outside of the ½-neighborhood of c. Therefore c cannot be a limit of S. Since $c \in \mathcal{R}$, is arbitrary, we deduce that S is divergent. □

The Existence of Monotone Subsequences

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then* there *is* a subsequence of X that *is monotone*.

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \cdots, x_{m_k}, \cdots$. Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots.$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \cdots, x_{m_r}$. Let $s_1 := m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing (x_{s_k}) of X. Q.E.D.

It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing.

The Bolzano-Weierstrass Theorem

We will now use the Monotone Subsequence Theorem to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence. Because of the importance of this theorem we will also give a second proof of it based on the Nested Interval Property.

3.4.8 The Bolzano-Weierstrass Theorem *A bounded sequence of real numbers has a* convergent subsequence.

First Proof: It follows from the Monotone Subsequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{n_k})$ that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 3.3.2 that the subsequence is convergent. Q.E.D.

Second Proof. Since the set of values $\{x_n : n \in \mathcal{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.

We now bisect I_1, into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathcal{N} : n > 1\}$ into two parts:

$$A_1 := \{n \in \mathcal{N} : n > n_1, x_n \in I'_1\} \qquad B_2 := \{n \in \mathcal{N} : n > n_1, x_n \in I''_1\}$$

If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_2. (See 1.2.1.) If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I''_1$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I'_2 and I''_2 and divide the set $[n \in \mathcal{N} : n > n_2)$ into two parts:

$$A_2 := \{n \in \mathcal{N} : n > n_2, x_n \in I'_2\}, \qquad B_2 := \{n \in \mathcal{N} : n > n_2, x_n \in I''_2\}$$

If A_2 is infinite, we take $I_3 := I'_2$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, arid we take $I_3 := I''_2$ and let n_3 be the smallest natural number in B_2,

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathcal{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathcal{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \leq (b-a)/2^{k-1}$$

whence it follows that the subsequence (x_{n_k}) of X converges to ξ Q.E.D.

Theorem 3.4.8 is sometimes called the Bolzano-Weierstrass Theorem *for sequences*, because there is another version of it that deals with bounded sets in $\mathcal{R}$ (see Exercise 11.2.6).

4.0 CONCLUSION

It is readily seen that a bounded sequence can have various subsequences that converge to different limits or even diverge. For example, the sequence $((-1)^n)$ has subsequences that converge to -1, other subsequences that converge to +1, and it has subsequences that diverge.

Let X be a sequence of real numbers and let X' be a subsequence of X. Then X' is a sequence in its own right, and so it has subsequences. We note that if X'' is a subsequence of X', then it is also a subsequence of X.

5.0 SUMMARY

3.4.9 Theorem *Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have* the *property* that every convergent subsequence *of X converges to x. Then the sequence X converges to x.*

Proof. Suppose $M > 0$ is a bound for the sequence X so that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If X does not converge to x, then Theorem 3.4.4 implies that there exist $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

(1) $$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}$$

Since X' is a subsequence of X, the number M is also a bound for X'. Hence the Bolzano-Weierstrass Theorem implies that X' has a convergent subsequence X''. Since X'' is also a subsequence of X, it converges to x by hypothesis. Thus, its terms ultimately belong to the ε_0-neighborhood of x, contradicting (1). Q.E.D.

6.0 TUTOR MARKED ASSIGNMENT

Exercises for Unit 3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.

2. Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.

3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

4. Show that the following sequences are divergent.
   (a) $(1 - (-1)^n + 1/n)$, (b) $(\sin n\pi/4)$.

5. Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1$, $z_2 := y_1$, $\cdots$, $z_{2n-1} := x_n$, $z_{2n} := y_n$, $\cdots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.

6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.
   (a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \geq 3$. (See Example 3.3.6.) Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.

(b) Use the fact that the subsequence (x_{2n}) also converges to x to conclude that x –1.

7. Establishthe convergence and find the limits of the following sequences:
(a) $((1+1/n^2)^{n^2})$, (b) $((1+1/2n)^n)$,

(c) $((1+1n^2)^{2n^2})$, (d) $((1+2/n)^n)$

8. Determine the limits of the following.
(a) $((3n)^{1/2n})$, (b) $((1+1/2n)^{3n})$.

9. Suppose that every subsequence of X -=(x_n) has a subsequence that converges to 0. Show that $\lim X = 0$.

10. Let (x_n) be a bounded sequence and for each $n \in \mathcal{N}$ let s_n: = $\sup\{s_k: k \geq n\}$and S := $\inf\{s_n\}$. Show that there exists a subsequence of (s_n)that converges to S.

11. Suppose that $x_n \geq 0$ for all $n \in \mathcal{N}$ and that $\lim((-1)^n x_n)$exists. Show that (x_n) converges.

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k})such that $\lim(1/x_{n_k})$ = 0.

13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take I_1 :- [-1, 1]

14. Let (x_n)be a bounded sequence and let $s := \sup\{x_n: n \in \mathcal{N}\}$. Show dial if $s \notin \{x_n.: n \in \mathcal{N}\}$,then there is a subsequence of (x_n) that converges to s.

15. Let (I_n)be a nested sequence of closed bounded intervals. For each $n \in \mathcal{N}$, let $x_n \in I_n$. Use the Bolzano- Weierstrass Theorem to give a proof of the Nested Intervals Property 2.5.2.

16. Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.

7.0 REFERENCES/FURTHER READINGS

Unit 5 The Cauchy Criterion

1.0 INTRODUCTION

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

2.0 OBJECTIVES

At the end of the Unit, readers should be able to

(i) know the usefulness of the Monotone Convergence Theorem

(ii) understand the Cauchy Criterion and its conditionality.

3.0 MAIN CONTENT

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n .x_m satisfy $|x_n - x_m| < \varepsilon$.

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

However, we will first highlight the definition of Cauchy sequence in the following examples.

3.5.2 Examples (a) The sequence $(1/n)$ is a Cauchy sequence.
If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \geq H$, we have $1/n \leq 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if $m, H \geq H$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.
The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \geq \varepsilon_0$.
For the terms $x_n := 1 + (-1)^n$, we observe that if n is even, then $x_n - 2$ and $x_{n+1} = 0$, If we take $\varepsilon_0 = 2$, then for any H we can choose an even number $n > H$ and let $m := n + 1$ to get

$$|x_n - x_{n+1}| = 2 = \varepsilon_0$$

We conclude that (x_n) is not a Cauchy sequence. □

Remark We emphasize that to prove a sequence (x_n) is a Cauchy sequence, we may not assume a relationship between m and n, since the required inequality $|x_n - x_m| < \varepsilon$ must hold for all $n, m \geq H(\varepsilon)$. But to prove a sequence is *not* a Cauchy sequence, we may specify a relation between n and m as long as arbitrarily large values of n and m can be chosen so that $|x_n - x_m| \geq \varepsilon_0$.

Our goal is to show that the Cauchy sequences are precisely the convergent sequences. We first prove that a convergent sequence is a Cauchy sequence.

3.5.3 Lemma *If $X - (x_n)$ is a convergent* sequence *of real* numbers, then *X is a Cauchy sequence.*

Proof. If $x := \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$ then $|x_n - x_m| < \varepsilon_0/2$. Thus, if $H(\varepsilon) := K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$\begin{aligned}|x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |(x_n - x) + |x_m - x < | \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a Cauchy sequence. Q.E.D

In order to establish that a Cauchy sequence is convergent, we will need the following result. (See Theorem 3.2.2.)

3.5.4 Lemma *A Cauchy sequence of real numbers is bounded.*

Proof: Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup\{|x_1|, |x_2|, \cdots |x_{H-1}|, |x_H| + 1\},$$

then it follows that $|xn| \leq M$ for all $n \in \mathscr{N}$. Q.E.D

We now present the important Cauchy Convergence Criterion.

4.0 CONCLUSION

3.5.5 Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof: We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence, we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{nk})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\varepsilon/2)$ such that if $n, m \geq H(\varepsilon/2)$ then

(1) $$|x_n - x_m| < \varepsilon/2$$

Since the subsequence $X' = (x_{nk})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \cdots\}$ such that

$$|x_k - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_k| < \varepsilon/2 \qquad \text{for} \quad n \geq H(\varepsilon/2)$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent.

Q.E.D.

We will now give some examples of applications of the Cauchy Criterion.

3.5.6 Examples (a) Let $X = (x_n)$ be defined by

$x_1 := 1, x_2 := 2,$ and $x_n := ½ (x_{n-2} + x_{n-1})$ for $n > 2$.

Can be shown by Induction that $1 \leq x_n \leq 2$ for all $\in \mathcal{N}$. (Do So.) Some calculation shows that the sequence X is not monotone. However, since the terms are formed by averaging, it readily seen that

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}} \quad \text{for } n \in \mathcal{N}$$

Prove this by Induction.) Thus, if $m > n$, we may employ the Triangle Inequality to obtain

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \quad + |x_{m-1} - x_m| \\ &= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \quad + \frac{1}{2^{m-2}} \\ &= \frac{1}{2^{n-1}}\left[1 + \frac{1}{2} + \quad + \frac{1}{2^{m-n-1}}\right] < \frac{1}{2^{n-2}} \end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $1/2^n < \varepsilon/4$ and if $m \geq n$, then it follows that $|x_n - x_m| < \varepsilon$. Therefore, X is a Cauchy sequence in $\mathcal{R}$. By the Cauchy Criterion 3.5.5 we infer that the sequence X converges to a number x.

To evaluate the limit x, we might first "pass to the limit" in the rule of definition = ½ $(x_{n-1} + x_{n-2})$ to conclude that x must satisfy the relation $x = ½(x + x)$, which is true, but not informative. Hence we must try something else.

Since X converges to x, so does the subsequence X' with odd indices. By Induction, we reader can establish that [see 1.2.4(f)]

$$\begin{aligned} x_{2n+1} &= 1 + \frac{1}{2} + \frac{1}{2^3} + \quad + \frac{1}{2^{2n-1}} \\ &= 1 + \frac{2}{3}\left(1 - \frac{1}{4^n}\right) \end{aligned}$$

It follows from this (how?) that $x = \lim X = \lim X' = 1 + {}^2/_3 = {}^5/_3$

(b) Let $Y = (y_n)$ be the sequence of real numbers given by

$$y_1 := \frac{1}{1!}, \quad y_2 := \frac{1}{1!} - \frac{1}{2!}, \ldots, y_n := \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}, \ldots$$

Clearly, Y is not a monotone sequence. However, if $m > n$, then

$$y_m - y_n = \frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}$$

Since $2r-1 \leq r!$ [see 1.2.4(e)], it follows that if $m > n$, then (why?)

$$\begin{aligned} |y_m - y_n| &\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\ &\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}} \end{aligned}$$

Therefore, it follows that (y_n) is a Cauchy sequence. Hence it converges to a limit y. At the present moment we cannot evaluate y directly; however, passing to the limit (with respect to m) in the above inequality, we obtain.

$$|y_n - y| \leq 1/2^{n-1}$$

Hence we can calculate y to any desired accuracy by calculating the terms y_n for sufficiently large n. The reader should do this and show that y is approximately equal to 0.632120559. (The exact value of y is $1 - 1/e$.)

(c) The sequence $\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ diverges

Let $H := (h_n)$ be the sequence defined by

$$h_n := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \quad \text{for} \quad n \in \mathcal{N},$$

which was considered in 3.3.3(b). If $m > n$, then

$$h_m = h_n = \frac{1}{n\text{-}1} + \cdots + \frac{1}{m}$$

Since each of these $m - n$ terms exceeds $1/m$, then $h_m - h_n > (m - n)/m = 1 - n/m$. In particular, if $m = 2n$ we have $h_{2n} - h_n > ½$. This shows that H is not a Cauchy sequence (why?); therefore H is *not* a convergent sequence. (In terms that will be introduced in Unit 3.7, we have just proved that the "harmonic series" $\sum_{n=1}^{\infty} 1/n$ is divergent.)

□

3.5.7 Definition We say that a sequence $X = (x_n)$ of real numbers is **contractive** If there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$$

for all $n \in \mathcal{N}$. The number C is called the **constant** of the contractive sequence.

3.5.8 Theorem Every *contractive sequence is a Cauchy sequence, and therefore is convergent.*

Proof. If we successively apply the defining condition for a contractive sequence, we can work our way back to the beginning of the sequence as follows:

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n| \le C^2|x_n - x_{n-1}|$$

$$\le C^3|x_{n-1} - x_{n-2}| \le \cdots \le C^n|x_2 - x_1|$$

For $m > n$, we estimate $\backslash x_m - x_n \backslash$ by first applying the Triangle Inequality and then using the formula for the sum of a geometric progression (see 1.2.4(f)) This gives

$$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$\le (C^{m-2} + C^{m-3} + \cdots + C^{n-1})\,|x_2 - x_1|$$

$$= C^{n-1}|x_2 - x_1| \left(\frac{1 - C^{m-n}}{1 - C}\right)$$

$$\le C^{n-1}\left(\frac{1}{1 - C}\right)|x_2 - x_1|.$$

Since $0 < C < 1$, we know $\lim(C^n) = 0$ [see 3.1.11(b)]. Therefore, we infer that (x_n) is a Cauchy sequence. It now follows from the Cauchy Convergence Criterion 3.5.5. that (x_n) is a convergent sequence. Q.E.D.

5.0 SUMMARY

In the process of calculating the limit of a contractive sequence, it is often very important to have an estimate of the error at the nth stage. In the next result we give two such estimates: the first one involves the first two terms in the sequence and n; the second one involves the difference $x_n - x_{n-1}$

3.5.9 Corollary If $X := (x_n)$ *is a contractive sequence with constant* C, $0 < C < 1$, *and if* $x^* :=$ lim X, *then*

(i) $|x^* - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$

(ii) $|x^* - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$.

Proof. From the preceding proof, if $m > n$, then $|x_m - x_n| \leq (C^{n-1}/(1 - C))|x_2 - x_1|$. If we let $m \to \infty$ in this inequality, we obtain (i).

