NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 417

COURSE TITLE: ELECTROMAGNETIC THEORY

COURSE GUIDE

MTH 417
ELECTROMAGNETIC THEORY

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Published by
National Open University of Nigeria

Printed 2011

Reprinted 2014

ISBN: 978-058-732-2

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INTRODUCTION

Electromagnetic theory is the basis of many of the technologies which surround us today; and indeed, the basis of contemporary civilisation.

You are expected to be familiar with basic electromagnetism and concepts in the prerequisite to this course, as you are encouraged to develop an enquiring attitude towards electromagnetically abundant universe around you which abounds with, and with which you interact every single day.

It is the objectives of this course to build upon the lessons learnt in the prerequisite course, and to revisit from a more elevated perspective, the principles of electromagnetism with the view to strengthening your understanding of its concepts upon which developmental work and research in the sciences and technology are based.

WHAT YOU WILL LEARN IN THIS COURSE

This course comprises a total of seven units categorised into three modules as follows:

Module 1 is composed of 2 units
Module 2 is composed of 3 units
Module 3 is composed of 2 units

Module 1 is exclusively devoted to Maxwell's equations and in this module; you will learn the basic concepts upon which Maxwell's equations are founded. In Unit 1 you will undergo a broad introduction tour where you will be shown how Maxwell's equations comprise of partial differential equations which are combined with the Lorentz force law to form the basis of classical electrodynamics, classical optics, and electric circuits. In Unit 2 you will be shown how the differential and integral formulations of Maxwell's equations are mathematically equivalent.

Module 2 treats electromagnetic waves with particular reference to the visible portion of the electromagnetic spectrum. Specifically; Unit 1 will tech you about electromagnetic wave equation and the theory of light. You will be taught how electromagnetic waves comprise two perpendicular vectors; which are representative of an electric electromagnetic field component and a perpendicular magnetic electromagnetic field component. You will also study the wave equation for both of these components. Unit 2 will expose you to the treatment of lights as transverse waves and you will learn to recognise that different media assert different, yet specific influences on electromagnet waves. In Unit 3; and with good analogy to light waves which represent the visible portion of the spectrum; and the same principles of which by extension can be extrapolated to the other parts of the electromagnetic spectrum ranging from the extremely low frequency waves through to and beyond hard radiation gamma rays, you will be able to appreciate the principles behind reflections and refraction of plane electromagnetic waves at plane boundaries, study the characteristics of the electromagnetic boundary conditions viz-a-viz the normal, reaction of waves at boundaries, laws of reflection and refraction, reflection and refraction at a boundary between dielectrics, reflection and refraction at the surface of a conductor.

Module 3 will impress upon you that electromagnetic waves represent an energy transport system and has associated momentum. You will learn about the energy theorem as well as the momentum theorem in Maxwell's theory of electromagnetism in Unit 1 whereas Unit 2 will explain to you all you need to know about radiations from extended sources and radiation from charges moving in matter. This unit, this module and this course ends with the all important of the Lorentz transformation and here we will show you how the transform is derived.

COURSE AIM

The aim of MTH 417 is to further intimate you with the electromagnetic theory and re-tool you for a better understanding of the world around you where you will now be able to establish a correlation between the theoretical foundation of electromagnetism and its multifaceted practical application as well as its permeating influence on virtually everything we do every day of our lives.

COURSE OBJECTIVES

You in turn shall be required to conscientiously and diligently work through this course which upon completion you should be able to:

- state the four Maxwell’s equations of electromagnetism
- describe the Lorentz force law
- explain why parallel currents attract and why anti-parallel currents repel
- use the right hand thumb rule
- understand the constitutive relations in electromagnetic theory

- relate Maxwell's equations to the electromagnetic properties of a material
- understand the genesis of Maxwell's equations
- analyse each of Maxwell's equations in detail
- work with the differential form of Maxwell's equation
- know why the third equation is known as Faraday's law
- explain the continuity equation
- see why Ampere's law is also the last of Maxwell's equations
- distinguish between rotation-free vector field and source-free vector field
- establish the relationship between electromagnetic wave and light
- state the wave equation for electric field vector $\vec{E}$
- describe the wave equation for the magnetic field vector $\vec{B}$
- recognise the three dimensional nature of wave equation for each component of the electric and magnetic fields
- establish conclusively that light waves are transverse waves
- know the ratio referred to as relaxation time of conducting medium
- understand propagation of plane electromagnetic waves in non-conducting media
- describe the relationship between the wave number and the amplitudes of electric and magnetic components of electromagnetic waves
- investigate the mechanism guiding the propagation of plane electromagnetic waves in conducting media
- see why electric charges move almost instantly to the surface of perfect conductors when subjected to electromagnetic influence (skin effect)
- qualify the phenomenon of reflection and refraction of plane waves at boundaries
- distinguish between reflection and refraction at boundary between dielectrics and at the surface of conductors
- solve problems involving electromagnetic boundary conditions
- derive the laws of reflection and refraction
- study the characteristics of monochromatic plane wave on a boundary
- understand polarisation by reflection on a boundary between two dielectrics
- comfortably work with reflection coefficient
- explain the energy theorem in Maxwell's electromagnetic theory
- know why the energy theorem is also known as Poynting's theorem
- quantify the momentum theorem in Maxwell's electromagnetic theory

- explain the term electric volume force
- write an expression for the Maxwell stress tensor
- qualify relative electric permittivity
- appreciate that there are radiation from extended sources

- write down the macroscopic Maxwell equations
- explain collision interaction between charge carriers
- describe anisotropic and birefringent medium
- know that charges moving in matter radiate electromagnetic waves
- derive the Lorentz transformation
- know why the combination of two Lorentz transformations must be a Lorentz transformation.

WORKING THROUGH THIS COURSE

This course requires you to spend quality time to read. Whereas the content of this course is quite comprehensive, it is presented in clear language that you can easily relate to. The presentation style is graphical descriptive and adequate; and is deliberately to ensure that your attention remains focused to the course content and remains sustained throughout.

You should take full advantage of the tutorial sessions because this is a veritable forum for you to “rub minds” with your peers – which provides you valuable feedback as you have the opportunity of comparing knowledge with your course mates.

COURSE MATERIALS

You will be provided course material prior to commencement of this course, which will comprise your course guide as well as your study units. You will receive a list of recommended textbooks which shall be an invaluable asset for your course material. These textbooks are however not compulsory.

STUDY UNITS

You will find listed below the study units which are contained in this course and you will observe that there are three modules. Module 2 comprises three units while Modules 1 and 3 comprise two each.

Module 1

Unit 1 Introduction to Maxwell’s Equations
Unit 2 Maxwell’s Equations

Module 2

Module 3

TEXTBOOKS AND REFERENCES

Banesh, H. (1973). *Relativity and its Roots.* New York: Freeman.

Charles, F. S. (1995). *The Six Core Theories of Modern Physics.* MIT Press.

