NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 381

COURSE TITLE: MATHEMATICAL METHODS III

COURSE GUIDE

MTH 381
MATHEMATICAL METHODS III

Course Team Dr. S.O. Ajibola (Course Developer/Writer) – NOUN
Dr. Bankole Abiola (Course Coordinator) – NOUN
Prof. Adebanjo A. (Programme Leader) – NOUN

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island, Lagos

Abuja Office
No. 5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@nou.edu.ng
URL: www.nou.edu.ng

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INTRODUCTION

The course, Mathematical Methods 111 is meant to provide essential methods for solving mathematical problems.

In scientific problems, often times are discovers that a factor depends upon several other related factors. For instance, the area of solid depends on its length and breadth. Potential energy of a body depends on gravity, density and height of the body e.t.c. Moreover, the strength of a material depends on temperature, density, isotropy and softness e.t.c.

WHAT YOU WILL LEARN IN THIS COURSE

This is a 3-credit course, it is grouped into 4 modules i.e. module1, 2, 3 and 4. Module 1 has 2 units; module 2 also has 2 units as well as module 3 with only one unit while module 4 has 3 units. In summary, the course has 4 modules and 8 units in all.

The course guide gives a brief summary of the total contents contained in the course material. Functions of several variables streamline the relationship between function and variables, the application of Jacobian, down to functional dependence and independence. Also discussed here is the multiple, line, improper integrals and tensor calculus.

COURSE OBJECTIVES

At the end of this unit, you should be able to:

- identify functions of two or more variablesthe ideal of Jacobian to be extended to three variables
- use of Jacobian to change variables in multiple integral and
- determine whether two or more functions are linearly depended or independent respectively.

WORKING THROUGH THE COURSE

This course involves that you would be required to spend lot of time to read. The content of this material is dense and requires that you spend great time to study it. This accounts for the great effort put into its development in the attempt to make it readable and comprehensible. Nevertheless, the effort required of you is still tremendous.

I would advice that you avail yourself the opportunity of attending the tutorial sessions where you would have the opportunity of comparing knowledge with your peers.

COURSE MATERIALS

You will be provided with the following materials:

1. Course Guide
2. Study Units

In addition, the course comes with a list of recommended textbooks, which though are not compulsory for you to acquire or indeed read, but are necessary as supplements to the course material.

STUDY UNITS

The following are the study units contained in this course. The units are arranged into four identifiable but related modules.

Module 1 Functions of Several Variables

Unit 1 Some Basic Concepts
Unit 2 Vector Field Theory

Module 2

Unit 1 Functions of Complex Variables
Unit 2 Integration of Complex Plane

Module 3

Unit 1 Residue Integration Method

Module 4

Unit 1 Integral Transform
Unit 2 Fourier Series and its Application
Unit 3 The Laplace Transform

TEXTBOOK AND REFERENCES

The following editions of these books are recommended for further reading.

Advance Engineering Mathematics by KREYSZIC.
Generalized Functions by R. F. Hoskins.
Complex Variables by Murray R. Spiegel.
Engineering Mathematics by K. A. Stroud.
Advance Calculus for Applications by F. B. Hildraban

ASSESSMENT

There are two components of assessment for this course. The Tutor-Marked Assignment (TMA), and the end of course examination.

TUTOR-MARKED ASSIGNMENT

The (TMA) is the continuous assessment component of your course. It accounts for 30 per cent of the total score. You will be given four (4) TMAS' to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs' would be given to you by your facilitate and returned after you have done the assignment.

FINAL EXAMINATIONS AND GRADING

This examination concludes the assessment for the course. It constitutes 70 per cent of the whole course. You will be informed of the time the examination. It may or may not coincide with the University Semester Examination.

SUMMARY

The students have been taught how to use Jacobian method to change the variable – multiple integral, also to determine whether two functions are linearly dependent or independent.

Solve line, multiple and improper integrals.
The use of Fourier transform to solve some differential equation, boundary values problems and e.t.c. Also talked about is Laplace transformation to solve some initial and boundary value problem, which are difficult to handle. After which Convolution theory is applied. And the result's then retrains-formed back to physical or mechanical problems.

So far, about three methods have been thoroughly dealt with in this course. In Mathematical Methods IV, we shall still talk about several other methods to handle any category of problem, provided the problem can be modeled into mathematical problems.

MAIN COURSE

CONTENTS PAGE

MODULE 1 FUNCTIONS OF SEVERAL VARIABLES

UNIT 1 SOME BASIC CONCEPTS

CONTENTS

1.0 INTRODUCTION

In scientific problems, often times one discovers that a factor depends upon several other related factors. For instance, the area of rectangle depends on its length and breath, hence can say that area is the function of two variables i.e. its length and breadth. Potential energy of a body depends on gravity, density and height of the body, hence, we can also say that potential energy is a function of three variables i.e gravity, density and height etc. The strength of a material depends upon temperature, density, isotropy softness etc., here we can say that the strength of material is a function of many variables i.e. temperature, density, isotropy softness etc.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use Jacobian change variables in multiple integral
- determine whether two or more functions are linearly dependent or independent
- identify the functions of two or more variables.

3.0 MAIN CONTENT

3.1 Functions of Several Variables

A function is composed of a domain set, a range set and a rule of correspondence that assigns exactly one element of the range to each element of the domain u, is called a function of two variables x and y if u has one definite value for every pair of variables of x and y. Symbolically, it is written as

$u = f(x, y)$.

The variables x and y are called independent variables while u is called the dependent variable.

Similarly, we can define u as a function of more than two variables.

In summary, we have that

$u(x) \Rightarrow$ a function of a single variable
$u(x_1, x_2) \Rightarrow$ a function of two variables
$u(x_1, x_2, x_3 \cdot \cdot \cdot \cdot \cdot \cdot x_n) \Rightarrow$ a function of several variables.

Example 1

If $f(x, y) = x^2 - 3xy + 6y$, find : (a) f(-1,1) and f(2,3).

(a) $f(x, y) = x^2 - 3xy + 6y$

$$f(-1,1) = (-1)^2 - 3(-1)(1) + 6(1)$$
$$f(-1,1) = 1 + 3 + 6 = 10$$

(b) $f(2,3) = 2^2 - 3(2)(3) + 6(3)$

$$f(2,3) = 4 - 18 + 18 = 4$$

3.2 Jacobian

Jacobian is a functional determinant (whose elements are functions) which is very useful in transformation of variables from Cartesian to polar, cylindrical and spherical coordinates in multiple integras. Let u(x,y) and v(x,y) be two given functions of two independent variables x and y.

The Jacobian of u and v with respect to x,y denoted by $J\left(\frac{u \quad v}{x \quad y}\right) or \frac{\partial(u,v)}{\partial(x,y)}$ is a second order functional determinant defined as

$$J\left(\frac{u \quad v}{x \quad y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Properties of Jacobians

If u and v are the functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} x \frac{\partial(x,y)}{\partial(u,v)} = 1$$

If u,v are the functions of r,s where r,s are functions of x, y, then, $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} x \frac{\partial(r,s)}{\partial(x,y)}$

If functions u, v, w of three independent variables x,y,z are not independent, then, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Example 2

Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ in each of the following:

(i) $u = x + \frac{y^2}{x}, \ v = \frac{y^2}{x}$

(ii) $u = x^2 + y^2, \ v = 2xy$

Solution.

$$u = x + \frac{y^2}{x},\ v = \frac{y^2}{x}, \text{ using } J\left(\frac{u \quad v}{x \quad y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J\left(\frac{u \quad v}{x \quad y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} (1 - \frac{y^2}{x^2}) & (\frac{2y}{x}) \\ (-\frac{y^2}{x^2}) & (\frac{2y}{x}) \end{vmatrix}$$

$$= \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

Solution

$$u = x^2 + y^2,\ v = 2xy, \text{ using } J\left(\frac{u \quad v}{x \quad y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

= (2x)(2x)-(2y)(-2y)

$= 4x^2 + 4y^2$

$= 4(x^2 + y^2)$

Example 3

If u=xyz, $v = x^2 + y^2 + z^2$, w=x+y+z ,find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution

Since u, v,w are explicitly given, so , first we evaluate

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

= yz(2y-2z)-zx(2x-2z)+xy(2x-2y)

=2[yz(y-z)-zx(x-z)+xy(x-y)]

$= 2[x^2y - x^2z - xy^2 + xz^2 + y^2z - yz^2]$

$= 2[x^2(y - z) - x(y^2 + z^2) + yz(y - z)]$

$= 2(y - z)[x^2 - x(y + z) + yz]$

$= 2(y - z)[y(z - x) - x(z - x)]$

$= 2(y - z)(z - x)(y - x)$

$= -2(x - y)(y - z)(z - x)$ The idea can be easily extended to three or several variables thus:

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example 4

Jacobian can be applied to polar coordinate r and θ, thus, $x = rCos\theta$ and $y = rSin\theta$.

Then,

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \qquad (1)$$

But $\frac{\partial x}{\partial r} = Cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ (2)

$\frac{\partial y}{\partial r} = Sin\theta$ and $\frac{\partial y}{\partial \theta} = rCos\theta$

Substituting equation (2) into (1) gives

$$J = \begin{vmatrix} Cos\theta & -rSin\theta \\ Sin\theta & rCos\theta \end{vmatrix}$$

$$= rCos^2\theta - (-rSin^2\theta)$$

$$= r[Coc^2\theta + Sin^2\theta] = r$$

Since $Cos^2\theta + Sin^2\theta = 1$

$$\therefore \quad J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

3.3 Function Dependence and Independence

Two functions $u(x)$ and $v(x)$ defined on an interval $0 < x < 1$ are said to be functionally (linearly) dependent on $0 < x < 1$ if there exist '$\exists$' two constants k_1 and k_2 where not both zero, such that'$\exists$'

$$k_1 u(x) + k_2 v(x) = 0 \text{ for } x, \forall x. \qquad \text{(i)}$$

However, the two functions $u(x)$ and $v(x)$ defined on interval $0 < x < 1$ are said to be functionally (linearly) independent on $0 < x < 1$, if the only constants k_1 and k_2 such that'$\exists$' for all x in the interval where both constants k_1 and k_2 are zeros i.e, when u or v can not be expressed as proportional to the other. Otherwise, u and v are linearly dependent if (i) holds for some k_1 and k_2 not both zero.

Example 5

Show that the functions $v(x) = e^{ax}$ and $u(x) = e^{bx}$ are linearly dependent on the interval. $0 < x < 1$.

Solution

Suppose $k_1 e^{ax} + k_2 e^{bx} = 0 \quad \forall x$ in $0 < x < 1$ (1)

Multiplying equation (1) by e^{-ax}, we obtain

$$k_1 e^{ax} e^{-ax} + k_2 e^{bx} e^{-ax} = 0 \qquad (2)$$

$$k_1 + k_2 e^{(b-a)x} = 0 \qquad (3)$$

differentiating equation (3) we obtain

$$(b-a)k_2 e^{(b-a)x} = 0 \qquad (4)$$

$(b-a)e^{(b-a)x} \neq 0$ since $b - a \neq 0$ then it implies that

$$b = 0 \qquad (5)$$

Substituting (5) into (1), and differentiating w.r.t.x, we obtain

$$k_1 a e^{ax} = 0 \qquad (6)$$

$\Rightarrow \quad a = 0$, since $e^{ax} \neq 0$.

Example 6

Show that the functions $v(x) = e^{ax}$ and $u(x) = e^{bx}$ are linearly independent on the interval. $0 < x < 1$.

Solution: If

$$k_1 e^{ax} + k_2 x e^{ax} = 0 \quad (1)$$

$$(k_1 + k_2 x) e^{ax} = 0 \quad (2)$$

Since $e^{ax} \neq 0$, $\Rightarrow \quad k_1 + k_2 x = 0$ (3)

Differentiating equation (3) we obtain

$$k_1 = 0 \quad (4)$$

Substituting (5) into (1), however

$$k_1 e^{ax} = 0 \quad \Rightarrow \quad k_1 = 0 \text{.Since } e^{ax} \neq 0 \quad (5)$$

3.3.1 Testing For Linear Dependence or Otherwise

A method called Wronskian of the function could also be used to test for linear dependence or otherwise. Thus, consider the functions $u(x)$ and $v(x)$ and the first derivatives $u'(x)$ and $v'(x)$, therefore we can define the Wronski determinant or Wroskian.:

$$\text{Wronskian } = W\big(v(x).u(x)\big) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix}$$
$$= v(x)u'(x) - u(x)v'(x)$$

Results:
v(x), u(x) are linearly independent if $W \neq 0$
Otherwise linearly dependent when W=0.

Example 7

Determine whether the following functions v(x) and u(x) are linearly dependent or independent.

v(x) =cosbx, u(x) = sinbx with $b \neq 0$
v(x) = e^{ax}, $u(x) = e^{-ax}$ with $a \neq 0$.

Solution

$v(x) = \cos bx$, $v'(x) = -b\sin bx$, $u(x) = \sin bx$ and $u'(x) = \cos bx$.

(a) $$W(v(x).u(x)) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix} = \begin{vmatrix} \cos bx & \sin bx \\ -\sin b & b\cos bx \end{vmatrix}$$

$$= b(\cos^2 bx + \sin^2 bx)$$
$$= b \neq 0$$

So v(x) and u(x) are linearly independent.

$v(x) = e^{ax}$, $v'(x) = -ae^{ax}$, $u(x) = e^{-ax}$ and $u'(x) = -ae^{-ax}$.

(b) $$W(v(x).u(x)) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix} = \begin{vmatrix} e^{ax} & e^{-ax} \\ ae^{ax} & -ae^{-ax} \end{vmatrix}$$
$$= -ae^0 - ae^0$$
$$= -a(e^0 + e^0)$$

With $a \neq 0$. So v(x) and u(x) are linearly dependent.

SELFASSESSMENT EXERCISE

Determine whether the following pair of functions are linearly dependent as the case may be

i. (a) $u(x) = x, \quad v(x) = e^{2x}$
(b) $u(x) = 2Sinhx, \quad v(x) = Cosx$
(c) $u(x) = x^3, \quad v(x) = 3x^3$

ii. (a) Show that the function $u(x)$ and $v(x)$ defined by are linearly
$u(x) = x^2, \quad v(x) = x|x|$ Independent for the interval $0 < x < 1$.
Compute the Wronskian of these functions.

iii. If $f(x, y) = x^4 - 2xy + 4y^2$,
Find (a) $f(1,-1)$, (b) $f(0,-3)$ and
(c) $\dfrac{f(x, y+k) - f(x, y)}{12}$

iv. If $f(x, y) = \dfrac{4x + 2y}{2 - 2xy}$, Find (a) $f(1,-3)$, (b) $\frac{f(2+h,3)-f(2,3)}{h}$

v. If $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$.

Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$.

vi. If $u = x^2, v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$

3.4 Multiple Integral

3.4.1 Double Integral

Definition: In this case the integrand is a function $f(x,y)$ that is given for all (x,y) in a closed bounded region R of the $x-y$ plane.

Let $f(x,y)$ be a single valued continuous function within a region R bounded by a close curve C . Then the region R is called

The region of integration. However, double integral can be defined thus:

$$\int_c^d \int_a^b f(x,y)dxdy \text{ or } \iint_r f(x,y)dA \qquad (1)$$

3.4.4.1 Evaluation of Double Integrals

Consider $a \le x \le b$ and $g(x) \le y \le h(x)$ so that $y = g(x)$ and $y = h(x)$ represents the boundary of R . Then

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\right]dx \qquad (2)$$

Similarly, if R can be described thus

$c \le y \le d$, $v(y) \le x \le u(y)$

So that $x = v(y)$ and $x = u(y)$. Then

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_{v(y)}^{u(y)} f(x,y)dx\right]dy \qquad (3)$$

In this case, one first calculates the integral within the square brackets. Then further integration is then performed.

Properties of Double Integrals

1. $\iint_D af(x,y)ds = a\iint_D f(x,y)ds$, a =constant

2. $\iint_D [f(x,y) + g(x,y)]ds = \iint_D f(x,y)ds + \iint_D g(x,y)ds$

3. $\iint_D f(x,y)ds = \iint_{D1} f(x,y)ds + \iint_{D2} f(x,y)ds$

Were D is the union of disjointed domains D1 and D2

Example 5

Evaluate the integrals

$$\int_0^1\int_0^1 (x^2 + y^2)dydx$$

Solution

$$\int_0^1 \left[\int_0^1 (x^2 + y^2)dy\right]dx$$

$$=\int_0^1 \left[x^2 y + \tfrac{1}{3}y^3\right]_0^1 dx$$

$$= \int_0^1 \left[(x^2 + \tfrac{1}{3}) - 0\right]dx = \int_0^1 (x^2 + \tfrac{1}{3})dx$$

$$= \frac{1}{3}x^3 + \frac{1}{3}x\Big|_0^1 = \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

3.4.4.2 Double Integral in Polar Coordinates

This is defined by

$$\int_{\theta_2}^{\theta_1}\int_{r_2}^{r_1} f(r,\theta)drd\theta$$

Example 6

Evaluate the integrals

$$\int_{-\pi/2}^{\pi/2}\int_0^{2Cos\theta} r^2 drd\theta.$$

Solution

$$\int_{-\pi/2}^{\pi/2}\left(\int^{2Cos\theta} r^2\,dr\right)d\theta = I \qquad (1)$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2Cos\theta} d\theta \qquad (2)$$

$$= \int_{-\pi/2}^{\pi/2}\frac{(2Cos\theta)^3}{3}d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{8}{3}Cos^3\theta\, d\theta \qquad (3)$$

Using trigonometric identity to simplify $Cos^3\theta$

Thus $Cos3\theta = Cos(2\theta+\theta) = Cos2\theta Cos\theta - Sin2\theta Sin\theta$

$$= (Cos^2\theta - Sin^2\theta)Cos\theta - (2Sin\theta Cos\theta)Sin\theta$$

$$= Cos^3\theta - Sin^2\theta Cos\theta - 2Sin^2\theta Cos\theta$$

$$= Cos^3\theta - 3Sin^2\theta Cos\theta$$

$$= Cos^3\theta - 3\left[1 - Cos^2\theta\right]Cos\theta$$

$$= Cos^3\theta - 3Cos\theta + Cos^3\theta$$

$$= 4Cos^3\theta - 3Cos\theta$$

$$\therefore \quad Cos^3\theta = \frac{1}{4}Cos3\theta + \frac{3}{4}Cos\theta \qquad (4)$$

Hence, substituting (4) into (3) we obtain

$$I = \frac{8}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{4}Cos3\theta + \frac{3}{4}Cos\theta\right)d\theta$$

$$I = -\frac{2}{3}\left[\frac{1}{3}Sin3\theta + 3Sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{2}{3}\left[\left(\frac{1}{3}Sin\tfrac{3}{2}\pi + 3Sin\tfrac{\pi}{2}\right) - \left(\frac{1}{3}Sin(-\tfrac{3}{2}\pi) + 3Sin(-\tfrac{\pi}{2})\right)\right] \qquad (5)$$

But $Sin\,^3\!/_2\pi = -1, \quad Sin\,^\pi\!/_2 = 1$

Similarly, $Sin - {}^3\!/_2\pi = -1$ and $Sin - {}^\pi\!/_2 = -1 \qquad (6)$

Substituting (6) into (5)

$$I = -\frac{2}{3}\left[\left(\frac{1}{3}(-1) + 3\right) - \left(\frac{1}{3}(1) + 3(-1)\right)\right]$$

$$= - \frac{3}{\ } + 3 - \frac{1}{\ } + 3$$

$$3 \qquad 3$$

$$3$$

$$= -\frac{2}{3}\left(\frac{8}{3}+\frac{8}{3}\right) = -\frac{2}{3}\left(\frac{16}{3}\right) = -\frac{32}{9}$$

$$I = -3\frac{5}{9}$$

3.4.4.3 Triple Integral

Definition: A function of three variables is involved in triple integral. However, in triple integral, integration is carried out thrice. It is then define as:

$\iiint_v f(x,y,z)dxdydz$ over the region v

$\int_v f(x,y,z)dv$. This can also be used to find the volume of any shape.

Example 7

Evaluate

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz$$

Solution

$$\int_{-1}^{1}\left[\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(xy+\tfrac{1}{2}y^2zy\right)_{x-z}^{x+z}dx\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(\left[x(x+z)+\tfrac{1}{2}(x+z)^2+z(x+z)\right]-\left[x(x-z)+\tfrac{1}{2}(x-z)^2+z(x-z)\right]\right)\right]dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z}\left(4xz+2z^2\right)dx\right]dz$$

$$= \int_{-1}^{1}\left[2x^2z+2xz^2\right]_{0}^{z} dz$$

$$= \int_{-1}^{1}4z^3dz = z^4\Big|_{-1}^{1} = 1-1=0$$

$$= 0$$

Example 8

Evaluate

$I = \iiint_v \left(3x^2 + 3y^2 + 3z^2\right) dv$ by changing to polar coordinate.

Thus $x = rSin\theta Cos\varphi$, $y = rSin\theta Cos\varphi$ and $z = rCos\theta$.

Solution

$$I = 24\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2 dr(rSin\theta d\varphi)(rd\theta)$$
$$= \frac{24}{5}\int_0^{\pi/2}\int_0^{\pi/2} a^5 Sin\theta d\theta d\varphi$$
$$= \frac{24}{5}a^5\int_0^{\pi/2}(-Cos\theta)_0^{\pi/2} d\varphi$$
$$= \frac{24}{5}a^5 \cdot \frac{\pi}{2} = \frac{24}{5}a^5\pi .$$

4.0 CONCLUSION

In conclusion, the student should be able to use Jacobian method to change the variable in multiple integral and to determine whether two functions are linearly dependent or independent. Also to solve integral, multiple.

5.0 SUMMARY

The following are discussed in the unit:

Functions of variable defined thus, $u(x_1, x_2, x_3 \cdots x_n)$. Jacobian of (uv) was discussed and extend it to three or several variables, thus

$$J\left(\frac{u\ \ v}{x\ \ y}\right) = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{(\partial x,y,z)}$$

Jacobian was also applied to polar coordinate thus

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r .$$

The functional dependence of two functions $u(x)$ and $v(x)$ was discussed thus:

$k_1u(x) + k_2v(x) = 0$, $\forall x$ where k_1 and k_2 are constants and are not zero.

While the functional independence of two functions $u(x)$ and $v(x)$ was also discussed thus:

$$k_1u(x) + k_2v(x) = 0 \quad \forall x, \text{ where } k_1 = k_2 = 0.$$

Testing for linear (independence) dependent was discussed using Wronskian method which involves the determinant thus

$$W(v(x), u(x)) = v(x)u'(x) - u(x)v'(x) = \begin{vmatrix} v(x) & u(x) \\ v'(x) & u'(x) \end{vmatrix}$$

Lastly, multiple integral was discussed.

6.0 TUTOR-MARKED ASSIGNMENT

i. Evaluate the double integrals

(a) $\int_{-\pi}^{\pi} \int_{-1}^{1} xydxdy$

(b) $\int_{1}^{2} \int_{-x}^{x} e^y Coshxdydx$

(c) $\int_{1}^{2} \int_{y}^{y^2+1} x^2ydxdy$

ii. Evaluate the following triple integral

(a) $\int_{-\pi}^{\pi} \int_{0}^{2} \int_{x-z}^{x+z} (x + y + z)dxdydz$

(b) $\int\int\int \frac{dxdydz}{x^2 + y^2 + z^2}$ where $x^2 + y^2 + z^2 = a$

(c) Compute the volume of the solid enclosed by

(i) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad x = 0, y = 0, z = 0$

(ii) $x^2 + y^2 - 2ax = 0, \quad z = 0, \; x^2 + y^2 = z^2$

iii. Determine whether the following pair of functions are linearly dependent or independent as the case may be.

(a) $u(x) = x, v(x) = e^{2x}$

(b) $u(x) = 2Sinhx, v(x) = Cosx$

(c) $u(x) = x^3, v(x) = 3x^3$

iv. (a) show that the functions $u(x)$ and $v(x)$ defined by $u(x) = x^2, u(x) = x|x|$ are linearly independent for the interval $0 \le x \le 1$.

(b) Compute the Wronskian of the function in 4(a)

v. Evalute $\iint_R (x+y)^2 dxdy$, where R is a region bounded by the parallelogram x+y=0, x+y=2, 3x-2y=0, and 3x-2y=3.

vi. Evalute $\iint_R (x^2+y^2)dxdy$, where R is a region in the first quadrant bounded by $x^2-y^2=a$, $x^2-y^2=b$, 2xy=d, 0<a<b, 0<c<d

7.0 REFERENCES/FURTHER READING

Dass, H.K. (1988). *Advanced Engineering Mathematics*. New Delhi: S. Chand and Company Limited.

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spiegel (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Ramana B.V (2008). *Higher Engineering Mathematics*. New Delhi: Tata McGraw-Hill Publishing Company Limited.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 2 VECTOR FIELD THEORY

CONTENTS

1.0 INTRODUCTION

Vector function represents vector fields which have various physical and geometrical applications.

The basic concepts of differential calculus can be extended to vector function in a simple and natural fashion.

Vector functions are useful for representing and investigating curves and application in mechanics as path of moving bodies.

Integral theorems will be considered in the later path of this unit's i.e Line Integral, Gauss, Stokes and Greens theorems.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- appreciate vector field and vector function
- understand the vector field theory, using vector function to investigate curves and their applications in mechanics and
- use integral theorem to solve some physical problems. Study of Line Integral, Gauss, Stokes and Greens theorems and their applications.

3.0 MAIN CONTENT

3.1 Vector Field Theory

A scalar function is a function that is defined at each point of a certain set of points in space and whose values are real numbers depending only on the points in real space but not on the particular choice of the coordinate system.

Furthermore, the distance of $f(x, y, z)$ of any point p from a fixed point p_0 in space is a scalar function whose domain of definition D is the whole space. $f(x, y, z)$ defines a scalar field in space. Introducing a Cartesian coordinate x_0, y_0, z_0. Then the distance

$$f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

3.2 Relations between Vector Field and Functions

A vector $v(p)$ is a function that is defined on some point set D in space i.e. the set of points of a curve, a surface or a three dimensional region and associates with each point p in D a vector $v(p)$.

While a vector field is given in D. We introduce Cartesian coordinates x, y, z then we may write our vector function in terms of compound function.

$$v(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$$

or using i, j, k,. Thus

$$v(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$$

But we should keep in mind that v depends only on that points of its domain of definition, and at the point defines the same vector for every choice of the coordinate system. The velocity of a moving fluid, gravitational force are the examples of vector point function.

Our notation in simple scalar and vector quantities in the pre-requisite course mathematical methods I and II are the same with that under discussion. The only difference is that the components v_1, v_2, v_3 of v now becomes functions of x, y, z since v is a function of x, y, z.

3.2.1 Example of Vector Field (Velocity Field)

At any instant, the velocity vectors $v(p)$ of a rotating body B constitute a vector field, the so called velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotations then

$$v(x,y,z) = w \times [z,y,z] = w \times (xi + yj + zk)$$

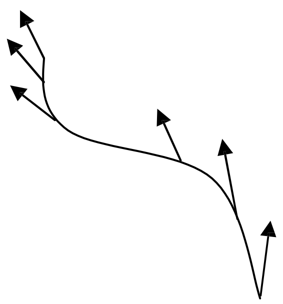


Fig. 1 Field of Tangent Vectors of a Curve

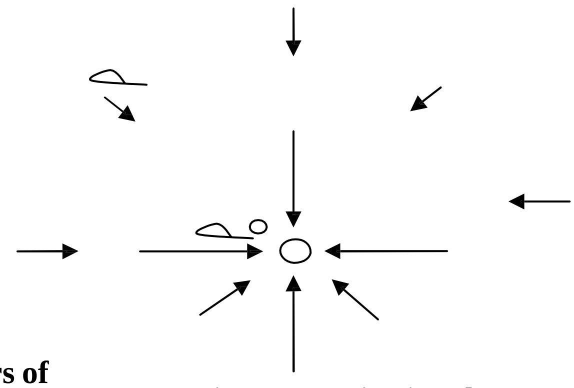


Fig. 2: Gravitational Field

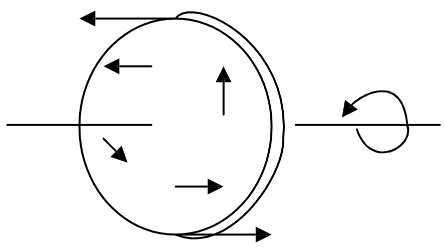


Fig. 3: A Rotating Body and the Corresponding Velocity Field

where x, y, z are the coordinates of any point p of B at the instant under consideration. If the coordinates are such that the z-axis of rotation and w points in the positive direction, then w = wk and

$$v = \begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = w(-yi + xj) = w[-y, x, 0]$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 3.

Example of Vector Field (Field of Force)

a. If the velocity at any point (x,y,z) within a moving fluid is known at a certain time, then a vector field is defined.

b. $v(x, y, z) = xyi - yz^2kj + x^2zk$ defines a vector field. A vector field which is independent of time is called a stationary steady-state vector field.

c. Let a particle A of mass M be fixed at a point p_0 and let a particle B of mass M to be free to take up various positions p in space. Then A attracts B. According to Newton's Law of gravitation, the corresponding gravitational force p is directed from p to p_0, and its magnitude is proportional to $1/r^2$ where r is the distance between p and p_0 say.

d. $$|p| = \frac{GM_A M_B}{r^2}$$

where G is the gravitational constant.

Hence p defines a vector field in space. If we introduce Cartesian coordinate such that p_0 has the coordinates x_0, y_0, z_0 and p has the coordinates x, y, z, then by Pythagoras theorem.

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \qquad (2)$$

Introducing the vector assuming $r > 0$ then

$$r = (x - x_0)i + (y - y_0)j + (z - z_0)k \qquad (3)$$

we have $|r| = r$ and $\left(-1/r\right)$ is a unit vector in the direction of p; the minus sign indicates that p is directed from $p + p_0$ .Fig. 2.

Hence substituting (1) into (3) we obtain

$$p = |p|\left(-\frac{1}{r}r\right) = -\frac{GM_A M_B}{r^3} r$$

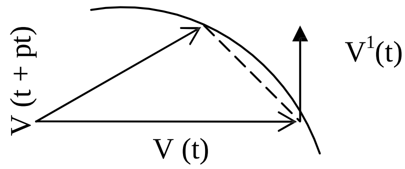
$$= -\frac{GM_A M_B}{r^3}[(x - x_0)i + (y - y_0)j + (z - z_0)k] \qquad (4)$$

Hence, this vector function describes the gravitational force acting on B.