To prove (ii), recall that if $m > n$, then

$$|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$

Since it is readily established, using Induction, that

$$|x_{n+k} - x_{n+k-1}| \leq C^k |x_n - x_{n-1}|$$

we infer that

$$|x_m - x_n| \leq (C^{m-n} + \cdots + C^2 + C|x_n - x_{n-1}|$$

$$\leq \frac{C}{1-C}|x_n - x_{n-1}|.$$

We now let $m \to \infty$ in this inequality to obtain assertion (ii). Q.E.D

3.5.10 Example We are told that the cubic equation $x^3 - 7x + 2 = 0$ has a solution between 0 and 1 and we wish to approximate this solution. This can be accomplished by means of an iteration procedure as follows. We first rewrite the equation as $x = (x^3 + 2)/7$ and use this to define a sequence. We assign to x_1 an arbitrary value between 0 and 1, and then define

$$x_{n+1} := {}^1/_7\,(x_n^3 + 2) \qquad \text{for} \qquad n \in \mathcal{N}$$

Because $0 < x_1 < 1$, it follows that $0 < x_n < 1$ for all $n \in \mathcal{N}$. (Why?) Moreover, we have

$$|x_{n+2} - x_{n+1}| = |{}^1/_7\,(x_{n+1}^3 + 2) - {}^1/_7(x_n^3 + 2)| = {}^1/_7|x_{n+1}^3 - x_n^3|$$

$$= {}^1/_7|x_{n+1}^2 + x_{n+1}x_n + x_n^2||x_{n+1} - x_n| \leq {}^3/_7\,|x_{n+1} - x_n|.$$

Therefore, (x_n) is a contractive sequence and hence there exists r such that $\lim(x_n) = r$. If we pass to the limit on both sides of the equality $x_{n+1} = (x_n^3 + 2)/7$, we obtain $r = (r^3 + 2)/7$ and hence $r^3 - 7r + 2 = 0$. Thus r is a solution of the equation.

We can approximate r *by* choosing x_1, and calculating $x_2, x_3, \cdots$ successively. For example, if we take $x_1 = 0.5$, we obtain (to nine decimal places):

$$x_2 = 0.303571429, \qquad x_3 = 0.289710830,$$

x_4=0,289188016, x_5 = 0.289 169244,
x_6= 0.289168571, etc.

To estimate the accuracy, we note that $\backslash x_2 – x_1| < 0.2$. Thus, after n steps it follows from Corollary 3.5.9(i) that we are sure that $|x^* - x_n\backslash \leq 3^{n-1}/(7^{n-2}\cdot 20)$. Thus, when $n = 6$, we are sure that

$$|x^* - x_6 \leq 3^5/(7^4 \cdot 20) = 243/48020 < 0.0051.$$

Actually the approximation is substantially better than this. In fact, since $|x_6 — x_5\backslash <$ 0.0000005, it follows from 3.5.9(ii) that $\backslash x^* - x_6| \leq ¾ |x_6 - x_5| < 0.0000004$. Hence the first five decimal places of x_6 are correct. □

6.0 TUTOR MARKED ASSIGNMENT

Exercise for Unit 3.5

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. Show directly from the definition that the following are Cauchy sequences.

   (a) $\left(\frac{n+1}{n}\right)$ (b) $1 + \left(\frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

3. Show directly from the definition that the following are not Cauchy sequences.

   (a) $\left((-1)^n\right)$ (b) $\left(n + \frac{(-1)^n}{n}\right)$, (c) (In n).

4. Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and (x_ny_n) are Cauchy sequences.

5. If $x_n := \sqrt{n}$, show that (x_n) satisfies lim $|x_{n+1} – x_n| = 0$, but that is not a Cauchy sequence.

6. Lt p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies lim $|x_{n+p} – x_n| = 0$.

7. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathcal{N}$. Show that (x_n) is ultimately constant.

8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.

9. If $0 < r < 1$ and $|x_{n+1} – x_n| < r^n$ for all $n \in \mathcal{N}$, show that (x_n) is a Cauchy sequence.

10. If $x_1 < x_2$ are arbitrary real numbers and $x_n := ½ (x_{n-2} + x_{n-1})$ for $n > 2$, show that (x_n) is convergent. What is its limit?

11. If $y_1 < y_2$ are arbitrary real numbers and $y_n := {}^1/_3 y_{n-1} + {}^2/_3 y_{n-2}$ for $n – 2$, show that (y_n) is convergent. What is its limit?

12. If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \geq 1$, show that (x_n) is a contractive sequence. Find the limit

13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \geq 1$, show that (x_n) is a contractive sequence. What is its limit?

14. The polynomial equation $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r within 10^{-4}.

7.0 REFERENCES/FURTHER READINGS

Unit 6 Properly Divergent Sequence

1.0 Introduction

For certain purposes it is convenient to define what is meant for a sequence (x_n) of real numbers to "tend to $\pm\infty$".

2.0 Objectives

At the end of the Unit, readers should be able to

(i) understand what is meant for a sequence of real numbers

(ii) understand the Properly divergent sequence in the concept of monotone sequence.

3.0 Main Content

6.1 Definition Let (x_n) be a sequence of real numbers

(i) We say that (x_n) **tend to** $\pm\infty$, and write $\lim(x_n) - \pm\infty$, if for every $\alpha \in \mathcal{R}$ there exists a natural number $K(\alpha)$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$.

(ii) We say that (x_n) **tend to** $-\infty$, and write $\lim(x_n) = -\infty$, if every $\beta \in \mathcal{R}$ there exists a natural number $K(\beta)$ such that if $n \geq K(\beta)$, then $x_n < \beta$.

We say that (x_n) is a **properly divergent** in case we have either $\lim(x_n) = \pm\infty$ or $\lim(x_n) = -\infty$.

The reader should realizethat we are usingthesymbols $+\infty$and $-\infty$purely as a convenient *notation* in theabove expressions. Results that have been proved in earlier sections for conventional limits $\lim(x_n) = L$ (for $L \in \mathcal{R}$) *may nor* remain true when $\lim(x_n) = +\infty$.

3.6.2 Examples (a) $\lim(n) = +\infty$.

In fact, if $\alpha \in \mathcal{R}$is given, let$K(\alpha)$ be any natural number such that $K(\alpha) > \alpha$.

(b) $\lim(n^2) = +\infty$.

If $K(\alpha)$ is a natural number such that $K(\alpha) > \alpha$, and if $n \geq K(\alpha)$ then we have $n^2 \geq n > \alpha$

(c) If $c > 1$, then $\lim(c^n) = +\infty$.

Let $c = 1 + b$, where $b > 0$. If $\alpha \in \mathcal{R}$is given, let $K(\alpha)$ be a natural number such that $K(\alpha) > \alpha/b$. *If* $n \geq K(\alpha)$it follows from Bernoulli's Inequality that

$$c^n = (1 + b)^n \geq 1 + nb > 1 + \alpha > \alpha.$$

Therefore $\lim(c^n) = +\infty$. □

4.0 CONCLUSION

Monotone sequences are particularly simple in regard to their convergence. We have seen in the Monotone Convergence Theorem 3.3.2 that a monotone sequence is convergent if and only if it is bounded. The next result is a reformulation of that result.

3.6.3 Theorem A monotone *sequence of real numbers is properly divergent if and only if it is unbounded.*

(a) *If* (x_n) is an *unbounded increasing sequence, then* $\lim(x_n) = +\infty$.

(b) *If* (x_n) *is an unbounded decreasing sequence, then* $\lim (x_n) = -\infty$.

Proof (a) Suppose that (x_n) is an increasing sequence. We know that if (x_n) is bounded, then it is convergent. If (x_n) is unbounded, then for any $\alpha \in \mathcal{R}$ there exists $n(\alpha) \in \mathcal{N}$ such that $\alpha < x_{n(\alpha)}$. But since (x_n) is increasing, we have $\alpha < x_n$ for all $n \geq n(\alpha)$. Since α is arbitrary, it follows that $\lim(x_n) = +\infty$.

Part (b) is proved in a similar fashion, Q.E.D.

The following "comparison theorem" is frequently used in showing that a sequence is properly divergent. [In fact, we implicitly used it in Example 3.6.2(c).]

3.6.4 Theorem *Let* (x_n) *and* (y_n) *be two sequences of real numbers and suppose that*

(1) $$x_n \leq y_n \text{ for all } n \in \mathcal{N}.$$

(a) If $\lim(x_n) = +\infty$, *then* $\lim(y_n) = +\infty$.

(b) If $\lim(y_n) = -\infty$, *then* $\lim (x_n) = -\infty$.

Proof. (a) If $\lim(x_n) = +\infty$, and if $\alpha \in \mathcal{R}$ is given, then there exists a natural number $K(\alpha)$ such that if $n > K(\alpha)$, then $\alpha < x_n$. In view of (1), it follows that $\alpha < y_n$ for all $n \geq K(\alpha)$. Since α is arbitrary, it follows that $\lim(y_n) = +\infty$.

The proof of (b) is similar. Q.E.D.

Remarks (a) Theorem 3.6.4 remains true if condition (1) is ultimately true; that is, if there exists $m \in \mathcal{N}$ such that $x_n \leq y_n$ for all $n \geq m$.

(b) If condition (I) of Theorem 3.6.4 holds and if $\lim(y_n) = +\infty$, it does *not* follow that $\lim (x_n) = +\infty$. Similarly, if (1) holds and if $\lim(x_n) = -\infty$, it does *not* follow that $\lim(y_n) = +\infty$. In using Theorem 3.6.4 to show that a sequence tends to $+\infty$ [respectively, $-\infty$] we need to show that the terms of the sequence are ultimately greater [respectively, less] than or equal to the corresponding terms of a sequence that is known to tend to $+\infty$ [respectively, $-\infty$].

Since it is sometimes difficult to establish an inequality such as (1), the following "limit comparison theorem" is often more convenient to use than Theorem 3.6.4.

5.0 SUMMARY

3.6.5 Theorem *Let* (x_n) *and* (y_n) *be* two sequences *of positive real numbers and suppose that for* $L \in \mathcal{R}$, $L > 0$, we have

(2) $$\lim(x_n/y_n) = L.$$

Then $\lim(x_n) = +\infty$ *if and only if* $\lim(y_n) = +\infty$.

Proof. If (2) holds, there exists $K \in \mathcal{N}$ such that

$$\tfrac{1}{2}L < x_n/y_n < \tfrac{3}{2}L \text{ for all } n \geq K$$

Hence we have $(\tfrac{1}{2}L)\, y_n < x_n < (\tfrac{3}{2}L)\, y_n$ for all $n \geq K$. The conclusion now follows from a slight modification of Theorem 3.6.4. We leave the details to the reader. Q.E.D.

The reader can show that the conclusion need not hold if either $L = 0$ or $L = +\infty$. However, there are some partial results that can be established in these cases, as will be seen in the exercises.

6.0 TUTOR MARKED ASSIGNMENT

1. Show that if (x_n) is an unbounded sequence, then there exists a properly divergent subsequence.

2. Give examples of properly divergent sequences (x_n) and (y_n) with $y_n \neq 0$ for all $n \in \mathcal{N}$ such that:
   (a) (x_n/y_n) is convergent, (b) (x_n/y_n) is properly divergent.

3. Show that if $x_n > 0$ for all $n \in \mathcal{N}$, then $\lim(x_n) = 0$ if and only if $\lim(1/x_n) = +\infty$.

4. Establish the proper divergence of the following sequences.
   (a) $(\sqrt{n})$, (b) $(\sqrt{n+1})$,
   (c) $(\sqrt{n-1})$, (d) $(\sqrt[n]{n+1})$.

5. Is the sequence $(n \sin n)$ properly divergent?

6. Let (x_n) be properly divergent and let (y_n) be such that $\lim(x_n y_n)$ belongs to $\mathcal{R}$. Show that (y_n) converges to 0.

7. Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = 0$.
   (a) Show that if $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
   (b) Show that if (y_n) is bounded, then $\lim(x_n) = 0$.

8. Investigate the convergence or the divergence of the following sequences:
   (a) $(\sqrt{n^2+2})$, (b) $\sqrt{n/(n^2+1)})$,
   (c) $\sqrt{n^2+1}/\sqrt{n})$, (d) $(\sin \sqrt{n})$.

9. Let (x_n) and (y_n) be sequences of positive numbers Such that $\lim(x_n/y_n) = +\infty$.
   (a) Show that if $\lim(y_n) = +\infty$, then $\lim(y_n) = +\infty$.
   (b) Show chat if (x_n) is bounded, then $\lim(y_n) = 0$.

10. Show that if $\lim(a_n/n) = L$, where $L > 0$, then $\lim(a_n) = +\infty$.

7.0 REFERENCES/FURTHER READINGS

Unit 7 Introduction to Infinite Series

1.0 INTRODUCTION

We will now give a brief introduction to infinite series of real numbers. We will establish a few results here. These results will be seen to be immediate consequences of theorems we have met in thisModule.

MAIN CONTENT

In elementary texts, an infinite series is sometimes "defined" to be "an expression of the form"

(1) $x_1 + x_2 + \cdots + x_n + \cdots$.

However, this"definition" lacks clarity, sincethere is *a priori* no particular value that we can attach to this array of symbols, which calls for an *infinite* number of additions to be performed.

3.7.1 Definition If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (orsimply the series) **generated by** X is thesequence $S := (s_k)$ defined by

$$s_1 := x_1$$
$$s_2 := s_1 + x_2 \quad (= x_1 + x_2)$$
$$\cdots$$
$$s_k := s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k)$$

The numbers $x_n$are called the **terms** of the series and the numbers s. are called the **partial sums** of thisseries. If lim$S$ exists, we say that thisseries is **convergent** and call this limit the **sum** or the **value** of thisseries. If this limit does not exist, we say that the series S is **divergent**.