Charles, W. *et al.* (1970). (1970). *Gravitation (A Treatment of Maxwell's Equations in Terms of Differential Forms).* New York: W. H. Freeman.

David, H. S, Ann, W. M, & Jin, Au Kong. (1994). *Electromagnetic Waves.* Prentice-Hall.

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Tipler, P. (2004). *Physics for Scientists and Engineers: Electricity, Magnetism, Light, and Elementary Modern Physics.* (5th ed.). W. H. Freeman.

Wheeler, J. A, Misner, C. & Thorne. K.S. (1973). *Gravitation.* W. H. Freeman & Co.

ASSESSMENT

Assessment of your performance is partly through tutor-marked assignments which you can refer to as TMA, and partly through the final examination.

TUTOR-MARKED ASSIGNMENT

This is basically continuous assessment which accounts for 30% of your total score. During this course you will be given four tutor-marked assignments and you must answer three of them to qualify to sit for the end of year examination. Tutor-marked assignments are provided by your course facilitator and you must return the answers to your course facilitator within the stipulated period.

FINAL EXAMINATION AND GRADING

You must sit for the final examination which accounts for 70% of your score upon completion of this course. You will be notified in advance of the date, time and the venue for the examinations which may, or may not coincide with National Open University of Nigeria semester examination.

SUMMARY

Each of the three modules of this course has been designed to stimulate your interest in electromagnetic theory through fundamental conceptual building blocks in the study and practical application of electromagnetism.

Module 1 takes you on an introductory tour of Maxwell's equations, intimating you with the Lorentz force law and the constitutive relations which place you in a comfortable environment to understand subsequent material. Module 2 further cements the lessons learnt in Module 1 by showing you that Maxwell's equations come in both integral and differential form, and that the law of conservation of energy applies to electromagnetism through the continuity equation.

Module 2 relates visible light to electromagnetic theory through the wave equation and further stresses that the two perpendicular components of electromagnetic waves mutually perpendicular to the direction of wave propagation are the electric and the magnetic vectors – and both vectors are quantified through the wave equation for each of these components. Light is transverse waves and exhibits specific characteristics in both conducting and non-conducting media. Its reflection and refraction at plane boundaries between adjacent media are guided by strict rules while the propagation of electromagnetic waves through media depends on the constitution of the media.

Module 3 visualises electromagnetic waves from the perspective of an energy transport with which it ascribes energy through the energy theorem in Maxwell's theory; and momentum through the momentum theorem in Maxwell's theory. Module 3 delves into radiation from extended sources, radiation from charges moving in matter and investigates the conceptual basis for, and the derivation of the Lorentz transformation.

Indeed, this course will change and broaden the way you hitherto perceived electromagnetism and my advice is: make sure you have enough referential and study materials at your disposal, and devote sufficient quality time to your study.

Good luck.

MAIN COURSE

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MODULE 1

UNIT 1 INTRODUCTION TO MAXWELL'S EQUATIONS

CONTENTS

1.0 INTRODUCTION

Maxwell's equations comprise partial differential equations which combined with the Lorentz force law form the basis of classical electrodynamics, classical optics, and electric circuits. Modern electrical and communications technologies depend on Maxwell's equations which describe how electric and magnetic fields are generated and how they are altered by each other, and by electric charges and currents.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the four Maxwell equations of electromagnetism
- describe the Lorentz force law
- explain why parallel currents attract and why anti-parallel currents repel
- use the right hand thumb rule
- understand the constitutive relations in electromagnetic theory
- relate Maxwell equations to the electromagnetic properties of a material.

3.0 MAIN CONTENT

3.1 Introduction to Maxwell's Equations

The basic equations of electromagnetism are the four Maxwell's equations and the Lorentz force law. In principle, these together with Newton's second law of motion are enough to completely determine the motion of an assembly of charges given the initial positions and velocities of all the charges. It is well known that light is a form of electromagnetic radiation, so it is instructive to review some of the properties of electricity and magnetism leading to the derivations of the Maxwell's equations.

The original studies of electricity and magnetism date back to at least the early Greek times. By the start of the nineteenth century, it was known that some objects could possess an electrical charge, and that these charges could exert a force on each other even through a vacuum. This force could be described mathematically as

$$\vec{F}_E = q\vec{E} \tag{1.1}$$

where q is the electrical charge on the object in question and $\vec{E}$ is the electric field produced by all the other charges in the universe. The charge was discovered to take on a discrete set of values, one of the first examples of quantisation. In its turn, the electric field can be described by a scalar potential field V, which is related to the electric field by

$$\vec{E} = -\nabla V \tag{1.2}$$

The vector, differential-operator ∇ in these equations is defined as

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

In addition, it was also noted that a moving charge may experience another force which is proportional to its velocity $\vec{v}$. This led to the definition of another field; namely the magnetic field $\vec{B}$, such that

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{1.3}$$

As with the electric field, the magnetic field is generated by all the other currents in the universe. The magnetic field can be described in terms of a vector potential field $\vec{A}$, which is related to the magnetic field by

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1.4}$$

3.2 The Lorentz Force Law

We now begin to consider how things change when charges are in motion. A simple apparatus demonstrates that something strange happens when charges are in motion:
If we run currents next to one another in parallel, we find that they are attracted when the currents run in the same direction; they are repulsed when the currents run in opposite directions. This is despite the fact the wires are completely neutral: if we put a stationary test charge near the wires, it feels no force.

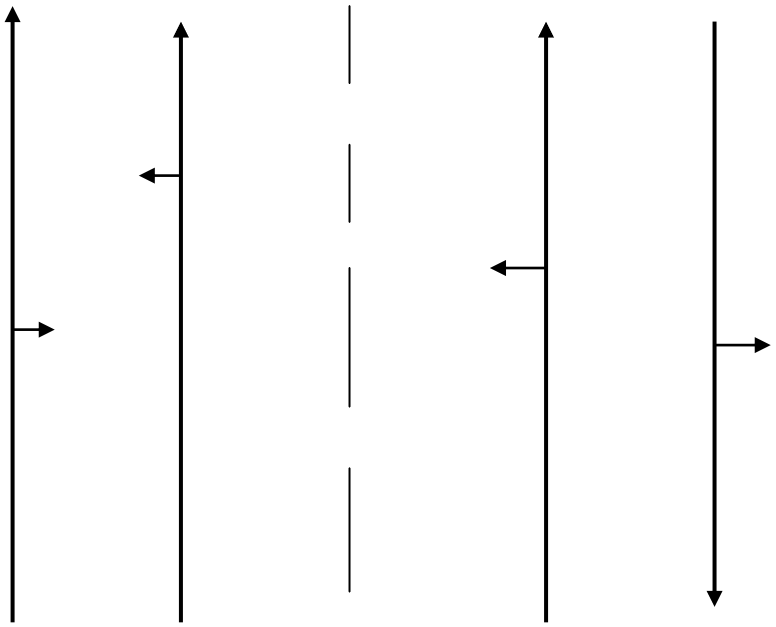

Fig. 1.1: Left: Parallel Currents Attract. Right: Anti-Parallel Currents Repel

Furthermore, experiments show that the force is proportional to the currents - double the current in one of the wires, and you double the force. Double the current in both wires and you quadruple the force. This indicates a force that is proportional to the velocity of a moving charge; and, that it points in a direction perpendicular to the velocity. These conditions suggest a force that depends on a cross product.