Derivative of a Vector Function

A vector function $v(t)$ is said to be differentiable at a point t if the limit exists. The vector is called the derivative of $v(t)$.

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$



Partial Derivatives of a Vector Function

The way of introducing partial derivation to vector analysis is obvious. Indeed, let the components of a vector function.

$v = v_1 i + v_2 j + v_3 k$ be differentiable functions of n variables $t_1, t_2, t_3 \cdots t_n$. Then the partial derivative of v with respect to t is denoted by $\frac{\partial v}{\partial t}$ and is defined as the vector function.

$$\frac{\partial v}{\partial t} = \frac{\partial v_1}{\partial t} i + \frac{\partial v_2}{\partial t} j + \frac{\partial v_3}{\partial t} k$$

Example 1

Let $r(t_1, t_2) = aCot_1 i + aS\,\text{int}_1\, j + 3t_2 k$

$$\frac{\partial r}{\partial t_1} = -aS\,\text{int}_1 + aCot_1 i,$$

$$\frac{\partial r}{\partial t_2} = 3k$$

3.2.2 Line Integrals

Definition: Let $f(x)$ be a single real valued function in the interval $a \le x \le b$. Thus, we can define line integral as

$$\int_a^b f(x)dx$$

3.2.3 Evaluation of Line Integral

Evaluation of line integral $\int_a^b f(x)dx$ can be accomplished by two methods. Thus:

a. A line integral of a vector function $F(r)$ over a curve c is defined by

$$\int_c F(r)dr = \int_a^b F\big(r(t)\big).\frac{dr}{dt}dt \qquad (1)$$

b. In term of components, with $dr = dxi_i + dy_j + dz_k$

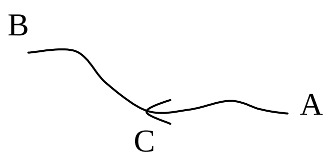
Then we obtain

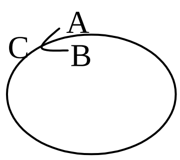
$$\int_c F(r)dr = \int_c (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_c (F_1 x' + F_2 y' + F_3 z')dt \qquad (2)$$

Where $x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad z' = \frac{dx}{dt}$ (3)

It is worth to mention that if the path of integration C in equation (1) above is a close curve that is





then.

Then instead of $\int_c$ we can also write $\oint_c$

3.2.4 General Properties of Line Integral

a. $\int_c kF.dr = k\int_c F.dr$ where k is a constant .

b. $\int_c (F + G)\cdot .dr = \int_c F\cdot .dr + \int_c G\cdot dr$

c. $\int_c F\cdot dr = \int_{c1} F\cdot .dr + \int_{c2} F\cdot .dr$

Where $c = c_1 + c_2$

3.2.5 Examples on Line Integrals

If $A = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int_c A..dr$ from (0,0,0) to (1,1,1) along the following parts C:

$x = t, y = t^2, z = t^3.$

The straight lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1).

The straight line joining (0,0,0) and (1,1,1).

Solution:

$$\int_c A..dr = \int_c [(3x^2 + 6y)i - 14yzj + 20xz^2k].(dxi + dyj + dzk)$$

$$= \int_c [(3x^2 + 6y)dx - 14yzdy + 20xz^2dz]$$

If $x = t, y = t^2, z = t^3$.,points (0,0,0) and (1,1,1) correspond to t=0 and t=1 respectively. Then

$$\int_c A..dr = \int_{t=0}^{t=1} (3t^2 + 6t^2)dt - 14(t^2)(t^3)d(t^2) + 20t(t^3)^2 d(t^3)$$

$$= \int_{t=0}^{t=1} (9t^2 - 28t^6 + 60t^9)dt$$

$$= [3t^3 - 4t^7 + 6t^{10}]_0^1 = 5$$

Along the straight line from (0,0,0) to (1,0,0) , y=0, z=0, dy=0 and dz=0 while x varies from 0 to 1. Then the integral over this point of the path is

$$\int_{x=0}^{t=1} (3x^2 + 6(0))dx - 14(0)(0)(0) + 20x(0)^2(0)$$

$$\int_{x=0}^{t=1} 3x^2dx = [x^9]_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) , x=1, z=0, dx=0 while y varies from 0 to 1. Then the integral over this point of the path is

$$\int_{y=0}^{t=1} (3(1)^2 + 6(y))0 - 14y(0)dy + 20(1)(0)^2(0) = 0$$

Along the straight line from (1,1,0) to (1,1,1) , x=1, y=1, dx=0 , dy=0 while z varies from 0 to 1. Then the integral over this point of the path is

$$\int_{z=0}^{t=1} \left(3(1)^2 + 6(1)\right)0 - 14(1)z(0)dy + 20(1)(z)^2\, dz$$

$$\int_{z=0}^{t=1} \left(20z^2\right)dz = \left[\frac{20t^3}{3}\right]_0^1 = \frac{20}{3}$$

Adding $\int_c A..dr = 1 + 0 + \frac{20}{3} = \frac{23}{3}$

The straight line joining (0,0,0) and (1,1,1) is giving in parametric form by x=t, z=t. Then

$$\int_c A..dr = \int_{t=0}^{t=1} \left(3t^2 + 6t\right)dt - 14(t)(t)d(t) + 20(t)(t^2)dt$$

$$= \int_{t=0}^{t=1} \left(3t^2 + 6t - 14t^2 + 20t^3\right)dt$$

$$= \int_{t=0}^{t=1} \left(6t - 11t^2 + 20t^3\right)dt = \frac{13}{3}$$

3.3 Integral Theorem

3.3.1 Divergence Theorem of Gauss

For simplicity, divergence theorem of Gauss can be used to transform triple integral into surface integral over the boundary surface of a region in space. This is obvious because surface integral is simpler and easier to handle compared to triple integral.

Therefore, let T be closed bounded in a region space whose boundary is a piecewise smooth orient table surface S.

Let $f(x, y, z)$ be a vector function that is continuous and has continuous first partial derivative in some domain containing T. However, the transformation is done by the so called divergence theorem which involves the divergence of a vector function F.

Where divergence of F

$$\Rightarrow \quad \text{div}\, F = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv = \iint_s F \cdot ndA \qquad (2)$$

But
$$\iint_s F \cdot \; ndA = \iint_s \left(F_1 dydz + F_2 dxdz + F_3 dxdy\right) \qquad (3)$$

Where $'n'$ is the outer unit normal vector of S.

but
$$F = F_1 i + F_2 j + F_3 k \qquad (4)$$

and
$$n = Cos\alpha i + Cos\beta j + Cos\gamma k \qquad (5)$$

where $\alpha, \beta,$ and γ are the angle between $'n'$ and the positive $x, y,$ and z axes respectively.

Next, we substitute equation (3) and (4) into (2) so we can obtain

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S \left(F_1 Cos\alpha i + F_2 Cos\beta j + F_3 Cos\gamma k\right) dA \qquad (6)$$

But
$$Cos\alpha = dzdy, Cos\beta = dzdx, Cos\gamma = dxdy$$
$$\therefore \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) cdz = \iint_S F_1 dydz + F_2 dxdz + F_3 dydx \qquad (7)$$

Example 2

Application of the Divergence Theorem

Harmonic Function

The theory of solution of Laplace gives thus:
$$\nabla^2 f = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 \qquad (8)$$
and equation (1) is called **potential theory**.

Now, from the divergence theorem formula
$$\iiint_T divFdv = \iint_S f \cdot ndA \qquad (9)$$

Where $F = \nabla f$ (10)

is gradient of scalar function.

$$divF = \nabla^2 f \quad (11)$$

and $F \cdot n = n \cdot \nabla f$

Hence,

$$\iiint_T \nabla^2 f dv = \iint_S \frac{\partial f}{\partial n} dA \quad (12)$$

Where

$$n \cdot gradf = \frac{\partial f}{\partial n} dA \quad (13)$$

we denote the directional derivative of f in the outer normal direction of S by $\frac{\partial f}{\partial n}$

However,

$$f \cdot n \equiv n \cdot \nabla f \equiv \frac{\partial f}{\partial n} \cdot dA \quad (14)$$

3.3.2 Green's Theorem

This theorem gives the relation between the integral over the boundary surface which encloses the volume. If F_1, F_2, F_3 are three functions of x, y, z and their derivatives $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ are continuous and single valued functions in a region V bounded by a closed surface S, then

$$\iint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \iint_S (PCos\alpha + QCos\beta + RCos\gamma) dA$$

As in (6) above

Where $Cos\alpha$, $Cos\beta$ and $Cos\gamma$ are the direction cosines normal to the surface S.

Example 3

Evaluate the surface integral

$$I = \iint_S (x^3 dydz + x^2 ydzdx + x^2 zdxdy)$$

where is the surface bounded by $z = 0, z = b, x^2 + y^2 = a^2$.

Solution

Using Green's theorem

$$I = \iiint_V \left(3x^2 + x^2 + x^2\right) dxdydz$$

$$= 4\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\int_0^b dz\right) dy\right] 5x^2\, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} (b)\, dy\right] 5x^2\, dx$$

$$= 20b\int_0^a x^2\sqrt{a^2 - x^2}\, dx$$

Substituting $x = aSin\theta$ or $x = aCos\theta$ we have $dx = aCos\theta d\theta$

$$= 20b\int_0^a \left(a^2 Sin^2\theta \sqrt{a^2 - a^2 Sin^2\theta}\right) Cos\theta d\theta$$

$$= 20a^4 b\int_0^a \left(Sin^2\theta \sqrt{1 - Sin^2\theta}\right) Cos\theta d\theta$$

but $\sqrt{1 - Sin^2\theta} = \sqrt{Cos^2\theta} = Cos\theta$

$$I = 20ba^4 \int_0^a \left(Sin^2\theta - Cos^2\theta\right) d\theta$$

$$= -20ba^4 \int_0^a \cos 2\theta d\theta$$

$$= 20ba^4 \left(\frac{\pi}{16}\right)$$

$$= \frac{5}{4}\pi a^4 b$$

3.3.3 Stoke's Theorem

This is the transformation between surface integrals and line integrals. Stoke's theorem involves the curl.

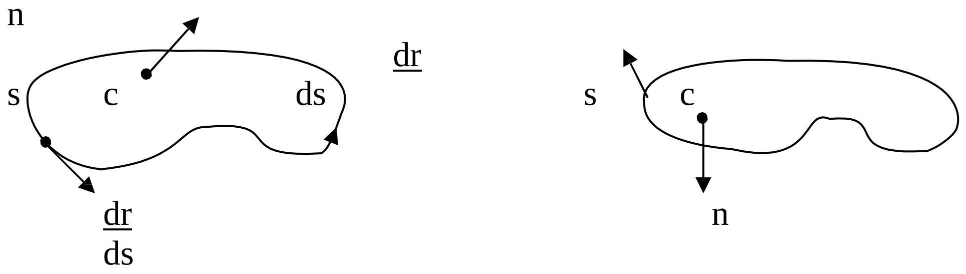
$$\text{Curl}\, F = \Delta x F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad (1)$$

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple close curve C.

Let $F(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\Delta x F) \cdot n\, dA = \oint_C F \cdot \frac{dr}{ds} \qquad (2)$$

where n is a unit normal vector of S and, also $\frac{dr}{ds}$ is the unit tangent vector and S the arc length of C.



$$\therefore \iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3\right] du\,dv$$

$$= \oint_C (F_1 dx + F_2 dy + F_3 dz) \qquad (3)$$

3.3.4 Green's Theorem in the Plane as a Special Case of Stoke's Theorem

Let $F = F_1 i + F_2 j + F_3 k$ be a vector function that is continuously differentiable in a domain in the $x - y$ plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple close curve.

Then from equation (1)

$$(\Delta\, xF) \cdot\ n = (\Delta\, xF) \cdot\ k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}$$

Then the formula in Stoke's theorem now takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\right) dA = \oint_C (F_1 dx + F_2 dy)$$

Hence, Green's theorem in space is s special case of Stoke's theorem.

Example 4

Evaluation of line integral by Stoke's theorem.

Evaluate $\int_C \left(F \cdot \frac{dr}{ds} \right) ds$, where C is the circle $x^2 + y^2 = 4$, $z = -3$, oriented counterclockwise as seen by a person standing at the origin, and with respect to right-handed Cartesian coordinates $F = yi + xz^3 j - zy^3 k$.

Solution

As a surface S bounded by C we can take the plane circular disc $x^2 + y^2 = 4$ in the plane $z = -3$. Then n in Stoke's theorem points in the positive z-direction; thus n=k,. Hence $(\Delta x F) \cdot n$ is simply the component of curl $(\Delta x F)$ in the positive z-direction. Since F with $z = -3$ has the components $F_1 = y, F_2 = -27x$ and $F_3 = 3y^3$, we thus obtain

$$(\Delta x F) \cdot n = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} = 17 - 1 = 128$$

Hence, the integral over S in Stoke's theorem equals 128times the area 4π of the disk S.

$$\therefore [(\Delta x F) \cdot n] 4\pi = -28 \cdot 4\pi = -112\pi$$
$$= -352$$

4.0 CONCLUSION

In conclusion, the students must have understood vector field theory and also be able to relate vector field and vector function together respectively.

However, the Line Integral, Gauss's, Stoke's, and Green's theorem were discussed using the knowledge acquired from vector field theory.

5.0 SUMMARY

In summary, double integrals over a region in the plane can be transformed into line integrals over the boundary C of R by Green's theorem in the plane using

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

Also Triple integrals taken over a region T in space can be transformed into surface integrals over the boundary surface S of T by the divergence theorem of Gauss using,

$$\iiint_T divF dv = \iint_S f \cdot n dA$$

where n is the outer unit normal vector to S which implies Green's formulas.

Likewise, surface integrals over a surface with boundary curve c can be transformed into line integrals over C by Stokes's theorem.

$$\iint_S (\Delta n F \cdot n dA) = \circ \int_C \left(F \cdot \frac{dr}{ds} \right) ds$$

6.0 TUTOR-MARKED ASSIGNMENT

i. Compute $\int_c F(r) \cdot dr$ where

(a) $F = y^2 i - x^4 j, \quad c : r = ti + t^{-1}$, for $1 \le t \le 3$

(b) $F = x^2 i - y^2 j, \quad c : y = 1 - x^2$, for $-1 \le x \le 1$

ii. Find the work done by the force $F = xi - zj + 2yk$ in the displacement;

(a) Along the y axis from 0 to 1

(a) Along the curve $z = y^4$, $x = 1$, from (1,0,1) to (1,1,1).

iii. Evaluate $\int_c (x^2 + y^2) \cdot ds$

(a) Over the path $y = 2x$ from (0,0) to (1,2)

(a) Over the path $y = -x$ from (1,-1) to (2,-2)

iv. Evaluate the relations between vector fields and vector functions.

v. State one example of a rotating body and the corresponding velocity field.

vi. Let the components of a vector function $r(t_1, t_2) = aCost_1 i + aS\text{int}_1\ j + 3t_2 k$ be differentiable functions on variables t_1 and t_2. Then find the partial derivatives of $r(t_1, t_2)$ with respect to t_1 and t_2 denoted by $\frac{\partial r}{\partial t_1}$ and $\frac{\partial r}{\partial t_2}$.

vii. Evaluate the surface integral

$$I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy \right)$$

where is the surface bounded by $z = 0, z = b, x^2 + y^2 = a^2$

viii. State and prove Stoke's theorem.

xiv. Evaluate $\int_C \left(F \cdot \frac{dr}{ds} \right) ds$, where C is the circle $x^2 + y^2 = 4$, $z = -3$, oriented counterclockwise as seen by a person standing at the origin, and with respect to right-handed Cartesian coordinates $F = yi + xz^3 j - zy^3 k$.

x. Show that vector function $F = (x^2 + yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational. Find the scalar potential

xi. Verify divergence theorem for the function
$F = 4xzi - y^3 j + yz$
over the unit cube $x = 0, x = 1, y = 1$ and $z = 0$ and $z = 1$.

xii. Prove that div $(\underline{u} \times \underline{v}) = \underline{v} \cdot Curl\underline{u} - uCurl\underline{v}$

xiii. Evaluate $\int_L \Phi \cdot dr$, where $\Phi = xyi + yzj + zxk$ and curve L
$r = ti + t^2 j + t^3 k$ where $-1 \le t \le 1$.

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. Advanced Calculus for Application.

Murray, R. S. (1974). Schaums Outline Series or Theory and Problem of Advanced Calculus. Great Britain: McGraw–Hill Inc.

Murray,R.S.(1974). "Schaums Outline Series on Vector Analysis". *In*: An Introduction to Tension Analysis. Great Britain: McGraw-Hill Inc.

Stephenson, G. (1977). Mathematical Methods for Science Students. London: Longman. Group Limited.

Stroud, K.A. (1995). Engineering Maths. 5th Edition Palgraw

Verma, P.D.S. (1995). Engineering Mathematics. New Delhi: Vikas Publishing House PVT Ltd.

MODULE 2

UNIT 1 FUNCTIONS OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

CONCEPTS OF SETS IN THE COMPLEX PLANE

Definition: The term set of points in the complex plane is the collection of finite or infinite points. Examples: the points on a line, the solution of quadratic equation and the points in the interior of a circle made up of sets respectively.

A set is called open if every point of S has a neighbourhood consisting entirely of points that belongs to S. that is the points in the interior of a circle or a square from an open set, and so do the points of the "right half – plane" $\operatorname{Re} z = 0 > 0$.

An open set S is to be connected if any two of its points can be joined by a broken line of finitely many straight line segments all of where points belong to S.

Likewise, an open connected set is called a domain. Thus, an open disk annulus is domain. An open square with a diagonal removed is not a domain since this set is not connected.

The complement of a set S in the complex plane is defined to be the set of all points of the complex plane that do not belong to S. A set is said to be closed if its complements is open. Example: the point on and inside the unit circle form a closed set.

A boundary point of a set S is a point every neighbourhood of which contains both points that belong to S and points that do not belong to S.

Example: if a set S is open, then no boundary point belongs to S, if S is closed, then every boundary point belongs to S.

A region is a set consisting of a domain plus, perhaps, some or all of its boundary points.

Next we shall consider functions of complex variables but before this we introduce complex functions first.

Complex functions

Definition: A real function F defined on a set S of real numbers is a rule that assigns to every X in S a real number f(x), called the value of f at x. Now in complex, S is a set of complex numbers and a function f

defined on S is a rule that assigns to every Z in ρ a complex number w, called the value of f at z. we write
$w = f(z)$

Here z varies in S and is called a Complex Variable. The set S is called the domain of definition of f.

Example 1

$w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the range of f. w is a complex, and we write $w = u + iv,$ where u and v are the real and the imaginary parts, respectively. Now w is depends on $z = x + iy.$ Hence, u becomes a real function of x and y. and so does v. we may thus write:

$w = f(z) = u(x, y) + iv(x, y).$

This shows that a complex function f(z) is equivalent to a pair of real functions u(x,y) and v(x,y), each depending on the two real variables x and y.

Example 2

Function of a complex variable.

Let $w = z^2 + 3z.$ Find u and v and calculate the values of f at z = 1 + 3i and

z = 2 – i.

Let the real part of w be defined thus $u = x^2 - y^2 + 3x$ and the imaginary part of w i.e. $v = 2xy + 3y.$

$\therefore f(1+3i) = (1+3i)^2 + 3(1+3i) = -5 + 15i$

Recall that $i^2 = -1$.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- complex numbers
- complex analytical function
- Cauchy – Riemann equation

- Cauchy's theorem and inequality
- integral transforms vis a vis: Fourier and Laplace transforms
- convolution theory and their applications.

3.0 MAIN CONTENT

3.1 Complex Numbers

It was observed early in history that there are equations which are not satisfied by any real number. Examples are:

$x^2 = -3$ *or* $x^2 - 10x + 40 = 0$

This led to the invention of complex numbers.

Definition

A complex number z is an ordered pair (x, y) of real numbers x, y and we write

$$z = (x, y).$$

We call x the real part of z and y the imaginary part of z and write

$$\operatorname{Re} z = x, \qquad lm\, z = y$$

Example 3

Re (4, -3) = 4 and lm (4, -3) = -3, furthermore, we defined two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ to be equal if and only if their real parts are equal and their imaginary parts are equal.

$z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Addition of complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

1. $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Multiplication of complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

2. $z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

We shall say more about these arithmetic operations and discuss examples below, but we first want to introduce a much more convenient form of writing them as points in the plane.

3.1.1 Representation in the Form z = x + iy

A complex number whose imaginary part is zero is of the form (x, 0). For such numbers we simply have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$
and
$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for the real numbers. This suggests that we identify (x, 0) with the real number x. hence the complex number system is an extension of the real number system.

The complex number (0,1) is denoted by i.

$$i = (o,1)$$

and is called the imaginary unit. We show that it has the property.

3. $i^2 = -1$

Indeed, from (2) we have

$i^2 = (0.1)(0.1) = (-1,0) = -1$ *futheremore, for every real y we obtain from* (2)

$$iy = (0,1)(y,0) = (0, y)$$

Combining this with the above x = (x, 0) and using (1), that is,

$$(x, y) = (x,0) + (0, y),$$

We see that we can write every complex number z = (x, y) in the form

$$z = x + iy$$

or z = x + yi. This is done in practice almost exclusively.

Example 4

Complex Numbers, their Real and Imaginary Parts

$z = (4, -3) = 4 - 3i,$ $\quad \mathrm{Re}(4 - 3i) = 4,$ $\quad \lim(4 - 3i) = -3$

$z = \left(\frac{-1}{2}, 0\right) = \frac{-1}{2} + 0i,$ $\quad \mathrm{Re}\left(\frac{-1}{2}\right) = \frac{-1}{2},$ $\quad \lim\left(\frac{-1}{2}\right) = 0$

$z = (0, \pi) = 0 + \pi i,$ $\quad \mathrm{Re}(\pi i) = 0,$ $\quad \lim(\pi i) = \pi$

3.1.2 Complex Plane

This is a geometric representation of complex numbers as points in the plane. It is of great importance in applications. This idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal x – axis, called the real axis, and the vertical y – axis called the imaginary axis. On both axes we choose the same unit of length (Fig. 4). This is called a **Cartesian coordinate system.** We now plot z = (x, y) = x + iy as the point P with coordinates x, y. The xy – plane in which the complex numbers are represented in this way is called the **complex plane** or *Argand diagram.* Figure 5 shows an example.

Instead of staying ***"the point represented by z in the complex plane"*** we say briefly and simply "the point z in the complex plane" this will cause no misunderstandings.

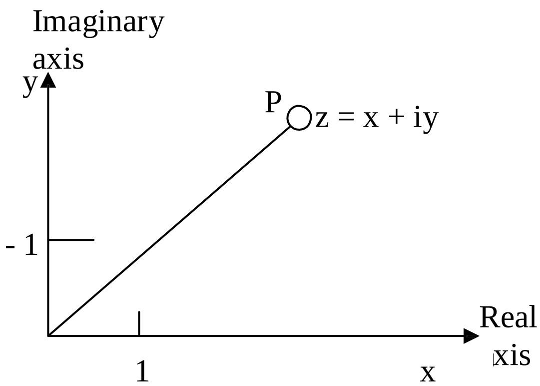


Fig.4 295**: The Complex Plane**

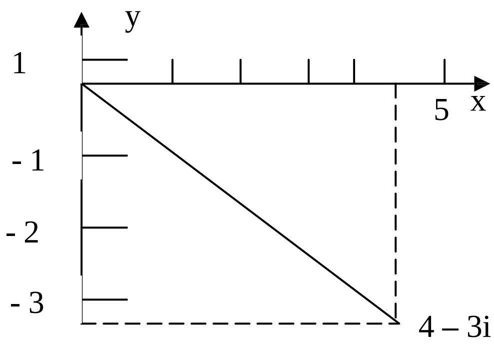


Fig. 5: The number 4 – 3i in the Complex Plane

3.1.3 Arithmetic Operations

We can make use of the notations z = x + iy and of the complex plane. Addition of the sum of $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ can now be written

4. $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$.

$$z_1 + z_2 = (x_1 + x_2) + (iy_1 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Example 5

$$(5 + i) + (1 + 3i) = (5 + 1) + (i + 3i) = 6 + 4i.$$

We see that addition of complex numbers is in accordance with the "parallelogram law" by which forces are added in mechanics.

Subtraction is defined to be the inverse operation of addition. That is the difference $z = z_1 - z_2$.

5. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

Example 6

$(5+i) - (1+3i) = (5-1) + (i-3i) = 4-2i$

Multiplication: The Product z_1z_2 in (2) can now be written

6. $$z_1z_2 = (x_1+iy_1)(x_2+iy_2) = x_1(x_2+iy_2) + iy_1(x_2+iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

This is easy to remember since it is obtained formally by the rules of arithmetic for real numbers and using (3), that is $i^2 = -1$

Example 7

$(5+i)(1+3i) = 5+15i + i+3i^2 = 2+16i$

Division is defined to be the inverse operation of multiplication. That is, the quotient $z = z_{1} / z_2$ is the complex number $z = x + iy$ for which

7. $z_1 = zz_2 = (x+iy)(x_2+iy_2)$ $(z_2 \neq 0)$.

We show that for $z_2 \neq 0$ *the qutotient* $z = x + iy = z_1 / z_2$ is given by

8. $$z = \frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_1+iy_2)(x_2-iy_2)}$$

where $(x_2 - iy_2)$ is the conjugate of $(x_2 + iy_2)$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

Example 8

If $z_1 = 9 - 8i$ and $z_2 = 5 + 2i$, then

$$z = \frac{z_1}{z_2} = \frac{9-8i_1}{5+2i} = \frac{(9-8i)(5-2i)}{(5_1+2i)(5-2i)}$$

$$= \frac{45 - 18i \ - 40i \ - 16}{25 \ + \ 4} = \frac{29 - 58i}{29} = 1 - 2i.$$

The reader may check this result by showing that

$$zz_2 = (1 - 2i)(5 + 2i) = 9 - 8i = z_1.$$

3.1.4 Properties of the Arithmetic Operations

From the familiar laws for real numbers we obtain for any complex numbers z_1, z_2, z_3, z the following laws (where z = x +iy):

$z_1 + z_2 = z_2 + z_1$………..commutative law of addition
$z_1 z_2 = z_2 z_1$ ……………… commutative law of multiplication

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$...associative law of addition
$(z_1 z_2) z_3 = z_1 (z_2 z_3)$……………………associative law of multiplication

9. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ ……distributive law

$$0 + z = z + 0 = z$$
$$z + (-z) = (-z) + z = +z - z = 0$$
$$z \, . \, 1 = z$$

3.1.5 Complex Conjugate Numbers

Let z = x + iy be any complex number. Then x – iy is called the conjugate of z and is denoted by $\bar{z}$, thus,

$$z = x + iy, \qquad \bar{z} = x - iy.$$

Example 9

The conjugate of $z = 5 + 2i$ is $\bar{z} = 5 - 2i$.

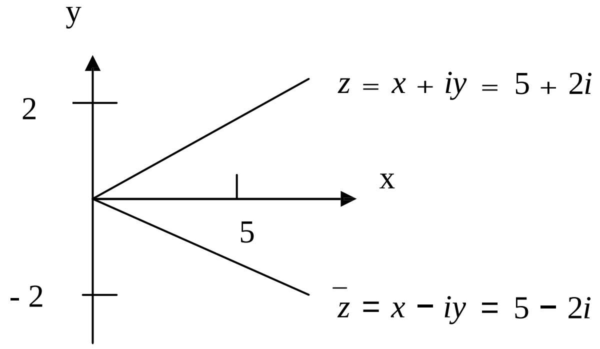


Fig. 6: Complex Conjugate Numbers

Conjugates are useful since $z\bar{z} = x^2 + y^2$ is real, a property we have used in the above division. Moreover, addition and subtraction yields $z + \bar{z} = 2x, \quad z - \bar{z} = 2iy,$ so that we can express the real part and the imaginary part of z by the important formulas.

10. $\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{lm} z = y = \frac{1}{2i}(z - \bar{z})$

Example 10

If z = 6 – 5i, then we have $\bar{z}$ = 6 + 5i and from (10) we obtain

$$x = \frac{1}{2}(6 - 5i + 6 + 5i) = 6 \text{ and}$$

$$y = \frac{1}{2i}(6 - 5i - 6 - 5i) = \frac{1}{2i}(0 - 10i)$$

$$= \frac{-10i}{2i} = -5$$

z is real if and only if y = 0, hence $\bar{z}$ = z by (10).
z is said to be pure imaginary if and only if x = 0, hence $\bar{z}$ = -z. Then working with conjugates is easy, since we have

11.
$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}, \quad \overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}$$
$$\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

In this section we were mainly concerned with complex numbers, their arithmetic operations and their representation as points in the complex plane. The next section we shall discuss the use of polar coordinates in the complex plane and situations in which **polar coordinates** are advantageous.

3.2 Polar Form of Complex Number Powers and Roots

It is often practical to express complex numbers z = x + iy in terms of polar coordinates r, θ, these are defined by:

1. $x = r\cos\theta, \; y = r\sin\theta$

By substituting this we obtain the polar form of z,

2. $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

r is called the absolute value or modulus of z and is denoted by $|z|$.Hence

3. $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 7).

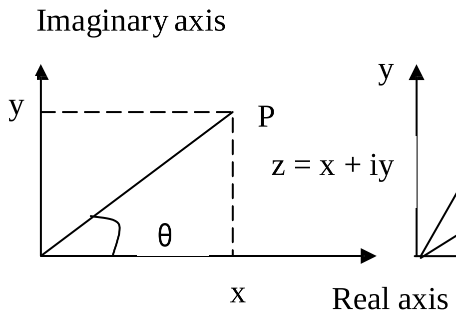
Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 301).

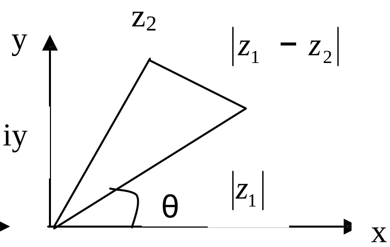
θ is called the **argument** of z and is denoted by arg z. thus (Fig. 7).