It is convenient to use symbols such as

(2) $$\sum (x_n) \quad \text{or} \quad \sum x_n \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

to denote both the infinite series S generated by the sequence $X = (x_n)$ and also to denote the value lim S, in case this limit exists. Thus the symbols in (2) may be regarded merely as a way of exhibiting an infinite series whose convergence or divergence is to be investigated. In practice, this double use of these notations does not lead to any confusion, provided it is understood that the convergence (or divergence) of the series must be established.

Just as a sequence may be indexed such that its first element is not x_1, but is x_0, or x_5 or x_{99}, we will denote the series having these numbers as their first element by the symbols

$$\sum_{n=0}^{\infty} x_n \quad \text{or} \quad \sum_{n=5}^{\infty} x_n \quad \text{or} \quad \sum_{n=99}^{\infty} x_n$$

It should be noted that when the first term in the series is x_N, then the first partial sum is denoted by s_N.

Warning The reader should guard against confusing the words "sequence" and "series". In nonmathematical language, these words are interchangeable; however, in mathematics, these words are not synonyms. Indeed, a series is a sequence $S = (s_k)$ obtained from a given sequence $X = (x_n)$ according to the special procedure given in Definition 3.7.1.

∞

3.7.2 Examples (a) Consider the sequence $X := (r^n)_{n=0}$ where $r \in \mathcal{R}$, which generates the **geometric series**:

(3) $$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots$$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. (See also Example 1.2.4(f).) Indeed, if $s_n := 1 + r + r^2$for $n \geq 0$, and if we multiply $s_n$by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1 - r) = 1 - r^{n+1}$$

Therefore, we have

$$s_n - \frac{1}{1-r} = -\frac{r^{n+1}}{1-r}$$

from which it follows that

$$|s_n - \frac{1}{1-r}| \leq \frac{|r|^{n+1}}{1-r}$$

Since $|r|^{n+1} \rightarrow 0$ where $|r| < 1$, it follows that the geometric series (3) converges to $1/(1 - r)$ when $|r| < 1$.

(**b**) Consider the series generated by $((-1)^n)_{n=0}^{\infty}$ that is, the series:

(4) $$\sum_{n=0}^{\infty} (-1)^n = (+1) + (-1) + (+1) + (-1) + \cdots$$

It is easily seen (by Mathematical Induction) that $s_n - 1$ if $n \geq 0$ is even and $sn = 0$ if n is odd; therefore, the sequence of partial sums is $(1,0,1,0,\cdots)$. Since this sequence is not convergent, the series (4) is divergent.

(c) Consider the series.

(5) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \frac{1}{1.2} + \frac{1}{2.3} + \cdots \frac{1}{3.4}$$

By a stroke of insight, we note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Hence, on adding these terms from $k = 1$ to $k = n$ and noting the telescoping that takes place, we obtain

$$s_n = -\frac{1}{1} \quad \frac{1}{n+1}$$

Whence it follows that $s_n \rightarrow 1$. Therefore the series (5) converges to 1.

We now present a very useful and simple *necessary* condition for the convergence of a series. It is far from being sufficient, however.

3.7.3The *nth* Term Test If the series $\sum$ x_n*converges, then* $\lim(x_n) = 0$.

Proof. By Definition 3.7.1, the convergence of $\sum x_n$ requires that $\lim(s_k)$ exists. Since $x_n = s_n - s_{n-1}$, then $\lim(x_n) = \lim(s_n) - \lim(s_{n-1}) = 0$. Q.E.D.

Since the following Cauchy Criterion is precisely a reformulation of Theorem 3.5.5, we will omit its proof.

3.7.4 Cauchy Criterion for Series *The series* $\sum x_n$ *converges if and only if for every* $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathcal{N}$ such *that if* $m > n \geq M(\varepsilon)$, *then*

(6) $$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon \ldots$$

The next result, although limited in scope, is of great importance and utility.

3.7.5 Theorem *Let* (x_n) *be a sequence of nonnegative real numbers. Then the series* $\sum x_n$ *converges if and only if the sequence* $S = (s_k)$ *of partial sums is bounded. In this case,*

$$\sum_{n=1}^{\infty} x_n = \lim(s_k) = \sup\{s_k : k \in \mathcal{N}\}$$

Proof. Since $x_n > 0$, the sequence S of partial sums is monotone increasing:

$$s_1 \leq s_2 \leq \cdots \leq s_k \leq \cdots.$$

By the Monotone Convergence Theorem 3.3.2, the sequence $S = (s_k)$ converges if and only if it is bounded, in which case its limit equals $\sup\{s_k\}$ Q.E.D.

3.7.6 Examples (a) The geometric series (3) diverges if $|r| \geq 1$.
This follows from the fact that the terms r^n do not approach 0 when $|r| \geq 1$.

(b) The **harmonic series** $\sum_{n=1}^{\infty} x_n \frac{1}{n}$ diverges

Since the terms $1/n \to 0$, we cannot use the nth Term Test 3.7.3 to establish this divergence. However, it was seen in Examples 3.3.3(b) and 3.5.6(c) that the sequence (s_n) of partial sums is not bounded. Therefore, it follows from Theorem 3.7.5 that the harmonic series is divergent.

(c) The **2-series** $\sum_{n=1}^{\infty} x_n \frac{1}{n^2}$ is convergent.

Since the partial sums are monotone, it suffices (why?) to show that some subsequence of (s_k) is bounded. If $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then

$$s_{k_2} = \frac{1}{1} + \left(\frac{1}{2^2} \quad \frac{1}{3^2}\right) < 1 + \frac{2}{2^2} = 1 + \frac{1}{2}$$

and if $k_3 := 2^3 - 1 = 7$, then we have

$$s_{k_3} = s_{k_2} + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) < s_{k_2} + \frac{4}{4^2} < 1 + \frac{1}{2} \quad \frac{1}{2^2}$$

By Mathematical Induction, we find that if $k_j := 2_j - 1$, then

$$0 < s_{k_j} < 1 + ½ + (½)^2 + \cdots + (½)^{j-1}$$

Since the term on the right is a partial sum of a geometric series with r = ½ , it is dominated by 1/(I — ½) = 2, and Theorem 3.7.5 implies mat the 2-series converges.

(d) The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

Since the argument is very similar to the special case considered in part (c), we will leave some of the details to the reader. As before, if $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $K_2 := 2^2 - 1 = 3$, then since $2^P < 3^P$, we have

$$s_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

Further, if $k_3 := 2^3 - 1$, then (how?) it is seen that

$$s_{k_3} < s_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}$$

Finally, we let $r := 1/2^{P-1}$; since $p > 1$, we have $0 < r < 1$. Using Mathematical Induction, we show that *if* $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + r + r^2 + \cdots + r^{j-1} < \frac{1}{1-r}$$

Therefore, Theorem 3.7.5 implies that the *p*-series converges when $p > 1$.

(e) The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \le 1$.

We will use the elementary inequality $n^p \le n$ when $n \in \mathcal{N}$ and $0 < p < 1$. It follows that

$$\frac{1}{n} \le \frac{1}{n^p} \quad \text{for } n \in \mathcal{N}$$

Since the partial sums of the harmonic series are not bounded, this inequality shows that the partial sums of the *p*-series are not bounded when $0 < p \le 1$. Hence the p-series diverges for these values of *p*.

(f) The **alternating harmonic series,** given by

(7) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent.

The reader should compare this series with the harmonic series in (b), which is divergent. Thus, the subtraction of some of the terms in (7) is essential if this series is to converge. Since we have

$$s_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

it is clear that the "even" subsequence (s_{2n}) is increasing. Similarly, the "odd" subsequence (s_{2n+1}) is decreasing since

$$s_{2n+1} = \frac{1}{1} - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

Since $0 < s_{2n} < s_{2n} + 1/(2n+1) = s_{2n+1} \leq 1$, both of these subsequences are bounded below by 0 and above by 1. Therefore they are both convergent and to the same value. Thusthe sequence (s_n) of partial sums converges, proving that the alternating harmonic series(7) converges. (It is far from obvious that the limit of this series is equal to In 2.) □

4.0 CONCLUSION

Comparison Tests

Our first testshows that if the terms of a nonnegative series are dominated by the corresponding terms of a *convergent series*, then the first series is convergent.

3.7.7 Comparison Test*Let* $X := (x_n)$ and $Y := (y_n)$ *be real sequences and suppose that for some* $K \in \mathcal{N}$ we have

(8) $$0 \leq x_n \leq y_n \qquad \text{for} \qquad n \geq \varepsilon$$

(a) *Then the convergence of* $\sum y_n$ *implies the convergence of* $\sum x_n$

(b) *The divergence of* $\sum x_n$ *implies the divergence of* $\sum y_n$

Proof(a) Suppose that $\sum y_n$ converges and, given $\varepsilon > 0$, let $M(\varepsilon) \in \mathcal{N}$ be such that if $m > n \geq M(\varepsilon)$, then

$$y_{n+1} + \cdots + y_m < \varepsilon$$

If $m > sup[K, M(\varepsilon)]$, then it follows that

$$0 \leq x_{n+1} + \cdots + x_m \leq y_{n+1} + \cdots + y_m < \varepsilon,$$

from which the convergence of $\sum x_n$ follows.

(b) This statement is the contrapositive of (a). Q.E.D.

Since it is sometimes difficult to establish the inequalities (8), the next result is frequentlyvery useful.

3.7.8Limit Comparison Test *Supposethat* $X := (x_n)$ and $Y := (y_n)$ *are strictly positive sequences and suppose that the following limit exists in* $\mathcal{R}$:

(9) $$r := \lim \left(\frac{x_n}{y_n}\right)$$

(a) If $r \neq 0$ *then* $\sum x_n$ *is convergent if and only if* $\sum y_n$ *is convergent.*

(b) If $r = 0$ and if $\sum y_n$ is *convergent, then* $\sum x_n$ *is convergent.*

Proof,..(a) It follows from (9) and Exercise 3.1.17 that there exists $K \in \mathcal{N}$ such that $\frac{1}{2}r \leq x_n/y_n \leq 2^r$ for $n \geq K$, whence

$$(^1/_2 r)y_n \le x_n \le (2r)y_n \text{ for } n \ge K.$$

If we apply the Comparison Test 3.7.7 twice, we obtain the assertion in (a),
(b) If $r = 0$, then there exists $K \in \mathcal{N}$ such that

$$0 < x_n \le y_n \qquad \text{for } n \ge K$$

so that Theorem 3.7,7(a) applies. Q.E.D.

5.0 SUMMARY

Remark The Comparison Tests 3.7.7 and 3.7.8 depend on having a stock of series that one knows to be convergent (or divergent). The reader will find that the *p*-series is often useful for this purpose.

3.7.9 Examples (a) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

It is clear that the inequality

$$0 < \frac{1}{n^2+n} < \frac{1}{n^2} \quad \text{for } n \in \mathcal{N}$$

is valid. Since the series $\sum 1/n^2$ is convergent (by Example 3.7.6(c)), we can apply the Comparison Test 3.7.7 to obtain the convergence of the given series.

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ is convergent.

If the inequality

(10) $$\frac{1}{n^2-n+1} \le \frac{1}{n^2}$$

were true, we could argue as in (a). However, (10) is *false* for all $n \in \mathcal{N}$. The reader can probably show that the inequality

$$0 < \frac{1}{n^2-n+1} \le \frac{1}{n^2}$$

Is valid for all $n \in \mathcal{N}$, and this equality will work just as well. However, it might take some experimentation to think of such an inequality and then establish it.

Instead, if we take $x_n := 1/(n^2 - n + 1)$ and $y_n := 1/n^2$, then we have

$$\frac{x_n}{y_n} = \frac{n^2}{n^2-n+1} = \frac{1}{1-(1/n)+(1/n^2)} \to 1.$$

Therefore, the convergence of the given series follows from the *Limit Comparison Test* 3.7.8(a).

(c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

This series closely resembles the series $\sum 1/\sqrt{n}$ which is a *p*-series with $p = ½$; by Example 3.7.6(e), it is divergent. If we let $x_n := 1/\sqrt{n+1}$ and $y_n := 1/\sqrt{n}$, then we have

$$\frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1+1/n}} \to 1.$$

Therefore the Limit Comparison Test 3.7.8(a) applies.

(d) The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

It would be possible to establish this convergence by showing (by Induction) that $n^2 < n!$ for $n \geq 4$, whence it follows that

$$0 < \frac{1}{n!} < \frac{1}{n^2} \text{ for } n \geq 4.$$

Alternative, if we let $x := 1/n!$ and $y_n := 1/n^2$, then (when $n \geq 4$) we have

$$0 \leq \frac{x_n}{y_n} = \frac{n^2}{n!} = \frac{n}{1.2\cdots(n-1)} < \frac{1}{n-2} \to 0.$$

Therefore the Limit Comparison Test 3.7.8(b) applies. (Note that this test was a bit troublesome to apply since we do not presently know the convergence of any series for which the limit of x_n/y_n is really easy to determine.) □

6.0 TUTOR MARKED ASSIGNMENT

1] Let $\sum_{n=1}^{\infty} a_n$ be a given series and let $\sum b_n$ be the series in which the terms are the same and in the same order as in $\sum a_n$ except that the terms for which $a_n = 0$ have been omitted. Show that $\sum a_n$ converges to A if and only if $\sum b_n$ converges to A.

2] Show that the convergence of a series is not affected by changing a *finite* number of its terms. (Of course, the value of the sum may be changed.)

3] By using partial fractions, show that

(a) $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$, (b) $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

4] If $\sum x_n$ and $\sum y_n$ are convergent, show that $\sum (xn + y_n)$ is convergent.

5] Can you give an example of a convergent series $\sum x_n$ and a divergent series $\sum y_n$ such that $\sum (x_n + y_n)$ is convergent? Explain.

6] (a) Show that the series $\sum_{n=1}^{\infty} \cos n$ is divergent.

(b) Show that the series $\sum_{n=1}^{\infty} (\cos n)/n^2$ is convergent.