What we say is that some kind of field $\vec{B}$ the "magnetic field" - arises from the current. The direction of this field is kind of odd: it wraps around the current in a circular fashion, with a direction that is defined by the right-hand rule: We point our right thumb in the direction of the current, and our fingers curl in the same sense as the magnetic field (Figure 1. 2).

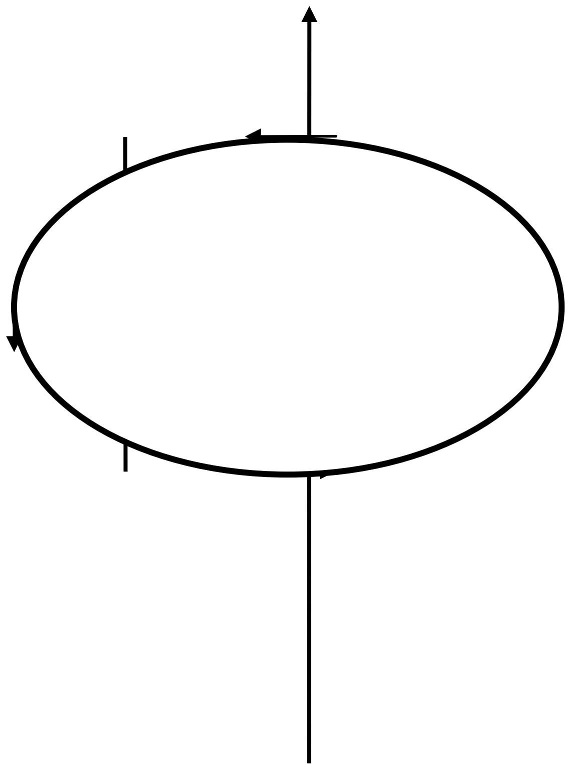

Fig. 1.2: Right Hand Rule

With this sense of the magnetic field defined, the force that arises when a charge moves through this field is given by

$$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}$$

where c is the speed of light. The appearance of c in this force law is a hint that special relativity plays an important role in these discussions.

If we have both the electric and magnetic fields, the total force that acts on a charge is of course given by:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \quad (1.5)$$

This combined force law is known as the **Lorentz force**.

3.3 The Constitutive Relations

Similar to the constitutive relations in continuous medium mechanics, there are also constitutive relationships in electromagnetic theory. Constitutive relations describe the medium's properties and effects when two physical quantities are related. It can be viewed as the description of response of the medium as a system to certain input. For example, in continuous medium mechanic, the response of a linear-elastic medium to strain can be described by the Hooke's law, and the resultant is the stress.

The relationship between stress and strain is the Hooke's law. In another word, Hooke's law is the constitutive relations for linear elasticity. In electromagnetic theory, there are four fundamental constitutive relationships to describe the response of a medium to a variety of electromagnetic input. Two of them describe the relationship between the electric field $\vec{E}$ and the conductive current $\vec{J}$, and the electric displacement $\vec{D}$, and the other two describe the relationship between the magnetic field $\vec{H}$ and the magnetic induction $\vec{B}$, and the magnetic polarisation $\vec{M}$. Quantitatively, these four constitutive relationships are

$$\begin{aligned} \vec{J} &= \sigma \vec{E} \quad \text{(Ohm's law)} && \text{(i)} \\ \vec{D} &= \varepsilon \vec{E} && \text{(ii)} \\ \vec{B} &= \mu \vec{H} && \text{(iii)} \\ \vec{M} &= \chi \vec{H} && \text{(iv)} \end{aligned} \qquad (1.50)$$

where σ is the electric conductivity, ε the dielectric permittivity, μ the magnetic permeability, and χ the magnetic susceptibility. It is possible to discuss the electromagnetic properties of earth material in terms of these four parameters. It is noteworthy that the first relation is the well-known Ohm's law in a microscopic form. These four parameters exclusively describe the electromagnetic properties of a material. It is necessary to point out that some of them are inter-related (to be seen later). To understand the behaviour of these electromagnetic parameters are the central piece to understand the geophysical response when geophysical surveys are employed to solve any engineering, exploration, and environmental problems.

4.0 CONCLUSION

In this unit we have been able to state the four Maxwell equations and apply the Lorentz law s of electromagnetism. We have discovered why parallel currents attract and anti-parallel currents repel in conductors as well as use the right hand thumb rule to determine the direction of the magnetic force field associated with his current.

The constitutive relations in electromagnetic theory and the relationship between Maxwell's equations and the electromagnetic properties of a material have been established.

5.0 SUMMARY

Maxwell's equations are fundamental to the understanding of the behaviour of electromagnetic waves in vacuum and in matter.

6.0 TUTOR-MARKED ASSIGNMENT

1. State Maxwell's equations?
2. Explain what is constitutive relations in electromagnetic.
3. Describe a brief experiment to demonstrate Lorentz force law.

7.0 REFERENCES/FURTHER READING

Banesh, H. (1983). *Relativity and its Roots.* New York: Freeman.

Charles, F. S. (1995). *The Six Core Theories of Modern Physics.* MIT Press.

Charles, W, Misner, K. S. & Thorne, J. A. W. (1970). *Gravitation (A Treatment of Maxwell's Equations in Terms of Differential Forms*). New York: W. H. Freeman.

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UNIT 2 MAXWELL'S EQUATIONS

CONTENTS

1.0 INTRODUCTION

Maxwell's equations represent a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits and while the differential and integral formulations of the equations are mathematically equivalent, both the differential and integral formulations are useful.

Whereas the integral formulation can often be used to simply and directly calculate fields from symmetric distributions of charges and currents, the differential formulation is a more natural starting point for calculating the fields in more complicated situations.
The conservation law that electric charge is conserved is expressed by the continuity equation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss the genesis of Maxwell's Equations
- analyse each of Maxwell's Equations in detail
- work with the differential form of Maxwell's Equation
- know why the third equation is known as Faraday's Law
- explain the Continuity Equation
- see why Ampere's Law is also the last of Maxwell's Equations
- distinguish between rotation-free vector field and source-free vector field.