4. $\theta = \arg z = arc\tan\frac{y}{x}$ $(z \neq 0)$.

Geometrically, θ is the directed angle from the positive x – axis to OP in fig. 7. Here, as in calculus, all angles are measured in radians and positive in the counterclockwise series.

$|z| = r$





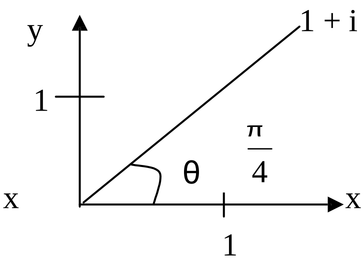


Fig. 7: Complex Plane, Polar Form of a Complex Number

Fig. 8: Distance between two points Complex Number

Fig. 9: Example 1

For z = 0 this angle θ is undefined. (Why?) For given z ≠ 0 it is determined only up to integer multiples of 2π. The value of θ that lies in the interval $-\pi < \theta \leq \pi$ is called the principal value of the argument of z (≠ 0) and is denoted by Arg. z. Thus θ = Arg z satisfies by definition.

$-\pi < \text{Arg } z \leq \pi$.

Polar Form of Complex Numbers Principal Value

Example 11

Let z = 1 + I (cf. Fig. 9). Then

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), |z| = \sqrt{2}, \quad \arg z = \frac{\pi}{4} \pm 2n\pi \quad (n = 0,1, ...0.)$$

The principal value of the argument is arg z = π/4, other values are -7π/4, 9π/4, etc.

Example 12

Let $z = 3 + 3\sqrt{3}i$, then $z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, the absolute value of z is $|z| = 6$, and the principal value of arg z is Arg z = π/3.

Caution! In using (4), we must pay attention to the quadrant in which z lies, since tan θ has period π, so that the arguments of z and –z have the same tangent. Example: for θ_1 = arg (1+i) and θ_2 = arg (-1 – i) we have tan θ_1 = tan θ_2 = 1.

Triangle Inequality

For any complex numbers we have the importance **triangle inequality**

5. $|z_1 + z_2| \leq |z_1| + |z_2|$ (*Fig*. 303)

Which we shall use quite frequently, this inequality follows by nothing that

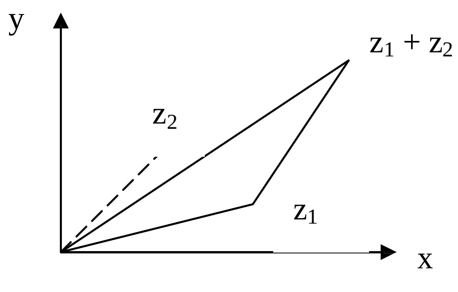


Fig 10: Triangle Inequality

The three points 0, z_1 and $z_1 + z_2$ are the vertices of a triangle (fig. 10) with sides $|z_1|, |z_2|$ *and* $|z_1 + z_2|$, and the side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob.45).

Example 13

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then

$|z_1 + z_2| = |-1+4i| = \sqrt{17} = 4.123 \angle \sqrt{2} + \sqrt{13} = 5.020.$

By induction the triangle inequality can be extended to arbitrary sums:

6. $|z_1 + z_2 + ...+ z_n| \le |z_1| + |z_2| + ...+ |z_n|;$

That is, the absolute value of a sum cannot exceed the sum of the absolute values of the terms.

3.2.1 Multiplication and Division in Polar Form

This will give us a better understanding of multiplication and division. Let:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad and \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then, by (6), sec. 12.1, the product is at first

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta 2)].$$

The addition rules for the sine and cosine (6) in appendix 3.1) now yield

7. $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Taking absolute values and arguments on both sides, we thus obtain the important rules

8. $|z_1 z_2| = |z_1|\,|z_2|$
and

9. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (*up to multiples of* 2π).

We now turn to division. The quotient $z = \frac{z_1}{z_2}$ is the number z satisfying $zz_2 = z_1$. Hence $|zz_2| = |z|\,|z_2| = |z_1|, \arg(zz_2) = \arg z + \arg z_2 = \arg z_1$.

This yield

10. $\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$ $(z_2 \ne 0)$

and

11. $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$ $\quad(up\ to\ multiples\ of\ 2\pi).$

By combining these two formulas (10) and (11) we also have

12. $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$

Example 13

Illustration of Formulas (8) – (11)

Let $z_1 = -2 + 2i$ *and* $z_2 = 3i.$ *Then* $z_1 z_2$
$= -6 - 6i,\ z_1 / z_2 = 2/3 + (2i/3)$
and for the arguments we obtain Arg $z_1 = 3\pi/4$, Arg $z_2 = \pi/2$.

$$Arg\ z_1 z_2 = \frac{-3\pi}{4} = Arg\ z_1 + Arg\ z_2 - 2\pi$$

$$Arg\ (z_1 / z_2) = \frac{\pi}{4} = Arg\, z_1 - Arg\ z_2$$

Integer power of z

From (7) and (12) we have

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta),$$

$$z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)]$$

and more generally, for any integer n,

13. $z^n = r^n(\cos n\theta + i\sin n\theta).$

Example 14

Formula of De Moivre

For $|z| = r = 1$, formula (3) yields the so – called formula of De Moivre

(13*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\ .$

This formula is useful for expressing cos nθ in terms of cos θ and sin θ. For instance when n = 2 and we take the real and imaginary parts on both sides of (13*), we get the familiar formulas.

$\cos 2\theta = \cos^2\theta - \sin^2, \quad \sin 2\theta = 2\cos\theta\sin\theta.$

3.2.2 Roots

If $z = w^n \ (n = 1, 2, \ldots)$, then to each value of w there corresponds one value of z, we shall immediately see that to a given z ≠ 0 there correspond precisely *n* distinct values of w. each of these values is called an nth root of z, and we write:

14. $w = \sqrt[n]{z}.$

Hence this symbol is multivalued, namely, n – valued, in contrast to the usual conventions made in real calculus. The n value of $\sqrt[n]{z}$ can easily be determined as follows. In terms of polar forms for z and

$$w = R(\cos\varphi + i\sin\varphi),$$

The equation $w^n = z$ becomes

$$w^n = R^n(\cos n\varphi + i\sin n\varphi) = z = r(\cos\varphi + i\sin\theta)$$

By equating the absolute values on both sides we have

$$R^n = r, \ thus \ R = \sqrt[n]{r}$$

Where the root is real positive and thus uniquely determined. By equating the arguments we obtain

$$n\varphi = \theta + 2k\pi, \qquad thus \ \varphi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

Where k is an integer. For k = 0,1, …, n – 1 we get n distinct values of w. further integers of k would give values already obtained. For instance, k = n gives 2kπ/n = 2π, hence the w corresponding to k = 0, etc. consequently, $\sqrt[n]{z}$, for z ≠ 0, has the n distinct values

15. $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad k = 0, 1, \ldots n-1.$

These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides.

The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and k = 0 in (15) is called the principal value of $w = \sqrt[n]{z}$

Example 15

Square Root

From (15) it follows that $w = \sqrt{z}$ has the two values

16a. $$w_1 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and

16b. $$w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1$$

Which lie symmetric with respect to the origin. For instance, the square root of 4i has the values $\sqrt{4i} \quad \pm 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm\left(\sqrt{2} + i\sqrt{2}\right)$.

From (16) we can obtain the much more practical formula

17. $$\sqrt{z} = \pm\left[\sqrt{\frac{1}{2}\left(|z|\right) + x} + (sign\ y)i\sqrt{\frac{1}{2}\left(|z| - x\right)}\right]$$

Where sign y = 1 if y ≥0, sign y = - 1 if y < 0, and all square toots of positive numbers are taken with the positive sign. This follows from (16) if we use the trigonometric identities.

$$\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1+\cos\theta)}n \qquad \sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(1-\cos\theta)}.$$

Multiply them by $\sqrt{r}$.

$$\sqrt{r}\cos\frac{1}{2}\theta = \sqrt{\frac{1}{2}(r\,r\cos\theta)}, \qquad \sqrt{r}\sin\frac{1}{2}\theta = \sqrt{\frac{1}{2}(r - r\cos\theta)},$$

Use r cos θ = x, and finally choose the sign of 1m $\sqrt{z}$ so that sign

$$[(\mathrm{Re}\sqrt{z})(1m\sqrt{z})] = \text{sign } y \text{ (why?)}.$$

Example 16

Complex Quadratic Equation

Solve $z^2 - (5+i)z + 8 - i = 0$

Solution

$$z = \frac{1}{2}(5+i) \pm \sqrt{\frac{1}{4}(5+i)^2 - 8 - i} \quad = \frac{1}{2}(5+i) \pm \sqrt{-2+\frac{3}{2}i}$$

$$= \frac{1}{2}(5+i) \pm \left[\sqrt{\frac{1}{2}\left(\frac{5}{2}+(-2)\right)} + i\sqrt{\frac{1}{2}\left(\frac{5}{2}-(-2)\right)} \right]$$

$$= \frac{1}{2}(5+i) \pm \left(\frac{1}{2}+\frac{3}{2}i\right)$$

$$= \begin{cases} 3+2i \\ 2-i \end{cases}$$

Example 17

Cube Root of a Positive Real Number

If z is positive real, then $w = \sqrt[3]{z}$ has the real value $\sqrt[3]{r}$ and the complex values

$$\sqrt[3]{r}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt[3]{r}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\text{and} \quad \sqrt[3]{r}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \sqrt[3]{r}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right).$$

For instance $\sqrt[3]{1} = 1, \frac{-1}{2} \pm \frac{1}{2}\sqrt{3}i$ (*fig*.304). These are the roots of the equation $w^3 = 1$.

Example 18

***n*th Root of Unity**

Solve the equation $z^n = 1$.

Solution

From (15) we obtain

18. $$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n} \qquad k = 0,1,\ldots,n-1.$$

If w denotes the value corresponding to k = 1, then the n values of $\sqrt[n]{1}$ can be written as 1, w, w^2, …, w^{n-1}. These values are the vertices of a regular polygon of n sides inscribed in the unit circle, with one vertex at the point 1. Each of these n values is called an nth root of unity. For instance, $\sqrt[4]{1}$ has the values 1, i, -1 and –i (Fig. 12 shows $\sqrt[5]{1}$). If w_1 is any nth root of an arbitrary complex number z, then the n values of $\sqrt[n]{z}$ are w_1, w_1w, w_1w^2, …. w_1w^{n-1}

Multiplying w_1 by w^k corresponds to increasing the argument of w_1 by $2k\pi/n$.

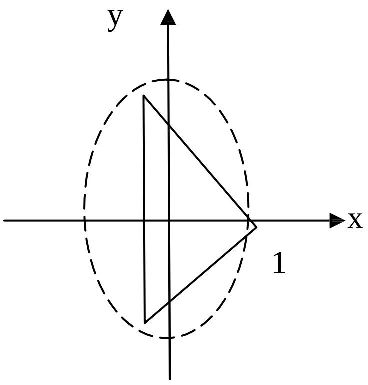


Fig 11. $\sqrt[3]{1}$

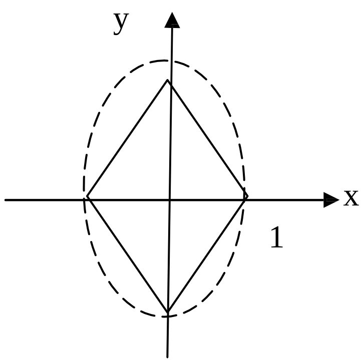


Fig 12. $\sqrt[4]{1}$

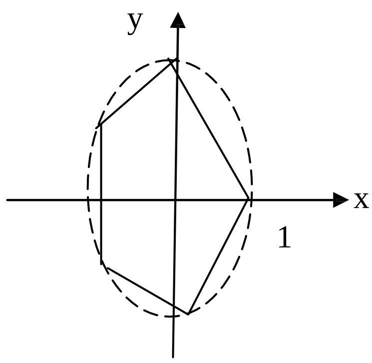


Fig 13. $\sqrt[5]{1}$

The student should be familiar with the problems related to the polar representation with particular care, since we shall need this representation quite often in our work. In the next section, we discuss some curves and regions in the complex plane which we shall also need in the chapters on complex analysis.

3.3 Curves on Regions in the Complex Plane

In this section we consider some important curves and regions and some related concepts we shall frequently need. This will also help us to become more familiar with the complex plane.

The distance between two points z and a is $|z-a|$. Hence a circle C of radius ρ and center at a (fig. 14) can be represented by;

1. $|z-a| = \rho$.

In particular, the so-called unit, that is the circle of radius 1 and center at the origin a = 0 (fig. 308), is given by;

$|z| = 1$.

Furthermore, the inequality

2. $|z - a| < \rho$

holds for every point z inside C: that is, (2) represents the interior of C. Such a region is called a circular disk or, more precisely, an open circular disk, in contrast to the closed circular disk.

$|z - a| \leq \rho.$

This consists of the interior of C and C itself. The open disk (2) is also called a neighborhood of the point a. Obviously, a has infinitely many such neighborhoods, each of which corresponds to a certain value of p (> 0); and a belong to each of these neighborhoods, that is a, is a point of each of them.

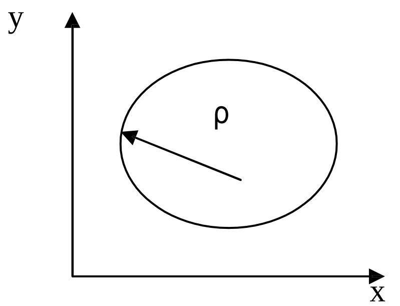


Fig 14. Circle in the Complex Plane

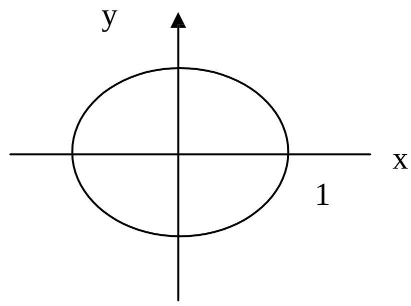


Fig 15. Unit Circle

Similarly, the inequality

$$|z - a| \ \rho.$$

represents the exterior of the circle C. Furthermore, the region between two concentric circles of radii ρ_1 and ρ_2 (> ρ_1) can be represented in the form

3. $\rho_1 < |z - a| < \rho.$

Where a is the center of the circles. Such a region is called an open circular ring or open annulus (Fig. 16).

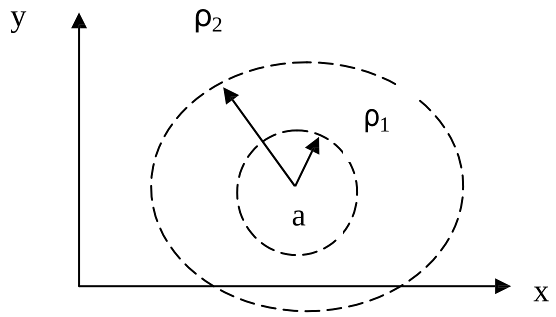


Fig 16. Annulus in the Complex Plane

Example 19

lar Disk

Determine the region in the complex plane given by $|z-3+i| \leq 4$.

Solution: the inequality is valid precisely for all z whose distance from a = 3 – i does not exceed 4. Hence this is a closed circular disk of radius 4 with center at 3 – i.

Example 20

Unit Circle and Unit Disk

Determine each of the regions

(a) $|z| < 1$ $(b)\ |z| \leq 1$ $(c)|z| > 1.$

Solution

(a) The interior of the unit circle. This called the open unit disk.
(b) The unit circle and its interior. This is called the closed ad disk.
(c) The exterior of the unit circle.

By the (open) upper half we mean the set of all points z = x + iy such that $y > 0$. Similarly, the condition y < 0 defines the lower half – plane, $x > 0$ the right half – plane and x < 0 the left half – plane.

3.3.1 Some Concepts Related to Sets in the Complex Plane

We finally list a few concepts that are of general interest and will be used in our further work.

The term set of points in the complex plane means any sort of collection of a quadratic equation. The points on a line and the points in the interior of a circle are sets.

A set S is called open, if every point of S has a neighborhood consisting entirely of points that belong to S. for example, the neighborhood consisting entirely of points that belong to S. For example, the points in the interior of a circle or a square form an open set, and so do the points of the "right half – plane" Re z = x > 0.

An open set S is said to be connected if any two of its points can be joined by a broken line of finitely many straight line segments all of whose points belong to S. an open connected set is called a domain. Thus an open disk (2) and an open annulus (3) are domains. An open square with a diagonal removed is not a domain since this set is not connected. (Why?).

The complement of a set S in the complex plane is defined to be the set of all points of the complex plane that do not belong to S. A set is called closed if its complement is open. For example, the points on and inside the unit circle form a closed set ("closed unit disk" cf. example 2) since its complement $|z| >$ is open.

A boundary point of a set S is points every neighbourhood of which contains both points that belong to S and points that do not belong to S. For example; the boundary points of an annulus are the points on the two bounding circles.

Clearly, if a set S is open, then no boundary point belongs to S; is closed, and then every boundary point belongs to S.

A region is a set of a domain plus, perhaps, some or all of its boundary points. (The reader is warned that some authors use the term "region" for what we call a domain (following the modern standard terminology) and others make no distinction between the two terms.)

So far, we have been concerned with complex numbers and the complex plane (just as at the beginning of calculus, one talks about real numbers and the real line). In the next section, we start doing complex calculus: we introduce complex functions and derivatives. This will generalise familiar concepts of calculus.

SELF ASSESSMENT EXERCISE 1

Determine and sketch the sets represented by

1.	$\|z-2i\| = 2$	2.	$1 \leq \|z+1-i\| \leq 3$
3.	$\text{Re}\left(z^2\right) \leq 1$	4.	$\|\arg z\| < \frac{\pi}{4}$
5.	$-\pi < 1m\, z \leq \pi$	6.	$\left\|\frac{1}{z}\right\| < 1$
7.	$\left\|\frac{z+1}{z-1} = 1\right\|$	8.	$\left\|\frac{z+3i}{z-i}\right\| = 1$

9. $lm\frac{2z+1}{4z-4}\leq 1$ 10. $z\bar{z}+(1+2i)z+(1-2i)\bar{z}+1=0.$

3.4 Limit, Derivative and Analytic Functions

The functions with which complex is concerned are complex functions that are differentiable. Hence, we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be quite similar to that in calculus.

3.4.1 Complex Function

Recall from the calculus that a real function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number f(x) called the value of f at x.

Now in complex, S is a set of complex numbers. And a function f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z. write

$w = f(z)$

Here z varies in S and is called a complex variable. The set S is called the domain of definition of f.

Example 21

$w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the range of f.

W is complex, and we write w = u + iv, where u and v are the real and imaginary parts, respectively. Now w depends on z = x + iy. Hence u becomes a real function; of x and y, and so does v. We may thus write:

$w = f(z) = u(x, y) + iv(x, y).$

This shows that a complex function f(z) is equivalent to a pair of real functions u(x,y) and depending on the two real variables x and y.

Example 22

Function of a Complex Variable

Let w = f(z) = z^2 +3z. Find u and v and z = 2 –i.

Solution

$u = \text{Re}\ f(z) = x^2 + y^2 + 3x\ and\ v = 2xy + 3y$, also,

$f(1+3i) = (1+3i)^2 + 3(1+3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i$

This shows that u(1,3) = -5 and v (1,3) = 15, similarly.

$f(2-i) = (2-i)^2 + 3(2-i) = 4i + 6 - 3i = 9 - 7i.$

Example 23

Function of a Complex Variable

Let $w = f(z) = 2iz + 6\bar{z}$. .Find u and v and the value for f at $z = \frac{1}{2} + 4i$

Solution $f(z) = 2i(x+iy) + 6(x-iy)$

gives

$u(x,y) = 6x - 2y\ and\ v(x,y) = 2x - 6y.$

Also

$$f\left(\frac{1}{2} + 4i\right) = 2i\left(\frac{1}{2} + 4i\right) + 6\overline{\left(\frac{1}{2} - 4i\right)} = i - 8 + 3 - 24i = -5 - 23i.$$

Limit, Continuity

A function f(z) is said to be limit l as z approaches a point z_0, written

1. $\lim_{z \to z_0} (f(z) = l$

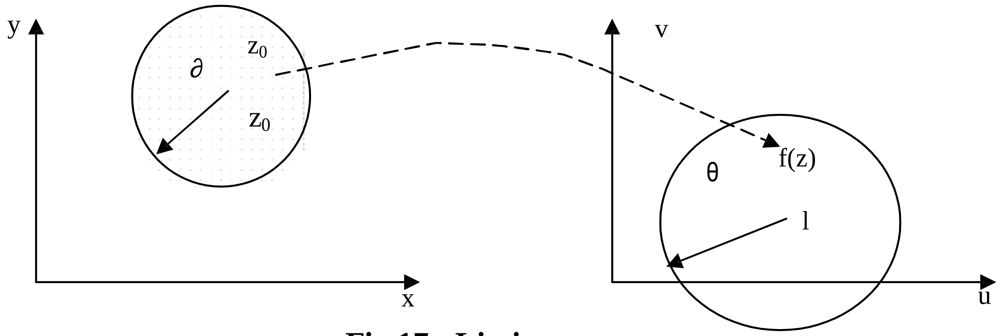


Fig 17: Limit

If f is defined in a neighborhood of z_0 (except itself) and if the values of f are "close" to l for all z "close" to z_0; that is, in precise terms, for every positive real $\in$ we can find a positive real ∂ such that for $z \neq z_0$ in the disk $|z - z_0| < \partial$ $(Fig. 310)$ we have

2. $\quad |f(z) - l| < \in;$

That is, for every $z \neq z_0$ in that the value of f lies in the disk (2).
Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real line, here, by definition, z may approach z_0 from any direction in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique. (Cf. Prob. 30)
A function f(z) is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and

3. $\quad \lim_{z \to z_0} f(z) = f(z_0).$

Note that by the definition of a limit this implies that f(z) is defined in some neighbourhood of z_0.

f(z) is said to be continuous in a domain if it is continuous at each point of this domain.

3.4.3 Derivative

The derivative of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

4. $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. Then f is said to be differentiable at z_0. if we write $\Delta z = z - z_0$ we also have

(4') $$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remember that this definition of a limit implies that f(z) is defined (at least) in a neighborhood of z_0. Also by that definition, z may approach z_0 from any direction. Hence differentially at z_0 means that, along whatever path z approaches z_0, the quotient in (4') always approaches a certain value and all these values are equal. This is important and should be kept in mind.

Example 24

Differentiability Derivatives

The function f(z) = z^2 is differentiate for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z.$$

The differentiation rules are the same as in real calculus, since their proofs are literally the same. Thus,

$$cf' = cf' \; (f + g)' = f' + g', \; (fg)' = f'g + fg', \; \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

As well as the chain rule and power rule $(z^n)' = nz^{n-1}$ (n integer) hold. Also, if f (z) is differentiable at z_0. It is continues s at z_0. (Cf. Prob. 34).

Example 25

$\bar{z}$ **not differentiable**

It is important to note that there are many simple functions that do not have a derivative at any point. For instance, $f(z) = \bar{z} = x - iy$ is such a function? Indeed, we write $\Delta z = \Delta x + i\Delta y, we\,have$

5. $$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

but -1 along path II. Hence, by equation of (5) at $\Delta z \to 0$ does not exit at any z.

This example may be surprising, but it merely illustrates that differentiability of a complex function is a rather serve requirement.
The idea of proof approach form different directions is based and will be discussed again in the next section.

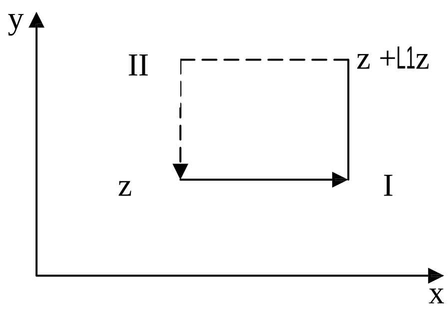


Fig. 18: Paths in (5)

3.4.1 Analytic Functions

These are the functions that are differentiable in some domain, so that we can do "calculus in complex." They are the main concern of complex analysis. Their introduction is our main goal in this section;

Definition (Analyticity)

A function f(z) is said to be analytical in a domain D if f(z) is defined and differentiable at all points of D. The function f(z) is said to be analytic at a point $z = z_0$ in D if f(z) is analytic in a neighbourhood (cf. sec. 12.3) of z_0.

Also, by analytical function we mean a function that is analytical in some domain.
Hence, analytical of f(z) at z_0 means that f (z) has a derivative at every point in some neighbourhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighbourhood). This concept is motivated by the fact that it is of no practical interest when a function is differentiable merely at a single point z_0 but not throughout some neighbourhood of z_0. Problem 28 gives an example.
An older term for analytical in D is regular in D, and a more modern term is holomorphic in D.

Example 26

Polynomids Rational Functions

The integer power 1, z, z^2, … and more generally, polynomials, that is function of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots c_n z^n$$

Where c_i, and i=1,2,3…. are complex constants, are analytical in the entire complex plane. The quotient of two polynomials g(z) and h(z).

$$f(z) = \frac{g(z)}{h(z)}.$$

is called a rational function. This f is analytic except at the points where $h(z)= 0$ here we assume that common factors of g and h have been cancelled partial fractions

$$\frac{c}{(z-z_0)^m} \qquad (c \neq 0)$$

(c and z_0 complex, m is a positive integer) are special rational functions, they are analytic except at z_0. It is in algebra that every rational function can be written as a sum of a polynomial (which may be 0) and finitely partial fractions.

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function. Indeed, complex analysis is concerned exclusively with analytic functions and although many will yield a branch of mathematics, that is most beautiful from the theoretical point of view and most useful for practical purposes.

Before we consider special analytic functions (exponential functions, cosine, sine etc.) let us give equations by means of which we can readily decide whether a function is analytic or not. These are the famous Cauchy–Riemann equation, which we shall discuss in the next section.

3.5 Cauchy – Riemann Equations

We shall now derive a very important criterion (a test) for the analyticity of a complex function.

$$w = f(z) = u(x, y) + i(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two equations

1. $u_2 = u_{y,} \qquad u_y = -v_2.$

Everywhere in D, here $u_x = \partial u / \partial x$, $u_y = \partial u / \partial y$ and similarly for u_x and u_y which are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorem 1 and 2 below. The equation (1) is called the Cauchy – Riemann equations. They are the most important equations in the whole unit.

Example 27

$f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z, and

$u = x^2 - y^2 \text{ and } v = 2xy$

Satisfy (1), namely, $u_x = 2x = v_y \text{ and } u_y = -2y = -v_x$ more examples will follow.

3.5.1 Theorem 1 (Cauchy Riemann Equations)

Let f(z) = u(x,y) + iv(x,y) be defined and continuous in some neighbourhood of a point z = x + iy and differentiable at z itself. Then at the point, the first – order partial derivatives of u and v exist and satisfy the Cauchy Riemann equations (1).

Hence if f(z) is analytic in a domain f' (z) at z exists. It is given by (1) at all points of D.

Proof

By assumption, the derivative f' (z) at z exists. It given by

2. $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

The idea of the proof is very simple, by the definition of a limit in complex (cf. sec. 12.4) we can let Δz approaches zero along any path in a neighbourhood of z. Thus, we may choose the two paths I and II in fig. 312 and equate the results. By comparing the real parts we shall obtain the first Cauchy Riemann equation and by comparing the imaginary parts we shall obtain the other equation in (1). The technical details are as follows.

We write Δz = Δx +iΔy. In terms of u and v, the derivative in (2) becomes

3. $$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}$$

We first choose path I in fig. 312. Thus we let $\Delta y \to 0$ *first and then* $\Delta x \to 0$.

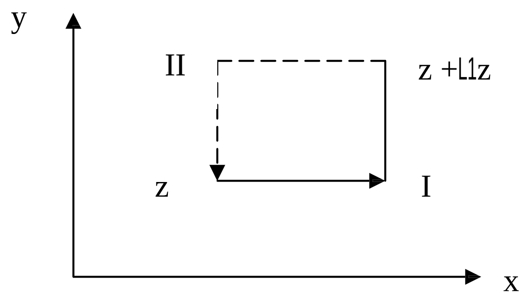


Fig. 19: Paths in (2)

After Δy becomes zero, Δz = Δx. then (3) becomes, if we first write the two u – terms and then two v-terms.

$$f^{'}(z)=\lim_{\Delta x\to 0}\frac{u(x+\Delta x, y)-u(x,y)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{v(x+\Delta x, y)-v(x,y)}{\Delta x}$$

Since f'(z) exists, the two real limits on the right exist. By definition, they are the partial derivatives of u and v with respect to x. hence the derivative f' (z) of f(z) can be written

4. $f^{'}(z) = u_x + iv_x$

Similarly, if we choose path II in fig 312, we let $\Delta x \to 0$ first and then $\Delta y \to 0$. After Δx becomes zero, Δz = iΔy, so that from (3) we now obtain

$$f^{'}(z)=\lim_{\Delta y\to 0}\frac{u(x , y+\Delta y)-u(x,y)}{i\Delta y}+i\lim_{\Delta y\to 0}\frac{v(x, y+\Delta y)-v(x,y)}{i\Delta y}$$

Since f'(z) exists, the limits on the right exist and yield partial derivatives with respect to y; noting that 1/i = -i, we obtain:

5. $f^{'}(z) = -iu_y + v_y$

The existence of the derivatives $f^{'}$(z) thus implies the existence of the four partial derivatives in (4) and (5). By equating the real parts u_x and v_y in (4) and (5) we obtain the first Cauchy – Riemann equation (1). Equating the imaginary part yields the other. This proves the first statements of the theorem and implies the second because of the definition of analyticity.

Formulas (4) and (5) are also quite practical for calculating derivatives $f^{'}$(z), as we shall see.

Examples 28

Cauchy – Riemann Equations

$f(z) = z^2$ is analytic for all z. it follows that the Cauchy – Riemann equations must be satisfied (as we have verified above).

For $f(z) = \bar{z} = x - iy$ we have u = x, v = -y and see that the second Cauchy-Riemann equation is satisfied, $u_y = -v_x = 0$, but the first is not: $u_x = 1 \neq v_y = -1$. We conclude that $f(z) = \bar{z}$ is not analytic, confirming example 4 of sec. 12.4. Note the savings in calculation!