7] Use an argument similar to that in Example 3.7.6(f) to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

8] If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a^2_n$ always convergent? Either prove it or give a counter example.

9] If $\sum a_n$ with $a_n > 0$ is converget, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counter example.

10] If $\sum a_n$ with $a_n > 0$ is convergent, th is $\sum \sqrt{\ }$ always convergent? Either prove it or give a counter example.

11] If $\sum a_n$ with $a_n > 0$ ia convergent, and if $b_n := (a_1 + \cdots + a_n)/n$ for $n \in \mathcal{N}$, show that $\sum b_n$ is always divergent.

12] Let $\sum_{n=1}^{\infty} a(n)$ be such that (a(n)) is a decreasing sequence of strictly positive numbers. If $s(n)$ denotes the *nth* partial sum, show (by grouping the terms in $s(2n)$ in two different ways) that ½ $(a(a) + 2a(2) + \cdots + 2^n a(2^n)) \leq s(2^n) \leq (a(1) + 2a(2) + \cdots + 2n\text{-}1)) + a(2^n)$.

Use these inequalities to show that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges.

This result is often called the **Cauchy Condensation Test**; it is very powerful.

13] Use the Cauchy Condensation Test to discuss the p-series $\sum_{n=1}^{\infty} (1/n^p)$ for $p > 0$.

14] Use the Cauchy Condensation Test to establish the divergence of the series:

(a) $\sum \frac{1}{n 1\text{n}\, n}$ (b) $\sum \frac{1}{n(1\text{n}\, n)(1\text{n}\ 1\text{n}\, n)}$

(c) $\sum \frac{1}{n(1\text{n}\, n)(1\text{n}\ 1\text{n}\, n)(1\text{n}\ 1\text{n}\ 1\text{n}\, n)}$

15] Show that if $c > 1$, then the following series are convergent:

(a) $\sum \frac{1}{n(1\text{n}\, n)^c}$ (b) $\sum \frac{1}{n(1\text{n}\, n)(1\text{n}\ 1\text{n}\, n)^c}$

7.0 REFERENCES/FURTHER READINGS

MODULE 4

LIMITS

“Mathematical analysis” is generally understood to refer to that area of mathematics in which systematic use is made of various limiting concepts. In the preceding chapter we studied one of these basic limiting concepts: the limit of a sequence of real numbers. In this chapter we will encounter the notion of the limit of a function.

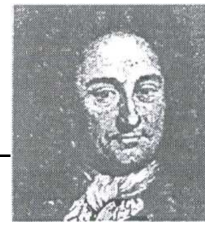
The rudimentary notion of a limiting process emerges in the 1680s as Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) struggled with the creation of the Calculus. Though each person’s work was initially unknown to the other and their creative insights were quite different, both realized the need to formulate a notion of function and the idea of quantities being “close to” one another. Newton used the word “fluent” to denote a relationshipbetween variables, and in his major work *Principia* in 1687 he discussed limits “to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *infinitum*”. Leibniz introduced the term “function” to indicate a quantity that depended on a variable, and he invented “infinitesimally small” numbers as a way of handling the concept of a limit.The term “function” soon became standard technology, and Leibniz also introduced the term “calculus” for this new method of calculation.

In 1748, Leonhard Euler (1707-1783) published his two-volume treatise *Introductio in Analysin Infinitorum*, in which he discussed power series, the exponential and logarithmic functions, *Calculi Differentalis* in 1755 and the three-volume *Institutiones Calculi Integralis* in 1780-70. These works remained the standard textbooks on calculus for many years. But the concept of limit was very intuitive and its looseness led to a number of problems. Verbal descriptions of the limit concept were proposed by other mathematicians of the era, but none was adequate to provide the basis for rigorous proofs.

In 1821, Augustin-Louis Cauchy (1789-1857) published his lecture on analysis in his *Cours d’Analyse*, which set standard for mathematical exposition for many years. He was concerned with rigor and in many ways raised the level of precision in mathematical discourse. He formulate definitions and presented arguments with greater than his predecessor, but the concept of limit still remained elusive. In an early chapter he gave the following definition:

> If the successive values attributed to the same variable approach indefinitely a fixed value, such that they finally differ from it by as little as one wishes, this latter is called the limit of all the others.

The final steps in formulating a precise definition of limit were taken by Karl Weierstrass (1815-1897). He insisted on precise language and rigorous proofs, and his definition of limit is the one we use today.



Gottfried Leibniz
Gottfried Wilhelm Leibniz (1646-1716) was born in Leipzig, Germany. He was six years old when his father, a professor of philosophy, died and left his son the key to his library and a life of books and learning. Leibniz entered the University of Leipzig at age 15, graduate at age 17, and received a Doctor of law degree from the University of Altdorf four years later. He wrote on legal matters, but was more interested in philosophy. He also developed original theories about language and the nature of the universe. In 1672, he went to Paris as a diplomat for four years. While there he began to study mathematics with the Dutch mathematician Christiaan Huygens. His travels to London to visit the Royal Academy further stimulated his interest in mathematics. His background in philosophy led him to very original, though not always rigorous, results.

Unaware of Newton's unpublished work, Leibniz published papers in the 1680s that presented a method of finding areas that is known today as the Fundamental Theorem of Calculus. He coined the term "calculus" and invented the *dy/dx* and elongated *S* notations that are used today. Unfortunately, some followers of Newton accused Leibniz of plagiarism, resulting in a dispute that lasted until Leibniz's death. Their approaches to calculus were quite different and it is now evident that their discoveries were made independently. Leibniz is now renowned for his work in philosophy, but his mathematical fame rests on his creation of the calculus.

Unit 1 Limits of Functions

1.0 INTRODUCTION

In this section we will introduced the important notion of the limit of a function. The intuitive idea of the function *f having a limit L at the point* c *is* that the values *f (x)* are close to *L* when *x*is close to (but different from) *c*. But it is necessary to have a technical way of working with the idea of "close to" and this is accomplished in the in the ε-δ definition given below.

In order for the idea of the limit of a function *f* at a point *c* to be meaningful, It is necessary that *f* be defined at points near *c*. It need not be defined at the point *c*, but it should be defined at enough point s close to *c*make the study interesting. This is the reason for the following definition.

2.0 OBJECTIVES

At the end of the unit, readers should be able to:-

(i) Understand the limit of a function.
(ii) Understand sequential criterion for limits.
(iii) Understand Divergence Criteria with examples.

3.0 MAIN CONTENT

4.1.1Definition Let $A \subseteq \mathcal{R}$. A point $c \in \mathcal{R}$ is a **cluster point**of *A*if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

This definition is rephrase in the language of neighborhoods as follows: A point *c* is a cluster point of the set *A* if every δ-neighborhood $V_\delta(c) = (c - \delta, c + \delta)$ of a *c* contains at least one point of *A* distinct from *c*.

Note: The point *c* may or may not be a member of *A*, but even it is in *A*, it is ignored when deciding whether it is a cluster point of *A* or not, since we explicitly require that there be points in $V_\delta(c) \cap A$ distinct from *c* in order for *c* to be a cluster point *A*.

For example, if $A := \{1, 2\}$, then the point 1 is not a cluster point of *A*, since choosing $\delta := {}^1/_2$ gives a neighborhood of 1 that contains no points of *A* distinct from 1. The same is true for the point 2, so we see that *A* has no cluster points.

4.1.2 Theorem A number $c \in \mathcal{R}$ is a cluster point of a subset *A* of $\mathcal{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathcal{N}$.

Proof. If *c* is a cluster point of *A*, then for any $n \in \mathcal{N}$ the $(1/n)$- neighborhood $V_{1/n}(c)$ contains at least one point $a_n$in *A* distinct from *c*. Then $a_n \in A, a_n \neq c$, and $|a_n - c| < 1/n$implies $\lim(a_n) = c$.

Conversely, if there exists a sequence (a_n) in $A \setminus \{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$ there exists *K*such that if $n \geq K$, *than* $a_n \in V_\delta(c)$.Therefore the δ-neighborhood $V_\delta(c)$ of *c* contains the points a_n, for $n \geq K$, which belong to *A* and are distinct from *c*. Q.E.D.

The next examples emphasize that a cluster point of a set may or may not belong to the set.

4.1.3 Examples **(a)** For the open interval $A_1 := (0, 1)$, every point of the closed interval [0, 1] is a cluster point of A_1. Note that the points 0, 1 are cluster points of A_1, but do not belong to A_1. All the points of $A_1$are cluster points of A_1.

(b) A finite set has no cluster points.

(c) The infinite set N has no cluster points.

(d) The set $A_4 := \{1/n : n \in \mathcal{N}\}$ has only the point 0 as a cluster point. None of the points in $A_4$is a cluster point of A_4.

(e) If $I := [0, 1]$, then the set $A_5 := I \cap \mathcal{Q}$ consists of all the rational numbers in *I*. It follows from the Density Theorem 2.4.8 that every point in 1 is a cluster point of A_5. □

Having made this brief detour, we now return to the concept of the concept of the limit of a function at a cluster point of its domain.

The Definition of the Limit

We now state the precise definition of the limit of a function *f* at a point *c*. it is important to note that in this definition, it is immaterial whether *f* is defined at *c* or not. In any case, we exclude *c* from consideration in the determination of the limit.

4.1.4 Definition ***Let*** $A \subseteq \mathbb{R}$, and let c be a cluster point of A. for a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of** f **at** c if, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Remarks (a) Since the value of δ usually depends on ε, we will sometimes write $\delta(\varepsilon)$ instead of δ to emphasize this dependence.
(b) The inequality $0 < |x - c|$ is equivalent to saying $x \neq c$.

If L is a limit of f at c, then we also say that f **converges** to L at c. We often write

$$L = \lim_{x \to c} f(x) \text{ or } L = \lim_{x \to c} f.$$

We also say that "$f(x)$ approaches L as x approaches c". (But it should be noted that the points do not actually move anywhere.) The symbolism

$$f(x) \to L \quad as \quad x \to c$$

is also used sometimes to express the fact that f has limit L at c.

If the limit of f at c does not exist, we say that f **diverges** at c.

Our first result is that the value L of the limit is uniquely determined. This uniqueness is not part of the definition of limit, but must be deduced.

4.1.5 Theorem If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only *one limit* at c.

Proof. Suppose that numbers L and L^1 satisfy 4.1.4 For any $\varepsilon > 0$, there exists $\delta(\varepsilon/2) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta(\varepsilon/2)$, then $|(x) - L| < \varepsilon/2$. Also there exists $\delta^1(\varepsilon/2)$ such that if $x \in A$ and $0 < |x - c| < \delta^1(\varepsilon/2)$, then $|f(x) - L^1| < \varepsilon/2$. Now let $\delta := \inf\{\delta(\varepsilon/2), \delta^1(\varepsilon/2)\}$. Then if $x \in A$ and $0 < |x - c| < \delta$, the Triangle Inequality implies that

$$|L - L'| \leq |L - f(x)| + |f(x) - L'| > \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L - L' = 0$, so that $L = L'$. Q.E.D.

The definition of limit can be very nicely described in terms of neighborhoods. (See Figure 4.1.1.) We observe that because

$$V_\delta(c) = (c - \delta, c + \delta)\{x : |x - c| < \delta\},$$

The inequality $0 < |x - c| < \delta$ is equivalent to saying that $x \neq c$ and x belong to the δ-neighborhood $V_\delta(c)$ of c. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to saying that $f(x)$ belongs to the ε-neighborhood $V_\varepsilon(L)$ of L. In this way, we obtain the following result. The reader should write out a detailed argument to establish the theorem.

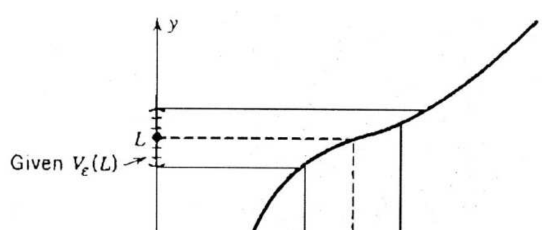


Figure 4.1.1 The limit of f at c is L.

4.1.6 Theorem Let $f\ A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

(i) $\lim\limits_{x \to c} f(x) = L$.

(ii) Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs $V_\varepsilon(L)$.

We now give some examples that illustrate how the definition of limit is applied.

4.1.7 Examples (a) $\lim\limits_{x \to c} b = b$

To be more explicit, let $f(x) := b$ for all $x \in$ R. We want to show that $\lim\limits_{x \to c} f(x) = b$.

If $\varepsilon > 0$ is given, we let $\delta := 1$. (in fact, any strictly positive δ will serve the purpose.) Then if $0 < |x - c| < 1$, we have $|f(x) - b|\ b - b| = 0 < \varepsilon$. Since $\varepsilon >$ is arbitrary, we conclude from Definition 4.1.4 that $\lim\limits_{x \to c} f(x) = b$.

(b) $\lim\limits_{x \to c} x = c$.

Let $g(x) := x$ for all $x \in \mathbb{R}$. If $\varepsilon > 0$, we choose $\delta(\varepsilon) := \varepsilon$. Then if $0 < |x - c| < \delta(\varepsilon)$, we have $|g(x) - c| = |x - c| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim\limits_{x \to c} g = c$.

(c) $\lim\limits_{x \to c} x^2 = c^2$.

Let $h(x) := x^2$ for all $x \in \mathbb{R}$. We have to make the deference

$$|h(x) - c^2| = |x^2 - c^2|$$

Less than a perassigned $\varepsilon > 0$ by taking x sufficiently close to c. To do so, we note that $x^{2\,-}c^2 = (x + c)(x - c)$. Moreover, if $|x - c| < 1$, then

$$|x| \leq |c| + 1 \text{ so that } \qquad |x + c| \leq |x| + |c| \leq 2|c| + 1.$$

Therefore, if $|x - c| < 1$, we have

$$(1) \qquad |x^2 - c^2| = |x + c|\ |x - c| \leq (2|c| + 1)\ |x^2 - c^2|.$$

Moreover this last term will be less than ε provided we take $|x - c| < \varepsilon / (2|c| + 1)$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{1, \frac{\varepsilon}{2|c|+1}\right\}$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < 1$ so that (1) is valid, and therefore, since $|x - c| < \varepsilon/(2|c| + 1)$ that

$$|x^2 - c^2| \le (2|c| + 1)\,|x - c| < \varepsilon.$$

Since w have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x\to c} h(x) = \lim_{x\to c} x^2 = c^2$.