3.0 MAIN CONTENT

3.1 Maxwell's Equations

The fact that the electric field was described in terms of stationary charges, while the magnetic field was described in terms of moving charges led people to suspect that some relationship existed between the two fields. This was confirmed when it was found that an electric current could be generated by changing the magnetic field. In the mid-1800's, the theories of electricity and magnetism were finally united by James Clerk Maxwell in four equations now known as **Maxwell's Equations**.

$$\oiint \vec{E}\cdot d\vec{S} = \iiint \frac{\rho_f}{\varepsilon} dV \qquad (1.6)$$

$$\oiint \vec{B}\cdot d\vec{S} = 0 \qquad (1.7)$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S} \qquad (1.8)$$

$$\oint \vec{B}\cdot d\vec{l} = \mu\iint \vec{j}_f \cdot d\vec{S} + \mu\varepsilon\frac{d}{dt}\iint \vec{E}\cdot d\vec{S} \qquad (1.9)$$

Each one of these can be understood separately.

The first of Maxwell's equations, equation (1.6), is known as **Gauss's Law**. It relates the flux of electric field intensity to the total charge enclosed by the surface. The flux is defined as

$$\Phi_E = \oiint \vec{E}\cdot d\vec{S} \qquad (1.10)$$

where ***dS*** is the vector outwardly normal to the surface and the integral is over the entire surface enclosing the region in question. In words, Gauss's law tells us that the total flux through a closed surface, i.e. the change in the number of field lines passing through a closed surface, is proportional to the total charge contained within the volume defined by the surface. Thus if there is no charge inside the surface, the net flux is zero. If there is a positive net charge, the enclosed region acts as a source; if the net charge is negative, the enclosed region acts as a sink.
The constant ε is called the **electric permittivity** of the medium. If the medium is a vacuum, then $\varepsilon = \varepsilon_0$, where ε_0 is known as the permittivity of free space and has a value of $\varepsilon_0 = 8.8542\times10^{-12}\ \frac{C^2}{N\cdot m^2}$. The electric permittivity was originally used to act as a medium dependent proportionality constant that connects a parallel plate capacitor's capacitance with its geometric characteristics. Conceptually, we can view the permittivity as encompassing the electrical behaviour of the

medium: in a sense, it is a measure of the degree to which the material is permeated by the electric field in which it is immersed. We can relate the electric permittivity to the dielectric constant by the following formula

$$\varepsilon = K_e \varepsilon_0 . \qquad (1.11)$$

The second equation is also a form of Gauss's law, this time applied to the magnetic field. The fact that the enclosed charge is zero tells us that, at least according to classical electromagnetic theory, there is no such thing as a magnetic monopole. In other words, whereas the electrical charge could be viewed as either a positive or negative charge individually, we can never find magnetic charges which do not include both a positive and negative pole. Since the total enclosed charge is the algebraic sum of the charges, this lack of magnetic monopoles automatically insures that the sum is zero.

The third equation is known as **Faraday's Law**. In a manner similar to the electric flux, the magnetic flux is defined as

$$\Phi_B = \iint \vec{B} \cdot d\vec{S} \qquad (1.12)$$

where the surface is now an open surface bounded by a conducting loop. Faraday found that if the induced emf (electromotive force) that was developed in the loop depended on the rate at which the magnetic flux changed,

$$\text{emf} = -\frac{d\Phi_B}{dt} \qquad (1.13)$$

However, the emf exists only as a result of the presence of an electric field, which is related to the emf by

$$\text{emf} = \oint \vec{E} \cdot d\vec{l} \qquad (1.14)$$

Combining (1.13) and (1.14), any direct reference to the induced emf is removed and we get Faraday's law. Physically, this shows us that if the magnetic flux changes, in other words if either the surface area or the magnetic field changes with time, an electrical field is produced as result. This electrical field creates an emf which acts in such a way as to resist the changes in the magnetic flux. Thus, a time varying magnetic field creates an electric field. Since there are no charges which act as a source or a sink, the field lines close on themselves, forming loops.

The last of Maxwell's equations is known as **Ampere's Law**. In its original form as expressed by Ampere, it related the number of magnetic field lines which passed through a surface formed by a closed loop to the total amount of current which was enclosed by the loop

$$\oint \vec{B}\cdot d\vec{l} = \mu\iint \vec{j}\cdot d\vec{S}\,, \qquad (1.15)$$

where $\vec{j}$ is known as the current density. The open surface is bounded by the loop, and the quantity μ is called the permeability of the medium. In a vacuum, $\mu = \mu_0$, where μ_0 is called the permeability of free space and has a value of $\mu_0 = 4\pi\times 10^{-7}\ \frac{\mathrm{N\cdot s^2}}{\mathrm{C^2}}$. We can relate the permeability of free space with the permeability via the equation

$$\mu = K_B\mu_0\,, \qquad (1.16)$$

where K_B is called the relative permeability. In a manner similar to the dielectric constant, the relative permeability can be viewed as a measurement of how well the magnetic field permeates a material.

While Ampere's law in its original formulation explained many important effects, such as the operation of a solenoid, it was found to also create larger problems. In particular, use of Ampere's law in the form of equation (1.15) led to violation of conservation of energy for the electric and magnetic fields. In order to correct this, Maxwell hypothesized the existence of an additional current, the **displacement current**, which is defined as

$$i_d = \varepsilon\iint \frac{\partial\vec{E}}{\partial t}\cdot d\vec{S}\,. \qquad (1.17)$$

When this is combined with Ampere's law in a region with no physical currents, we get

$$\oint \vec{B}\cdot d\vec{l} = \mu\varepsilon\frac{d\Phi_E}{dt}\,.$$

In other words, just as a time varying magnetic flux lead to the creation of a circular electric field, so too does a time varying electric flux lead to the creation of a linear magnetic field. If a physical current also exists, we again regain the last of Maxwell's equations.

3.2 Differential Form of Maxwell's Equation

In this section we derive the Maxwell's equations based of the differentiation form of a number of physical principles. Thus we recast Maxwell's Equations into a differential form. This form will be necessary later when we begin discussing the wave nature of light. In order to do this conversion, we first need two important results from vector calculus, **Gauss's divergence theorem** and **Stokes theorem**.
Gauss's divergence theorem tells us that the net flux of a vector field through a closed surface is equal to the integral of the divergence of that

field over the volume contained in the surface (i.e. conversion of integration over s closed Surface to Volume Integral)

$$\oiint \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{F} dV . \qquad (1.18)$$

Similarly, Stokes theorem states that the flux through a closed loop is equal the integral of the curl of the field over the area enclosed by the loop (integral over a closed curve to Surface integral)

$$\oint \vec{F} \cdot d\vec{l} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} . \qquad (1.19)$$

Let's start with Gauss's divergence theorem and apply it to the first two of Maxwell's equations. Then we get

$$\iiint \frac{\rho}{\varepsilon} dV = \oiint \vec{E} \cdot d\vec{S}$$
$$= \iiint \vec{\nabla} \cdot \vec{E} dV$$

and

$$0 = \oiint \vec{B} \cdot d\vec{S}$$
$$= \iiint \vec{\nabla} \cdot \vec{B} dV .$$

These relations must be equal for any volume, so the first two Maxwell's equations in MKSA system become

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon} \qquad (1.20)$$

and

$$\vec{\nabla} \cdot \vec{B} = 0 . \qquad (1.21)$$

From the above equations, (1.20) implies that electric charges whose density ρ is _ are the sources of the electric field $\vec{E}$, while (1.21) implies that Field lines of $\vec{B}$ are closed, which is equivalent to the statement that there are no magnetic monopoles(Figure 3a).