The Cauchy – Riemann equations are fundamental because they are not only necessary but also sufficient for a function to be analytic. More precisely, the following holds.

Theorem 2 (Cauchy – Riemann Equations)

If two real – valued continuous functions u(x,y) and v(x,y) of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy – Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

The proof of this theorem is more involved than the previous proof; Theorems 1 and 2 are of great practical importance, since by using the Cauchy – Riemann equations we can now easily find out whether or not a given complex function is analytic.

Example 29

Cauchy – Riemann Equations

Is $f(z) = z^3$ analytic?

Solution

We find $u = x^3 - 3xy$ and $v = 3x^2y$ and $v = 3x^2y - y^3$. next we calculate

$u_x = 3x_2 - 3y_2$, $v_y = 3x^2 - 3y^2$

$u_y = -6xy$, $v_x = 6xy$

We see that the Cauchy – Riemann equations are satisfied for every z, hence $f(z) = z^3$ is analytic for every z, by theorem 2.

Example 30

Determination of an Analytic Function with given Real Part

We illustrate another class of practical; that can be solved by the Cauchy – Riemann equations.

Find the most general analytic function f(z) whose real part is

$$u = x^3 – y^2 – x.$$

Solution

We have $u_z = 2x – 1 = v_y$ by the first Cauchy – Riemann equation. This we integrate with respect to y;

$$v = 2xy - y + k(x).$$

As an important point, since we integrated a partial derivative with respect to y, the "constant" of integration k may depend on the other variable, x. (To understand this, calculate v_y from the v.) and the second Cauchy – Riemann equation.

$$u_y = -v_x = -2y + \frac{dk}{dx}$$

On the other hand, from the given $u = x^2 – y^2 – x$ we have $u_y = -2y$. By comparison, dk/dx = 0. Hence k = constant, which must be real. (Why?).

The result is

$$f(z) = u + iv = x^2 - y^2 - x + i(2xy - y + k).$$

We can express in terms of z, namely, $f(z) = z^2 – z + ik$.

Example 31

An Analytic Function of Constant Absolute Value is Constant

The Cauchy – Riemann equations also help to establish general properties of analytic functions.

For example, show that if f (z) is analytic in a domain D and $|f(z)| = k =$ constant in D, then f(z) = constant in D.

Solution

By assumption, $u^2 + v^2 = k^2$ by differentiation.

$$uu_x - vu_x = 0. \qquad uu_y + vv_y = 0.$$

From this and the Cauchy – Riemann equations.

6. (a) $uu_x - uu_y = 0.$ (b) $uu_y + uu_x = 0$

To get rid of u_y multiply (6a) by u and (6b) by v and add. Similarly to eliminate u_x, multiply (6a) by – v and (6b) by u and add. This yield.

$$\left(u^2 + v^2\right)u_x = 0. \qquad \left(u^2 + v^2\right)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then u = v, hence f = 0. if $k \neq 0$, *then* $u_x = u_y = 0$, hence by the Cauchy – Riemann equations, also $v_x = v_y = 0.$ together, u = constant and v = constant, hence f = constant.

If we use polar form z = r(cos θ + isin θ) and set
f(z) = u(r, θ), then the Cauchy – Riemann equations are

7. $u_r = \frac{1}{r} v_\theta$ *and* $v_r = \frac{-1}{r} u_\theta$

The derivative can then be calculated from

8a. $f'(z) = \left(u_r + iv_r\right)\left(\cos\theta - i\sin\theta\right)$
or from

8b. $f'(z) = \left(v_\theta - iu_\theta\right)\left(\cos\theta - i\sin\theta\right)/r$.

Example 32 Cauchy – Riemann equations in polar form

let $f(z) = z^3 = r^3\left(\cos 3\theta + i\sin 3\theta\right).$

Then $u = r^3\cos 3\theta, v = r^3\sin 3\theta$

By definition,

$$u_r = 3r^2\cos 3\theta, \qquad v_\theta = 3r^3\cos 3\theta,$$
$$v_r\, 3r^2\sin 3\theta, \qquad u_\theta = 3r^3\sin 3\theta$$

We see that (7) holds for all z $\neq 0.$this confirms that z^3 is analytic for all z $\neq 0.$(and we know that it is also analytic at (z = 0). From (8b) we obtain the derivative as expected.

$$f'(z) = 3r^2(\cos 3\theta + i\sin 3\theta)(\cos\theta - i\sin\theta) = 3z^2.$$

Laplace's Equation: Harmonic functions

One of the main reasons for the great practical importance of complex analysis in engineering mathematics results form the fact that the real part of an analytic function f = u + iv satisfies the so – called Laplace's equation.

9. $\nabla^2 u = u_{xx} + u_{yy} = 0.$

(∇^2 read "nabla squared") and the same holds fort the imaginary part

10. $\nabla^2 v = v_{xx} + v_{yy} = 0.$

Laplace's equation is one of the most equations in physics, occurring in gravitation, electrostatics, fluid flow, etc. (cf. chaps. 11, 17) let us discover why this equation arises in complex analysis.

Theorem 3 (Laplace's Equation)

If f(z) = u(x,y) + iv(x,y) is analytic in a domain d, then u and v satisfy Laplace's equation (9) and (10) in d and have continuous second partial derivatives in D.

Proof:

Differentiating $u_x = v_y$ with respect to x and $u_y = v_x$ with respect to y, we obtain

11. $u_{xx} = v_{yx}$ $u_{yy} = -v_{xy}.$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in sec. 13.6). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal; $v_{yx} = v_{xy}$. By adding (11) we thus obtain (9). Similarly, (10), is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$.

Solutions of Laplace's equation having continuous second – order partial derivatives are called harmonic functions and their theory is called potential theory (cf. also sec. 11.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy – Riemann equations in a domain d, they are the real and imaginary parts of an analytic function f in d. Then v is said to be a conjugate harmonic function of u in d. (of course this use of the word "conjugate" has nothing to do with that employed in defining $\bar{z}$, the conjugate of a complex number z).

A conjugate of a given harmonic function can be obtained from the Cauchy – Riemann equations, as may be illustrated by the following example.

Example 33

Conjugate Harmonic Function

Verify that $u = x^2 - y^2 - y$ is harmonic in the complex plane and find a conjugate harmonic function of v of u.

Solution

$\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1.$ hence a conjugate v of u must satisfy

$$v = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain.

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}$$

A comparison with the second shows that dh/dx = 1. This gives h(x) = x + c. hence $v = 2xy + x + c\,(c \; any\; real\; constant)$ is the most general conjugate harmonic of the given u.

The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$

The Cauchy – Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shown in chapter 17. In the remainder of this chapter we discuss elementary functions, one after the other, beginning with e^z in the next section. Without knowing these functions and their properties we would not be able to do any useful practical work. This is just as in calculus.

3.6 Exponential Function

The remaining sections of this chapter will be devoted to the most important elementary complex function, logarithm, trigonometric functions, etc we shall see that these complex functions can easily be defined in such a way that, for real values of the independent variable, the functions become identical with the familiar real functions. Some of the complex functions have interesting properties. Which do not show when the independent variable is restricted to real values. The student should follow the consideration with great care, because these elementary functions will be frequently needed in applications.

We begin with the complex exponential function also written as one of most important analytic functions. The definition of e^z in terms of the real functions e^x cos y and sin y is $e^z = e^x(\cos y + \sin y)$. This definition is motivated by requirement that make e^z a natural extension of the real exponential function e^x, namely.

(a) e^z should reduce to the latter when z = x is real;
(b) e^z should be an entire function, that is analytic for all z, and resembling calculus, its derivative should be

2. $(e^z)^1 = e^z$

from (1) we see that (a) holds, since cos 0 = 1 and sin 0 = 0. that e^z is easily verified by the Cauchy-Riemann equations. Formula (2) then follows from (4) that

$(e^z)^1 = (e^z \cos y)_z + i(e^x \sin y)_x = e^z \cos y + ie^z \sin y = e^{z.}$

$e^{z.}$ has further interesting properties. Let us first show that, as in real, we have the functional relations

3. $e^{z_1+z_2} = e^{z_1 z_2}$

For any

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, indeed, by (1).

$= e^{x_1}\left(\cos y_1 + i\sin y_1\right)e^{x_2}(\cos y_2 + i\sin y_2\right).$

Since $e^{x_1}e^{x_2} = e^{x_1} + e^{x_2}$ for these real functions, by an application of the addition formulas for the cosine and sine functions (similar to that in sec. 12.2) we find that this equals

$e^{z_1 z_2} = e^{x_1}\left(\cos\left(y_1 + y_2\right) + i\sin(y_1 + y_2)\right) = e^{z_1 + z_2}$

As asserted. An

4. $\left|e^{iy}\right| = \left|\cos y + i\sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1.$

That is, for pure imaginary exponents the exponential function has absolute value one, a result the student should remember. From (7) and (1),

5. $\left|e^{z}\right| = e^{x}$. Hence arg $e^{z} = y + 2n\pi$ $(n = 0,1,2,\cdots)$

since $\left|e^{z}\right| = e^{x}$.shows that (1) is actually e^{x} in polar form.

Example 34

Illustration of Some Properties of the Exponential Function

Computation of values from (1) provides no problem. For instance, verify that

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i\sin 0.6) = 4.055(0.825 - 0.565i) = 3.347 - 2.290i,$$
$$\left|e^{1.4-0.6i}\right| = e^{1.4} = 4.055, \qquad \text{Arg} e^{1.4-0.6i} = -0.6.$$

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we have from (5)

6. $e^{2\pi i} = 1$

Furthermore use (1), (5) or (6) to verify these important special values:

7. $e^{\pi i/2} = i, \quad e^{\pi i} = -1, \quad e^{-\pi i/2} = -i, \quad e^{-\pi i} = -1.$

To illustrate (3), take the product of

$$e^{2+i} = e^{2}(\cos i + i\sin 1) = e^{4-i}e^{4}(\cos 1 - i\sin 1)$$

and verify that equals

$$e^{2}e^{4}(\cos^2 1 + \sin^2 1) = e^{6} = e^{(2+i)-(4-i)}.$$

Finally, conclude from $\left|e^{z}\right| = e^{x} \neq 0$ in (8)that

8. $e^{x} \neq 0$ for all z

So here we have an entire function that never vanishes, in contrast to (non-constant) polynomials, which are also entire (Example 5 in

Sec.2.4) but always have zero, as is proved in algebra. [Can you obtain (11) from (3) ?]

Periodicity of e^x with period $2\pi i$,

9. $e^{z+2\pi i} = e^z$ all z

is a basic property that follows from (1) and the periodicity of cosy and siny. It also follows from (3) and (9).] Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π .

10. $-\pi < y \leq \pi$

This infinite strip is called a **fundamental region** of e^x .

Example 35

Solution of an Equation

Find all solution of $e^x = 3 + 4i$

Solution

$\left|e^x\right| = e^x = 5, x = In\,5 = 1.609$ is a real part of all solutions. Furthermore, since $e^x = 5$,

$e^x \cos y = 3,\quad e^x \sin y = 4,\quad \cos y = 0.6,\quad \sin y = 0.8,\quad y = 0.927$.

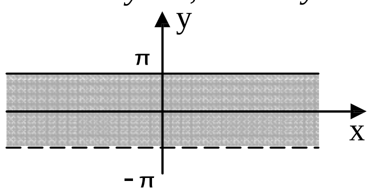


Fig. 20: Fundamental Region of the Exponential Function e^z I in the z-plane

Ans. $z = 1.609 + 0.927i \pm 2n\pi i \quad (n = 0,1,2,\cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line x=1.609 at a distance 2π from their neighbours.

To summarise: many properties of $e^z = \exp z$ parallel to those of e^x; an exception is the periodicity of e^x with $2\pi i$, which suggested the concept of a fundamental region and causes the periodicity of cosz and sinz with the *real* period 2π , as we shall see in the next section. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

3.7 Trigonometric Functions, Hyperbolic Functions

Just as e^z extends e^x to complex, we want the complex trigonometric functions to extend the familiar real trigonometric functions. The idea of making the connection is the use of the Euler formulae.

$$e^{ix} = \cos x + i\sin x, \qquad e^{-ix} = \cos x - i\sin x.$$

By addition and subtraction we obtain

$$\cos x = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin x = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad x \text{ real}$$

This suggests the following definitions for complex values $z = x + iy$

1. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$

Furthermore, in agreement with the definition from the real calculus we define

2. $\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$

and

3. $\sec z = \frac{1}{\cos z}, \qquad \operatorname{cosec} z = \frac{1}{\sin z}.$

Since e^z is entire, cosz and sinz are entire functions. Tanz and secz are not entire; they are analytic except at the point where cosz is zero; and $\cot z$ and $\csc z$ are analytic except, where $\sin z = 0$. Formulas for the derivatives follows readily from $(e^z)' = e^z$ and (1)-(3); as in calculus,

4. $(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z,$

etc. Equation (1) also shows that **Euler's formula** *is valid in complex:*

5. $e^{iz} = \cos z + i\sin z \qquad$ for all z.

Real and imaginary parts of cosz and sinz are needed in computing values, and they also help in displaying properties of our functions. We illustrate this by typical example.

Example 36

Real and Imaginary Parts. Absolute Value. Periodicity

Show that

(a) $\cos z = \cos x \cosh y - i \sin x \sinh y$
6. (b) $\sin z = \sin x \cosh y + i \cos x \sinh y$

and

7. (a) $|\cosh z|^2 = \cos^2 x + \sinh^2 y$
(b) $|\sinh z|^2 = \sin^2 x + \sinh^2 y$

And give some application of these formulas.

Solution

From (1)

$$\cos z = \frac{1}{2}(e^{i()x+iy} + e^{-i(x+iy)})$$
$$= \frac{1}{2}e^{-y}(\cos x + i \sin y) + \frac{1}{2}e^{y}(\cos x - i \sin y)$$
$$= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2}i(e^{y} - e^{-y})\sin x.$$

This yields (6a) since, as is known from calculus,

8. $\cosh y = \frac{1}{2}(e^{y} + e^{-y}),$ $\sinh y = \frac{1}{2}(e^{y} - e^{-y});$

(6b) is obtained similarly. From and $\cosh^2 y = 1 + \sinh^2 y$ we obtain $|\cos|^2 = \cos^2 x(1 + \sinh^2 y) + \sin^2 x + \sinh^2 y.$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2 + 3i \cos 2 \cosh 3 - i \sin 2 \sinh 3 = -4.190 - 9.109i$.

From (6) we see that cosz and sinz are *periodic with period* 2π , just as in real. Periodicity of tan z and cot z with period π now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine: whereas $|\cos x| \le 1$ and $|\sin x| \le 1$, the complex cosine and sine functions are no longer bounded but approach infinity in absolute value as $y \to \infty$, since $\sinh y \to \infty$.

Example 37

Solution of Equations. Zeros

Solve

(a) $\cos z = 5$ (which has no real solution),
(b) $\cos z = 0$
(c) $\sin z = 0$

Solution

(a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solution (3D-values)

$$e^{iz} = e^{-y+ix} 5 \pm \sqrt{25-1} = 9.899 \text{ and } 0.101.$$

Thus $e^{-y} = 9.899$ or 0.101, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$

Ans. $z = \pm 2n\pi \pm 2.292i (n = 0,1,2, \cdots)$, can you obtain this by using (6a)?

(b) $\cos x = 0$, $\sinh y = 0$, by (7a), $y = 0$.

Ans. $z = \pm \frac{1}{2}(2n+1)\pi \quad (n = 0,1,2, \cdots)$.

(c) $\sin sx = 0$, $\sinh y = 0$, by (7b), $y = 0$.

Ans. $z = 2n\pi \quad (n = 0,1,2, \cdots)$.

Hence the only zeros of cos*z* and sin*z* are those of the real cosine and sine functions.

From the definition it follows immediately that all the familiar formulas for the real trigonometric functions continue to hold for complex values.

We mention in particular the addition rules

9. $$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$

and the formula

10. $\cos^2 z + \sin^2 z = 1.$

Some further useful formulas are inclined in the problem set.

HYPERBOLIC FUNCTIONS

The complex **hyperbolic cosine** and **sine** are defined by the formulas

11. $\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}).$

This suggested by the familiar definition for the real variable. These functions are shown below, with derivatives

12. $(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z,$

as in calculus. The other hyperbolic functions are defined by

$$\tan z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

13. $\sec hz = \dfrac{1}{\cosh z}, \qquad \csc hz = \dfrac{1}{\sin zh},$

Complex trigonometric and hyperbolic functions are related

If in (11), we replace z by *iz* and use (1), we obtain

14. $\cosh iz = \cos z, \qquad \sinh iz = i \sin z,$

From this, since cosh is even and sinh is odd, conversely

15. $\cos iz = \cosh z, \qquad \sin iz = i \sinh z,$

Apart from their practical importance, these formulas are remarkable in principle. Whereas in real calculus, the trigonometric and hyperbolic functions are of a different character, in complex these functions are intimately related. Moreover the Euler formula relates them to the exponential function. This situation illustrates that by working in complex, rather than in real, one can often gain a deeper understanding of **special functions.** This is one of the three main reasons of the practical importance of complex analysis, mentioned at the beginning of this chapter.

In the next section we discus the **complex logarithms,** which differ substantially from the real logarithm (which is simpler), and the student should work the next section with particular care.

4.0 CONCLUSION

To this end, we conclude by giving a summary of what we have covered.

5.0 SUMMARY

For arithmetic operations with **complex number**

1. $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$,
   $r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Sec 2.1 and 2.2
   A complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in domain *D* if it has a **derivative.**

2. $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$
   Everywhere in *D*. Also, $f(z)$ is analytic at a point $z = z_0$ if it has a derivative in a neighbourhood of z_0 (not merely at z_0 itself).
   If $f(z)$ is analytic in *D*, then *u(x, y)* and $v(x, y)$ satisfy the (very important!) **Cauchy-Riemann** equations (Sec. 2.5).

3. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
   everywhere in *D*. Then u and v also satisfy **Laplace's equation**

4. $$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$
   everywhere in *D* . If *u(x, y)* and $v(x, y)$ are continuous and have continuous partial derivatives in *D* that satisfy (3) in *D*, then $f(z) = u(x, y) + iv(x, y)$ is analytic in domain *D* .Sec. 2.5 the complex exponential function (Sec. 2.6)

5. $e^z = \exp z = e^z(\cos y + i\sin y)$
   is periodic with $2\pi i$, reduces to e^z when $z = x(y = 0)$ and has the derivative e^z. The **trigonometric function**s are (Sec.2.7)
   $$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y$$
   $$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \sin x \cosh y - i \cos x \sinh y$$

$\tan z = (\sin z)/\cos z, \cot z = 1/\tan z,$ etc.

6.0 TUTOR-MARKED ASSIGNMENT

i. Let $z_1 = 3 + 4i$ and $z_2 = 5 - 2i$
Find in the form $x + iy$

(a) $(z_1 - z_2)^2$ (b) $z_2 / 2z$

ii. Show that z is pure imaginary if and only if $\bar{z} = -z$.

iii. Find; (a) $|1 - i|^2$ (b) $\left|\frac{(3+4i)^4}{(3-4i)^3}\right|$

iv. Represent in polar form

(a) $i\sqrt{2} / 3 + 3i$ (b) $4i$

v. Determine the principal value of the arguments of

(a) $-2 + 2i$ (b) $1 - i\sqrt{3}$

vi. Represent in form $x + iy$

(a) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\sqrt{50}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

vii. Determine and sketch the sets represented by

(a) $|z - 2i| = 2$ (b) $z\bar{z} + (1+2i)z + (1-2i) + 1 = 0$

viii. Find $f(2+i)$, $f(-4+i)$ where $f(z)$ equals

(a) $3z^2 + z$ (b) $\frac{(z+1)}{(z-1)}$

ix. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

x. Prove the product rule $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$

xi. Are the following functions analytic?

(a) $f(z)z^4$ (b) $f(z)e^x(\cos y + i\sin y)$.

xii. Let v be a conjugate harmonic of u in some domain *D*. Show that then $h = u^2 - v^2$ is harmonic in *D*.

xiii. Derive the Cauchy-Riemann equations in polar form equation from equation 1.

xiv. Using the Cauchy-Riemann equations, show that e^x is analytic for all z.

xv. Compute e^z (in the form $(u + iv)$ and $|e^z|$) when z equals

(a) $\pi - i/2$ (b) $-1 - \frac{7\pi i}{4}$

xvi. Show that $u = e^{xy}\cos\left(\frac{x^2}{2} - \frac{y^2}{2}\right)$ is harmonic and find a conjugate.

xvii. Prove that $\cos z, \sin z, \cosh z,$ and $\sinh z$ are entire functions.
xviii. What is the idea that led to the Cauchy-Riemann equations?
xix. State the Cauchy-Riemann equations from memory.
xx. What is an analytic function? Can a function be differentiable at a point z_0 without being analytic at z_0.

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5^{th} Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 2 INTEGRATION OF COMPLEX PLANE

CONTENTS

1.0 INTRODUCTION

In this unit we defined and explained complex integrals. The most fundamental result in the whole unit is Cauchy's integral theorem. It implies, the importance of Cauchy integral formula.

We prove that if a function is analytic, it has derivatives of all orders. Hence, in this respect, complex analytic functions behave much more simply than real-valued functions of real variables. Interpretation by means of residues and applications to real integrals will be considered in Module 3.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- in applications there occur real integrals that can be evaluated by complex integration, whereas the usual methods of real integral calculus are not successful and
- some basic properties of analytic function can be established by integration, but would be difficult to prove by other methods. The existence of higher derivatives of analytic functions is a striking property of this type.

3.0 MAIN CONTENT

3.1 Line Integral in the Complex Plane

As in real calculus, we distinguish between definite integrals, and indefinite integrals or ant derivatives. An **indefinite integral** is a function whose derivative equals a given analytic function in a region. By inverting known differentiation formulas we may find many types of indefinite integrals.

We shall now define *definite integrals*, or line integrals, of complex function $f(z)$, where $z = x + iy$ as follows;.

Path of Integration

In real calculus, a definite integral is taken over an interval (a segment) of the real line. In the case of a complex definite integral we integrate along a curve C in the complex plane, which will be called the *path of integration.*

Now a curve C in the complex plane can be represented in the form

$$z(t) = x(t) + iy(t) \qquad (a \le t \le b) \qquad (1)$$

where t is a real parameter. For example,

$$z(t) = t + 3it \qquad (0 \le t \le 2)$$

represent a portion of the line $y = 3x$ (sketch it!),

$$z(t) = 4\cos t + 4i\sin t \qquad (-\pi \le t \le \pi)$$

represent the circle $|z| = 4$, etc. (More example below)

C is called a smooth curve if C has a derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each of its points which is continuous and nowhere zero. Geometrically this means that C has a continuous turning tangent. This follow directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

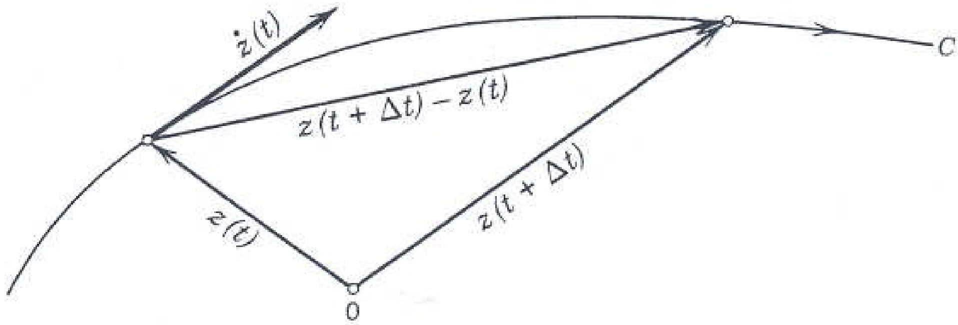


Fig. 21: Tangent vector z(t) of a curve C in the complex plane given by z(t). The arrow on the curve indicates the positive sense (sense of increasing t).

3.1.1 Definition of the Complex Line Integral

This will be similar to the method used in calculus. Let C be a smooth curve in the z-plane represented in the form (1). Let $f(z)$ be a continuous function defined (least) at each point of C. We subdivided ("partition") the interval $(a \le t \le b)$ in (1) by points of

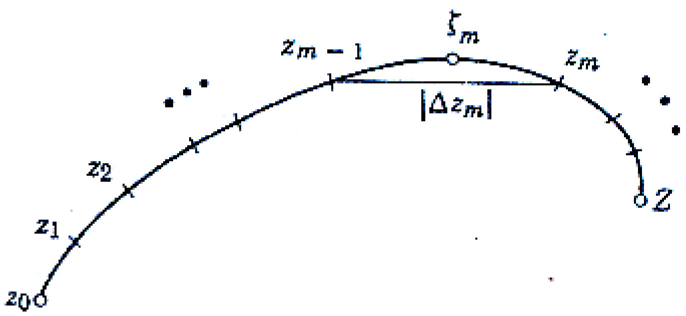


Fig. 22: Complex Line Integral

$t_0(=a), t_1, \cdot \cdot \cdot \cdot t_{n-1}, t_n(=b)$

Where $t_0 \langle t_1 \langle \cdots t_n$. To do this subdivision there corresponds a subdivision of C by points

$$z_0, z_1, \cdots .z_{n-1}, z_n (= z),$$

where $z_j = z(t_j)$. On each portion of subdivision of C we choose an arbitrary point, say, a point ξ_1 between z_0 and z_1 (that is, $\xi_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$, a point ξ_1 between z_0 and z_1 (that is, $\xi_1 = z(t)$) where t satisfies $t_0 \le t \le t_1$, a point ξ_2 between z_1 and z_2 etc. Then we form the sum

$$S_n = \sum_{m=1}^{n} f(\xi_m) \qquad (2)$$

where

$$\Delta z_m = z_m - z_m - 1.$$

This we do for each $n = 1,2,3,\cdots$ in a completely independent manner, but in such a way that the greatest $|\Delta z_m|$ approaches zero as n approaches infinity. This gives a sequence of complex numbers $S_2, S_3, \cdots$. The limit of these sequence is called the **line integral** (or simply the integral) of $f(z)$, along the oriented curve C and is denoted by

$$\int_C f(z)dz \qquad (3)$$

The curve C is called t**he path of integration.** C is called a **closed path** if $z = z_0$, that is, if its terminal point coincides with its initial point.

(Example: a circle, a curve shaped like an 8, etc.) Then also writes

$$\oint_C \text{ instead of } \int_C$$

Examples follow in the next section.

General Assumption

All path of integration for complex line integral will be assumed to be **piecewise smooth,** *that is, to consist of finitely many smooth curves joined end to end.*

3.1.2 Existence of the Line Integral

From our assumption that $f(z)$ is continuous and C is piecewise smooth, the existence of the line integral (3) follows, as in the previous chapter let us write $f(z) = u(x, y) + iv(x, y)$. We also set

$$\xi_m = \xi_m + i\eta_m \text{ and } \Delta z_m = \Delta x_m + i\Delta y_m$$

then (2) may be written

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m) \qquad (4)$$

Where $u = u(\xi_m, \eta_m)$ and $v = v(\xi_m, \eta_m)$ we sum over m from1 to n. We may now split up S_n into four sums:

$$S_n = \sum u\Delta x_m - \sum v\Delta y_m + i[\sum u\Delta y_m + \sum v\Delta x_m]$$

These sums are real. Since f is continuous, u and v are continuous. Hence, if we let n approach infinity in the aforementioned way, then the greatest Δx_m and Δy_m will approach zero and each sum on the right becomes a real line integral:

$$\lim_{n\to\infty} S_n = \int_C f(z)dz = \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right] \qquad (5)$$

This shows that under our assumption (f continuous on C_1 and C_2 piecewise smooth) the line integral (3) exist and its value is independent of the choice of subdivisions and intermediate points ξ_m.

3.1.3 Three Basic Properties of Complex Line Integrals

We list three properties of complex line integrals that are quite similar to those of real definite integrals (and real line integrals) and follow immediately from the definition.

Integration is a linear operation, that is, a sum of two (or more) functions can be integrated term by term, and constant factors can be taken out from under the integral sign:

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz \qquad (6)$$

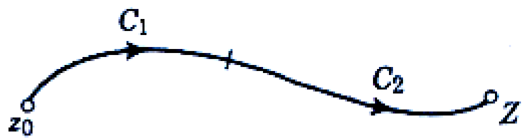


Fig. 23: Subdivision of Path (Formula (7)

Decomposing C into two portions C_1 and C_2 (Fig), we get

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad (7)$$

3. Reversing the sense of integration, we get the negative of the original value:

$$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz \qquad (8)$$

here the path C with endpoint z_0 and Z is the same; on the left we integrate from z_0 to Z, on the right from z_0 to Z.

Applications follow in the next section and problems at the end of it.

3.2 Two Integration Methods

Complex integration is rich in methods for evaluating integrals. We discuss first two of them, and others will follow later in this chapter.

3.2.1 First Method: Use of Representation of the Path

This method applies to any continuous complex function.

Theorem 1 (Integration by the use of the path)

Let C be a piecewise smooth path, represented by z = z(t), where $a \le t \le b$ *Let f(z) be a continuous function on C. Then*

$$\int_C f(z) = \int_a^b f[z(t)]\dot{z}(t)dt \qquad \left(\dot{z} = \frac{dz}{dt}\right) \qquad (1)$$

Proof

The left-hand side of (1) is given by (5), Sec, 13.1, in terms of real integrals, and we show that the right-hand side of (1) also equals (5).

We have $z = x + iy$, hence $\dot{z} = \dot{x} + i\dot{y}$. We simply write u for $u[x(t), y(t)]$ and v for $v[x(t), y(t)]$. We also have $dx = \dot{x}dt$ and $dy = \dot{y}dt$. Consequently, in (1),

$$\int_a^b f[z(t)]\dot{z}(t)dt = \int_a^b (u + iv)(\dot{x} + i\dot{y})dt$$
$$= \int_C [udx - vdy + i(udy + vdx)],$$

Which is the right-hand side of (5), as claimed.

Steps in applying Theorem 1

Represent the path C in the form $z(t)$ $a \le t \le b$
Calculate the derivative $\dot{z}(t) = dz/dt$
Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y)
Integrate $f[z(t)]\dot{z}(t)$ over t from a to b

Example 1

A Basic Result: Integral of $1/z$ around the unit circle

Show that

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i} \qquad (C \text{ the unit circle, clockwise}) \qquad (2)$$

The important result will be frequently needed.