(d) $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$.

Let $\varphi(x) := 1/x$ for $x > 0$ and let $c > 0$. To show that $\lim_{x\to c} \varphi = 1/c$ we wish to make the difference

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $c > 0$. We first note that

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c - x)\right| = \frac{1}{cx}|x - c|$$

for $x > 0$. It is useful to get an upper bound for the term $1/(cx)$ that holds in some neighborhood of c. In particular, if $|x - c| < \frac{1}{2}c$, then $\frac{1}{2}c < x < \frac{3}{2}c$ (why?), so that

$$0 < \frac{1}{cx} < \frac{2}{c^2} \quad \text{for } |x - c| < \tfrac{1}{2}c.$$

Therefore, for these values of x we have

(2) $$\left|\varphi(x) - \frac{1}{c}\right| \le \frac{2}{c^2}|x - c| < \varepsilon$$

In order to make this last term than ε it suffices to take $|x - c| < \frac{1}{2}c^2\varepsilon$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\{\tfrac{1}{2}c, \tfrac{1}{2}c^2\varepsilon\}$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < \frac{1}{2}c$ so that (2) is valid, and therefore, since $|x - c| < (\frac{1}{2}c^2)\,\varepsilon$, that

$$\left|\quad 1 \quad\right| \left|1 \quad 1\right|$$

$\varphi(x) - \frac{}{c\,x} = \frac{}{c} - - < \varepsilon.$

Since we have a way of choosing δ (ε) > 0 for an arbitrary choice of ε > 0, we infer that $\lim_{x \to c}$ $\varphi = 1/c$.

(e) $\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$

Let $\psi(x) := (x^3 - 4) / (x^2 + 1)$ for $x \in$ R. Then a little algebraic manipulation gives us

$$\left|\Psi(x) - \frac{4}{5}\right| = \frac{|5x^3 - 4x^2 - 24|}{5(x^2+1)}$$

$$= \frac{|5x^2 + 6x + 12|}{5(x^2+1)} \cdot |x - 2|.$$

To get a bound on the coefficient of $|x - 2|$, we restrict x by the condition $1 < x < 3$.
For x in this interval, we have $5x^2 + 6x + 12 \le 5\cdot3^2 + 6\cdot3 + 12 = 75$ and $5(x^2 + 1) \ge 5(1 + 1) = 10$, so that

$$\left|\Psi(x) - \frac{4}{5}\right| \le \frac{75}{10} |x - 2| = \frac{15}{2} |x - 2|.$$

Now for given ε> 0, we choose

$$\delta(\varepsilon) := \inf\{1, \tfrac{2}{15}\varepsilon\}.$$

Then if $0 < |x - 2| < \delta(\varepsilon)$, we have $|\psi(x), - (4/5)| \le (15/2)\,|x - 2| < \varepsilon > 0$ is arbitrary, the assertion is proved. □

4.0 CONCLUSION

Sequential Criterion for Limits

The following important formulation of limit of a function is in terms of limits of sequences. This characterization permits the theory of chapter 3 to be applied to the study of limits of functions.

4.1.8 Theorem*(Sequential Criterion)* Let $f: A \to$ R and let c be a cluster point of A. Then the following are equivalent.

(i) $\lim_{x \to c} f = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in$ N, the sequence $(f(x_n))$ converges to L.

Proof. (i)⇒(i). Assume f has limit L at c, and suppose (x_n) is a sequence in A with $\lim(x_n) = c$ and $x_n \neq c$ for all n. We must prove that the sequence $(f(x_n))$ converges to L. Let ε > 0 be given. Then by Definition 4.1.4, there exists δ > 0 such that if $x \in A$ satisfies $0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) = L| < \varepsilon$. We now apply the definition of convergent sequence for the given δ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|x - c| < \delta$. But for each such x_n we have $|f(x_n) - L| < \varepsilon$. This if $n > K(\delta)$, then $|f(x_n) - L| < \varepsilon$. Therefore, the sequence $(f(x_n))$ converges to L.

(ii) ⇒ (i). [The proof is a contrapositive argument.] If (i) is not true, then there exists an ε_0-heighborhood $V_\varepsilon(L)$ such that no matter what δ-neighborhood of c we pick, there will be at least one number x_δ in $A \cap V_\delta(c)$ with $x_\delta \neq c$ such that $f(x_\delta) \notin V_{\varepsilon_0}(L)$. Hence for every $n \in \mathcal{N}$, the $(1/n)$-neighborhood of c contains a number x_n such that

$$0 < | x_n - c | < 1/n \quad \text{and } x_n \in A,$$

but such that

$$| x(x_n) - L | \geq \varepsilon_0 \quad \text{for all } n \in \mathcal{N}.$$

We conclude that the sequence (x_n) in $A \setminus \{c\}$ converges to c, but the sequence $(f(x_n))$ does not converge to L. Therefore we have shown that if (i) is not true, then (ii) is not true. We conclude that (ii) implied (i). Q.E.D.

We shall in the next unit that many of the basic limit properties of functions can be established by using corresponding properties for convergent sequences. For example, we know from our work with sequences that if (x_n) is any sequence that converges to a number c, then (x^2_n) converges to c^2. Therefore, by the sequential criterion, we can conclude that the function $h(x) := x^2$ has limit $\lim_{x \to c} h(x) = c^2$.

5.0 SUMMARY

Divergence Criteria

It is often important to be able to show (i) that a certain number is *not* the limit of a function at a point, or (ii) that the function *does not have* a limit at a point. The following result is a consequence of (the proof of) Theorem 4.1.8. We leave the details of its proof as an important exercise.

4.1.9 Divergence Criteria Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$ and let $c \in \mathcal{R}$ be a cluster point of A

(a) If $L \in \mathcal{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathcal{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge to L.

(b) The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathcal{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathcal{R}$.

We now give some applications of this result to show how it can be used.

4.1.10 Example **(a)** $\lim_{x \to 0} (1/x)$ does not exists in $\mathcal{R}$.

As in Example 4.1.7 (d), let $\varphi(x) := 1/x$ for $x > 0$. However, here we consider $c = 0$. The argument given Example 4.1.7 (d) break down if $c = 0$ since we cannot obtain a bound such as that in (2) of that example. Indeed, if we take the sequence (x_n) with $x_n := 1/n$ for $n \in \mathcal{N}$, then $\lim(x_n) = 0$, but $\varphi(x_n) = n$. As we know, the sequence $(\varphi(x_n)) = (n)$ is not convergent in $\mathcal{R}$, since it is not bounded. Hence, by Theorem 4.1.9(b), $\lim_{x \to 0}(1/x)$ does not exist in $\mathcal{R}$.

(b) $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

Let the **signum function** sgn be defined by

$$\operatorname{Sgn}(x) := \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \end{cases}$$

-1 for $x < 0$.

Note that $\operatorname{sgn}(x) = x / |x|$ for $x \neq 0$. (See Figure 4.1.2.) We shall show the sgn does not have a limit at $x = 0$. We shall do this by showing that there is a sequence (x_n) such that $\lim(x_n) = 0$, but such that $(\operatorname{sgn}(x_n))$ does not converge.

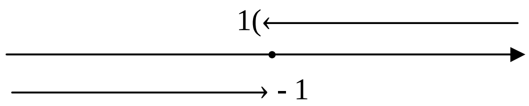


Figure 4.1.2 The signum function.

Indeed, let $x_n := (-1)^n / n$ for $n \in$ N so that $\lim(x_n) = 0$. However, since

$$\operatorname{Sgn}(x_n) = (-1)^n \qquad \text{for } n \in \mathcal{N},$$

if follows from Example 3.4.6 (a) that $(\operatorname{sgn}(x_n))$ does not converge. Therefore $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

(c) $\lim_{x \to 0} \operatorname{sgn}(1/x)$ does not exist in $\mathcal{R}$.

Let $g(x) := \sin(1/x)$ for $x \neq 0$. (See Figure 4.1.3.) We shall show that g does not have limit at $c = 0$, by exhibiting two sequences (x_n) (y_n) with $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathcal{N}$ and such that $\lim(x_n) = 0$ and $\lim(x_n) = 0$, but such that $\lim(g(x_n)) \neq \lim(g(y_n))$. In view of theorem 4.1.9 this implies that lim g cannot exist. (Example why.)

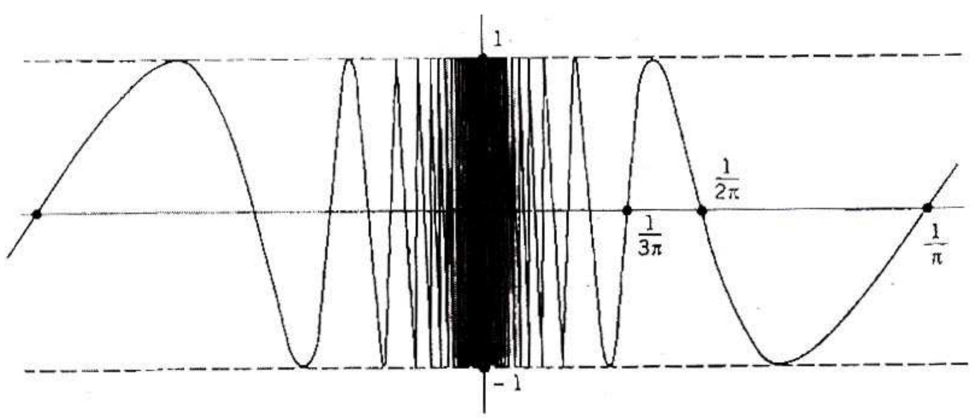


Figure 4.1.3 The function $g(x) = \sin(1/x)$ $(x \neq 0)$.

Indeed, we recall from calculus that $\sin t = 0$ if $t = n\pi$ for $n \in \mathcal{Z}$, and that $\sin t = +1$ if $t = \frac{1}{2}\pi + 2\pi n$ for $n \in \mathcal{Z}$. Now let $x_n := 1/n\pi$ for $n \in \mathcal{N}$; then $\lim(x_n) = 0$ and $g(x_n) = \sin n\pi = 0$ for all $n \in \mathcal{N}$, so that $\lim(g(x_n)) = 0$. On the other hand, Let $y_n := (\frac{1}{2}\pi + 2\pi n)^{-1}$ for $n \in \mathcal{N}$; then $\lim(y_n) =$

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0 and $g(y_n) = \sin(\frac{1}{2}\pi + 2\pi n) = 1$ for all $n \in \mathcal{N}$, so that $\lim(g(y_n)) = 1$. We conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist.

In order to have some interesting applications in this and later examples, we shall make use of well-known properties of trigonometric and exponential that will be established in Module 8.

6.0 TUTOR MARKED ASSIGNMENT

1. Determine a condition on $|x - 1|$ that will assure that:
   (a) $|x^2 - 1| < \frac{1}{2}$, (b) $|x^2 - 1| < 1/10^{-3}$
   (c) $|x^2 - 1| < 1/n$ for a given $n \in N$, (d) $|x^3 - 1| < 1/n$ for a given $n \in N$

2. Determine a condition on $|x - 4|$ that will assure that:
   (a) $|\sqrt{x} - 2| < \frac{1}{2}$, (b) $|\sqrt{x} - 2| < 10^{-2}$.

3. Let c be a cluster point of $A \subseteq \mathcal{R}$ and let $f: A \to \mathcal{R}$. Prove that $\lim_{x \to c} f(x) = L$ if and if $\lim_{x \to c} |f(x) - L| = 0$.

4. Let $f: \mathcal{R} \to \mathcal{R}$ and let $c \in R$. Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

5. Let $1 := (0, a)$ where $a > 0$, and let $g(x) := x^2$ for $x \in 1$. For any points $x, c \in 1$, show that $|g(x) - c^2| \leq 2a\,|x - c|$. Use this inequality to prove that $\lim_{x - c} x^2 = c^2$ for any $c \in 1$.

6. Let 1 be interval in $\mathcal{R}$, let $f; 1 \to \mathcal{R}$, and let $c \in 1$. Suppose there exist constants K and L such that $|f(x) - L| \leq K\,|x - c|$ for $f \in 1$. Show that $\lim_{x \to c} f(x) = L$.

7. Show that $\lim_{x \to c} x^3 = c^3$ fro any $c \in R$.

8. Show that $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.

9. Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.
   (a) $\lim_{x \to 2c} \frac{1}{1 - x} = -1$, (b) $\lim_{x \to 1} \frac{x}{1 + x} = \frac{1}{2}$,
   (c) $\lim_{x \to c} \frac{x^2}{|x|} = 0$, (d) $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$

10. Use the definition of limit to show that
    (a) $\lim_{x \to 2}(x^2 + 4x) = 12$, (b) $\lim_{x \to -1} \frac{x + 5}{2x + 3} = 4$.

11. Show that the following limits do *not*exist.
    (a) $\lim \frac{1}{x^2}$ $(x > 0)$ (b) $\lim \frac{1}{\sqrt{x}}$ $(x > 0)$,

(c) $\lim_{x \to 0} (x + \text{sgn}(x))$ (d) $\lim_{x \to 0} \sin(1/x^2)$.

12. Suppose the function $f : R \to R$ has limit L at 0, and let $a > 0$. If $g : R \to R$ is defined by $g(x) := f(ax)$ for $x \in R$, show that $\lim_{x \to 0} g(x) = L$.

13. Let $c \in \mathcal{R}$ and let $f : \mathcal{R} \to \mathcal{R}$ be such that $\lim_{x \to c} ((x))^2 = L$.

(a)Show that if $L = 0$, then $\lim_{x \to c} f(x)$

(b) Show by example that if $L \neq 0$, then f may not have a limit at c.