We shall now obtain the last two Maxwell's equations using the Stokes theorem. First, we discuss the Ampere's law. Ampere's law describes the fact that an electric current can generate an induced magnetic field. It states that in a stable magnetic field the integration along a magnetic loop is equal to the electric current the loop enclosed. Mathematically, Ampere's law can be expressed as:

$$\oint \vec{H} \cdot d\vec{l} = \vec{j} \qquad (1.22)$$

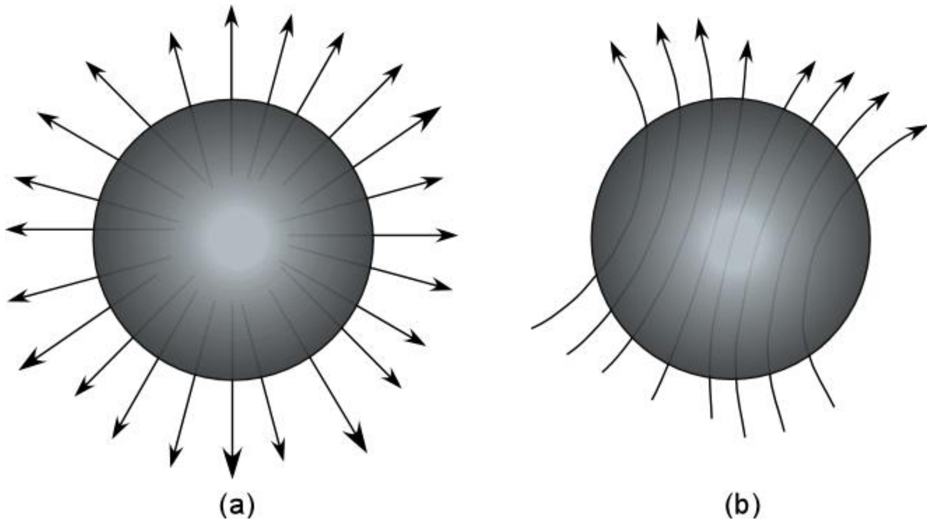

Fig. 1.3: Rotation-Free Vector Field (a) and Source-Free Vector Field (b)

Let us take a simple case to illustrate the Ampere's law, as shown in Figure 1.4. Recall that the curl of a vector field is defined as

$$Cur\vec{H} = \nabla \times \vec{H} = \lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot d\vec{l}}{\Delta S} \vec{n}. \qquad (1.23)$$

Consider the case of that the magnetic field is on the plane of the paper and the electric current is flowing out from the paper with the current normal to the paper we can have

$$Cur\vec{H} = \nabla \times \vec{H} = \lim_{\Delta S \to 0} \frac{\oint \vec{H} \cdot d\vec{l}}{\Delta S} \vec{n}. = \frac{\vec{j}}{\Delta S} = \vec{J} \qquad (1.24)$$

where $\vec{J}$ is the current density in an infinitesimal area. Meanwhile, if the electric field $\vec{E}$ is not stable, i.e., varying with respect to time, and the variation frequency is high enough and extends into the radar frequency, there will be another current in the medium known as the displacement current and is proportional to the variation of the electric field $\vec{E}$, and the proportional factor is the dielectric permittivity ε. Thus, there will be another contributor, $d\vec{D}$/dt, to induce the magnetic field $\vec{H}$. The displacement current works exactly the same way as the conductive current $\vec{J}$, so that the total current should be $\vec{J}$ + $d\vec{D}$/dt; put both contributors into the above equation ends up with the first equation of the Maxwell's equations:

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad (1.25)$$

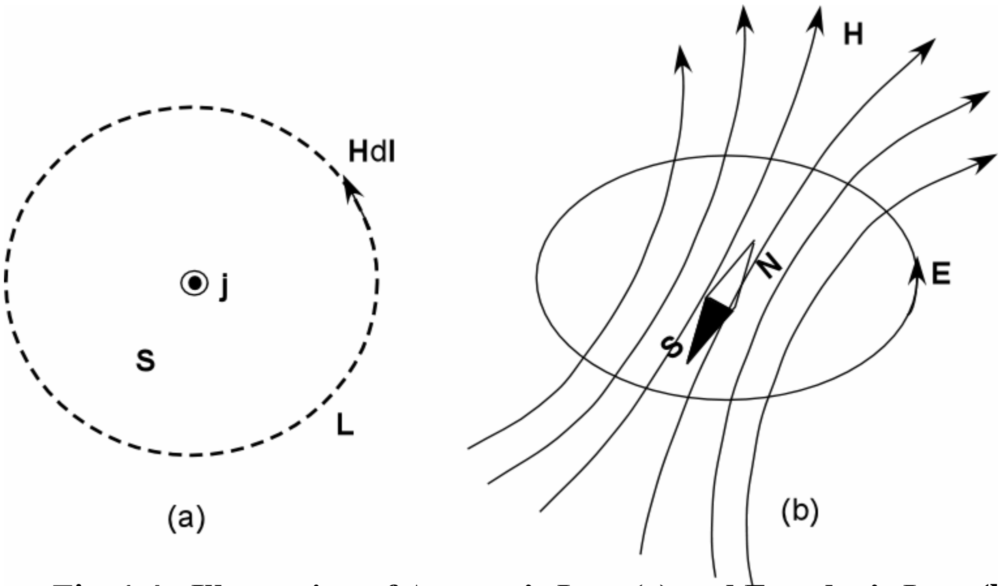

Fig. 1.4: Illustration of Ampere's Law (a) and Faraday's Law (b)

Second, we take a look of the Faraday's law. Faraday's law states that a moving magnet can generate an alternating electric field. Mathematically, the moving magnet can be represented by the variation of a vector magnetic potential ψ and the Faraday's law can be mathematically expressed as

$$\vec{E} = -\frac{\partial \vec{\Psi}}{\partial t} \tag{1.27}$$

by taking curl or cross product of both sides of the equation we have

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{\Psi}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\Psi}) = -\frac{\partial \vec{B}}{\partial t}. \tag{1.28}$$

These relations must hold for any surface bounded by a closed loop, so the last two Maxwell's equations become

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.29}$$

and

$$\vec{\nabla} \times \vec{B} = \mu\left(\vec{j} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \tag{1.30}$$

Within material media having polarisation $\vec{P}$ and magnetisation $\vec{M}$ the above laws still hold with the following replacements

$$\vec{\rho} \Rightarrow \vec{\rho} - \vec{\nabla} \times \vec{P}, \qquad \vec{j} \Rightarrow \mu\left(\vec{j} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \tag{1.31}$$

That is to the true charge density we have to add the polarisation charge density and to the true current density we have to add the contributions of the magnetisation current, the polarisation current and the displacement current introduced by Maxwell. In terms of the electric displacement and magnetic fields defined by

$$\vec{D} = \varepsilon \vec{E} + \vec{P}$$

and

$$\vec{H} = \frac{1}{\mu}\vec{B} - \vec{M}$$

Respectively, Maxwell equations can be brought into the following form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad M1$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad M2$$

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad M3$$

$$\vec{\nabla} \cdot \vec{B} = 0. \qquad M4$$

The set of equations Maxwell's equations expressed in terms of the derived field quantities $\vec{D}$ and $\vec{H}$ are called **Maxwell's macroscopic equations**. These equations are convenient to use in certain simple cases. Together with the boundary conditions and the constitutive relations, they describe uniquely (but only approximately) the properties of the electric and magnetic fields in matter.