Solution

We may represent the unit circle C in the form

$$z(t) = \cos t + i \sin t \, z \qquad (0 \le t \le 2\pi).$$

So that the counterclockwise integration correspond to an increase of t from 0 to 2π. By differentiation,

$$\dot{z}(t) = -\sin t + i\cos t$$

Also $f[z(t)] = \dfrac{1}{z(t)}$. Formula (1) now yields the desired result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{\cos t + i \sin t}(-\sin t + i\cos t)dt$$

$$= i\int_0^{2\pi} dt$$

$$= 2\pi i$$

The Euler formula helps us to save work by representing the unit circle simply in the form

$$z(t) = e^{it}$$

Then

$$\frac{1}{z(t)} e^{-it}, \quad dz = ie^{it}dt.$$

As before, we now get more quickly

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it} = i\int_0^{2\pi} dt$$
$$= 2\pi i .$$

Example 2

Integral of Integer Powers

Let $f(z) = (z - z_0)^m$ where m is an integer and z_0 is a constant. Integrate in the clockwise sense around the circle C of radius ρ with centre at z_0

Solution

We may represent the unit circle C in the form

$$z(t) = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \text{z} \qquad (0 \le t \le 2\pi).$$

Then we have
$(z - z_0)^m = \rho^m e^{imt}$, $\quad dz = i\rho e^{it} dt$,
and we obtain

$$\oint_C (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} dt$$
$$= \int_0^{2\pi} e^{i(m+1)t} dt.$$

By the Euler formula (5), the right-hand side equals

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t + i\int_0^{2\pi} \sin(m+1)t\right].$$

When $m = -1$, we have $\rho^{m+1} = 1, \cos 0 = 1, \sin 0 = 0$ and thus obtain $2\pi i$. For integer $m \neq 1$ each of the two integer is zero because we integrate over an interval of length $2\pi i$, equal to a period of sine and cosine. Hence the result is

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & (\text{m = -1}) \\ 0 & (\text{m} \neq \text{-1 and integer}). \end{cases} \qquad (3)$$

Let us now illustrate the following important fact. If we integrate a function $f(z)$, from a point z_0 to a point z_1 along different path, we generally get the values of the integral. In other words, a complex line integral generally depends not only on the end point of the path but also on the geometric shape of the path.

Example 3

Integral of Non-analytic Function

Integrate $f(z) = x$ from 0 to 1.
along C^* in fig. 325 below.
along C consisting of C_1 and C_2.

Solution

a. C^* can be represented by $z(t) = t + it \quad (0 \le t \le 1)$. Hence
$\dot{z}(t) = +i$ and $f[z(t)] = x(t) = 1$ (on C^*).

We now calculate

$$\int_C \operatorname{Re} z dz = \int_0^1 t(1+i)dt$$
$$= \frac{1}{2}(1+i).$$

b. C_1 can be represented by $z(t) = t \quad (0 \le t \le 1)$. Hence
$\dot{z}(t) = 1$ and $f[z(t)] = x(t) = 1$ (on C_1).
C_2 can be represented by $z(t) = t + it \quad (0 \le t \le 1)$. Hence
$\dot{z}(t) = 1$ and $f[z(t)] = x(t) = 1$ (on C_2).
Using (7) , we calculate

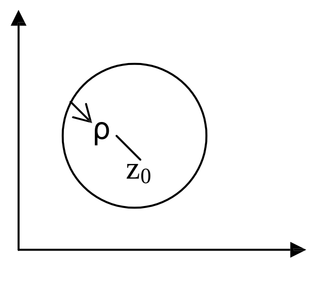


Fig. 24 Path in Example 2

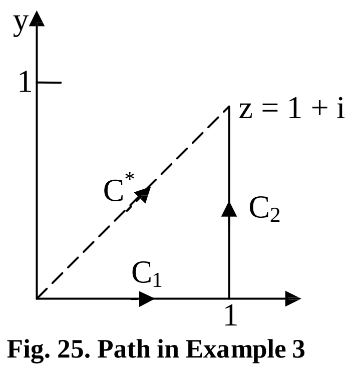


Fig. 25. Path in Example 3

$$\int_C \operatorname{Re} z dz = \int_{C_1} \operatorname{Re} z dz + \int_{C_2} \operatorname{Re} z dz = \int_0^1 t dt + \int_0^1 1 \cdot t dt$$
$$= \frac{1}{2} + i$$

Note that this result is differ from the result in (a).

3.2.2 Second Method: Indefinite Integration

In real calculus, if for given $f(x)$ we know an $F(x)$ such that $F'(x) = f(x)$,
then we can apply the formula

$$\int_a^b f(x)dx = F(b) - F(a)$$

This method extends to complex functions. We shall see that it is simpler than the previous method, but, of course, we have to find an $F(z)$ whose derivative $F'(z)$ equals the given function $f(z)$ that we want to integrate. Clearly, differentiation formulas will often helps us in finding such an $F(z)$, so that this method becomes of great practical importance.

Theorem 2 (Indefinite Integration of Analytic Functions)

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all path in D joining two point z_0 and z_1 in D we have

4. $$\boxed{\int_{z_0}^{z_1} f(x)dz = F(z_1) - F(z_0)} \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 *and* z_1 *instead of* C, *since we get the same value for all those* C *from* z_0 *and* z_1).

This theorem will be proved by using Cauchy's integral theorem which we discuss in the next section…

Example 4

$$\int_0^{1+i} z^3 dz = \frac{1}{3}z^3 \Big|_0^{1+i}$$

$$= \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

Example 5

$$\int_{-\pi i}^{\pi i} \cos z dz = \sin z \Big|_{-\pi i}^{\pi i}$$

$$= 2\sin \pi i = 2i \sinh \pi = 23.097i$$

Example 6

$$\int_{8+3\pi i}^{8-3\pi i} e^{z/2} dz = 2e^{z/2}\Big|_{8+3\pi}^{8-3\pi}$$
$$= 2(e^{4-3\pi i/2} - e^{4+3\pi i/2})$$
$$= 0$$

Since e^z is periodic with period $2\pi i$.

3.2.3 Bound for Absolute Value of Integrals

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

6. $\left|\int_C f(z)dz\right| \le ML$ **(ML -inequality)**;
here L is the length of C and M a constant such that $|f(z)| \le M$ everywhere on C.

Proof:

We consider S_n as given by (2). By the generalized triangle inequality (6), we obtain

$$|S_n| = \left|\sum_{m=1}^{n} f(\xi_m)\Delta z_m\right| \le \sum_{m=1}^{m} |f(\xi_m)||\Delta z_m|$$
$$\le M\sum_{m=1}^{n}|\Delta z_m|.$$

Now Δz_m is the length of the chord whose end points are z_{m-1} and z_m. Hence the sum on the right represents the length L^* of the broken line of the chord whose endpoints are $z_0, z_1, \cdot \cdot \cdot \cdot \cdot, z_n (n$ If n approaches $= Z)$.

infinity in such a way that the greatest $|\Delta z_m$ approaches zero, then L^* approaches the length L of the curve C , by the definition of the length of a curve. From this the inequality (6) follows.

We cannot see for (6) how close to the bound *ML* the actual absolute value of the integral is, but this will be no hardship in applying (6). For the time being we explain the practical use of (6) by a simple example.

Example 8

Find a upper bound for the absolute value of the integral

$\int_C z^2 dz,$ C the straight-line segment from 0 to $1+i$

Solution

$L = \sqrt{2}$ and $|f(z)| = |z^2| \leq 2$ on C gives by (6)

$\left|\int_C z^2 dz\right| \leq 2\sqrt{2} = 2.8284$

The absolute value of the integral is

$\left|-\frac{2}{3} + \frac{2}{3}i\right| = \frac{2}{3}\sqrt{2} = 0.9428$

In the next section we discuss the most important theorem of the whole chapter, **Cauchy's integral theorem,** which is the basic in itself and has far reaching consequences which we shall explore, above all the existence of all higher derivatives of an analytic function, which are themselves analytic functions.

3.3 Cauchy's Integral Theorem

Cauchy's integral theorem is very important in complex analysis and has various theoretical and practical consequences. To state this theorem, we shall need the following concepts.

A closed path C is called a **simple close path** if C does not intersect or touch itself (see diagram below). For example a circle is simple, an eight- shaped curve is not.

A domain D in the complex plane is called a **simply connected domain** if every closed path in D encloses only points of D. A domain that is not simply connected is called *multiply connected.*

For instance, the interior of a circle ("circular disk"), ellipse or square is

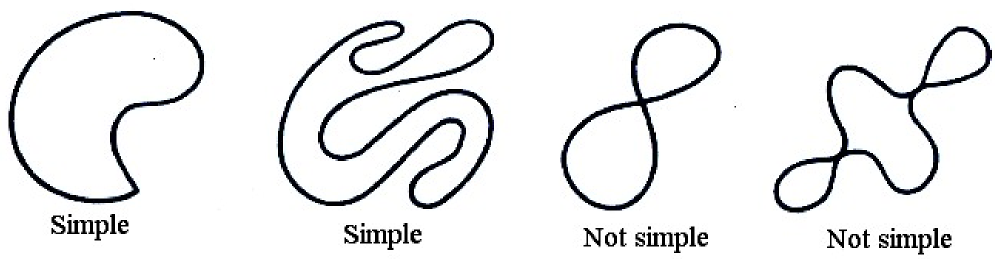


Fig. 326. Closed paths

simply connected. More generally, the interior of a simple closed curve is simply connected. A circular ring or annulus is multiply connected (more precisely: doubly connected). The figure below shows further examples.

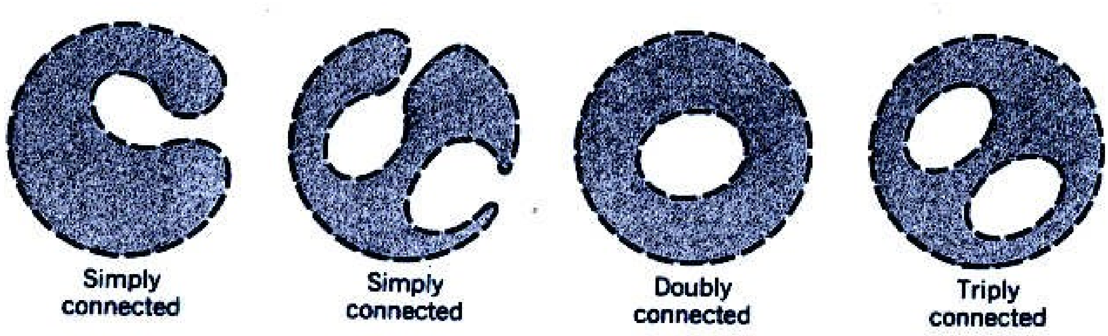


Fig. 27: Simply and Multiply Connected Domain

Recalling that, by definition, a function is a *single-valued* relation, we can now state Cauchy's integral theorem as follows. This theorem is sometimes also called the **Cauchy-Gaursat theorem.**

3.3.1 Cauchy's Integral Theorem

If $f(z)$ be analytic in a simply connected domain D, then for every simple close path C in D,

1. $$\int_C f(z)dz = 0$$

Proof

If we make assumption –as Cauchy did- that the derivative $f'(z)$ of $f(z)$ is continuous in *D* (existence of $f'(z)$ in *D* being a consequences of analyticity), then Cauchy's theorem follows from a basic theorem on real

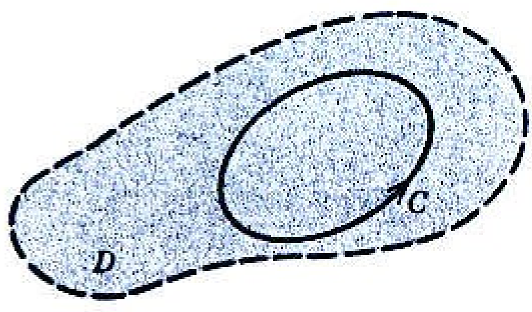


Fig. 28: Cauchy's Integral Theorem

line integrals (proof below). Goursat finally proved Cauchy's theorem without the assumption that $f'(z)$ is continuous (optional proof at the end of this chapter). Before we go into details, let us consider some example in order to really understand what is going on.

We mention that a closed path is sometimes called a contour and an integral over such a path a **contour integral.**

Example 9

$\int_C e^z dz = 0$, $\int_C \cos z dz = 0$ $\int_C z^n dz = 0$ $(n = 0, 1, \cdots)$

For any closed path, since these functions are (analytic for all z).

Example 10

$\int_C \sec z dz = 0$, $\int_C \frac{dz}{z^2 + 4} = 0$

where C is the unit circle. $\sec z = \frac{1}{\cos z}$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$ but all these points lie outside C ; none lie on C.

Similarly for the second integral, whose integrand is not analytic at $z = \pm 2\pi i$ outside C.

Example 11

$\int_C \bar{z} dz = 2\pi i$

(C the unit circle, counterclockwise) does not contradict Cauchy's theorem, since $f(z) = \bar{z}$ is not analytic, so that the theorem does not apply. (Verify this result!)

Example 12

$\int_C \frac{dz}{z^2} = 0,$

where C is the unit circle. This result does not follow from the Cauchy's theorem, because $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$. Hence the condition that f be analytic in D is sufficient rather than necessary for (1) to be true.

Example 13

$\int_C \frac{dz}{z^2} = 2\pi i,$

The integration being taken around the unit circle in the clockwise sense. C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $\frac{1}{z}$ is analytic, but this domain is not simply connected, so that Cauchy's theorem cannot be applied. Hence the condition that the domain D be simply connected is quite essential.

Example 14

$$\int_C \frac{7z-6}{z^2-2z}\,dz = \int_C \frac{7z-6}{z(z-2)}\,dz = \int_C \frac{3}{z}\,dz + \int_C \frac{4}{z-2}\,dz = 3\cdot 2\pi i + 0$$

$$= 6\pi i$$

(C the unit circle, counterclockwise) by partial fraction reduction.

Cauchy's proof under the condition that $f'(z)$ ***I continuous***

From (5) we have

$$\int_C f(z)dz = \int_C (udx - vdy) + \int_C (udy + vdx).$$

Since $f(z)$ is analytic in D, its derivative $F'(z)$ exists in D. Since $F'(z)$ is assume to be continuous, (4) and (5) in previous section imply that u and v have continuous partial derivatives in D. Hence Green's theorem with u and $-v$ instead of F_1 and F_2 is applicable and gives

$$\int_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

where R is the region bounded by C. The second Cauchy-Rieman integration shows that the integrand on the right is identically zero.

Hence, the integral on the left is zero. In the same fashion it follows by the use of the first Cauchy-Rieman equation that the last integral in the above formula is zero. This complete Cauchy's proof.

3.3.2 Independence of Path, Deformation of Path

We shall now discuss an important consequence of Cauchy's integral theorem that has great practical interest, proceedings as follows. If we subdivided the path, C in Cauchy's theorem into two arcs C_1^* and C_2, then (1) takes the form

$$(2') \qquad \int_{C_1} fdz + \int_{C_2} fdz = 0 .$$

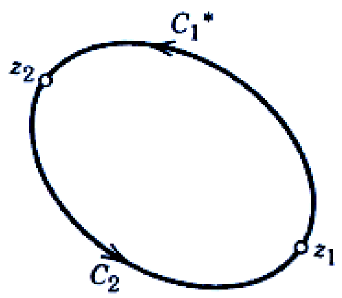


Fig. 29: Formula (2')

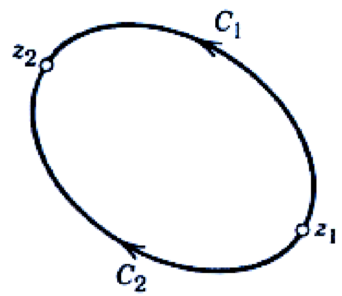


Fig. 30: Formula (2)

If we now reverse the sense of integration along C_1^*, then the integral over C_1^* is multiplied by -1. Denoting C_1^* with its new orientation by C_2, we thus obtain from $(6')$.

2. $$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz.$$

Hence, if *f* it is analytic in *D*, C_1^* and C_2 are any path in *D* joining two points in *D* and having no further points in common, then (2) holds.

If those paths C_1^* and C_2 have finitely many points in common, then (2) continues to hold. This follow by apply previous result to the portion of C_1 and C_2 between each pair of consecutives point of intersection.

If it is even true that (2) *holds for any paths that join ant points* z_1 *and* z_2 *and lie entirely in the simply connected domain D in which* $f(z)$ *is analytic.*

To express this we may say *that the integral of* $f(z)$ is **independent of path in *D*. (**Of course the value of the integral depends on the choice of z_1 *and* z_2.**)**

The proof may require additional consideration of the case in which C_1 and C_2 have infinitely many points of intersection, and is not presented here.

We may imagine that the path C_2 in (2) was obtained from C_1 by a continuous deformation. It follows that in a given integral we may impose a continuous deformation on the path of integration (keeping the endpoint fixed); as long as we do not pass through a point where $f(z)$ is not analytic, the value of the integral will not change under such deformation. This is often called the **principle of deformation of path.**

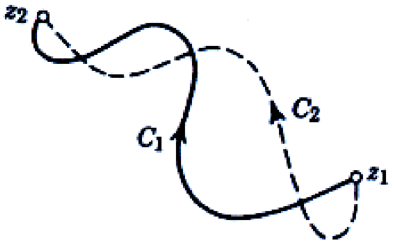


Fig. 31: Paths having finitely Many Intersections

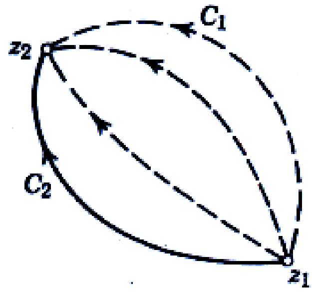


Fig. 32: Continuous Deformation of Path

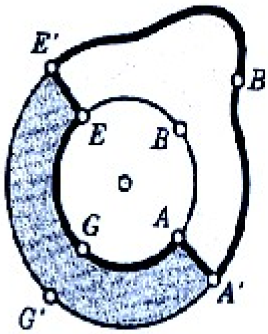


Fig. 33: Unit Circle and Path C

Example 15

$\int_C \frac{dz}{z} = 2\pi i$, (Counterclockwise integration) now follow from example (1), for any simple closed path C whose interior contains 0.The figure above gives the idea: first deform ABE continuously into the path $AA'B'E'E$. The heavy curve in the figure shows the resulting deformed path. Then deform $E'EGAA'$ and $E'G'A'$.

There is more general systemic approach to problem of this kind, as we shall now see.

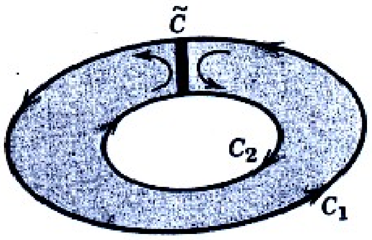
3.3.3 Cauchy Theorem for Multiple Connected Domains

A multiplys connected domain D^* can be cut so that the resulting domain (that is, D^* without the point of the cut or cuts) become simply connected.

For doubly connected domain D^* we need one cut $\tilde{C}$(figure below).If $f(z)$ is analytic in D^* and at each point of C_1 and C_2 then, since C_1, C_2 and C bound a simply connected domain, it follows from Cauchy's theorem that the integral of f taken over $C_1, \tilde{C}, C_2$ in the sense indicated by the arrows in the figure has the value zero. Since we integrate along $\tilde{C}$ in both directions, the corresponding integrals cancel out, and we obtain

(3*) $\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$

where one of the curve is traversed in the counterclockwise sense and the other in the opposite sense. Reversing the sense of integration on one of the curves, we may write this



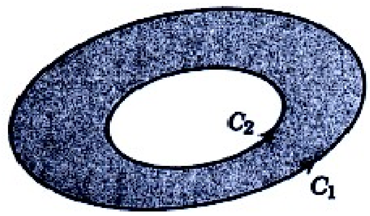


Fig 34: Doubly Connected Domain **Fig. 35: Paths in (3)**

3. $\int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

where curve now traversed in the same sense (the figure above). We remember that (3) holds under the assumption that $f(z)$ is analytic in the domain bounded by C_1 and C_2 and at each point of C_1 and C_2.

Can you see how the result in Example (7) now follows immediately from our present consideration?

For more complicated domains we may need more than one cuts, but the basic idea remains the same as before. For instance, for the triply connected domain in figure below,

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0$$

where C_2 and C_3 are traversed in the same sense and C_1 is traversed in the opposite sense.

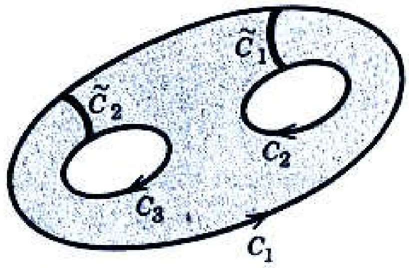


Fig. 36: Triply Connected Domain

Example 16

From (3), Example 2, it now follows that

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i (m = -1) \\ 0 (m \neq -1 \text{ and integer}) \end{cases}$$

For counterclockwise integration around any simple closed path containing z_0 in its interior.

In the next section, using Cauchy integral theorem, we prove the existence of indefinite integrals of analytic functions. This will also justify our earlier method of indefinite integration.

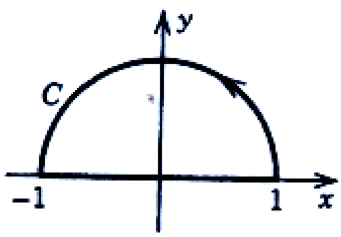


Fig. 39: Problem 29

3.4 Existence of Indefinite Integral

This section includes an application of Cauchy's integral theorem. It relates to Theorem 2 in section 3.2 on the evaluation of line integrals by indefinite integration and substitution of the limits of integration:

1. $\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$ $[F'(z) = f(z)]$,

Where $F(z)$ is an indefinite integral of *f(z)*, that is $F'(z) = f(z)$, as indicated.
In most applications, such a $F(z)$ can be found from differentiation formulas.

Theorem 1 (Existence of an Indefinite Integral)

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of f(z)in D, which is analytic in D joining two points z_0 and z_1 in D, the integral of f(z) from z_0 and z_1 can be evaluated by formula (1).

Proof

The conditions of Cauchy's integral theorem are satisfied. Hence the line integral of $f(z)$ from z_0 in D to any z *in D* is independent of path in D. We keep z_0 fixed. Then this integral becomes a function of *z*, which we denote by $F(z)$:

2. $F(z) = \int_{z_0}^{z_1} f(z^*)dz^*.$

We show that this $F(z)$ is analytic in D and that $F'(z) = f(z)$. The idea of doing this is as follows. We form the differential quotient

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{0}^{z+\Delta z} f(z^*)dz^* - \int_{z_0}^{z+\Delta z} f(z^*)dz^*\right]$$

$$= \frac{1}{\Delta z}\int_{z_0}^{z+\Delta z} f(z^*)dz^*,$$

Subtract $f(z)$ from it and show that expression obtained approaches zero as $\Delta z \to 0$; this is done by using the continuity of $f(z)$. We now give the details.

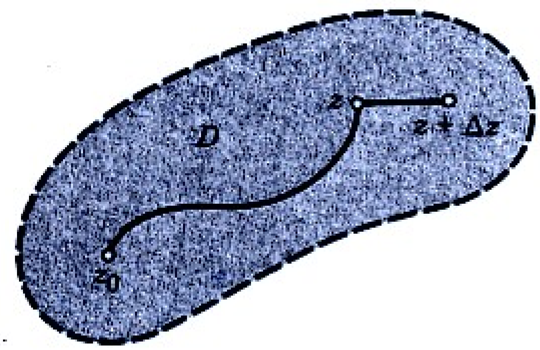


Fig. 38: Path of Integration

We keep *z* fixed. Then we choose $z + \Delta z$ in *D*. This is possible since *D* is a domain; hence *D* contains a neighbourhood of *z*. See figure above.

The segment we use as the path of integration in the previous formula. We now subtract $f(z)$. This is a constant, since z is kept fixed. Hence

$$\int_{z}^{z+\Delta z} f(z)dz^{*} = f(z)\int_{z}^{z+\Delta z} dz^{*} = f(z)\Delta z.$$

Thus

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^{*}$$

This trick permits us to write a single integral:

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z_0}^{z+\Delta z}[f(z^{*})-f(z)]dz^{*}$$

$f(z)$ is analytic, hence continuous. An ϵ>0 being given, we can thus find a $\delta > 0$ such that

$$\left|f(z^{*})-f(z)\right| < \epsilon \qquad \text{when } \left|z^{*}-z\right| < \delta$$

Consequently, letting $|\Delta z| < \delta$, we see that the *ML*-inequality yields

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z_0}^{z+\Delta z}[f(z^{*})-f(z)]dz^{*}\right| \leq \frac{1}{|\Delta z|}\epsilon\,|\Delta z| = \epsilon;$$

that is, by the definition of a limit and of the derivative,

$$F'(z) = \lim_{\Delta z}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z).$$

Since z is any point in D, this proves that $F(z)$ is analytic in D and is an indefinite integral or antiderivative of $f(z)$ in D, written

$$F(z) = \int f(z)dz.$$

Also, if $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in D; hence $F(z) - G(z)$ is constant in D. That is, two indefinite integrals of $f(z)$. This proves the theorem.

See section 3.2 for examples and problems on indefinite integration.

The theorem in this section followed from Cauchy's integral theorem. A much more fundamental consequence is **Cauchy's integral formula** for evaluating integrals over close curves, which we discuss in the next section.

3.5 Cauchy's Integral Formula

The most important consequences of Cauchy's integral theorem is Cauchy's integral formula. This formula is useful for evaluating integrals (see example below). More importantly, it plays a key role in providing the surprising fact that analytic function have derivative of all orders (see section 3.6), In establishing Taylor series representations and so on. Cauchy's integral formula and its conditions of validity may be stated as follows.

Theorem 1 (Cauchy's Integral Formula)

Let $f(z)$ *is analytic in a simply connected domain D. Then for any point* z_0 *in D and any simple closed path C in D which encloses* z_0 (fig. below),

1. $$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$ **(Cauchy's integral formula)**

The integration being taken in the counterclockwise sense.

Proof

By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. We insert this into (1) on the left and can take constant factor $f(z_0)$ out from under the integral sign. Then

2. $$\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz.$$

The first on the right hand equals $f(z_0) \cdot 2\pi i$ (see Example 8 in sec. 3.3, with m=-1). This proves this theorem, provided the second integral on the right is zero. This is what we are now going to show. It's integrand is analytic, except at z_0. Hence by the principle of deformation of path (sec. 3.3) we replace C by a small circle K of radius ρ and centre z_0 (figure below), without altering the value of the integral. Since $f(z)$ is analytic, it is continuous. Hence, an ϵ>0 being given, we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \qquad \text{for all } z \text{ in the disk } |z - z_0| < \delta$$

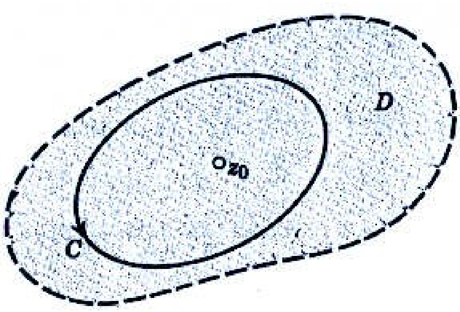


Fig. 39: Cauchy's Integral Formula

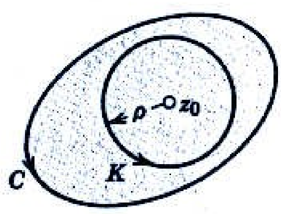


Fig. 40: Proof of Cauchy's Integral Formula

Choosing the radius ρ of k smaller than δ , we thus have the inequality

$$\left|\frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\varepsilon}{\rho}$$

At each point of k. The length of k is $2\pi\rho$ .Hence by *ML*-inequality in sec. 3.2,

$$\left|\oint_K \frac{f(z)-f(z_0)}{z-z_0}\right| dz < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon .$$

Since $e(>0)$ can be choosing arbitrarily small, it follows that the last integral on the right-hand side of (2) has the value zero, and the theorem is proved.

Example 17

Cauchy's Integral Formula

$$\oint_C \frac{e^z}{z-2}dz = 2\pi e^z \Big|_{z=2} = 2\pi e^2$$

For any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem).

Example 18

Cauchy's Integral Formula

$$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{z^3-3}{2z-\frac{1}{2}i}dz = 2\pi\,[\tfrac{1}{2}z^3-3]\Big|_{z=i}$$ [redacted] 2

$$= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside C}).$$

Example 19

Integration Around Different Contour

$$g(z) = \frac{z^2+1}{z^2-1}$$

in the counterclockwise sense around a circle of radius 1 with centre at the point

a. $z = 1$ (b) $z = \frac{1}{2}$ (c) $z = -1 + \frac{1}{2i}$, (d) $z = i$.

Solution

To see what is going on, locate the point where *g(z)* is not analytic and sketch them along with the contours (figure below) . These points are -1 and1. We see that (b) will give the same result as (a), by the principle of deformation of path. And (d) gives zero, By Cauchy's integral theorem. We consider (a) and afterward (c).

Here $z_0 = 1$, so that $z - z_0 = z - 1$ in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z^2-1} = (\frac{z^2+1}{z+1})(\frac{1}{z-1}); \quad \text{thus} \quad f(z) = \frac{z^2+1}{z^2-1},$$

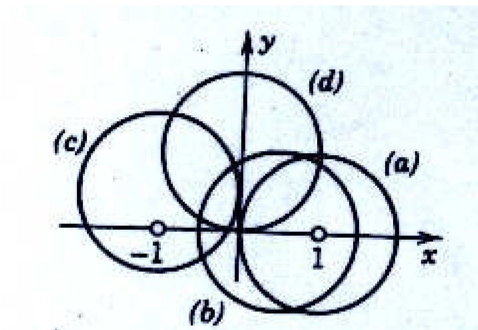


Fig. 41: Example 3

Looking back, we point to a chain of basic results. The beginning was Cauchy's integral theorem in sec. 3.3. From it followed Cauchy's integral formula (1) in this section. From it follows the existence of all higher derivatives of an analytic function, in the next section. This is the probably the most exciting link of our chain. From it follows in the Taylor series for analytic functions.