14. Let $f : \mathcal{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.
(a) Show that f has a limit at $x = 0$.
(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

15. Let $f : \mathcal{R} \to \mathcal{R}$, let 1 be an openinterval in $\mathcal{R}$, and let $c \in 1$. If $f_1$is the restriction of f to 1, show that $f_1$has a limit at c if and only if f has a limit at c, and that the limits are equal.

16. Let $f : \mathcal{R} \to \mathcal{R}$, let J be a closed interval in $\mathcal{R}$, and let $c \in J$. If $f_2$is the restriction of f to J, show that if f has a limit at c then $f_2$has a limit at c. Show by example that it does not follow that if $f_2$has a limit at c, then f has a limit at c.

7.0 REFERENCES / FURTHER READINGS

Unit 2: Limit Theorems

1.0 INTRODUCTION
We shall now obtain results that are useful in calculating limits of functions. These results are parallel to the limits theorems established in Unit 3.2 for sequences. In fact, in most cases these results can be proved by using Theorem 4.1.8 and results from Unit 3.2. Alternatively, the results in this section can be proved by using ε-δ arguments that are very similar to the ones employed in Unit 3.2.

2.0 OBJECTIVE
At the end of the unit, readers should be able to:
(i) Understand the concept of f is bounded on a neighborhood.
(ii) Define squeeze Theorem of a Cluster Point.

3.0 MAIN CONTENT
4.2.1 Definition Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and let $c \in \mathcal{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

4.2.2 Theorem If $A \subseteq \mathcal{R}$ and $f : A \to \mathcal{R}$ has a limit at $c \in \mathcal{R}$, then f is bounded on some neighborhood of c.

Proof. If $L := \lim_{x\to c} f$, then for ε = 1, there exists δ > 0 such that if $0 < |x - c| < \delta$, then

$|f(x) - L|$ 1; hence (by Corollary 2.2.4 (a)),

$$|f(x)| - |L| \leq |f(x) - L| < 1.$$

Therefore, if $x \in A \cap V_\delta(c)$, $x \neq c$, then $|f(x)| \leq |L| + 1$. If $c \notin A$, we take $M = |L| + 1$, while if $c \in A$ we take $M := \sup\{|f(c)|, |L| + 1\}\}$. If follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \leq M$. This shows that f is bounded on the neighborhood $V_\delta(c)$ of c. Q.E.D.

The next definition is similar to the definition for sums, differences, products, and quotients of sequences given in Unit 3.2.

4.2.3 Definition Let $A \subseteq \mathcal{R}$ and let f and g be functions defined on A to $\mathcal{R}$. We define the sum $f + g$, the difference $f - g$, and the product $f\,g$ on A to $\mathcal{R}$ to be the functions given by

$$(f + g)(x) := f(x) + g(x), \qquad (f - g)(x) := f(x) - g(x),$$

$$(fg)(x) := f(x)\,g(x)$$

for all $x \in A$. Further, if $b \in \mathcal{R}$, we define the **multiple** $b\,f$ to be the function given by

$$(bf)(x) := bf(x) \qquad \text{for all} \quad x \in A.$$

Finally, if $h(x) \neq 0$ for $x \in A$, we define the **quotient** f / h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \qquad \text{for all} \quad x \in A.$$

4.2.4 Theorem Let $A \subseteq \mathcal{R}$, let f and g be functions on A to R, and let $c \in$ R be a cluster point of A. Further, let $b \in \mathcal{R}$.

(a) $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then:

$$\lim_{x\to c} (f + g) = L + M, \quad \lim_{x\to c}(f - g) = L - M,$$

$$\lim_{x\to c} (fg) = LM, \quad \lim_{x\to c} (bf) = bL.$$

(b) IF $h: A \to$ R, if $h(x) \neq 0$ for all $x \in A$, and if $\lim_{x\to c} h = H \neq 0$, then

$$\lim_{x\to c}\left(\frac{f}{h}\right) = \frac{L}{H}.$$

Proof. One proof to this theorem is exactly similar to that of Theorem 3.2.3. Alternatively, it can be proof by making use of Theorems 3.2.3 and 4.1.8. For example, let (x_n) be any sequence in A such that $x_n \neq c$ for $n \in \mathcal{N}$, and $c = \lim(x_n)$. It follows from Theorem 4.1.8 that

$$\text{Lim}(f(x_n)) = L, \qquad \lim(g(x_n)) = M.$$

On the other hand, Definition 4.2.3 implies that

$$(fg)\,(x_n) = f(x_n)\,g(x_n) \qquad \text{for } n \in \mathcal{N}.$$

Therefore an application of Theorem 3.2.3 yields

$$\begin{aligned}\text{Lim}((fg)\,(x_n)) &= \lim(f(x_n)\,g(x_n)) \\ &= [\lim(f(x_n))]\,[\lim(g(x_n))] = LM.\end{aligned}$$

Consequently, it follows from Theorem 4.1.8 that

$$\text{Lim}_{x \to c}(fg) = \lim((fg)\,(x_n)) = LM.$$

The other parts of this theorem are proved in a similar manner. We leave the details to the reader. Q.E.D.

Remarks (1) We note that, in part (b), the additional assumption that $H = \lim_{x \to c} h \neq 0$ is made. If this assumption is not satisfied, then the limit

$$\lim_{x \to c} \frac{f(x)}{h(x)}$$

may or may not exist. But even if this limit does exist, we *cannot* use Theorem 4.2.4 (b) to evaluate it.

(2) Let $A \subseteq \mathcal{R}$, and let $f_1, f_2, \cdots, f_n$ be functions on A to R, and let c be a cluster point of A. If

$$L_k := \lim_{x \to c} f_k \qquad \text{for} \qquad k = 1, \cdots, n,$$

then it follows from Theorem 4.2.4 by an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x \to c}(f_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x \to c} f$ and $n \in \mathcal{N}$, then

$$L^n = \lim_{x \to c}(f(x))^n.$$

4.2.5 Examples (a) Some of the limits that were established in Unit 4.1 can be proved by using Theorem 4.2.4. For example, it follows from this result that since $\lim_{x \to c} x = c$,

then $\lim_{x \to c} x^2 = c^2$, and that if $c > 0$, then

$$\lim_{x\to c} \frac{1}{x} = \frac{1}{\lim_{x\to c} x} = \frac{1}{c}.$$

(b) $\lim_{x\to 2} (x^2 + 1)(x^3 - 4) = 20.$

It follows from theorem 4.2.4 that

$$\lim_{x\to 2} (x^2 + 1)(x^3 - 4) = \left(\lim_{x\to 2}(x^2 + 1)\right)\left(\lim_{x\to 2}(x^3 - 4)\right)$$

$$= 5.4 = 20.$$

(c) $\lim_{x\to 2} \left(\frac{x^3 - 4}{x^2 + 1}\right) = \frac{4}{5}.$

If we apply Theorem 4.2.4 (b), we have

$$\lim_{x\to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{\lim_{x\to 2}(x^2-4)}{\lim_{x\to 2}(x^2+1)} = \frac{4}{5}.$$

Note that since the limit in the denominator [i.e., $\lim_{x\to 2} (x^2 + 1) = 5$] is not equal to 0, then Theorem 4.2.4 (b) is applicable.

(d) $\lim_{x\to 2} = \frac{x^2 - 4}{3x - 6} = \frac{4}{3}.$

If we let $f(x) := x^2 - 4$ and $h(x) := 3x – 6$ for $x \in$ R, then we *cannot* use Theorem 4.2.4 (b) to evaluate $\lim_{x\to 2} (f(x)/h(x))$ because

$$H = \lim_{x\to 2} h(x) = \lim_{x\to 2} (3x - 6)$$

$$= 3 \lim_{x\to 2} x - 6 = 3\cdot 2 - 6 = 0.$$

However, if $x \neq 2$, then it follows that

$$\frac{x^2-4}{3x-6} = \frac{(x+2)(x-2)}{3(x-2)} = \frac{1}{3}(x + 2).$$

Therefore we have

$$\lim_{x\to 2} \frac{x^2-4}{3x-6} = \lim_{x\to 2} \frac{1}{3}(x + 2) = \frac{1}{3}\left(\lim_{x\to 2} x + 2\right) = \frac{4}{3}.$$

Note that the function $g(x) = (x^2 - 4) / (3x - 6)$ has a limit at $x = 2$ *even though it is not defined there.*

(e) $\lim_{x\to 0} \frac{1}{x}$ does not exist in R.

Of course lim 1 = 1 and $H := \lim x = 0$. However, since $H = 0$, we *cannot* use

$x\to 0$ $x \to 0$

Theorem 4.2.4 (b) to evaluate $\lim_{x\to 0} (1/x)$. In fact, as was seen in Example 4.1.10 (a), the

function $\varphi(x) = 1/x$ does not have limit at $x = 0$. This conclusion also follows from Theorem 4.2.2 since the function $\varphi(x) = 1/x$ is not bounded on a neighborhood of $x = 0$.
(Why?)

(f) If p is a polynomial function, then $\lim_{x\to c} p(x) = p(c)$.

Let p be a polynomial function on $\mathcal{R}$ so that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for all $x \in \mathcal{R}$. It follows from Theorem 4.2.4 and the fact that $\lim_{x\to c} x^k = c^k$, that

$$\begin{aligned}\lim_{x\to c} p(x) &= \lim_{x\to c} [a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0] \\ &= \lim_{x\to c} (a_n x^n) + \lim_{x\to c} (a_{n-1}x^{n-1}) + \cdots + \lim_{x\to c} (a_1 x) + \lim a_0 \\ &= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 \\ &= p(c).\end{aligned}$$

Hence $\lim_{x\to c} p(x) = p(c)$ for any polynomial function p.

(g) If p and q are polynomial function on $\mathcal{R}$ and if $q(c) \neq 0$, then

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

Since $q(x)$ is a polynomial function, it follows from a theorem in algebra in algebra that there are at most a finite number of real numbers $\alpha_1, \cdots, \alpha_m$ [the real zeroes of $q(x)$] such that $q(\alpha_j) = 0$ and such that if $x \notin \{\alpha_1, \cdots, \alpha_m\}$, then $q(x) \neq 0$. Hence, if $x \notin [\alpha_1, \cdots, \alpha_m]$, we can define

$$r(x) := \frac{p(x)}{q(x)}.$$

If c is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows from part (*f*) that $\lim_{x\to c} q(x) = q(c) \neq 0$. Therefore we can apply Theorem 4.2.4 (b) to conclude that

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} = \frac{p(c)}{q(c)}$$

The next result is a direct analogue of Theorem 3.2.6.

4.2.6. Theorem Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$ and let $c \in R$ be a cluster point of A. If

$$a \leq f(x) \leq b \quad \text{for all} \quad x \in A, x \neq c,$$

and if $\lim_{x \to c} f$ exists, then $a \leq \lim_{x \to c} f \leq b$.

Proof. Indeed, if $L = \lim_{x \to c} f$, then it follows from Theorem 4.1.8 that if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathcal{N}$ and if the sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to L. Since $a \leq f(x_n) \leq b$ for all $n \in \mathcal{N}$, it follows from Theorem 3.2.6 that $a \leq L \leq b$. Q.E.D.

We now state an analogue of the squeeze Theorem 3.2.7. We leave its proof to the reader.

4.0 CONCLUSION

4.2.7 Squeeze Theorem Let $A \subseteq \mathcal{R}$, let $f, g, h : A \to R$, and let $c \in \mathcal{R}$ be a cluster point of A. If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in A, x \neq c.$$

and if $\lim_{x \to c} f = L = \lim_{x \to c} h$, then $\lim_{x \to c} g = L$.

4.2.8 Examples (a) $\lim_{x \to 0} x^{3/2} = 0 \; (x > 0)$.

Let $f(x) := x^{3/2}$ for $x > 0$. Since the inequality $x < x^{1/2} \leq 1$ holds for $0 < x \leq 1$ (Why?), it follows that $x^2 \leq f(x) = x^{3/2} \leq x$ for $0 < x \leq 1$. Since

$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} x = 0,$$

it follows from the Squeeze Theorem 4.2.7 that $\lim_{x \to 0} x^{3/2} = 0$.

(b) $\lim_{x \to 0} \sin x = 0$.

It will be proved later (See Theorem 8.4.8), that

$$-x \leq \sin x \leq x \quad \text{for all } x \geq 0.$$

Since $\lim_{x \to 0} (\pm x) = 0$, it follows from the squeeze Theorem that $\lim_{x \to 0} \sin x = 0$.

(c) $\lim_{x \to 0} \cos x = 1$.

It will be proved later (see Theorem 8.4.8) that

(1) $1 - \frac{1}{2}x^2 \leq \cos x \leq 1$ fro all $x \in \mathcal{R}$.

Since $\lim_{x\to 0} (1 - \frac{1}{2}x^2) = 1$, it follows from the squeeze Theorem that $\lim_{x\to 0} \cos x = 1$.

(d) $\lim_{x\to 0} \left(\frac{\cos x - 1}{x}\right) = 0$.

We cannot use Theorem 4.2.4 (b) to evaluate this limit. (Why not?) However, it follows from the inequality (1) in part (*c*) that

$$-\frac{1}{2}x \leq (\cos x - 1) / x \leq 0 \qquad \text{for } x > 0$$

and that

$$0 \leq (\cos x - 1) / x \leq -\frac{1}{2}x \qquad \text{for } x < 0.$$

Now let $f(x) := -x/2$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$, and let $h(x) := 0$ for $x \geq 0$ and $h(x) := -x/2$ for $x < 0$. Then we have

$$f(x) \leq (\cos x - 1)/x \leq h(x) \qquad \text{for } x \neq 0.$$

Since it is readily seen that $\lim_{x\to 0} f = 0 = \lim_{x\to 0} h$, it follows from the squeeze theorem that $\lim_{x\to 0} (\cos x - 1)/x = 0$.