In some materials (linear media) it happens that $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ where the quantities ε and μ are called the **dielectric constant and magnetic permeability** of the medium respectively.

3.3 The Continuity Equation

The electric charge is conserved. Actually we have never observed in the laboratory a violation of this conservation law. This conservation law is expressed by the following continuity equation

$$\vec{\nabla}.\vec{j} + \frac{\partial \vec{\rho}}{\partial t} = 0 \qquad \text{(M5)}$$

where j is the charge density and $\vec{j} = \rho \vec{u}$ is the current density. This equation follows from Maxwell's equations and it is not an independent hypothesis.

The quantity $\int_S \vec{j}.\vec{dS}$ S represents the charge flowing out of surface S per unit time (this is measured in Amperes in the system MKSA). If the charge density is time independent then from the continuity equation it follows that $\vec{\nabla}.\vec{j} = 0$. In this case we say that we have steady currents.

Remark

In this system the first three equations (M1) − (M3) are independent, equation (M4) has the character of an initial condition, and the continuity equation (M5) follows from (M2) and (M3). Indeed, equation (M1) implies that

$$\vec{B}\left(\vec{x},t\right) = \vec{B}\left(\vec{x},t_0\right) - \vec{\nabla}\times\int_{t_0}^{t} \vec{E}(\vec{x},\tau)d\tau,$$

where

$$\vec{\nabla}.\vec{B}\left(\vec{x},t\right) = \vec{\nabla}.\vec{B}\left(\vec{x},t_0\right),$$

so that equation (M4) holds for all *t* if it holds at some fixed (say, initial) time *to* .

Similarly, the continuity equation (M5) follows by taking the divergence of (M2) and by applying (M3). In its turn, equation (M3) can be used to eliminate the unknown ρ by defining the electric volume charge density in terms of $\vec{D}$ as

$$\rho := \vec{\nabla}.\vec{D}$$

In this way, the Maxwell system reduces to the two vector equations (M1), (M2) , valid in any material medium, conducting or non-conducting, for the five unknown vector functions $\vec{E}, \vec{D}, \vec{B}, \vec{H}, \vec{J}$ of (*x*, *t*). These two vector equations are complemented by three additional vector relations, called constitutive equations, and so the count is right. These constitutive relations are not universally valid but depend upon the properties of the materials under consideration. We can assume to start with that they have the form of local relations

$$\vec{J} = \vec{J}(\vec{E}, \vec{H})$$
$$\vec{D} = \vec{D}(\vec{E}, \vec{H})$$
$$\vec{B} = \vec{B}(\vec{E}, \vec{H})$$

and in fact for many purposes we will take the very simple linear constitutive relations

$$\vec{J} = \gamma \vec{H} \quad \text{(Ohm's law)} \qquad \text{(C1)}$$

$$\vec{D} = \varepsilon \vec{E} \qquad \text{(C2)}$$

$$\vec{B} = \varepsilon \vec{H} \qquad \text{(C3)}$$

where $\gamma = \gamma(x) \geq 0$, is the electric conductivity, γ^{-1} the resistivity, $\varepsilon = \varepsilon(x) \geq \varepsilon_0 > 0$ is the electric permittivity and $\mu = \mu(x) \geq 0$ the magnetic permeability of the material. These relations apply to empty space with $\varepsilon = \varepsilon_0, \mu = \mu_0, \gamma = 0$ and the more common materials can be classified according to the values of the scalar coefficients $\varepsilon, \mu_0, \gamma = 0$ as follows:

$$\begin{cases} \gamma = 0 : \text{dielectrics} \\ 0 < \gamma < \infty : \text{conductors} \\ \gamma = +\infty : \text{perfect conductors} \end{cases}$$

$$\begin{cases} \mu > \mu o : \text{paramagnetic bodies} \\ 0 < \mu < \mu o : \text{diamagnetic bodies} \\ \mu = 0 : \text{superconductors} \end{cases}$$

Where ε_0, μ_0 are the (constant) permittivity and permeability of empty space.

We will exclude in the sequel the case of superconductors 8 and we will always assume that there exists $\bar{\mu} > 0$ such that $\mu(x) = \bar{\mu} > 0$.
For homogeneous media the coefficients γ, ε and μ are constant. They depend on physical parameters such as temperature: for example, the conductivity of metals decreases with increasing temperature.

4.0 CONCLUSION

This unit further explains the role of Maxwell's equations in media. It has been shown that while the integral form of Maxwell's equations serves as a veritable tool for the analysis of electromagnetic waves, the differential form of Maxwell's equation is often a more natural and more useful tool when investigating wave properties in the physical domain.

The continuity equation shows us that Maxwell’s equations do not contradict the laws of conservation.

5.0 SUMMARY

Maxwell’s equations can be applied in either the integral form or the differential form and Maxwell’s equations resonate with the principle of conservation of energy.

6.0 TUTOR-MARKED ASSIGNMENT

1. Write down Maxwell’s equations in both integral and differential form?
2. What does the continuity equation state?
3. Relate Gauss's law to Maxwell’s equations.
4. Which of Maxwell’s equations is known as Faraday's law?
5. Explain Ampere's law. What role does it play in electromagnetic theory?

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MODULE 2

UNIT 1 ELECTROMAGNETIC WAVE EQUATION AND THEORY OF LIGHT

CONTENTS

1.0 INTRODUCTION

Electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum where the homogeneous form of the equation can be written in terms of either the electric field E or the magnetic field B.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- establish the relationship between electromagnetic wave and light
- state the wave equation for electric field vector $\vec{E}$
- describe the wave equation for the magnetic field vector $\vec{B}$
- recognise the three dimensional nature of wave equation for each component of the electric and magnetic fields.