3.6 Derivative of Analytic Functions

From the assumption that a real function of a real variable is once differentiable, nothing follows about the existence of derivatives of higher order**.** We shall now see that from the assumption that a complex function has a first derivative in a domain *D*, there follows the existence of derivative of all orders in *D*. This means that in this respect complex analytic functions behave much more simply than real functions that are once differentiable.

Theorem 1 (Derivative of Analytic Function)

If $f(z)$ *is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic function in D. The value of these derivatives at a point* z_0 *in D are given by the formulas*

$$(1') \qquad f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$(1'') \qquad f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$$

and in general

$$(1) \qquad f^{n}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1,2,\cdots);$$

here C is any simple closed path in D that encloses z_0 *and whose full interior belongs to D; And we integrate counterclockwise around C(*figure below*).*

Comment

For memorizing (1), it is useful to observe that these formulas are obtained formally by differentiating the Cauchy formula (1), Sec. 3.5, under the integral sign with respect to z_0.

Proof of Theorem

We prove $(1')$.
We start from the definition

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

On the right we represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula (1), sec. 3.5; we can combine the two integrals into a single integral by taking the common denominator and simplifying the numerator (where $z - z_0$ drops out and only $f(z)\Delta z$ remains):

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\left[\oint_C \frac{f(z)}{z-(z_0 + \Delta z)}dz - \oint_C \frac{f(z)}{z - z_0}dz\right]$$

$$= \frac{1}{2\pi i \Delta z}\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}dz$$

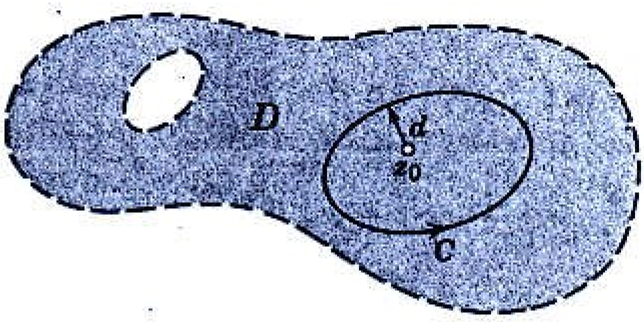


Fig. 42: Theorem 1 and its Proof

Clearly, we can now establish $(1')$ by showing that, as $\Delta z \to 0$, the integral on the right approaches the integral in $(1')$. To do this, we consider the difference between these two integrals. We can write this difference as a single integral by taking the common denominator and simplifying. This gives

$$\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}dz - \oint_C \frac{f(z)}{(z - z_0)^2}dz$$

$$= \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}dz$$

We show by *ML*-inequality (Sec. 3.2) that this difference approaches zero as $\Delta z \to 0$, .

Being analytic, the function $f(z)$ is continuous on C , hence bounded in absolute value, say, $|f(z)| \le K$. Let d be the smallest distance from z_0 to the points of *C*(see fig. below). Then for all $|z|$ on *C*,

$$|z - z_0|^2 \ge d^2,$$

hence

$$\frac{1}{|z - z_0|^2} \le \frac{1}{d^2}.$$

Furthermore, if $|\Delta z| \le d/2$, then for all z on C we also have

$$|z - z_0 - \Delta z| \ge \frac{d}{2}, \quad \text{hence} \quad \frac{1}{|z - z_0 - \Delta z|} \le \frac{2}{d}.$$

Let L be the length of C. Then by ML-inequality, if $|\Delta z| \le d/2$,

$$\left| \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)^2} dz \right| \le K|\Delta z| \frac{2}{d} \cdot \frac{1}{d^2}.$$

This approaches zero as $\Delta z \to 0$, Formula $(1')$ is proved.

Note that we used Cauchy's integral formula (1), Sec. 3.5, but if all we had known about $f(z_0)$ is the fact that it can be represented by (1), Sec. 3.5, our argument would have established the existence of the derivative $f'(z_0)$ of $f(z)$. This is essential to continuation and completion of this proof, because it implies that $(1'')$ can be proved by similar argument, with f replaced by f', and that the general formula (1) then follows by induction.

Example 20

Evaluation of Line Integrals

From $(1')$, for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z - \pi i)^2} dz = 2\pi i (\cos z)' \Big|_{z = \pi i}$$
$$= 2\pi i \sin \pi i = 2\pi \sinh \pi$$

Example 21

From $(1'')$, for any contour enclosing the point -1(counterclockwise)

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z + i)^3} dz = \pi i (z^4 - 3z^2 + 6)'' \Big|_{z = -i}$$
$$= \pi i [12z^2 - 6]_{z=-i} = -18\pi i$$

Example 22

By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz = 2\pi i \left(\frac{e^z}{z^2+4} \right)' \Bigg|_{z=1}$$

$$= 2\pi i \frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2} \Bigg|_{z=1}$$

$$= \frac{6e\pi}{25} i = 2.050i.$$

3.6.1 Moreras's Theorem

If $f(z)$ *is continuous in a simply connected domain D and if*

2. $\oint_C f(z)dz = 0$

for every closed path in D, then $f(z)$ *is analytic in D.*

Proof

In sec.3.4 it was shown that if $f(z)$

$$F(z) = \int_{z_0}^{z} f(z^*)dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we use only the continuity of $f(z)$ and the property that its integral around every close path in D is zero; from the assumptions we concluded that $F(z)$ is analytic. By theorem 1, the derivative of $F(z)$ is analytic, that is $f(z)$ is analytic in D, and Morera's theorem is proved.

Theorem 1 also yields a basic inequality that has many applications. To get it, all we have to do is to choose for C in (1) a circle of radius r and centre z_0 and apply ML-inequality (Sec. 3.2); with $|f(z) \le M|$ on C we obtain from (1)

$$\left| f^{(n)} z_0 \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{)z - z_0)^{n+1}} dz \right| \le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r.$$

This yields **Cauchy's inequality**

3. $\left|f^{(n)} z_0\right| \le \frac{n! M}{r^n}$.

To gain first impression of the importance of this inequality, let us prove a famous theorem on entire functions (functions that are analytic for all z; cf.Sec. 2.6)

3.6.2 Liouville's Theorem

If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be a constant.

Proof

By assumption, $|f(z)|$ is bounded, say, $|f(z)| < K$ for all z. Using (3), we see that $|f'(z_0)| < K/r$ .Since this is true for every *r*, we can take *r* as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, and $f(z)$ is a constant.

This completes the proof.

This is the end of section on complex integration, which gave us a first impression of the methods that have no counterpart in real integral calculus. We have seen that these methods result directly or indirectly from Cauchy's integral theorem (Sec.3..3) More on integration follows in the next section.

In the next section, we consider **power series,** which play a great role in complex analysis, and we shall see that the Taylor series of calculus have a complex counterpart, so that e^z, $\cos z, \sin z$ etc. have Maclaurin series that are quite similar to those in calculus.

4.0 CONCLUSION

In conclusion, we state that if a function is analytic, it has derivative of all orders.

5.0 SUMMARY

The complex line integral of a function $f(z)$ taken over a path C is donated by (sec. 3.1)

$\int_C f(z)dz$ or, if C is closed, also by $\oint_C f(z)dz$.

Such an integral can be evaluated by using the equation *z=z(t)* of *C*, where $a \le t \le b$ (se. 3.2):

1. $\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)$ $\quad \left(\dot{z} = \frac{dz}{dt}\right)$

As another method, if $f(z)$ is analytic (sec.2.4) in a simply connected domain *D*, then there exists an $F(z)$ in *D* such that $F'(z) = f(z)$ and for every path *C* in D from a point z_0 to a point z_1 we have

2. $\int_C f(z)dz = F(z_1) - F(z_0)$ $\quad [F'(z) = f(z)]$.

Cauchy integral theorem states that if $f(z)$ is analytic in as simply connected domain *D*, then for every closed path *C* in *D*

3. $\oint_C f(z)dz = 0$.

If $f(z)$ is as in Cauchy's integral theorem, then for any z_0 in its interior we have C**auchy integral formula**

4. $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$

Furthermore, then $f(z)$ has derivative of all orders in *D* that are themselves analytic functions in *D* and (sec. 3.6)

5. $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz.$ $\quad (n = 1,2,\cdots).$

6.0 TUTOR-MARKED ASSIGNMENT

i. Show that $\oint_C \frac{dz}{z} = 2\pi i$ (*C* the unit circle clockwise)

ii. Evaluate $\oint_C e^z dz$ by the method in theorem 1 and compare the result by method in theorem 2.

(*C* is the line segment from 0 to $1 + \frac{\pi i}{2}$)

iii. For what contour *C* will it follow from Cauchy's theorem that

(a) $\oint_C \frac{dz}{z} = 0,$ (b) $\oint \frac{e^{-z}}{(z^5 - z)} dz = 0$?

iv. Evaluate the following integrals

(a) $\int_i^{2i} (z^2 - 1)^3 dz$ (b) $\int_0^{\pi i} z \cos z dz$

v. State and prove Morera's theorem

vi. State and prove Liouville's theorem

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. S. (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

MODULE 3

Unit 1 Residue Integration Method

UNIT 1 RESIDUE INTEGRATION METHOD

CONTENTS

1.0 INTRODUCTION

Since there are various methods of determining the coefficients of a Laurent series, without using the integral formulas. We intend (may) use the formula for b_1 for evaluating complex integrals in a very elegant and simple fashion. b_1 will be called the residue or $f(z)$ at $z = z_0$.The powerful method may also be applied for evaluation certain real integrals, as we shall see in section 3.3 and 3.4 of module 3 and unit 1.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine and explain Residue
- use Residue to evaluate integrals and
- show that the Residue integration method can be extended to the case of several singular points of $f(z)$ inside C.

3.0 MAIN CONTENT

3.1 Residues

Let us first explain what a residue is and how it can be used for evaluating Integrals

$$\oint_C f(z)dz.$$

There will be counter integral taken around a simple closed path C.
If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem and we are done.

If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots\cdots$$

That converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z = z_0| < R$. Now comes the key idea. The coefficient b_1 of the first negative power $\frac{1}{(z = z_0)}$ of this Laurent series is given by the integral formula, with n=1, that is,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz,$$

Since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients, we can find b_1 by one of these methods and then use the formula for b_1 for evaluating the integral:

1. $$\boxed{\oint_C f(z)dz = 2\pi i b_1.}$$

   Here we integrate in the counterclockwise sense around the simple closed path that contains $z = z_0$ in its interior.

   The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we shall denote it by

2. $$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)}$$

Example1

Evaluation of an Integral by Means of a Residue

Integrate the function $f(z) = z^{-4}$ around the unit circle C in the counterclockwise sense.

Solution

We obtain the Laurent series thus:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Which converges for $[z[> 0$ (that is for all $z \neq 0).)$ This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue of $f(z)$ at $z = 0$ is $b_1 = 1/3!$.

From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Example 2

Be Careful to use the *right* Laurent Series!

Integrate $f(z) = 1/(z^3 - z^4)$ around the circle C: $|z| = 1/2$ in the clockwise sense.

Solution

$z^3 - z^4 = z^3(1 - z)$ Shows $f(z)$ that $z = 0$ and $z = 1$. Now $z = 1$ lies outside C.

Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$ that

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad 0 < |z| < 1$$

We see it from this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i$$

Caution! Had we use the wrong series (II) say:

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots\cdots \qquad (|z| < 1),$$

We would have obtained the wrong answer 0. Explain!

3.1.1 Two Formulas for Residues at Simple Poles

Before we continue the integration, we ask the following question: To get a residue, a single coefficient of a Laurent series, must we divide the whole series or is there a more economical way? For poles, there is. We shall derive, once and for all, some formulas for residues at poles, so that in this case we no longer need the whole series.

Let $f(z)$ have a simple pole at $z = z_0$

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad 0 < |z - z_0| < R$$

Here $b_1 = 0$ (why?) Multiply both sides by $z - z_0$ we have

$$(z - z_0) f(z) = b_1 + (z - z_0)[a^0 + a_1(z - z_0) + \cdots\cdots]$$

We now let $z \to z_0$. The right hand side approaches b_1. This gives

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z=z_0}(z - z_0) f(z) \qquad (3)$$

Example 3

Residue at a Simple Pole

$$\operatorname*{Res}_{z=z_0} \frac{9i+1}{z(z^2+1)} = \lim_{z=i}(z-i)\frac{9i+1}{z(z+i)} = \left[\frac{9z+1}{z(z+i)}\right]_{z=1}$$

$$= \frac{10i}{-2} = -5i$$

Another, sometimes simpler formula for the residue at a simple pole is obtained by starting from

$$f(z) = \frac{p(z)}{q(z)}$$

with analytic $p(z)$ and $q(z)$ where we assume that $p(z_0) \neq 0$ and $q(z)$ has a simple zero at $z - z_0$ (so that $f(z)$ has a simple pole at $z - z_0$ ad wanted. By the definition of a simple zero, $q(z)$ has a Taylor series of the form

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots$$

This we substitute into $f = p/q$ and then f into (3), finding

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z=z_0}(z - i)\frac{p(z)}{q(z)} = \lim_{z=z_0}\left[\frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''(z_0)/2 + \cdots]}\right] \neq 1$$

We now see that on the right, a factor $z - z_0$ is cancelled and resulting denominator has the limit $q'(z_0)$. Hence our second formula for the residue at a pole is

$$\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q(z_0)}. \qquad (4)$$

Example 4

Residue at a Simple Pole Calculated by Formula (4)

$$\operatorname{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i$$

Example 5

Another Application of Formula (4)

$$f(z) = \frac{\cos \pi z}{z^4 - 1}.$$

Solution

$p(z) = \cos \pi z$ is entire, and $q(z) = z^4 - 1$ has a simple zero at1,i,-1,-i. Hence $f(z)$ has a simple pole at these points (and no further poles).

Since $q'(z) = 4z^3$, we see from (4) that the residue equal the value for $\frac{\cosh \pi i}{4z^3}$ at those points, that is,

$$\frac{\cosh \pi}{} \approx 2.8980, \quad \frac{\cosh \pi i}{} = \frac{\cos \pi}{} = -\frac{i}{}, \quad -\frac{\cosh \pi}{}, \quad \frac{\cosh(-\pi i)}{} = \frac{i}{}.$$

$$4 \qquad \overline{4i^3} \quad \overline{-4i} \qquad 4 \qquad 4 \qquad 4(-i)^3 \qquad 4$$

3.1.2 Two Formulas for Residues at Simple Poles

Let $f(z)$ be analytic function that has pole of any order m>1 at a point $z = z_0$. Then, by the definition of such pole, the Laurent series of $f(z)$ converging near $z = z_0$ (except $z = z_0$) is

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

where $b_m \neq 0$. Multiplying both sides by $(z-z_0)^m$, we have

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1}$$
$$+ a_0(z-z_0)^m + +a_1(z-z_0)^{m+1} + \cdots.$$

We see that the residue b_1 of $f(z)$ at $z = z_0$ is now the coefficient of the power $(z-z_0)^{m-1}$ in the Taylor series of the function

$$g(s) = (z-z_0)^m f(z)$$

On the left, with center $z = z_0$. Thus by Taylor's theorem,

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$$

Hence if $f(z)$ has a pole of mth order at $z = z_0$, the residue is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)} \lim_{z=z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}. \qquad (5)$$

In particular, for a second-order pole (m=2),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z=z_0} \{[(z-z_0)2 f(z)']\}.$$

Example 6

Residue at a Pole of Higher Order

The function

$$f(z) = \frac{50z}{(z+4)(z-1^2)}$$

has a pole pole of second order at $z = 1$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1} \frac{d}{dz}\left(\frac{50z}{z+4}\right) = 8$$

Example 7

Residue from a Partial Fraction

If $f(z)$ is rational, we can also determine its residue from partial fractions. In Example 6,

$$f(z) = \frac{50z}{(z+4)(z-1)^2} = \frac{-8}{z+4} + \frac{8}{z-1} + \frac{10}{(z-1)^2}.$$

This shows that the residue t $z = 1$ is 8 (as before),and at z=-4 (simple pole) it is -8. Why is this so? Consider $z = 1$. There the Laurent has two fractions as its principal part and the first fraction as the sum of the other part. This first fraction is analytic at $z = 1$, so that it has a Taylor series with centre $z = 1$, as it should be. Similarly, at z=4 the first fraction is the principal part of the Laurent series.

Example 8

Integration around a Second-order Pole

Counterclockwise integration around any simple closed path C such that $z = 1$ is inside C and z=4 outside C yields

$$\oint_C \frac{z}{(z+4)(z-1)^2} dz = \operatorname*{Res}_{z=1} 2\pi i \frac{z}{(z+4)(z-1)^2} = 2\pi i \frac{8}{50} \approx 1.0053$$

So far we can evaluate integrals of analytic functions $f(z)$ over closed curve C when $f(z)$ has only *one* singular point inside C. In the next section we show that the residue integration method can be readily extended to the case of several singular points of $f(z)$ inside C.

3.2 Residue Theorem

So far we are in a position to evaluate contour integrals whose integrands have only a single isolated singularity inside the contour of integration. We shall now see that our simple method may be extended to the case when the integrand has several isolated singularity inside the contour. This extension is surprisingly simple, as follows

Residue Theorem

Let $f(z)$ be a function that is analytic inside a simple closed path C and on C, except for finitely many singular point $z_1, z_2, \cdots, z_k$ inside C. Then

$$\oint_C f(z) = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=1} f(z), \qquad (1)$$

The integral being taken in the clockwise sense around the path C

Proof: We enclose each of the singular points z_j in a circle C_j with radius small enough that k circles and C are all separated (fig. 43). Then

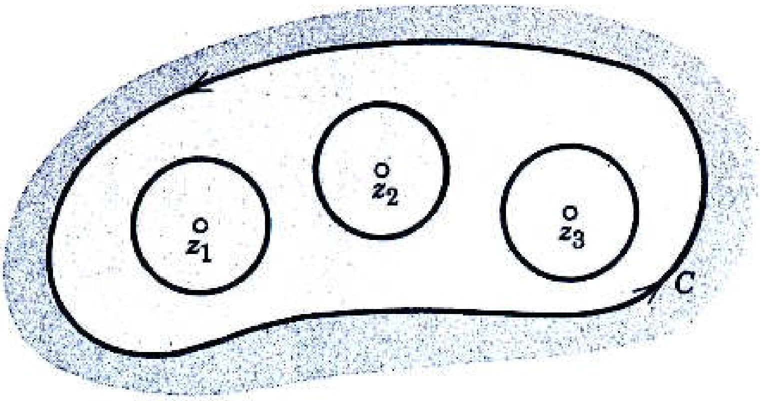


Fig. 43: Residue Theorem

$f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1 \cdots C_n$ and on the entire boundary of D. From the Cauchy's integral theorem we have

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_k} f(z)dz = 0 \qquad (2)$$

the integral along C being taken in the counterclockwise sense and the other integrals in the clockwise sense. We now reverse the sense of integration along $C_1 \cdots C_n$. Then the signs of the values of these integrals change, and we obtain from (2)

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_k} f(z)dz \qquad (3)$$

All these integrals are now taken in the clockwise sense. By (1) in the previous section

$$\oint_{C_j} f(z)dz = \operatorname*{Res}_{z=z_j} f(z),$$

So that (3) yields (1), and the theorem is proved.

This important theorem has various applications with complex and real integrals. We shall first consider some complex integrals.

Example 9

Integration by the Residue Theorem

Evaluate the following integral counterclockwise around any simple close path such that:

a. 0 and 1 are inside C
b. 0 is inside, 1outside,
c. 1 is inside, 0 outside,
d. 0and1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}$$

Solution

The integrand has simple poles at 0 and 1, with residues

$$\operatorname{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4, \qquad \operatorname{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

Confirm this by (4) Ans.(a). ($2\pi i(-4+1) = -6\pi i,$) (b). $-8\pi i$ (c). $2\pi i$ (d). 0

Example 10

Integration by the Residue Theorem

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right)dz$$

Solution

Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues (note: $e^{2\pi i} = 1$)

$$\operatorname{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16}, \quad \operatorname{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16},$$

and simple poles at ± 2 which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{z} \cdot \frac{1}{z} + \cdots.$$

Ans. $2\pi i(-6/-1/6 + \pi^2/2) = \pi(\pi^2 - 1/4)i = 30.221i.$ by the residue theorem.

Example 10

Confirmation of an Earlier Result

Integrate $\frac{1}{(z - z_0)^m}$ (m a positive integer) in the clockwise sense around and simple close path C enclosing point $z = z_0$.

Solution

$\frac{1}{(z - z_0)^m}$ in its own Laurent series with centre $z = z_0$ consisting of this one- term principal path, and

$$\operatorname*{Res}_{z=z_0} \frac{1}{z - z_0} = 1, \qquad \operatorname*{Res}_{z=z_0} \frac{1}{(z - z_0)^m} = 0 \qquad (m = 2,3 \cdots).$$

In agreement with Example (2), we thus obtain

$$\oint_C \frac{dz}{(z - z_0)^m} = \begin{cases} 2\pi i & \text{if } m = 1 \\ 0 & \text{if } m = 2,3,\cdots \end{cases}$$

It should be very surprising to hear that our present *complex* integration method can be used for evaluating **real integrals** (incidentally, some of them difficult to evaluate by other methods). In the next section we discuss two methods for accomplishing this goal.

3.3 Evaluation of Real Integral

We want to show that residue theorem also yields a very elegant and simple method for evaluating certain classes of complicated real integrals.

Integrals of Rational fractions of $Cos\theta$ **and** $Sin\theta$

We first consider integrals of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \qquad (1)$$

where $F(\cos\theta, \sin\theta)$ is a real rational fraction of $Cos\theta$ and $Sin\theta$ [for example, $(\sin^2\theta)/(5-4\cos\theta)$ and is finite on the interval of integration. Setting $e^{i\theta} = z$, we obtain

(2)

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{2}\right)$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{2}\right)$$

and we see that the integrand becomes a rational function of z, say, $f(z)$.

As θ ranges from 0 to 2π, the variable z ranges once around the unit circle $|z| = 1$ in the counterclockwise sense. Since we have $d\theta = dz/iz$, and the given integral takes the form

$$I = \oint_C f(z) \frac{dz}{iz}, \qquad (3)$$

The integration being taken counterclockwise around the unit circle.

Example 11

An Integral of the Type (1)

Show by the present method that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

Solution

We use $\cos\theta = (z + 1/z)$ and $d\theta = \dfrac{dz}{iz}$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2}+\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_C \frac{dz}{c+\frac{i}{2}(z^2+2\sqrt{2z}+1)} = \frac{2}{i}\oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

We see that the integrand has two simple poles, one at $z_1 = \sqrt{2}+1$, which lies outside the unit circle. C: $|z| = 1$ and is thus of no interest, and the other at $z_2 = \sqrt{2-1}$ inside C, where the residue is

$$\operatorname{Res}_{z=z_2} \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = \left[\frac{1}{z-\sqrt{2}-1}\right]_{z=\sqrt{2}-1} = -\frac{1}{2}.$$

Together with the factor $-2\pi i$ in front of the integral this yields the desired result $2\pi i(-2/i)(-1/2) = 2\pi$

3.3.1 Improper Integrals of Rational Function

We now consider the real integral of the type

$$\int_{-\infty}^{\infty} f(x)dx \tag{4}$$

Such an integral, for which the interval of integration is not finite, is called an **improper integral**, and it has the meaning

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to-\infty}\int_0^b f(x)dx. \tag{5a}$$

If both limit exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{5b}$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of denominator. Then the limit in (5a) exists, and we may start from (5b). We may consider the corresponding contour integral

$$\oint_C f(z)dz \tag{5c}$$

Around a path C on the diagram below. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper-half plane, and if we choose R large enough, then

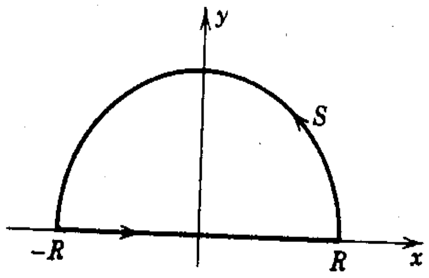


Fig. 44: Path C of the Contour Integral in (5*)

C encloses all these poles .By the residue theorem we then obtain

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

When the sun consists of all the residues, of $f(z)$ at the point in the upper half-plane at which $f(z)$ has a pole. From this we have

(6) $$\int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

We prove that $R \to \infty$,the value of the integral over the semicircle S approaches zero. If we set $z = Re^{i\theta}$, then S is represented by $R = const$, and as z ranges along S, the variable θ ranges from0 to π .Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0. By the ML-inequality

$$\left|\int_S f(z)dz\right| < \frac{k}{R^2}\pi R = \frac{k\pi}{R} \qquad (R > R_0)$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

(7) $$\boxed{\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)}$$

the sum being extended over the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper half-plane.

Example 12

An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Solution

Indeed, $f(z) = \frac{1}{(1+z^4)}$ has four simple poles at the points

$$z_1 e^{\pi i/4}, \quad z_2 e^{3\pi i/4}, \quad z_3 e^{-3\pi i/4}, \quad z_4 e^{-\pi i/4}$$

The first two of these poles lie in the upper-half plane. We find

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4} e^{-3\pi i/4},$$

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4} e^{-9\pi i/4}$$

By (1) and (7), in the current section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{4}(-e^{\pi i/4} + e^{-\pi i/4}) = \pi \sin\frac{\pi}{4} = \frac{\pi}{2\sqrt{2}}.$$

Since $1/(1 + x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

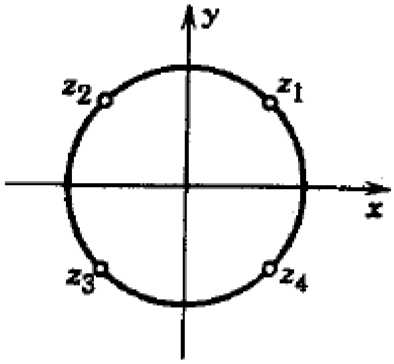


Fig. 45: Example 2

Example 13

Another Improper Integral

Using (7) show that

$$\int_{-\infty}^{\infty} \frac{x^2-1}{x^4+5x^2+4}\,dx = \frac{\pi}{6}.$$

Solution

The degree of denominator is two units higher than that of the numerator, so that our method again applies. Now

$$f(z) = \frac{p(z)}{q(z)} = \frac{z^2-1}{(z^2+4)(z^2+1)} = \frac{z^2-1}{(z-2i)(z+2i)(z-i)(z+i)}$$

has simple poles at $2i$ and i in the upper-plane (and at $-2i$ and $-i$ in the lower half-plane, which are of no interest here). We calculate the residues from (4), noting that $q'(z) = 4z^3 + 10z$,

$$\operatorname*{Res}_{z=2i} f(z) = \left[\frac{z^2-1}{4z^3+10z}\right]_{z=2i} = \frac{5}{12i}, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{z^2-1}{4z^3+10z}\right]_{z=i} = \frac{-2}{6i}$$

$Ans.\ 2\pi i(5/12i - 1/3i) = \frac{\pi}{6}$, as asserted.

Looking back, we realise that the key ideas of our present methods were these. In the first method we mapped the interval of integration on the real axis onto a closed curved in the complex plane (the unit circle). In the second method we attached to an interval on the real axis a semi circle such that we got a closed curve in the complex plane, which we then "blew up." This second method can be applied to further types of integrals, as we show in the next section, the last in the chapter.

3.4 Further Types of Real Integrals

There are further classes of integrals that can be evaluated by applying the residue theorem to suitable complex integrals. In application suchintegral may arise in connection with integral transformations or representation of special functions. In the present section we shall consider two such classes of integrals. One of them is important in the problems involving the Fourier integral representation. The other class consists of real integral whose integrand is finite at some point in the interval of integration.

3.4.1 Fourier Integral

Real integral of the form

1. $\int_{-\infty}^{\infty} f(x) = \cos sxdx$ and $\int_{-\infty}^{\infty} f(x) = \sin sxdx$ (s real)

occur in connection with the Fourier integral.

If $f(x)$ is a rational function satisfying the assumptions on the degree stated in connection with (4), then the integral (1) may be evaluated in a similar to that used for the integral in (4) of the previous section. In fact, we may then consider the corresponding integral

$$\oint_C f(z)e^{isz}dz \qquad (s \text{ real and positive})$$

Over the contour C in sec 3.3 instead of (7), sec. 3.3, we get

$$\int_{-\infty}^{\infty} f(z)e^{isz}dz = 2\pi i \sum \operatorname{Re} s[f(z)e^{isz}] \qquad (s > 0) \qquad (2)$$

where the sum consists of the residue of $f(z)e^{isz}$ as its pole in the upper half-plane. Equating the and imaginary parts on both sides of (2), we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\cos sxdx &= -2\pi i \sum \operatorname{Im} \operatorname{Re} s[f(z)e^{isz}], \\ \int_{-\infty}^{\infty} f(x)\sin sxdx &= 2\pi i \sum \operatorname{Re} \operatorname{Re} s[f(z)e^{isz}] \end{aligned} \qquad (s > 0)(3)$$

We remember that (7), was established by proving that the value of the integral over the semicircle S in fig. approaches zero as $R \to \infty$.

To establish (2) we should now prove the same fact for our present contour integral. This can be done as follows, Since S lies in the upper half-plane $y \geq 0$ and $s > 0$, we see that

$$|e^{isz}| = |e^{isx}||e^{-isy}| = e^{-sy} \leq 1 \qquad (s > 0, \quad y \geq 0)$$

From this obtain the inequality

$$|f(z)e^{isz}| =) = |f(z||e^{isz}| \leq |f(z| \qquad (s > 0, \quad y \geq 0)$$

which reduces our present problem to that in previous section.

Continuing as before, we see that the value of the integral under consideration approaches zero as R approaches infinity. This establishes (2), which implies (3).