(e) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right) = 1$.

Again we cannot use Theorem 4.2.4 (b) to evaluate this limit. However, it will be proved later (see Theorem 8.4.8) that

$$x - \frac{1}{6}x^3 \leq \sin x \leq x \qquad \text{for } x \geq 0$$

and that

$$x \leq \sin x \leq x - \frac{1}{6}x^3 \qquad \text{for } x \leq 0.$$

Therefore it follows (why?) that

$$1 - \frac{1}{6}x^2 \leq (\sin x)/x \leq 1 \quad \text{for } x \neq 0.$$

But since $\lim_{x\to 0} (1 - \frac{1}{6}x^2) = 1 - \frac{1}{6} \cdot \lim_{x\to 0} x^2 = 1$, we infer the Squeeze Theorem that $\lim_{x\to 0} (\sin x)/x = 1$

(f) $\lim_{x\to 0} (x \sin(1/x)) = 0$

Let $f(x) = x \sin(1/x)$ for $x \neq 0$. Since $-1 \leq \sin z \leq 1$ for all $z \in$ R, we have the inequality

$$-|x| \leq f(x) = x \sin(1/x) \leq |x|$$

for all $x \in \mathcal{R}$, $x \neq 0$. Since $\lim_{x \to 0} |x| = 0$, it follows from the squeeze Theorem that $\lim_{x \to 0} f = 0$. For a graph, see Figure 5.1.3 or the cover of this book. □

There are results that are parallel to Theorem 3.2.9 and 3.2.10; however, we will leave them as exercises. We conclude this unit with a result that is, in some sense, a partial converse to Theorem 4.2.6.

5.0 SUMMARY

4.2.9 Theorem Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$ and let $c \in \mathcal{R}$ be a cluster point of A. If

$$\lim_{x \to c} f > 0 \qquad \left[\text{respectively, } \lim_{x \to c} f < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c)$, $x \neq c$.

Proof. Let $L := \lim_{x \to c} f$ and suppose that $L > 0$. We take $\varepsilon = \frac{1}{2}L > 0$ in Definition 4.1.4, and

obtain a number $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \frac{1}{2}L$. Therefore (why?) it follows that if $x \in A \cap V_\delta(c)$, $x \neq c$, then $f(x) > \frac{1}{2}L > 0$.

If $L < 0$, a similar argument applies. Q.E.D.

6.0 TUTOR MARKED ASSIGNMENT

1. Apply Theorem 4.2.4 to determine the following limits:

   (a) $\lim_{x \to 1} (x + 1)(2x + 3)$ $(x \in \mathcal{R})$, (b) $\lim_{x \to 1} \frac{x^2 + 2}{x^2 - 2}$ $(x > 0)$,

   (c) $\lim_{x \to 2} \left(\frac{1}{x+1} - \frac{1}{2x}\right)$ $(x > 0)$, (d) $\lim_{x \to 0} \frac{x+1}{x^2 + 2}$ $(x \in \mathcal{R})$.

2. Determine the following limits and state which theorems are used in each case. (You may wish to use Exercise 14 below.)

   (a) $\lim_{x \to 2} \sqrt{\frac{2x+1}{x+3}}$ $(x > 0)$, (b) $\lim_{x \to 2} \frac{x^{2-4}}{x-2}$ $(x > 0)$,

   (c) $\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$ $(x > 0)$, (d) $\lim_{x \to 1} \frac{\sqrt{x-1}}{x-1}$ $(x > 0)$.

3. Find $\lim_{x \to 0} \frac{\sqrt{1 + 2x} - \sqrt{1 + 3x}}{x + 2x^2}$ where $x > 0$.

4. Prove that $\lim_{x \to 0} \cos(1/x)$ does exist but that $\lim_{x \to 0} x \cos(1/x) = 0$.

5. Le f, g be defined on $A \subseteq \mathcal{R}$ to $\mathcal{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x \to c} g = 0$. Prove that $\lim_{x \to c} fg = 0$
6. Use the definition of the limit to prove the first assertion in Theorem 4.2.4 (a).
7. Use the sequential formulation of the limit to prove Theorem 4.2.4 (b).
8. Let $n \in$ N be such that $n \geq 3$. Derive the inequality $-x^2 \leq x^n \leq x^2$ for $1 < x < 1$. Then use the fact that $\lim_{x \to 0} x^2 = 0$ to show that $\lim_{x \to 0} x^n = 0$.
9. Let f, g be defined on A to $\mathcal{R}$ and let c be a cluster point of A.
   (a) Show that if both $\lim_{x \to c} f$ and $\lim_{x \to c} (f + g)$ exist, then $\lim_{x \to c} g$ exists.
   (b) If $\lim_{x \to c} f$ and $\lim_{x \to c} fg$ exist, does it follow that $\lim_{x \to c} g$ exists?
10. Give examples of function f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.
11. Determine whether the following limits exist in $\mathcal{R}$.
    (a) $\lim_{x \to 0} \sin(1/x^2)$ $(x \neq 0)$, (b) $\lim_{x \to 0} x \sin(1/x^2)$ $(x \neq 0)$,
    (c) $\lim_{x \to 0} \operatorname{sgn} \sin(1/x)$ $(x \neq 0)$, (d) $\lim_{x \to 0} \sqrt{x} \sin(1\ x^2)$ $(x > 0)$.
12. Let $f : \mathcal{R} \to \mathcal{R}$ be such that $f(x + y) = f(x) + f(y)$ for all x, y in R. Assume that $\lim_{x \to 0} f = L$

exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathcal{R}$. [*Hint:* First note that $f(2x) = f(x) + f(x) = 2 f(x)$ for $x \in \mathcal{R}$. Also note that $f(x) = f(x - c) = f(c)$ for x, c in $\mathcal{R}$.]

13. Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$ and let $c \in \mathcal{R}$ be a cluster point of A. If $\lim_{x \to c} f$ exists, and if $|f|$

denotes the function defined for $x \in A$ by $|f|(x) := |f(x)|$, prove that $\lim_{x \to c} |f| = |\lim_{x \to c} f|$

14. $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and let $c \in \mathcal{R}$ be a cluster point of A. In addition, suppose that $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$. If $\lim f$ exists, prove that $\lim_{x \to c} \sqrt{f} = \sqrt{\lim_{x \to c} f}$.

7.0 REFERENCES / FURTHER READINGS

Unit 3 Some Extension Of The Limit Concept

1.0 INTRODUCTION

In this unit, we shall present three types of extensions of the notion of a limit of a function that often occur. Since all the ideals here are closely parallel to the ones we have already encountered, this unit can be read easily.

2.0 OBJECTIVES

At the end of the Unit, readers should be able to:

(i) know three types of extensions of the notion of a limit of a function

One-sided Limits ___

There are times when a function f may not possess a limit at a point c, yet a limit does exist when the function is restricted to an interval on one side of the cluster point c.

For example, the signum function considered in Example 4.1.10 (b), and illustrated in Figure 4.1.2, has no limit at $c = 0$. However, if we restrict the signum function to the interval $(0, \infty)$, the resulting function has a limit of 1 at $c = 0$. Similarly, if we restrict the signum function to the interval $(-\infty, 0)$, the resulting function has a limit of -1 at $c = 0$.
These are elementary examples of right-hand and left-hand limits at $c = 0$.

4.3.1 Definition Let $A \in \mathcal{R}$ and let $f : A \to \mathcal{R}$.
(i) If $c \in R$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $L \in \mathcal{R}$ is a **right-hand limit of** f **at** c and we write

$$\lim_{x \to c+} f = L \qquad \text{or} \qquad \lim_{x \to c+} f(x) = L$$

†This unit can be largely omitted on a first reading of this chapter

if given any $\epsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.
(ii) If $c \in \mathcal{R}$ is a cluster point of the set $A \cap (-\infty, c) = (x \in A : x < c)$, then we say that $L \in R$ is a **left-hand limit of** f **at** c and we write

$$\lim_{x \to c-} f = L \qquad \text{or} \qquad \lim_{x \to c-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Notes (1) The limits $\lim_{x \to c+} f$ and $\lim_{x \to c-} f$ are called **one-sided limits of** f **at** c. It is possible that neither one-sided limit may exist. Also, one of them may exist without the other existing. Similarly, as is the case for $f(x) := \text{sgn}(x)$ at $c = 0$, they may both exist and be different.
(2) If A is an interval with left endpoint c, then it is readily such that $f : A \to \mathcal{R}$ has a limit at c if and only if it has a right-hand limit at c. Moreover, in this case the limit $\lim_{x \to c} f$ and the right-hand limit $\lim_{x \to c+} f$ are equal. (A similar situation occurs for the left-hand limit when A is an interval with right endpoint c.)

the reader can show that f can have only one right-hand (respectively, left-hand) limit at a point. There are results analogous to those established in Units 4.1 and 4.2 for two-sided limits. In particular, the existence of one-sided limits can be reduced to sequential considerations.

4.3.2 Theorem Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and let $c \in \mathcal{R}$ be a cluster of $A \cap (c, \infty)$. Then the following statements are equivalent:

(i) $\lim_{x \to c+} f = L$.

(ii) For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in$ N, the sequence $(f(x_n))$ converges to L.

We leave the proof of this result (and the formulation and proof of the analogous result for left-hand limits) to the reader. We will not take the space to write out the formulations of the one-sided version of the other results in Units 4.2 and 4.2.

The following result relates the notion of the limit of a function to one-sided limits. We leave its proof as an exercise.

4.3.3 Theorem Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and let $c \in$ R be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f = L$ if and only if $\lim_{x \to c+} f = L \lim_{x \to c+} f$.

4.3.4 Examples (a) Let $f(x) := \operatorname{sgn}(x)$.

We have seen in Example 4.1.10 (b) that sgn does not have a limit at 0. It is clear that $\lim_{x \to c+} \operatorname{sgn}(x) = +1$ and that $\lim_{x \to 0-} \operatorname{sgn}(x) = -1$. Since these one-sided limits are different, it also follows from Theorem 4.3.3 that sgn(x) does not have a limit at 0.

(b) Let $g(x) := e^{1/x}$ for $x \neq 0$. (See Figure 4.3.1.)

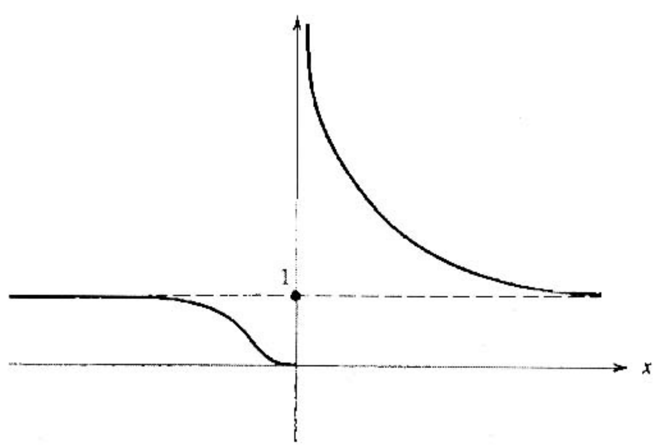

Figure 4.3.1 Graph of $g(x) = e^{1/x} (x \neq 0)$.

We first show that g does not have a finite right-hand limit at $c = 0$ since it is not bounded on any right-hand neighborhood $(0, \delta)$ of 0. We shall make use of the inequality

(1) $0 < t < e^1$ for $t > 0$,

which will be proved later (see Corollary 8.3.3). It follows from (10 that if $x > 0$, then $0 < 1/x < e^{1/x}$. Hence, if we take $x_n = 1/n$, then $g(x_n) > n$ for all $n \in \mathcal{N}$. Therefore $\lim_{x \to 0+} e^{1/x}$ does not exist in $\mathcal{R}$.

However, $\lim_{x \to 0-} e^{1/x} = 0$. Indeed, if $x < 0$ and we take $t = -1/x$ in (1) we obtain $0 < -1/x\, e^{-1/x}$,

Since $x < 0$, this implies that $0 < e^{1/x} < -x$ for all $x < 0$. It follows from this inequality that $\lim_{x \to 0-} e^{1/x} = 0$.

(c) Let $h(x) := 1/(e^{1/x} + 1)$ for $x \neq 0$. (See Figure 4.3.2.)
We have seen in part (b) that $0 < 1/x\, e^{1/x}$ for $x > 0$, whence

$$0 < \frac{1}{e^{1/x} + 1} < \frac{1}{e^{1/x}} < x,$$

Which implies that $\lim_{x \to 0+} h = 0$.

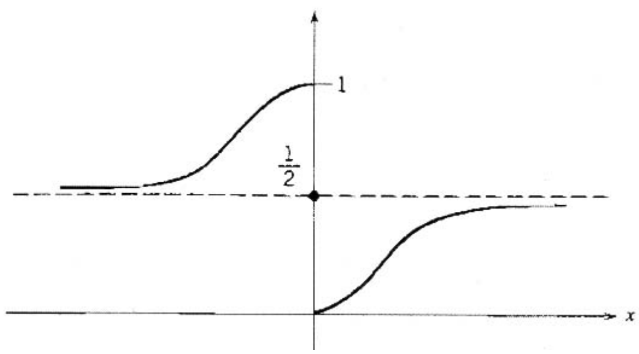

Figure 4.3.2 Graph of $h(x) = 1/(e^{1/x} + 1)$ $(x \neq 0)$.

Since we have in part (b) that $\lim_{x \to 0-} e^{1/x} = 0$, it follows from the analogue of Theorem 4.2.4 (b) for left-hand limits that

$$\lim_{x \to 0-}\left(\frac{1}{e^{1/x} + 1}\right) \frac{2}{\lim_{x \to 0-} e^{\frac{1}{x}} + 1} = \frac{1}{0+1} = 1.$$

Note that for this function, both one-sided l9mits exist in R, but they are unequal.