3.0 MAIN CONTENT

3.1 Electromagnetic Wave Equation and Theory of Light

A common question is, how are Maxwell's equations used to show wave motion? Consider the electric and magnetic fields in a charge free vacuum region. Then Maxwell's equations become

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}(x,t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad .$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

3.2 Wave Equation for $\vec{E}$

To derive the wave equation for the electric field, start with the third of Maxwell's equations and take the curl of both sides

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) . \qquad (1.32)$$

The left hand side can be simplified by using the vector relationship

$$\vec{a} \times \vec{b} \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b}) \qquad (1.33)$$

to get

$$\begin{aligned} \vec{\nabla} \times \vec{\nabla} \times \vec{E} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} \\ &= -\nabla^2 \vec{E} \end{aligned} \qquad (1.34)$$

where the last step used the fact that $\vec{\nabla} \cdot \vec{E} = 0$. To evaluate the right hand side of (1.32), we start with the fact that the spatial derivatives (∇) and the time derivative can be interchanged. We then use the last of Maxwell's equations to find

$$\begin{aligned} -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} &= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \\ &= -\frac{\partial}{\partial t}\left(\mu_0 \vec{j}(x,t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\ &= -\mu_0 \frac{\partial}{\partial t}\left(\vec{j}(x,t) + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\ &= -\mu_0 \frac{\partial}{\partial t}\left(\sigma \vec{E} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \end{aligned} \qquad (1.35)$$

Combining (1.34) and (1.35), (1.32) on rearrangement can be written as

$$-\nabla^2\vec{E} = -\mu_0\sigma\frac{\partial \vec{E}}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2},$$

$$\Rightarrow \qquad (1.36)$$

$$\nabla^2\vec{E} - \mu_0\sigma\frac{\partial \vec{E}}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} = 0,$$

which we recognise as the three dimensional wave equation for each component of the electric field ($\vec{E}$). Comparing (1.36) with the standard result for a wave whose velocity is v, we obtain

$$\begin{aligned} v &= \frac{1}{\sqrt{\mu_0\varepsilon_0}} \\ &= \frac{1}{\sqrt{\left(4\pi\times10^{-7}\ \frac{\mathrm{m\cdot kg}}{\mathrm{C}^2}\right)\left(8.85\times10^{-12}\ \frac{C^2}{J\cdot m}\right)}}. \qquad (1.37) \\ &= 3.00\times10^{8}\ \tfrac{\mathrm{m}}{\mathrm{s}}. \end{aligned}$$

Using the fact that the experimentally determined speed of light is also 3.00 x 10^8 m/s, we are lead to the inescapable conclusion that light is just one form of electromagnetic wave propagation. When the electromagnetic disturbance is moving in a vacuum, we denote its speed by a special symbol, *c*.

3.3 Wave Equation for $\vec{B}$

In a manner similar to those leading to equation (1.36), we can start with the last of Maxwell's equations to find the wave equation for the magnetic field. Thus,

$$\begin{aligned} \vec{\nabla}\times\vec{\nabla}\times\vec{B} &= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{B} \\ &= -\nabla^2\vec{B} \\ &= \vec{\nabla}\times\left(\mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \\ &= \mu_0\sigma\left(\vec{\nabla}\times\vec{E}\right) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right) \qquad (1.38) \\ &= -\mu_0\sigma\left(\frac{\partial\vec{B}}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{B}}{\partial t}\right). \end{aligned}$$

Finally, we have

$$\nabla^2 \vec{B} - \mu_0 \sigma \left(\frac{\partial \vec{B}}{\partial t} \right) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0, \qquad (1.39)$$

$$\nabla^2 \vec{B} - \mu_0 \sigma \left(\frac{\partial \vec{B}}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0.$$

This is the wave equation for the magnetic field. We notice that it is exactly the same form as the wave equation for the electric field equation (1.36).

4.0 CONCLUSION

Light waves are electromagnetic which comprise electric and magnetic vector fields perpendicular to the direction of propagation which implicitly is a three dimensional construct.

5.0 SUMMARY

Light is transverse electromagnetic waves with both electric and magnetic vectors.

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that light is electromagnetic waves?
2. What is the orientation of the electric and magnetic field vectors of an electromagnetic wave relative to its direction of propagation?
3. Are light waves one, two or three dimensional?
4. Is gravity related in any way with electromagnetism?
5. At what speed does electromagnetic wave travel through vacuum?

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UNIT 2 LIGHTS AS TRANSVERSE WAVES

CONTENTS

1.0 INTRODUCTION

Light waves consist of electric and magnetic field oscillations occurring at right angle to the direction of energy transfer which qualifies light as transverse wave; and for transverse waves in matter the displacement of the medium is perpendicular to the direction of propagation of the wave.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- establish conclusively that light waves are transverse waves
- know the ratio referred to as relaxation time of conducting medium
- understand propagation of plane electromagnetic waves in non-conducting media
- describe the relationship between the wave number and the amplitudes of electric and magnetic components of electromagnetic waves
- investigate the mechanism guiding the propagation of plane electromagnetic waves in conducting media
- see why electric charges move almost instantly to the surface of perfect conductors when subjected to electromagnetic influence (skin effect).

3.0 MAIN CONTENT

3.1 Lights as Transverse Waves

We can also determine whether light waves are longitudinal or transverse waves. Remember that longitudinal waves oscillate in the same direction as the direction of propagation, while transverse waves oscillate in a direction perpendicular to the direction of propagation. For simplicity, let the direction of propagation be in the *x* direction. Then $\vec{E} = \vec{E}\ (x,t)$. Now look at a Gaussian box oriented along the coordinate axes. The flux is through the faces in the *y*-*z* planes, so Gauss's law becomes

$$\frac{\partial E_x}{\partial x} = 0$$

From this, we see that the electromagnetic wave has no electric field component in the direction of propagation. Thus, the electric field is exclusively transverse. A similar argument can be used on Gauss's law for magnetic fields to show that it is also transverse to the direction of propagation. In particular, Faraday's law tells us that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \qquad (1.40)$$

In other words, the time dependent magnetic field can only have a component in the *z* direction when the electric field is exclusively in the *y* direction. From these, we see that, **in free space, the plane electromagnetic wave is transverse**.

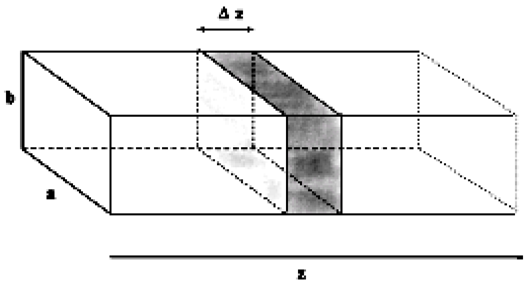

Fig. 2.1: A Rectangular Wave Guide

3.2 Plane Electromagnetic Waves in Non-Conducting Media $(\sigma = 0)$

In a medium with values ε, μ for the electric constant and the magnetic permeability respectively, we have derived the Maxwell laws in (1.32) and (1.39) as

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \sigma \frac{\partial \vec{E}}{\partial t},$$
$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \sigma \left(\frac{\partial \vec{B}}{\partial t}\right). \qquad (1.41)$$

In region where there are no charge and current distributions, the terms on the right hand sides of (1.41) are absent and the fields $\vec{E}$ and $\vec{B}$ satisfy the free wave equations.
The plane waves are particular solutions of (1.41) in regions where sources are absent. In the following we shall use complex notation and write the electric component of a plane wave as

$$\vec{E} = \vec{E}_0 \exp i(\vec{k}.\vec{x} - \omega t). \qquad (1.42)$$

The physical electric field measured in the laboratory is meant to be the real part of this expression. That is $\vec{E} = \vec{E}_0 \cos(\vec{k}.\vec{x} - \omega t)$. A similar expression holds for the magnetic field too with $\vec{E}, \vec{E}_0$ replaced with $\vec{B}, \vec{B}_0$ respectively. In this expression $\vec{E}_0$ is the amplitude of the electric field, $\vec{k}$ is its *wave vector* and ω ! its *frequency*. This monochromatic pulse is a solution when the frequency is linearly related to the magnitude $k \equiv |\vec{k}|$ of the wave vector $\vec{k}$, by the relationship $\omega = vk$, k is called the *wave number* and is related to the *wave length* by the relation $k = \frac{2\pi}{\lambda}$.