Example 14

An Application of (3)

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2} dx = \frac{\pi}{k} e^{ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2} dx = 0 \qquad (s > 0, \ \text{k} > 0)$$

Solution

In fact, $\frac{e^{isz}}{k^2 + x^2}$ has only one pole in the upper plane, namely, a simple pole at $z = ik$, and from (4) we obtain

$$\operatorname{Re} s_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left(\frac{e^{isz}}{2z} \right)_{z=ik} = \left(\frac{e^{-ks}}{2ik} \right).$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{isz}}{k^2 + z^2} dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{ks}.$$

Since $e^{isx} = \cos sx + i \sin sx,$ this yields the above results

3.4.2 Types of Real Improper Integrals

Another kind of improper integral is a definite integral

$$\int_A^B f(x)dx \qquad (4)$$

whose integral becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty$$

Then the integral (4) means

$$\int_A^B f(x)dx = \lim_{\tau \to a} \int_A^{a-\tau} f(x)dx + \lim_{\eta \to 0} \int_{a+\eta}^B f(x)dx \qquad (5)$$

where τ and η approaches zero independently and through positive values. It may happen that neither of these limits exists, if $\tau, \eta \to 0$ independently,

but

$$\lim_{\tau \to 0} \left[\int_A^{a-\tau} f(x)dx + \int_{a+\eta}^{B} f(x)dx \right] \qquad (6)$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pv.v.} \int_A^B f(x)dx.$$

For example,

$$\text{pv.v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim \left[\int_{-1}^{-\tau} \frac{dx}{x^3} + \int_{\tau}^{1} \frac{dx}{x^3} \right] = 0$$

the principal value exists although the integral itself has no meaning. The whole situation is quite similar to that discussed in the second part of the previous section.

To evaluate improper integral whose integrands have poles on the real axis, we use a part that avoids these singularities by following small semi-circles at the singular points; the procedure may be illustrated by the following example.

Example 15

An Application

Show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(This is the limit of sine integral Si(x) as $x \to \infty$)

Solution

a. We do not consider $\frac{(\sin z)}{z}$ because this function does not behave suitably at infinity. We consider $\frac{e^{iz}}{z}$, which has a simple pole at z=0, and integrate around the contour in figure below. Since $\frac{e^{iz}}{z}$ is analytic inside and on C Cauchy's integral theorem gives

$$\oint_C \frac{e^{iz}}{z} dz = 0 \qquad (7)$$

b. We prove that the value of the integral over the large semicircle C_1 approaches R as approaches infinity. Setting $z = R e^{i\theta} . dz = iR e^{i\theta} d\theta, \frac{dz}{z = id\theta}$ and therefore

$$\left| \int_C \frac{e^{iz}}{z} dz \right| = \left| \int_0^{\pi} e^{iz} id\theta \right| \leq \int_0^{\pi} \left| e^{iz} \right| d\theta \qquad (z = R e^{i\theta})$$

In the integrant on the right,

$$\left| e^{iz} \right| = \left| e^{iR(\cos\theta + i\sin\theta)} \right| = \left| e^{iR\cos\theta} \right| \left| e^{-R\sin\theta} \right| = e^{-R\sin\theta} .$$

We insert this, $\sin(\pi - \theta) = \sin\theta$ to get an integral from 0 to $\pi/2$, and then $\varpi \geq 2\theta/\pi$ (when $0 \leq \theta \leq \pi/2$); to get an integral that we can evaluate:

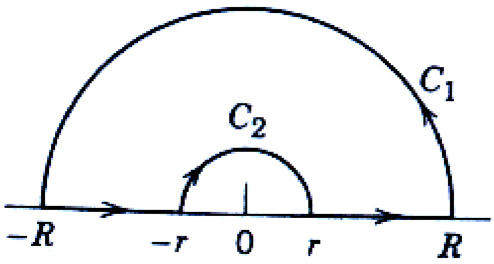


Fig. 46: Contour in Example 2

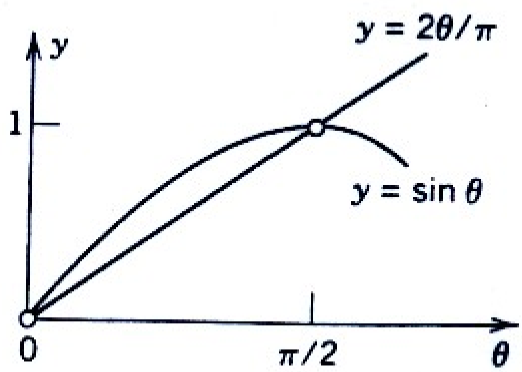


Fig. 47: Inequality in Example 2

$$\int_0^{\pi} \left| e^{iz} \right| d\theta = \int_0^{\pi} e^{-R\sin\theta} d\theta = \int_0^{\pi/2} e^{-R\sin\theta d\theta} d\theta$$

$$< 2\int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \frac{\pi}{R}(1 - e^{-R}) \to 0 \qquad \text{as} \quad R \to \infty C_1$$

Hence the value of the integral over C_1 approaches as $R \to \infty$

c. For the integral over small semicircle C_2 in figure above , we have

$$\int_{C_2} \frac{e^{iz}}{z} dz = \int_{C_2} \frac{dz}{z} + \int_{C_2} \frac{e^{iz} - 1}{z} dz$$

The first integral on the right equals $-\pi i$. The integral of the second integral is analytic and thus bounded, say, less than some constant M in absolute value for all z on C_2 and between C_2 and the x-axis. Hence by the $ML-$inequality, the absolute value of this integral cannot exceed $M\pi r$. This approaches $r \to 0$. Because of part (b), from (7) we thus obtain

$$\int_{C_2} \frac{e^{iz}}{z} dz = \text{pv.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + \lim_{r\to 0}\int_{C_2} \frac{e^{iz}}{z} dz$$

$$= \text{pv.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i = 0$$

Hence this principal value equals πi; its real part is 0 and its imaginary part is

$$\text{pv.v.}\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \qquad (8)$$

d. Now the integrand in (8) is not singular at $x = 0$. Furthermore, Since for positive x the function $1/x$ decreases, the area under the curve of the integrand between two consecutive positive zeros decreases in a monotone fashion, that is, the absolute value of the integrals

$$I_n = \int_{n\pi}^{n\pi+\pi} \frac{\sin x}{x} dx \qquad n = 0, 1, \cdots$$

From a monotone decreasing sequence, $|I_1|, |I_2|, \cdots$ and $I_n \to 0$ as $n \to \infty$. Since these integrals have alternating sign (why?), it follows from the Leibniz test that the infinite series $I_0 + I_1 + I_2 + \cdots$ converges. Clearly, the sum of the series is the integral

$$\int_0^{\infty} \frac{\sin x}{x} dx = \lim_{b\to\infty} \int_0^{b} \frac{\sin x}{x} dx$$

which therefore exists. Similarly the integral from 0 to $-\infty$ exists. Hence we need not take the principal value in (8), and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Since the integrand is an even function, the desired result follows.
In part (c) of example 2 we avoided the simple pole by integrating along a small semicircle C_2, and then we let C_2 shrink to a point. This process suggests the following.

3.4.3 Simple Poles on the Real Axis

If (z) has a simple pole at $z = a$ on the real axis, then

$$\lim_{r \to 0} \int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

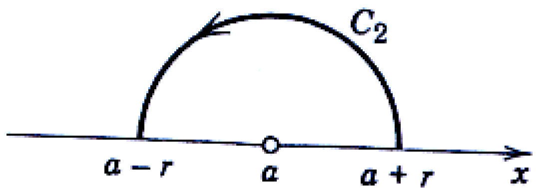


Fig. 48: Theorem 1

Proof

By the definition of a simple pole the integrand $f(z)$ has at $z = a$ the Laurent series

$$f(z) = \frac{b_1}{z - a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z)$$

where $g(z)$ is analytic on the semicircle of integration

$$C_2 : z = a + re^{i\theta}, \qquad 0 \le \theta = \pi$$

and for all z between C_2 and the x-axis. By integration,

$$\int_{C_2} f(z)dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} ire^{i\theta} d\theta + \int_{C_2} g(z)dz$$

The first integral on the right equals $-b_1\pi i$. The second cannot exceed $M\pi r$ in absolute value, by the ML-inequality and $M\pi r \to 0$ as $r \to 0$.

We may combine this theorem with (7) or (3) in this section.

Thus,

$$\text{pv.v.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z) \qquad (9)$$

(summation over all poles in the upper half-plane in the first sum, and on the x-axis in the second), valid for rational $f(x) = p(x)/q(x)$ with degree $q \geq$ degree $p + 2$, having simple poles on the x-axis.

This is the end of unit 1, which added another powerful general integration method to the methods discussed in the chapter on integration. Remember that our present residue method is based on Laurent series, which we therefore had to discuss first.

In the next chapter we present a systemic discussion of mapping by analytic functions (**"conformal mapping"**) .Conformal mapping will then be applied to potential theory, our last chapter on complex analysis.

4.0 CONCLUSION

In this unit, we have seen that our simple method have been extended to the case when the integrand has several isolated singularities inside the contour. We also proved the residue theorem.

5.0 SUMMARY

The **residue** of an analytic function $f(z)$ at a point $z = z_0$ is the coefficient of $\frac{1}{z - z_0}$ the power in the Laurent series

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$ of $f(z)$ which converges near z_0 (except at z_0 itself). This residue is given by the integral 3.1

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz \qquad (1)$$

but can be obtained in various other ways, so that one can use (1) for evaluating integral over closed curves. More generally, the **residue theorem** (sec.3.2) states that if $f(z)$ is analytic in a domain D such except at finitely many points z_j and C is a simple close path in D such that no z_j lies on C and the full interior of C belongs to D, then

$$\oint_{C_j} f(z)dz = \frac{1}{2\pi i} \sum_j \operatorname*{Res}_{z=z_j} f(z) \qquad (2)$$

(summation only over those z_j that lie inside C).

This integration method is elegant and powerful. Formulas for the residue at **poles** are ($m =$ order of the pole)

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}, \quad m = 1,2,\cdots \qquad (3)$$

Hence for a simple pole ($m = 1$),

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z) \qquad (3^*)$$

Another formula for the case of a simple pole of $f(z) = p(z)/q(z)$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z)}{q'(z)} \qquad (3^{**})$$

Residue integration involves closed curves, but the real interval of integration $0 \le \theta \le 2\pi$ is transformed into the unit circle by setting $z = e^{i\theta}$, so that by residue integration we can integrate **real integrals** of the form (sec. 3.3)

$$\int_0^{2\pi} F(\cos\theta \sin\theta)\, d\theta$$

where F is a rational function of $\cos\theta$ and $\sin\theta$, such as, for instance,

$$\frac{\sin^2\theta}{5 - 4\cos\theta}, \text{etc.}$$

Another method of integrating *real* integrals by residues is the use of a closed contour consisting of an interval $-R \le x \le R$ of the real axis and a semicircle $|z| = R$. From the residue theorem, if we let $R \to \infty$, we obtain for rational $f(x) = p(x)/q(x)$ (with $q(x) \neq 0$ and $q >$ degree $p + 2$)

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) \qquad \text{(sec.3.3)}$$

$$\int_{-\infty}^{\infty} \cos sx\,dx = -2\pi \sum \operatorname{Im} \operatorname{Res}[f(z)e^{isz}]$$

$$\int_{-\infty}^{\infty} \sin sx\,dx = 2\pi \sum \operatorname{Im} \operatorname{Res}[f(z)e^{isz}] \qquad \text{(sec.3.4)}$$

(sum of all residues at poles in the upper-half plane). In sec.3.4, we also extend this method to real integrals whose integrands become infinite at some point in the interval of integration.

6.0 TUTOR-MARKED ASSIGNMENT

i. Explain the term residues and how it can be used for evaluating integrals.
ii. Find the residues at the singular points of the following functions;

(a) $\frac{\cos 2z}{z^4}$ (b) $\tan z$ (c) $\frac{e^z}{(z+\pi i)^6}$

iii. Evaluate the following integrals where C is the unit circle (counterclockwise).

(a) $\oint_C \cot z dz$ (b) $\oint_C \frac{dz}{1-e^z}$ (c) $\oint_C \frac{z^2+1}{z^2-2z}$

iv. Show that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2-\cos\theta}} = 2\pi$$

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

MODULE 4 INTEGRAL TRANSFORMS

UNIT 1 INTEGRAL TRANSFORMS

CONTENTS

1.0 INTRODUCTION

The integral transform method is one of the best methods used in handling problems involving mechanical vibrations. The integral transform method is given by

$$F(p) = \int_a^b f(x)k(x,\rho)dx$$

With the inverse,

$$f(x) = \sum_{p=a}^{b} F(p)H(x,\rho)$$

$F(\rho)$ is the integral transform of $f(x)$ and $k(x,\rho)$ is called the kernel of the transformation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state various form of integral transform
- state Fourier Sine series and Fourier Cosine series
- apply Fourier transform to solve some fourth, third and second order differential equations
- develop techniques and methods through transformation or along with transform to be able to solve physical and mechanical problems (vibrations).

3.0 MAIN CONTENT

3.1 Finite Fourier Transform

Let $f(x)$ be a function defined in the interval $a \le x \le b$ i.e. $f(x)$ is defined on x-space. Let $k(x, \rho)$ be a function x of and some parameter ρ .

Then the integral transform method is given by,

$$F(p) = \int_a^b f(x)k(x, \rho)dx \qquad (1)$$

$F(\rho)$ is called an integral transform of $f(x)$ and $k(x, \rho)$ is called the kernel of the transform

Symbolically,

$$F = Tf \qquad (2)$$

where T is an integral operator which means multiply what follows T by $k(x, p)$ and integrate the product with respect to x between the limit of $'a'$ and $'b'$. The new function $F(\rho)$ can be regarded as the image of $f(x)$ produced by T .

$F(p)$ is defined on p-space/image-space.

For integral transform to be a useful concept, it is necessary that there should exist an inverse operator T^{-1} which yields a unique $F(t)$ from a given $F(\rho)$. From equation (2) we have that:

$$f = T^{-1}(F) \qquad (3)$$

Finding the operator T^{-1} is equivalent to solving equation (1) regardless an integral equation for $f(t)$

$$f(t) = \int_{\alpha}^{\beta} F(\rho) H(\rho, x) d\rho \qquad (4)$$

i.e. $F(t)$ is an integral transform of $F(\rho)$ with kernel $H(\rho, x)$.
A specification of the T^{-1} operator as in equation (4) is known as **Inversion Theorem**.

3.1 Finite Fourier Transforms

3.1.1 Half Range Fourier Sine Series

$$f(x) = \sum_{\rho=1}^{\infty} b_\rho Sin \frac{\rho\pi x}{L} \qquad 0 \le x \le L$$

Where

$$b\rho = \int_0^L f(x) \left(\frac{2}{L} \sin \frac{\rho\pi x}{L} \right) dx$$

$$k(x, \rho) = \frac{2}{L} \frac{\rho\pi x}{L}.$$

The image space is given by all the positive integral values of ρ. Hence $b\rho$ rather than $b(\rho)$.

3.1.2 Half Range Fourier Sine Series $0 \le x \le L$

$$f(x) = \frac{1}{2} a_0 \times \sum_{\rho=1}^{\infty} \cos \frac{\rho\pi x}{L}$$

Where

$$a\rho = \int_0^L f(x) \left(\frac{2}{L} \cos \frac{\rho\pi x}{L} \right) dx$$

3.1.3 Ordinary Fourier Series

$$f(x) = \sum_{\rho=-\infty}^{\infty} C_\rho e^{\left(i \frac{\rho\pi x}{L} \right)}$$

$$= \sum_{\rho=-\infty}^{\infty} C_\rho \exp\left(-\frac{\rho\pi x}{L} \right)$$

Where $-L \le x \le L$

$$C\rho = \int_{-L}^{L} f(x) \left(\frac{1}{2L} \exp\left(-i \frac{\rho\pi x}{L} \right) \right) dx$$

3.2 The Fourier Transform

3.2.1 Fourier Sine Transforms

$$F_S(\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(x)\sin\rho(x)\,dx \tag{5}$$

$0 \le x \le \infty$

With inversion

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty F_S(\rho)\sin\rho(x)\,d\rho$$

$0 \le \rho \le \infty$

Since kernel for operator and its inversion.

3.2.2 Fourier Cosine Transforms

$$F_c(\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(x)\cos\rho(x)\,dx \tag{7}$$

With the inversion

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} F_c(\rho)\cos\rho(x)\,dx \tag{8}$$

Same kernel $\cos\rho(x)$ for operator and its inversion.

3.2.3 Ordinary Fourier Transforms

$$F(\rho) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)e^{i\rho(x)}dx \tag{9}$$

The kernel $k(x,\rho) = e^{i\rho x}$

With inversion is

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(\rho)e^{-i\rho(x)}d\rho$$

Then $H = (\rho, x)e^{-i\rho(x)}$ (10)

have $k \neq H(\rho, x)$ (11)

If $f(x)$ is even then $f(-x) = f(x)$

and $F(\rho) = F_c(\rho)$ (12)

But if $f(x)$ is odd then $f(-x) = -f(x)$ and

Thus $F(\rho) = iF_c(\rho)$ (13)

From equation (9) above, we can deduce that;

$$(2\pi)^{-\frac{1}{2}} F(\rho) = \int_{-\infty}^{\infty} f(x) e^{-i\rho(x)} dx$$

$$= \int_{-\infty}^{0} f(x) e^{-i\rho(x)} dx + \int_{0}^{\infty} f(x) e^{-i\rho(x)} dx \quad (14)$$

But if

$x = -t$

$\Rightarrow x = 0 \Rightarrow t = 0$

$x = -\infty \Rightarrow t = 0$

$\therefore \quad dx = -dt$

Thus, we have

$$\int_{-\infty}^{0} f(x) e^{-i\rho(x)} dx = \int_{0}^{\infty} f(-t) e^{-i\rho(x)} dt \quad (15)$$

$$(2\pi)^{-\frac{1}{2}} F(\rho) = \int_{-\infty}^{0} f(x) e^{-i\rho(x)} dx + \int_{0}^{\infty} f(-x) e^{-i\rho(x)} dx \quad (16)$$

If $f(x)$ is even then $f(-x) = f(x)$

$\therefore$ Equation (16) becomes

$\int_{0}^{\infty} f(x)[e^{i\rho(x)} + e^{-i\rho(x)}]dx$ for even $f(x)$

$$2\int_{0}^{\infty} f(x) \cos \rho(x) dx = (2\pi)^{\frac{1}{2}} F(\rho) \quad (17)$$

But, for odd $f(x)$

$$\int_{0}^{\infty} f(x)[e^{i\rho(x)} - e^{-i\rho(x)}]dx$$

$$= 2i\int_{0}^{\infty} f(x) \sin \rho(x) dx \quad (18)$$

3.3 Fourier Integral Formular

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\cos\rho(x-t)dt \qquad (19)$$

Note that from (9) and (10) we have that:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)}d\rho \int_{-\infty}^{\infty} f(t)e^{i\rho(x)}dt \qquad (20)$$

We have now prove that equations (19) equals (20)
Consider equation (19)

$\int_{-\infty}^{\infty} f(t)\cos\rho(x-t)dx$ is a an even function of ρ

So that (19) can be re-written in the form

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\cos\rho(x-t)dt \qquad (21)$$

Since $\int_{0}^{\infty} g(\rho)d\rho = \frac{1}{2}\int_{-\infty}^{\infty} g(\rho)d\rho$
$g(\rho)$ is even

Hence $$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} f(t)\sin\rho(x-t)dt \qquad (22)$$

In other to arrive at equation (19), we have equation (21) equals (22) because

$$\cos\theta = i\sin\theta = e^{-i\theta}$$

$$\therefore \; F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} f(t)e^{-i\rho(x-t)}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)}d\rho \int_{-\infty}^{\infty} f(t)e^{i\rho(x)}dt$$

Which is equal to (20).

3.4 Transforms of Derivatives

$$F(\rho) = F(y(x)) = \left(\frac{1}{2\pi}\right)^{-1/2}\int_{-\infty}^{\infty} y(x)e^{i\rho(x)}dx \qquad (23)$$

We shall now transform $y'(x) = F(y'(x))$

$$\therefore F(y'(x)) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y'(x) e^{i\rho(x)} dx \tag{24}$$

Using integration by parts, we have

$$\left(\frac{1}{2\pi}\right)^{-1/2} \left[t(x) e^{i\rho(x)} \Big|_{-\infty}^{\infty} - i\rho \int_{-\infty}^{\infty} y(x) e^{i\rho(x)} dx \right] \tag{25}$$

suppose $y(x) \to 0$ as $x \to \pm\infty$

$$\therefore \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y'(x) e^{i\rho(x)} dx = i\rho \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y(x) e^{i\rho(x)} dx$$

$$= i\rho(Y(\rho))$$

$$\therefore \quad F(y'(x)) = i\rho(Y(\rho)). \tag{26}$$

$$y''(x) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y'(x) e^{i\rho(x)} dx \tag{27}$$

Integration by parts,

$$\left(\frac{1}{2\pi}\right)^{-1/2} \left[y'(x) e^{i\rho(x)} \Big|_{-\infty}^{\infty} - i\rho \int_{-\infty}^{\infty} y'(x) e^{i\rho(x)} dx \right] \tag{28}$$

suppose $y'(x) \to 0$

Then we have

$$-i\rho \int_{-\infty}^{\infty} y'(x) e^{i\rho(x)} dx.$$

Which $-\rho[F(y'(x))] = i\rho\left(-i\rho(Y(\rho))\right)$

$$= -i\rho^2[Y(\rho)]$$

$$= -\rho^2\left(y(x)\right) \tag{29}$$

Suppose we have

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = f(x)$$
$$y \to 0, \qquad y' \to 0 \text{ as } \quad x \to \pm\infty \qquad (30)$$

In other to arrive at equation (19), we equation (21)
Because $\cos\theta = i\sin\theta = e^{-i\theta}$.

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} f(t)e^{-i\rho(x-t)}dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\rho(x)}d\rho \int_{-\infty}^{\infty} f(t)e^{i\rho(x)}dt.$$

Which is equal to (20).

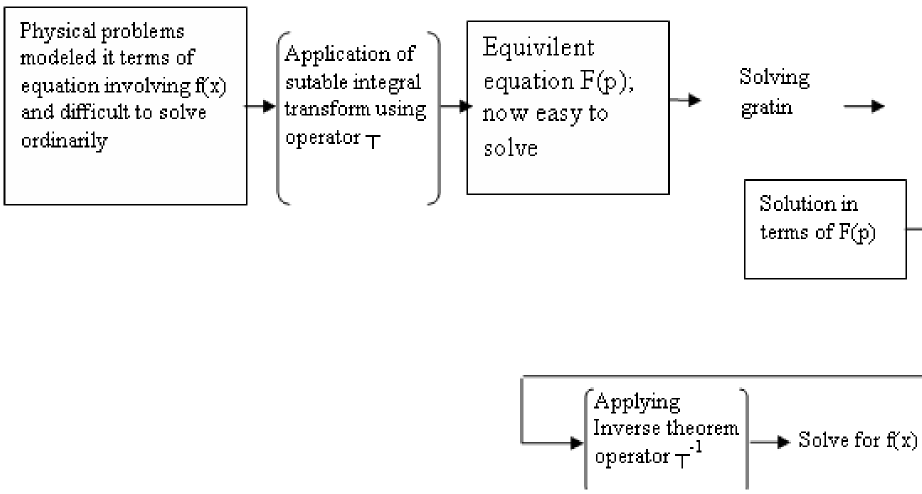
$$F(y'' + y' + y) = G(\rho)$$
$$\therefore -\rho^2 Y(\rho) - iY(\rho) + Y(\rho) = G(\rho)$$
$$Y(\rho)[-\rho^2 - i\rho + 1]$$
$$Y(\rho) = \left[\frac{G(\rho)}{-(\rho^2 + i\rho - 1)}\right] \quad (31)$$

4.0 CONCLUSION

In this unit, we treated the various forms of integral transform. The Fourier sine and cosine series representation were discussed. The inverse theorem was also considered.

5.0 SUMMARY

The general scheme of solving problem by integral transform is summarized below;



This is the diagrammatic expression of the summary.

6.0 TUTOR-MARKED ASSIGNMENT

i. State the method of integral transforms and its inverse. State also the Kernels of the method and its inverse
ii. Discuss briefly the inverse theorem.
iii. State the three theorems of finite Fourier transforms.
iv. If $F(\rho) = F(y(x)) = \left(\frac{1}{2\pi}\right)^{-1/2} \int_{-\infty}^{\infty} y(x)e^{i\rho(x)}dx$

use the transformation $y'(x) = F(y''(x))$, proof that $F(y'(x)) = i\rho[Y(\rho)]$.

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 2 FOURIER SERIES APPLICATION

CONTENTS

1.0 INTRODUCTION

Fourier series arises from the task of representing a given periodic function $f(x)$ by trigonometric series. The Fourier series coefficients are determined from $f(x)$ by Euler formula.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine Fourier coefficients
- find the convergence and sum of Fourier seriesand
- use Euler formula for the Fourier coefficients.

3.0 MAIN CONTENT

3.1 Fourier Series

3.1.1 Euler Formula for the Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π that can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (1)$$

That is to say, we assume the convergence of the series and has $f(x)$ as its sum.

In any function $f(x)$ of such, we shall determine the coefficients a_n and b_n of the corresponding series.

(1) To determine a_0, we shall integrate both sides of the equation 1, from $-\pi \le x \le \pi$

Thus, we have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]dx$$

$$= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nxdx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nxdx$$

$$= a_0 x\Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx\Big|_{-\pi}^{\pi} - \sum_{n=1}^{\infty}\frac{b_n}{n}\cos nx\Big|_{-\pi}^{\pi}$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty}\frac{1}{n}[a_n(\sin n\pi - \sin(-n\pi)) - (\cos n\pi - \cos(-n\pi))]$$

$$= 2\pi a_0 \qquad (2)$$

Hence

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (3)$$

To determine $a_1, a_2, \cdots a_n$ using the same procedure. However, multiplying equation (1) by $\cos mx$, when m is any fixed real number, and integrate from $-\pi \le x \le \pi$

$$\therefore \int_{-\pi}^{\pi} f(x)\cos mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\cos mxdx \qquad (4)$$

$$= a_0\int_{-\pi}^{\pi}\cos mxdx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nxdx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nxdx \qquad (5)$$

Evaluate (5) term by term, we have

$$a_0\int_{-\pi}^{\pi}\cos mxdx = a_0\left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi} = 0 \qquad (6)$$

Using trigonometric identities

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x] x dx \qquad (7)$$

Similarly,

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx dx = \frac{1}{2} \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} [\sin(n+m)x + \sin(n-m)x] x dx \qquad (8)$$

From (7), we have,

$$\int_{-\pi}^{\pi} \cos(n+m)x dx = \frac{\sin(n+m)x}{n+m}\Bigg|_{\pi}^{\pi} = 0 \qquad (9)$$

and

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \frac{\sin(n-m)x}{n-m}\Bigg|_{\pi}^{\pi} = 0 \qquad (10)$$

for $n \neq m$

but if $n = m$ we have that

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \int_{-\pi}^{\pi} \cos(0)x dx = \int_{-\pi}^{\pi} dx.$$

because $\cos 0 = 1$

$$\therefore \int_{-\pi}^{\pi} dx = x\Big|_{-\pi}^{\pi} = 2\pi \qquad (11)$$

From equation (8) we obtain thus

$$\int_{-\pi}^{\pi} \sin(n+m)x dx = -\frac{\cos(n+m)x}{n+m}\Bigg|_{\pi}^{\pi} = 0 \qquad (12)$$

and

$$\int_{-\pi}^{\pi} \sin(n-m)x dx = -\frac{\cos(n-m)x}{n-m}\Bigg|_{\pi}^{\pi} = 0 \qquad (13)$$

Substituting equations (9), (10), and (11) into (7), we have

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \qquad (14)$$

and substituting equations (12), (13), and (14) into (8) gives

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \quad (15)$$

Then, in view of equations (14), (15) and (6), equation (5) becomes:

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = a_n(0) + \sum_{n=m}^{\infty} a_n \pi + \sum_{n=1}^{\infty} b_n(0)$$
$$= a_m \pi \quad (16)$$

$$\therefore a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \quad (17)$$

$b_1, b_2, \cdots b_n$ can also be obtained in the same manner, by multiplying equation (1) by $\sin mx$ and integrate from $-\pi \le x \le \pi$.

Using the trigonometric identities and manipulation, we have

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} \left[a_0 + \left[\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right] \right] \sin mx dx \quad (18)$$

Integrating term by term, we see that the right hand side becomes

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} a_n \sin mx dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx \sin mx dx$$
$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx \sin mx dx \quad (19)$$

Using the same principle as before

$$\int_{-\pi}^{\pi} a_n \sin mx dx = 0 \quad (20)$$

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx dx = 0 \quad (21)$$

$for\ n = 1, 2, 3, \cdots\cdots$

but

$$\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx \sin mx dx = \frac{1}{2} \left[\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] x dx \right]$$

$$\frac{1}{2}\frac{(-1)\sin(n-m)x}{(n-m)} - \frac{1}{2}\frac{(-1)\sin(n+m)x}{(n+m)}0 \quad (22)$$

$n \neq m$

but for $n = m$

$$\frac{1}{2}\int_{-\pi}^{\pi}\cos(0)dx = \frac{1}{2}\int_{-\pi}^{\pi}dx = \pi$$

$$\therefore \int_{-\pi}^{\pi}\sin nx\cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \quad (23)$$

$\therefore$ substituting equation (23) into (19) we obtain thus

$$\int_{-\pi}^{\pi} f(x)\sin mx dx = b_n\pi$$

$$\Rightarrow \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \quad (24)$$

For $m = 1,2,\cdots$

Writing n in place of m in equation (17) and (24) respectively, we have

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\cos mx dx \\ &and \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \end{aligned}\right\} \quad (25)$$

This is called the Euler formula.

These numbers given in equation (25) are called the Fourier coefficients of $f(x)$. However, the trigonometric series in equation (1) with coefficients given by (25) is called the Fourier series of $f(x)$.

Example 1

Find the Fourier coefficients of the periodic function $f(x)$ where

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$$

and $\quad f(x+2\pi) = f(x)$.