Infinite Limits

The function $f(x) := 1/x^2$ for $x \neq 0$ (see Figure 4.3.3) is not bounded on a neighborhood of 0, so it cannot have a limit in the sense of Definition 4.1.4. While the symbols ∞ $(= +\infty)$ and $-\infty$ do not represent real numbers, it is sometimes useful to be able to say that "$f(x) = 1/x^2$ tends to ∞ as $x \to 0$". This use of $\pm\infty$ will not cause any difficulties, provided we exercise caution and *never* interpret ∞ or $-\infty$ as being real numbers.

Figure 4.3.3 Graph of $f(x) = 1/x^2$ $(x \neq 0)$

Figure 4.3.4 Graph of $g(x) = 1/x$ $(x \neq 0)$

4.3.5 Definition Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and let $c \in \mathcal{R}$ be a cluster point of A.

(i) We say that f **tends to** ∞ **as** $x \to c$, and write

$$\lim_{x \to c} f = \infty,$$

if for every $\alpha \in \mathcal{R}$ there exists $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

(ii) We say that f **tends to** $-\infty$ **as** $x \to c$, and write

$$\lim_{x \to c} f = -\infty$$

if for every $\beta \in \mathcal{R}$ there exists $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with 0 $|x - c| < \delta$, then $f(x) < \beta$.

4.3.6 Examples **(a)** $\lim (1/x^2) = \infty$.

For, if $\alpha > 0$ is given, let $\delta := 1/\sqrt{\alpha}$. It follows that if $0 < |x| < \delta$, then $x^2 > 1/\alpha$ so that $1/x^2 > \alpha$.

(b) Let $g(x) := 1/x$ for $x \neq 0$. (See Figure 4.3.4.)

The function g does *not*to either ∞ or $-\infty$ as $x \to 0$. For, if $\alpha > 0$ then $g(x) < \alpha$ for all $x <$ o, so that g does not tend to ∞ as $x \to 0$. Similarly, if $\beta < 0$ then $g(x) > \beta$ for all $x > 0$, so that g does not tend to $-\infty$ as $x \to 0$. □

Where many of the results in Units 4.1 and 4.2 have extensions to this limiting notion, not all of them do since $\pm\infty$ are not real numbers. The following result is an analogue of the Squeeze Theorem 4.2.7. (See also Theorem 3.6.4.)

4.3.7 Theorem Let $A \subseteq \mathcal{R}$, let $f, g : A \to \mathcal{R}$, and let $c \in \mathcal{R}$ be a cluster point of A. Suppose that $f(x) \leq g(x)$ for all $x \in A$, $x \neq c$.

(a) If $\lim_{x \to c} f = \infty$, then $\lim_{x \to c} g = \infty$.

(b) If $\lim_{x \to c} g = -\infty$ then $\lim_{x \to c} f = -\infty$.

proof. (a) If $\lim_{x \to c} f = \infty$ and $\alpha \in \mathcal{R}$ is given, then there exists $\delta(\alpha) > 0$ such that if $0 < |x - c|$

$<\delta(\alpha)$ and $x \in A$, then $f(x) > a$. But since $f(x) \le g(x)$ for all $x \in A$, $x \ne c$, it follows that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $g(x) > \alpha$. Therefore $\lim_{x \to c} g = \infty$.

The proof of (b) is similar. Q.E.D.

The function $g(x) = 1/x$ considered in Example 4.3.6 (b) suggests that it might be useful to consider one-sided infinite limits. We will define only right-hand infinite limits.

4.3.8 Definition Let $A \subseteq \mathcal{R}$ and let $f : A \to \mathcal{R}$. If $c \in \mathcal{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A: x > c\}$, then we say that f **tends to** ∞ [respectively, $-\infty$] as $x \to c+$, and we write

$$\lim_{x \to c+} f = \infty \left[\text{respectively, } \lim_{x \to c+} f = -\infty\right],$$

if for every $\alpha \in \mathcal{R}$ there is $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

4.3.9 Examples (a) Let $g(x) := 1/x \ne 0$. We have noted in Example 4.3.6 (b) that $\lim_{x \to 0} g$ does not exist. However, it is an easy exercise to show that.

$$\lim_{x \to 0+} (1/x) = \infty \quad \text{and} \quad \lim_{x \to 0-} (1/x) = -\infty$$

(b) It was seen Example 4.3.4 (b) that the function $g(x) := e^{1/x}$ for $x \ne 0$ is not bounded on any interval $(0, \delta)$, $\delta > 0$. Hence the right-hand limit of $e^{1/x}$ as $x \to 0+$ does not exist in the sense of Definition 4.3.1 (i). However, since

$$1/x < e^{1/x} \qquad \text{for} \qquad x > 0,$$

It is readily seen that $\lim_{x \to 0+} e^{1/x} = \infty$ in the sense of Definition 4.3.8 □

Limits at Infinity

It is also desirable to define the notion of the limit of a function as $x \to \infty$ The definition as $x \to -\infty$ is similar.

4.0 CONCLUSION

4.3.10 Definition Let $A \subseteq \mathcal{R}$ and let $f : A \to \mathcal{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathcal{R}$. We say that $L \in \mathcal{R}$ is **a limit of** f **as** $x \to \infty$, and write

$$\lim_{x \to \infty} f = L \qquad \text{or} \qquad \lim_{x \to \infty} f(c) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > a$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

The reader should note the close resemblance between 4.3.10 and the definition of a limit of a sequence.

We leave it to the reader to show that the limits of f as $x \to \pm\infty$ are unique whenever they exist. We also have sequential criteria for these limits; we shall only state the criterion as $x \to \infty$. This uses the notion of the limit of a properly divergent sequence (see Definition 3.6.1).

4.3.11 Theorem Let $A \subseteq \mathcal{R}$, let $f : A \to \mathcal{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathcal{R}$. Then the following statements are equivalent:

(i) $L = \lim_{x \to \infty} f$.

(ii) For every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim(x_n) = \infty$, the sequence $(f(x_n))$ converges to L.

We leave it to the reader to prove this theorem and to formulate and prove the companion result concerning the limit as $x \to -\infty$.

4.3.12 Examples (a) Let $g(x) := 1/x$for $x \neq 0$.
It is an elementary exercise to show that $\lim_{x \to \infty}(1/x) = 0 = \lim_{x \to -\infty}(1/x)$. (See Figure 4.3.4.)

(b) Let $f(x) := 1/x^2$ for $x \neq 0$.
The reader may show that $\lim_{x \to \infty}(1/x) = 0 = \lim_{x \to -\infty}(1/x^2)$. (See Figure 4.3.3.) One way to do this is to show that if $x \geq 1$ then $0 \leq 1/x^2 \leq 1/x$. In view of part (a), this implies that $\lim_{x \to \infty}(1/x^2) = .0$ □

Just as it is convenient to be able to say that $f(x) \to \pm\infty$ as $x \to c$ for $c \in \mathcal{R}$, it is convenient to have the corresponding notion as $x \to \pm\infty$. We will treat the case where $x \to \infty$.

4.3.13 Definition Let $A \subseteq \mathcal{R}$ and let $f : A \to \mathcal{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say that f **tends to** ∞ [respectively, $-\infty$] **as** $x \to \infty$, and write

$$\operatorname{Lim}_{x \to \infty} f = \infty \qquad \left[\text{respectively, } \lim_{x \to \infty} f = -\infty\right]$$

if given any $\alpha \in \mathcal{R}$ there exists $K = K(\alpha) > a$ such that for any $x > K$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

As before there is a sequential criterion for this limit.

4.3.14 Theorem Let $A \in \mathcal{R}$, let $f : A \to \mathcal{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathcal{R}$. Then the following statements are equivalent:

(i) $\lim_{x \to \infty} f = \infty$ [respectively, $\lim_{x \to \infty} f = -\infty$].

(ii) For every sequence (x_n) in (a, ∞) such that $\lim(x_n) = \infty$, then $\lim(f(x_n)) = \infty$ [respectively, $\lim(f(x_n)) = -\infty$].
The next result is an analogue of Theorem 3.6.5.

5.0 SUMMARY

4.3.15 Theorem Let $A \subseteq \mathscr{R}$, let $f, g : A \to \mathscr{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathscr{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathscr{R}$, $L \neq 0$, we have

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

(i) If $L > 0$, then $\lim_{x \to \infty} f = \infty$ if and only if $\lim_{x \to \infty} g = \infty$.

(ii) If $L < 0$, then $\lim_{x \to \infty} f = -\infty$ if and only if $\lim_{x \to \infty} g = \infty$.

Proof. (i) Since $L > 0$, the hypothesis implies that there exists $a_1 > a$ such that

$$0 < \tfrac{1}{2}L \leq \frac{f(x)}{g(x)} < \tfrac{3}{2}L \qquad \text{for} \qquad x > a_1.$$

Therefore we have $\left(\frac{1}{2}L\right)g(x) < f(x) < \left(\frac{3}{2}L\right)g(x)$ for all $x > a_1$, from which the conclusion follows readily.

The proof of (ii) is similar. Q.E.D.

We leave it to reader to formulate the analogous result as $x \to -\infty$.

4.3.16 Examples (a) $\lim_{x \to \infty} x^n = \infty$ for $n \in \mathscr{N}$.

Let $(x) := x^n$ for $x \in (0, \infty)$. Given $\alpha \in \mathscr{R}$, let $K := \sup\{1, \alpha\}$. Then for all $x > K$, we have $g(x) = x^n \leq x > \alpha$. Since $\alpha \in \mathscr{R}$ is arbitrary, it follows that $\lim_{x \to \infty} g = \infty$.

(b) $\lim_{x \to -\infty} x^n = \infty$ for $n \in \mathscr{N}$, n even, and $\lim_{x \to -\infty} x^n = -\infty$ for $n \in \mathscr{N}$, n odd.

We will treat the case n odd, say $n = 2k + 1$ with $k = 0, 1, \cdots$. Given $\alpha \in \mathscr{R}$, let $K := \inf\{\alpha, -1\}$. For any $x < K$, then since $(x^2)^k \geq 1$, we have $x^n = (x^2)^k\, x \leq x < \alpha$. Since $\alpha \in \mathscr{R}$ is arbitrary, it follows that $\lim x^n = -\infty$.

(c) Let $p : \mathscr{R} \to \mathscr{R}$ be the polynomial function

$$P(x) := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Then $\lim_{x \to \infty} p = \infty$ if $a_n > 0$, and $\lim_{x \to \infty} p = -\infty$ if $a_n < 0$.

Indeed, let $g(x) := x^n$ and apply Theorem 4.3.15. Since

$$\frac{p(x)}{g(x)} = a_n + a_{n-1}\left(\frac{1}{x}\right) + \cdots + a_1\left(\frac{1}{x^{n-1}}\right) + a_0\left(\frac{1}{x^n}\right),$$

It follows that $\lim_{x \to \infty}(p(x) / g(x)) = a_n$. Since $\lim_{x \to \infty} g = \infty$, the assertion follows from Theorem 4.3.15.

(d) Let p be the polynomial function in part (c). Then $\lim_{x \to \infty} p = \infty$ [respectively, $-\infty$] if n is even [respectively, odd] and $a_n > 0$.

We leave the details to the reader. □

6.0 TUTOR MARKED ASSIGNMENT

1. Prove Theorem 4.3.2.

2. Give an example of a function that has a right-hand limit but not a left-hand limit at a point.

3. Let $f(x) := |x|^{-1/2}$ for $x \neq 0$. Show that $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = +\infty$.

4. Let $c \in \mathbb{R}$ and let f be defined for $x \in (c, \infty)$ and $f(x) > 0$ for all $x \in (c, \infty)$. Show $\lim_{x \to c} f = \infty$ if and only if $\lim_{x \to c} 1/f = 0$

5. Evaluate the following limits, or show that they do not exist.

   (a) $\lim_{x \to 1+} \frac{1}{x-1}$ $(x \neq 1)$, (b) $\lim_{x \to 1} \frac{x}{x-1}$ $(x \neq 1)$,

   (c) $\lim_{x \to 0+} (x + 2) / \sqrt{x}$ $(x > 0)$, (d) $\lim_{x \to \infty} (x + 2) / \sqrt{x}$ $(x > 0)$,

   (e) $\lim_{x \to 0} (\sqrt{x+1}) / x$ $(x > -1)$, (f) $\lim_{x \to \infty} (\sqrt{x+1}) / x$ $(x > 0)$,

   (g) $\lim_{x \to \infty} \frac{\sqrt{x-5}}{\sqrt{x+3}}$ $(x > 0)$, (h) $\lim_{x \to \infty} \frac{\sqrt{x-x}}{\sqrt{x+x}}$ $(x > 0)$.

6. Prove Theorem 4.3.11.

7. Suppose that f and g have limits in $\mathbb{R}$ as $x \to \infty$ and that $f(x) \leq g(x)$ for all $x \in (a, \infty)$. Prove that $\lim_{x \to \infty} f \leq \lim_{x \to \infty} g$

8. Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Prove that $\lim_{x \to \infty} f(x) = L$ if and only if $\lim_{x \to 0+} f(1/x) = L$.

9. Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x \to \infty} x f(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x \to \infty} f(x) = 0$.

10. Prove Theorem 4.3.14.

11. Suppose that $\lim_{x \to c} f(x) = L$ where $L > 0$, and that $\lim_{x \to c} g(x) = \infty$. Show that $\lim_{x \to c} f(x) g(x) = \infty$. If $L = 0$, show by example that this conclusion may fail.

12. Find function f and g defined on $(0, \infty)$ such that $\lim_{x \to \infty} f = \infty$ and $\lim_{x \to \infty} g = \infty$, and $\lim_{x \to \infty} (f - g) = 0$. Can you find such functions, with $g(x) > 0$ for all $x \in (0, \infty)$, such that $\lim_{x \to \infty} f / g = 0$?

13. Let f and g be defined on (a, ∞) and suppose $\lim_{x \to \infty} f = L$ and $\lim_{x \to \infty} g = \infty$. Prove that $\lim_{x \to \infty} f \circ g = L$.

7.0 REFERENCES / FURTHER READINGS