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -dx + \int_{0}^{\pi} dx\right]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} -dx = \frac{1}{2\pi}(-x)\Big|_{-\pi}^{0} = \frac{1}{2\pi}[-0-(-\pi)]$$

$$= -\frac{1}{2}$$

and

$$\frac{1}{2\pi}\int_{0}^{\pi} dx = \frac{1}{2\pi}(x)\Big|_{0}^{\pi} = \frac{1}{2\pi}[\pi - 0]$$

$$= \frac{1}{2}$$

$$\therefore \; = -\frac{1}{2} + \frac{1}{2} = 0.$$

From equation (25) i.e.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{-\sin nx}{n}\Big|_{-\pi}^{0} + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

$$\therefore \qquad a_n = 0$$

Similarly for

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx dx + \int_{0}^{\pi} \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}[\cos 0 - \cos(-nx) - \cos nx + \cos 0]$$

$$= \frac{1}{\quad}[2 - 2\cos(nx)]$$

$n\pi$

N.B $\cos(-n\pi) = \cos(nx)$

$$= \frac{1}{n\pi}[1 - \cos(n\pi)]$$
$$= \frac{2}{n\pi}[1 - (-1)^n]$$

N.B $\cos nx = (-1)^n$

$$b_n = \frac{2}{n\pi}[1+1] = \frac{4}{n\pi}$$

for $n = 1,3,5,\cdots$

$$b_n = \frac{2}{n\pi}[0] = 0$$

for $n = 2,4,6,\cdots$

$$\therefore \quad b_1 = \frac{4}{\pi}, \quad b_3 = \frac{4}{3\pi}, \quad b_5 = \frac{4}{5\pi}, \text{ etc}$$
$$b_2 = b_4 = b_6 = 0$$

3.2 Even and Odd Numbers

Fourier coefficients of a function can be avoided if the function is odd or even. We say a function $y = g(x)$ is said to be even if

$$g(-x) = g(x) \quad \textit{for all } x. \tag{26}$$

While a function $h(x)$ is said to be odd if

$$h(-x) = -h(x) \quad \textit{for all } x. \tag{27}$$

However, it worth mentioning here that the function $\cos nx$ is even, while the function $\sin nx$ is odd.

If $g(x)$ is an even function, then

$$\int_{-L}^{L} g(xdx) = 2\int_{0}^{L} g(x)dx. \tag{28}$$

If $h(x)$ is an odd function, then

$$\int_{-L}^{L} h(x)dx = 0 \tag{29}$$

The product of both odd and even function is odd

$\therefore$ let $q(x) = g(x)h(x)$
and $q(-x) = g(-x)h(-x) = g(x)[-h(x)] = -q(x)$

3.2.1 Theorem 1 (Fourier Series of Even and Odd Functions)

The Fourier series of an even function $f(x)$ of periodic $2L$ is a "Fourier cosine series"

$$f(x) = a_0 + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \tag{30}$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$n = 1,2,\cdots$

Also the Fourier series of an odd function $f(x)$ of period $2L$ is a "Fourier sine series"

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{31}$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{32}$$

In particular, this theorem implies that the Fourier series of an even function $f(x)$ of period $2L = 2\pi$ Fourier cosine series.

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots\cdots$$

with coefficients (33)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

n 2, 1, 2 ………. (34)

Similarly, the Fourier series of an odd function $f(x)$ of period 2π is a Fourier sine series.

$$f(x) = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots\cdots$$

with coefficients (35)

$$b_n = \frac{2}{\pi}\int_0^L f(x)\sin nx\,dx \qquad (36)$$

3.2.2 Theorem 2 (Sum of Functions)

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_2 f_1 and f_2.

The Fourier coefficients of a cf are c times the corresponding Fourier coefficients of f.

Example 2

The function $f^*(x)$ is the sum of the function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$ as in example 1 and the constant 1.

Hence from example 1 and theorem 2, above, we conclude that

$$f^*(x) = 1 + \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{6}\sin 6x + \cdots\right)$$

Example 3

Find the Fourier series of the function

$f(x) = x + \pi$ if $-\pi < x < \pi$ and
$f(x + 2\pi) = f(x)$

Solution

Let $f = f_1 + f_2$ where $f_1 = x$ and $f_2 = \pi$.

The Fourier coefficients of f_2 are zero, except for the one (the constant term), which is π.

Hence, by theorem 2, the Fourier coefficients a_n, b_n are those f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$

and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

Integrating by parts we obtain

$$b_n = \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx\right]$$

$$= \frac{2}{n}\cos n\pi$$

$$= \frac{2}{n}(-1)^n = \frac{2}{n} \text{ for odd } n$$

$$= -\frac{2}{n} \quad \text{for even } n$$

Hence, $b_1 = 2, b_2 = -1, b_3 = \frac{2}{3}, b_4 = -\frac{1}{2} \ldots$

Therefore the Fourier series of $f(x)$ is given thus;

$$f(x) = \pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x\right]$$

4.0 CONCLUSION

The conclusion of this unit is embedded in the summary as discussed below.

5.0 SUMMARY

A Fourier series of a given function $f(x)$ of period 2π is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

With coefficients given as in equation (25).

Theorem 1 given conditions that is sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find the Fourier coefficients of the periodic function $f(x)$ where

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$

ii. Explain the term odd and even function of a Fourier series

iii. Find the Fourier series of the function

$f(x) = x + \pi$ if $0 < x < \pi$ and

$f(x+2\pi) = f(x)$

iv Find the smallest positive period p of the following function

(a) $\cos x, \sin x, \cos 2x, \sin 2x$

v. If $f(x)$ and $g(x)$ have period p, show that

$h = af + bg(a, b, cons\tan t)$ has the period p.

Thus all functions of period p from a vector space.

vi. Evaluate the following integrals when

$n = 0,1,2,\cdots$

(a) $\int_0^{\pi/2} \cos nx dx$ (b) $\int_{\pi/2}^{\pi/2} x \cos nx dx$

(c) $\int_0^{\pi/2} e^x \cos nx dx$ (d) $\int_0^2 x^2 \cos nx dx$

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 3 LAPLACE TRANSFORMS AND APPLICATION

CONTENTS

1.0 INTRODUCTION

The Laplace transform is a method for solving differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem.

The Laplace transform is the most important method used in solving engineering mathematics.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- undergo the three main steps of solving initial and boundary value problem.

3.0 MAIN CONTENT

3.1 The Classical Laplace Transform

Let f be a function of the real variable t which is defined for all $t \geq 0$ and which is either continuous or at least sectionally continuous.

The classical Laplace Transform † of f is the function $F_0(s)$ defined by the formula

$$F_0(s) \equiv \text{ı} \{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \qquad (1)$$

This definition of $F_0(s)$ clearly makes sense only for those values of s for which the infinite integral is convergent. For many applications it is enough to regard s as a real parameter, but in general it should be taken as complex, say $s = \sigma + i\omega$. Thus $F_0(s)$ is really a function of a complex variable defined over a certain region of the complex plane; the region of definition comprises just those values of s for which the infinite integral exists.

3.1.1 Elementary Applications of the Laplace Transform Depend Essentially on Three Basic Properties

i. **Linearity.** If the Laplace Transforms of *f* and *g* are $F_0(s)$ and $G_0(s)$ respectively, and if a_1 and a_2 are any (real) constants, then the Laplace Transform of the function *h* defined by

is $h(t) = a_1 f(t) + a_2 g(t)$

$$H_0(s) = a_1F_0(s) + a_2G_0(s). \qquad (2)$$

The proof is trivial.

ii. **Transform of a Derivative**. If f is differentiable (and therefore continuous) for $f \geq 0$, then

$$= sF^0(s) – f(0). \qquad (3)$$

Proof

Using integration by parts we have

$$\mathbf{t}[f'(t)] = \int_0^\infty e^{-st} f(t)dt = [e^{-st} f(t)]_0^\infty se^{-st} f(t)dt$$

$$= - f(0) + s \int_0^\infty e^{-st} f(t)dt$$

Since $\lim e^{-st} f(t) = 0$

Corollary. If f is n-times differentiable for $t \geq 0$, then

$$\mathbf{t}\,[f^{(n)}(t)] = s^nF_0(s) – s^{n-1}f(0) – s^{n-2} f'(0) \ldots\ldots – f^{(n-1)}(0).$$

iii. **The Convelution Theorem**. Let *f* and *g* have Laplace Transforms $F_0(s)$ and $G_0(s)$ respectively, and define h as follows:

$$H(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geq 0.$$

Then,

$$\mathfrak{L}[h(t)] = F_0(s)G_0(s). \qquad (4)$$

(Recall that h, as defined here, is the convolution of the functions u(t)f(t) and u(t)g(t). If f and g happen to be functions which vanish identically for all negative values of t then the above result can be expressed in the form:

The Laplace Transform of the convolution of f and g is the product of the individual Laplace Transform.

Proof

The Laplace Transform of h is given by

$$H_0(s) = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt.$$

Now,

$$\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau$$

because u(t - τ) = 1 for all τ such that τ < t
and u(t - τ) = 0 for all τ such that τ > t.

Hence

$$H_0(s) = \int_0^\infty e^{-st}\left[\int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau\right]dt.$$

Again,

$$\int_0^\infty g(t-\tau)u(t-\tau)e^{-st}dt = \int_\tau^\infty g(t-\tau)e^{-st}d\tau$$

because u(t -τ) = 1 for all t such that t > τ ,
and u(t -τ) = 0 for all t such that t < τ .

Thus,

$$H_0(s) = \int_0^\infty f(\tau)\left[\int_\tau^\infty g(t-\tau)e^{-st}dt\right]d\tau.$$

And so putting T = t - τ , we get

$$H_0(s) = \int_0^\infty f(\tau)\left[\int_0^\infty g(T)e^{-s(T+\tau)}dT\right]d\tau.$$

Since T = 0 when $t = \tau$.

That is,

$$H_0(s) = \int_0^\infty f(\tau)e^{-st}d\tau \int_0^\infty g(T)e^{-sT}dT = F_0(s)D_0(s).$$

Remark

The change in the order of integration in the proof given above is justified by the absolute convergence of the integrals concerned.

3.1.2 Applications of Laplace

The most immediate application of these properties is in the solution of ordinary differential equations with constants. Consider the case of the general second-order equation

$$a\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + cy = f(t) \quad (5)$$

Where y(0) = α and d'(0) = β . If $\mathcal{L}$[y(t)] = $Y_0(s)$ then

$$\mathcal{L}\left[\frac{dy}{dt}\right] = sY_0(s) - \alpha, \text{ and } \mathcal{L}\left[\frac{d^2y}{dt^2}\right] = 2Y0(s) - \alpha s - \beta .$$

Taking Laplace Transforms of both sides of (5.5) therefore gives

$$a[s^2Y_0(s) - \alpha s - \beta] + 2b[sY_0(s) - \alpha] + cY_0(s) = F_0(s).$$

That is,

$$Y_0(s) = \frac{F_0(s)}{as^2 + 2bs + c} + \frac{a\alpha s + (a\beta + 2b\alpha)}{as^2 + 2bs + c} \quad (6)$$

$Y_0(s)$ is thus given explicitly as a function of s, and what remains is an **inversion problem**; that is to say we need to determine a function y(t) whose Laplace Transform is $Y_0(s)$. The question of uniqueness which naturally arises at this point is not, in practice, a serious problem. In brief, if y_1 and y_2 are any two functions which have the same Laplace Transform $Y_0(s)$, then they can differ in value only on a set of points which is (in a sense which can be made precise) a negligibly small set. In fact, we have the following situation:

if $\mathcal{L}[y_1(t)] = \mathcal{L}[y_2(t)]$ then $\int_0^\infty |y_1(t) - y_2(t)|\, dt = 0.$

With this proviso in mind, we admit the slight abuse of notation involved, and write:

$$y(t) \equiv \mathcal{L}^{-1}[Y_0(s)] = \mathcal{L}^{-1}\left[\frac{F_0(s)}{as^2 + 2bs + c}\right] + \mathcal{L}^{-1}\left[\frac{a\alpha s + (a\beta + 2b\alpha)}{as^2 + 2bs + c}\right] \quad (7)$$

□ $as^2 + 2bs + c$ □

where y is defined for all t > 0.

A more serious problem from the practical point of view is that of implementing the required inversion; that is, of division effective procedures which allow us to recover a function f(t) given its Laplace Transform $F_0(s)$. In a large number of commonly occurring cases this can be done by expressing $F_0(s)$ as a combination of standard functions of s whose inverse transforms are known .

Note that with zero initial conditions, (y(0) = y'(0) = 0), the differential equation (5) can be regarded as representing a linear time-invariant system which transforms a given input signal f into a corresponding output y. This output function y is the **particular integral** associated with f and, using the Convolution Theorem, it can be expressed in terms of the appropriate impulse response function characterizing the system:

$$Y(t) = \int_0^t f(\tau)h_1(t-\tau)d\tau = \mathcal{L}^{-1}[F_0(s)H_0$$

Where

$$H_0(s) = \int_0^\infty e^{-st}h(t)\,dt = \frac{1}{as^2 + 2bs + c}$$

Non-zero initial conditions correspond to the presence of stored energy in the system at time t = 0. The response of the system to this stored energy is independent of the particular input f and is given by the **complementary function.** The complete solution (valid for all t > 0) of the equation (5) can be written in the form.

$$Y(t) = \mathcal{L}^{-1}[F_0(s)H_0(s)] + \mathcal{L}^{-1}[a\alpha s + (a\beta + 2b\alpha)]H_0(s)]. \qquad (8)$$

In applying the classical Laplace transform technique to (5) we are tacitly assuming that the system which it is being taken to represent is **unforced** for t < 0; that is, that the response which we compute from (5) is actually the response to the excitation f(t)u(t). This is sometimes expressed by saying that the input is **suddenly applied** at time t = 0.

3.2 Laplace Transforms of Generalised Functions

If a is any positive number then there is no specialty in extending the definition of the classical, one-sided, Laplace Transform to apply to the case of a delta function located at t = a, or to any of its derivatives located there; for a direct application of the appropriate sampling property gives immediately

$$\mathcal{L}\{\delta_a(t)\} = \mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa} \tag{9}$$

$$\mathcal{L}\{\delta'(t-a)\} = \int_0^\infty e^{-st}\delta'(t-a)dt = -\left[\frac{d}{dt}(e^{-st})\right]_{t=a} = se^{-sa} \tag{10}$$

and so on

Now take the case of a function f defined by a relation of the form

$$f(t) = \varphi_1(t)u(a-t) + \varphi_2(t)u(t-a) \tag{11}$$

where $a > 0$, and φ_1 and φ_2 are continuously differentiable functions. Using the notation

$$f'(t) = \varphi_1'(t)u(a-t) + \varphi_2'(t)u(t-a) \quad \text{(for all } t \neq a)$$

and

$$Df(t) = \varphi_1'(t)u(a-t) + \varphi_2'(t)u(a-t) + [\varphi_2(a) - \varphi_1(a)]\delta(t-a)$$
$$\equiv f'(t) + [f(a+) - f(a-)]\delta(t-a). \tag{12}$$

Using integration by parts to evaluate the Laplace integral we have

$$\int_0^\infty e^{-st}f'(t)dt = \int_0^a \varphi_1'(t)e^{-st}dt + \int_0^\infty \varphi_2'(t)e^{-st}dt$$

$$= \left[e^{-st}\varphi_1(t)\right]_0^a + s\int_0^a \varphi_1(t)e^{-st}dt + \left[e^{-st}\varphi_2(t)\right]_a^\infty + s\int_a^\infty \varphi_2(t)e^{-st}dt$$

$$= s\left[\int_0^a \varphi_1(t)e^{-st}dt + \int_a^\infty \varphi_2(t)e^{-st}dt\right] - e^{-as}[\varphi_2(a) - \varphi_1(a)] - \varphi_1(0)$$

$$\equiv sF_o(s) - f(0) - e^{-as}[f(a+) - f(a-)] \tag{13}$$

so that a modification of the derivative rule is required when we adhere to the classical meaning of the term “derivative” in the case of discontinuous functions.

On the other hand, from (12) we get

$$\int_0^\infty e^{-st}[Df(t)]dt = \int_0^\infty e^{-st}f'(t)dt + [f(a+) - f(a-)]\,e^{-as}$$
$$= sF_0(s) - f(0) \tag{14}$$

and the usual form of the derivative rule continues to apply.

The result (13) makes sense even when we allow a to tend to zero, for then we get

$$\mathcal{L}[f'(t)] = \int_0^\infty \varphi_2'(t)e^{-st}dt = s\int_0^\infty \varphi_2(t)e^{-st}dt - \varphi_2(0)$$
$$= sF_0(s) - f(0+). \qquad (15)$$

However, a complication arises with regard to $\mathcal{L}[Df(t)]$ when a = 0. If we have

then $$f(t) = \varphi_1(t)u(-t) + \varphi_2(t)u(t)$$
$$Df(t) = \varphi_1'(t)u(-t) + [\varphi_2(0) - \varphi_1(0)]\delta(t)$$

and so,

$$\mathcal{L}[Df(t)] = \mathcal{L}[\varphi_2'(t)] + [\varphi_2(0) - \varphi_1(0)]\mathcal{L}[\delta(t)]$$
$$= s\mathcal{L}[\varphi_2(t)] - \varphi_2(0) + [\varphi_2(0) - \varphi_1(0)]\Delta(s)$$
$$\equiv sF_0(s) - f(0+) + [f(0+) - f(0-)]\Delta(s). \qquad (16)$$

The difficulty is that, as remarked in Sec. 4.5, the Laplace Transform of the delta function (which we have denoted by $\Delta(s)$) is not defined by the Laplace integral

$$\int_0^\infty e^{-st}\delta(t)dt = \int_{-\infty}^{+\infty} e^{-st}u(t)\delta(t)dt.$$

The role of the delta function as a (generalized) impulse response function suggests that we should have $\Delta(s) = 1$ for all s, and this is the definition most usually adopted. However the discussion on the significance of the formal product $u(t)\delta(t)$ shows that there are grounds for taking $\Delta(s) = \frac{1}{2}$, for all s; other values for $\Delta(s)$ have also at the issue cannot be resolved simply by an appeal to the definition of δ as a limit, nor by means of the formulation as a (Riemann) Stieltjes integral. In the latter case, for example, we have for an arbitrary continuous integrand f

$$\int_0^\infty f(t)du_c(t) = (1 - c)f(0) \qquad (17)$$

We could therefore obtain $\Delta(s) \equiv 1$ by choosing c = 0 or, equally well, $\Delta(s) \equiv \frac{1}{2}$ by choosing $c = \frac{1}{2}$. Whatever value we choose for $\Delta(s)$ the relation (16) is bond to be consistent with the behaviour of δ as the derivative of the unit step function u. for, since

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st}dt = 1/s,$$

We have

$$\mathcal{L}\,[u'(t)] = [s\,\frac{1}{s} - u(0+)] + \Delta(s)[u(0+) - u(0-)]$$
$$= (1 - 1) + \Delta(s)\,(1 - 0) = \Delta(s).$$

On the other hand care must be taken to ensure that the correct form of (16) is used when a specific definition of $\Delta(s)$ has been decided on. Thus, for $\Delta(s) = 1$ we get

$$\mathcal{L}\,[Df(t)] = sF_0(s) - f(0-)$$
$$= sF_0(s) \qquad (18)$$

Whenever f(t) = 0 for all t < 0.

But for $\Delta(s) = \frac{1}{2}$ the result becomes

$$\mathcal{L}\,[Df(t)] = sF_0(s) - \frac{1}{2}[f(0+) + f(0-)].$$

In what follows, we shall adopt the majority view and define $\Delta(s)$ to be 1 for all values of s. Similarly, we shall take the Laplace Transform of δ' to be s; the analogue of (19) then becomes

$$\mathcal{L}\,[D^2f(t)] = s^2F_0(s) - sf(0-) - f'(0-) \qquad (19)$$
$$= s^2F_0(s)$$

whenever f(t) = 0 for all t < 0. The convenience of these definitions is readily illustrated by the following derivation of the Laplace Transform of a **periodic function:**

Let f be a function which vanishes identically outside the finite interval (0,T). The periodic extension of f, of period T, is the function obtained by summing the translates, f(t – kT), for k = 0, ± 1, ± 2,…., (see fig. 49)

$$f_T(t) = \sum_{k=-\infty}^{+\infty} f(t - kT) \qquad (20)$$

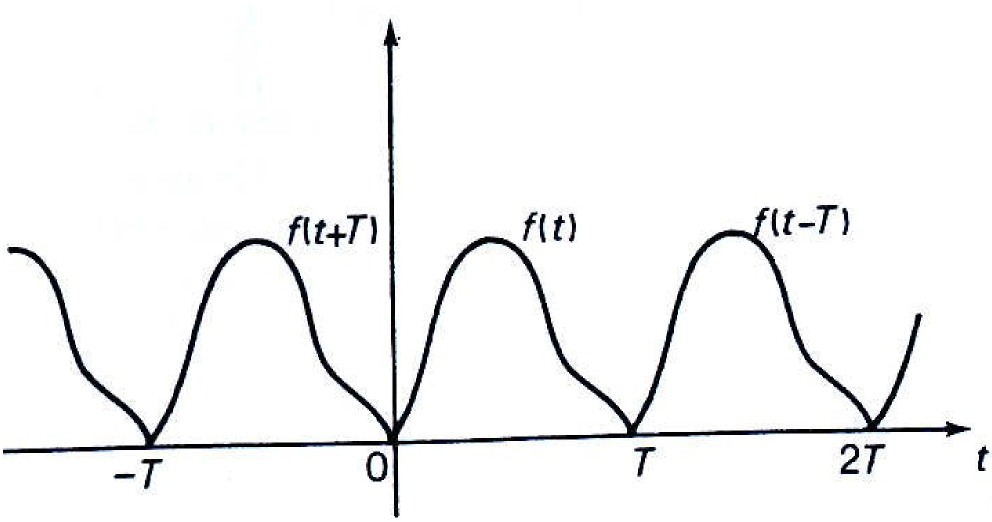


Fig. 49

We can write f_T as a convolution:

$$f_T(t) = \sum_{k=-\infty}^{+\infty}[f(t) * \delta(t-kT)] = f(t) * \sum_{k=-\infty}^{+\infty}\delta(t-kT). \qquad (21)$$

further, using the above definition of $\Delta(s)$, we obtain

$$\mathcal{L}\left[\sum_{k=-\infty}^{+\infty}\delta(t-kT).\right] = \mathcal{L}\left[\sum_{k=\infty}^{\infty}\delta(t-kT).\right]$$

$$= 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots = \frac{1}{1-e^{-sT}} \qquad (22)$$

The summation being valid provided that

$$|e^{-sT}| = |e^{-(\alpha+i\omega)T}| = e^{-\alpha T} < 1,$$

That is, for all s such that Re(s) > 0. Hence, appealing to the Conclusion

Theorem for the Laplace transform, (21) and (22) together yield

$$\mathcal{L}\left[\sum_{k=-\infty}^{+\infty}\delta(t-kT)\right] = \frac{F_0(s)}{1-e^{-sT}}. \qquad (23)$$

3.3 Computation of Laplace Transforms

If f is an ordinary function whose Laplace Transform exists (for some values of s) then we should be able to find that transform, in principle at least, by evaluating directly the integral which defines $F_0(s)$. It is usually simpler in practice to make use of certain appropriate properties of the Laplace integral and to derive specific transforms from them. The following results are easy to establish and are particularly useful in this respect:

(L.T.I) The first Translation Property. If $\mathcal{L}[f(t)] = F_0(s)$, and if a is any real constant, then

$$\mathcal{L}[e^{at}f(t)] = F_0(s - a).$$

(L.T.2) The Second Translation Property. If $\mathcal{L}[f(t)] = F_0(s)$, and if a is any positive constant, then

$$\mathcal{L}[u(t - a)f(t - a)] = e^{-as}F_0(s).$$

(L.T.3) Change of Scale. If $\mathcal{L}[f(t)] = F_0(s)$, and if a is any positive constant, then

$$\mathcal{L}[f(at)] = \frac{1}{a}F_0\left(\frac{s}{a}\right).$$

(L.T.4) Multiplication t. If $\mathcal{L}[f(t)] = F_0(s)$, then

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F_0(s) \equiv -F_0'(s).$$

(L.T.5) Transform of an Integral. If $\mathcal{L}[f(t)] = F_0(s)$, and if the function g is defined by

$$g(t) = \int_0^t f(\tau)\,d\tau$$

then

$$\mathcal{L}[g(t)] = \frac{1}{s}F_0(s).$$

The first three of the above properties follow immediately on making suitable changes of variable in the Laplace integrals concerned. For (L.T.4) we have only to differentiate with respect to s under the integral sign, while in the case of (L.T.5) it is enough to note that $g'(t) = f(t)$ and that $g(0) = 0$; the result then follows from the rule for finding the

Laplace Transform of a derivative. Using these properties, an elementary basic table of standard transforms can be constructed without difficulty (Table 1). This list can be extended by using various special techniques. In particular, the results for the transforms of delta functions derived in the preceding section are of considerable value in this connection.

Table 1: Basic Table of Standard Transforms

fu(t)(t)	**$F_0(s)$**	**Region of (absolute) convergence**
u(t)	1/s	Re(s) > 0
t	$1/s^2$	Re(s) > 0
t^n(n>1)	$n!/s^{n+1}$	Re(s) > 0
e^{at}	$\frac{1}{s-a}$	Re(s) > a
e^{-at}	$\frac{1}{s+a}$	Re(s) > - a
sinh at	$\frac{a}{s^2-a^2}$	Re(s) > | a |
cosh at	$\frac{s}{s^2-a^2}$	Re(s) > | a |
sin at	$\frac{a}{s^2+a^2}$	Re(s) > 0
cos at	$\frac{s}{s^2+a^2}$	Re(s) > 0

Example 1

Find the Laplace transform of the triangular waveform show in fig. 50.

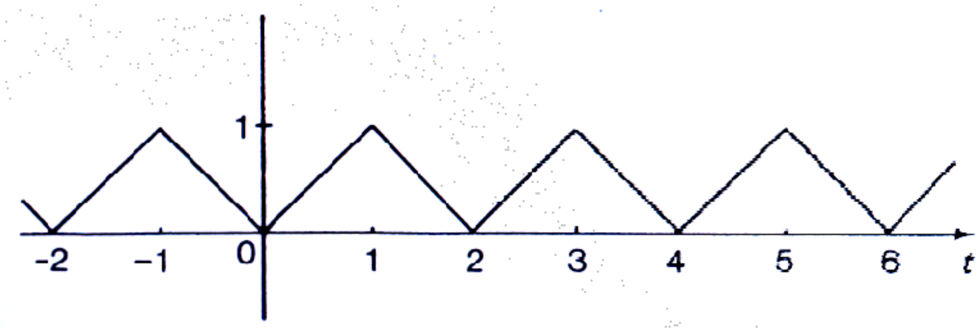
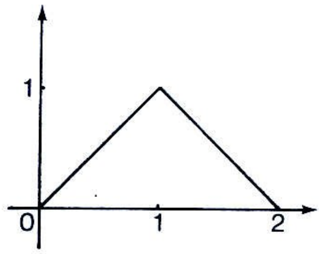


Fig. 50: Laplace Transform of the Triangle Waveform

We shall obviously expect to use the formula (23) for the Laplace Transform of the periodic extension of a function f, but the first need is to establish the transform of this function f itself. In fig. 51 there is shown a decomposition of the required function into a combination of ramp functions:

$$f(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

Fig. 5.4(a).



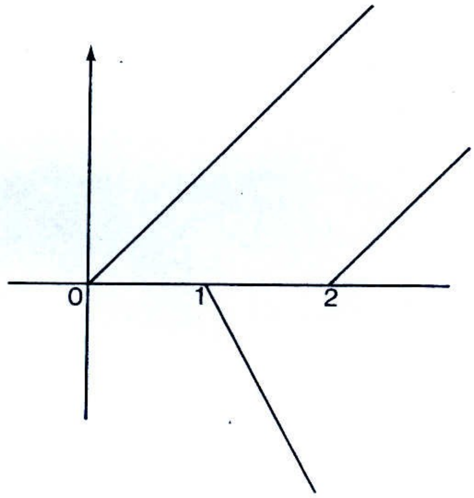


Fig. 5.4(b).

Fig. 51 (b)

A straightforward application of the second translation property (L.T.2) immediately gives

$$F_0(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{e^{-2s}}{s^2} = \left(\frac{1-e^{-s}}{s}\right)^2 = \frac{4}{s^2}e^{-s}\sinh^2\frac{s}{2}.$$

Hence, applying (5.23)

$$\mathcal{L}[f_T(t)] = \left(\frac{4}{s^2}e^{-s}\sinh^2\frac{s}{2}\right)\left(\frac{1}{1-e^{-2s}}\right) = \frac{2\sinh^2 s/2}{s^2\sinh s} = \frac{\tanh s/2}{s^2}.$$

4.0 CONCLUSION

In this unit we considerd the Laplace transform atum from practical point of view and illustrate its use by important engineering problems, many of them related to ordinary differential equations.

5.0 SUMMARY

The main purpose of the Laplace transformation is the solution of differential equations and systems of such equations, as well as corresponding initial value problems.

The Laplace transform f(s) = ı (f) of a function f(t) depend by.

F(s) ı (f) $\int_0^{\delta} e^{-st} f(t)dt$

Further, more discussion, the Laplace of the derivation such that.

ı (f)' = s ı (f) + f(0)
ı (f'') = s^2 ı (f) – sf(0) – f(0).

Hence, by taking the transform of a given differential equation $\frac{d^2y}{dx^2}$ + $a\frac{dy}{dx}$ + by = f(t).

∴ ı (y) = y(s)
Hence, the simple equation becomes

(s^2 x as x b) y = ı (δ) x s f(0) + f'(0) + a f(0).

Hence, ı (δ) the transformation back to hard problem can be gotten from the table 1 – unit 3.

6.0 TUTOR-MARKED ASSIGNMENT

i. Find the Laplace transform of the following function

a. $e^{at,}$
b. cos wt
c. cosh bt

ii. Use Laplace transforms to obtain, for t x0, the solution of the linear differential equation
$\frac{d^2y}{dx^2}$ xy = t., which satisfies the condition y(0) = 1, y'(0) = -2

iii. Use the convolution theorem for the Laplace Transform to solve the integral equation y(t) = cost + 2sint + $\int_0^t y(\tau)\sin(t-\tau)d\tau$
for t > 0.

iv. Identify the function whose Laplace Transforms are:

(a) $\dfrac{s^2+2}{s+1}$

(b) $\dfrac{\cosh s}{e^s}$.

7.0 REFERENCES/FURTHER READING

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995).. *Engineering Maths*. 5th Edition Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.