Using Gauss' law $\vec{\nabla} \cdot \vec{E} = 0$, and the Faraday's law, $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, one can immediately arrive at the following relations for the wave number and the amplitudes of the electric and magnetic components:

$$\vec{k}.\vec{E}_0 = 0,, \quad \vec{B}_0 = \frac{1}{\omega} \vec{k} \times \vec{E}_0. \qquad (1.43)$$

Equation (1.35) state that the electric and magnetic fields of a plane wave are perpendicular to each other and both perpendicular to the direction of the propagation $\vec{n} = \frac{1}{k} \vec{k}$.

3.3 Plane Electromagnetic Waves in Conducting Media $(\sigma \neq 0)$

Within a conductor the electric current density and the electric field are related by $\vec{j} = \sigma \vec{E}$, from which it follows that $\nabla . \vec{j} = \sigma \nabla . \vec{E} = \frac{\sigma}{\varepsilon} \rho$. Then from the continuity equation one has

$$\vec{\nabla}.\vec{j}+\frac{\partial \vec{\rho}}{\partial t}=\frac{\partial \vec{\rho}}{\partial t}+\frac{\sigma}{\varepsilon}\rho=0,$$

which is immediately solved to yield

$$\rho(\vec{x},t)=\rho(\vec{x},0)\exp\left(-\frac{\sigma}{\varepsilon}t\right) \tag{1.44}$$

For good conductors $\frac{\sigma}{\varepsilon}\approx 10^{14}\sec^{-1}$ 1 so that from the eqn. (1.36) we conclude that charges move almost instantly to the surface of the conductor. The ratio $\tau=\frac{\varepsilon}{\sigma}$ is called ***the relaxation time*** of the conducting medium. For perfect conductors, $\sigma=\infty$, so that the relaxation time is vanishing. For good, but not perfect conductors τ _ is small and of the order of 10^{-14} sec or so. For times much larger than the relaxation time there are practically no charges inside the conductor. All of them have moved to its surface where they form a charge density Σ. Within a conductor the wave equation for the vector field $\vec{E}$, see equation (1.41), becomes

$$\left(\nabla^2-\mu\ \varepsilon\ \frac{\partial^2}{\partial t^2}-\mu\sigma\frac{\partial}{\partial t}\right)\vec{E}=0 .$$

Notice the appearance of a "friction" term $\frac{\partial}{\partial t}$ which was absent in the free wave equation. If we seek for monochromatic solutions of the form $\vec{E}=\vec{E}(\vec{x})\exp(-i\omega t)$, then the equation above takes on the form

$$\left(\nabla^2+K^2\right)\vec{E}(\vec{x})=0,$$

where $K^2=\mu\omega(\omega\varepsilon+i\sigma)$. This can be immediately solved to yield, for a plane wave solution travelling along an arbitrary direction $\vec{n}$ ~n,

~E

= ~E

$$\vec{E}=\vec{E}(\vec{x})\exp\{i(\alpha\xi-\omega t)-\beta t\} \tag{1.45}$$

where $\xi=\vec{n}.\vec{x}$. The constants α,β, appearing in (1.45), have dimensions of $length^{-1}$ and are functions of σ. Their analytic expressions are not presented here. These can be traced in any standard book of electromagnetic theory. However we can distinguish two particular cases in which their forms are simplified a great deal. These regard the case of an isolator and the case of a very good conductor respectively.

For an insulator $\sigma = 0$ and $\alpha = k, \beta = 0$ In this case (1.45) reduces to an ordinary plane wave which is propagating with wave vector $\vec{k} = n\vec{k}$.

For a very good conductor, and certainly this includes the case of a perfect conductor, the conductivity is large so that the range of frequencies with $\sigma >> \varepsilon\omega$ is quite broad. In this case the constants α, β are given by $\alpha \approx \beta \approx \delta^{-1}$, where δ is a constant called the *Skin Depth* , given by the following expression

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \qquad (1.46)$$

Therefore we see from equation (1.45) that inside a good conductor:
The field is attenuated in the direction of the propagation and its magnitude decreases exponentially $\sim \exp\left(-\frac{\xi}{\delta}\right)$ as it penetrates into the conductor. The depth of the penetration is set by δ Æ and is smaller the higher the conductivity, the higher the permeability and the frequency.
As an example for copper $\sigma = 5.8 \times 10^7$ mho m^{-1} and the skin depth is $\delta \approx 0.7 \times 10^{-3} cm$ for a frequency $\omega = 100 MHz$. It is important to point out that the magnetic field within the conductor is related to the electric field by the relation

$$\vec{H} = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\sigma}{\mu\omega}}\,\vec{n} \times \vec{E} \ . \qquad (1.47)$$

As in the case of non-conducting materials both $\vec{E}$, $\vec{H}$ are perpendicular to each other and to the direction of propagation $\vec{n}$. From (1.47), it is evident that the magnetic field has a phase difference of 45^0 from its corresponding electric component $\vec{E}$, due to the pre-factor $1+i$.

4.0 CONCLUSION

The propagation of light waves is affected by the medium through which it propagates. It is transverse by nature with plane electromagnetic waves in non-conducting media possessing $\sigma = 0$ while plane electromagnetic waves in conducting media have a . $\sigma \neq 0$

5.0 SUMMARY

A parametric constant σ associated with material bear a distinct relationship with the propagation of light in different media, ranging from zero in non conducting media to a non-zero value in conducing media.

6.0 TUTOR-MARKED ASSIGNMENT

1. "Light waves are transverse waves." Explain.
2. What do you understand by "relaxation time" of a conducting medium?
3. Describe how plane electromagnetic waves are affected by perfectly non-conducting, and in perfectly conducting media.

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UNIT 3 REFLECTIONS AND REFRACTION OF PLANE BOUNDARY OF PLANE WAVES

CONTENTS

1.0 INTRODUCTION

In this unit we are going to take a close look at the behaviour of plane electromagnetic waves at the boundaries between media. At boundaries, electromagnetic wave is either reflected or refracted and in this unit we shall be looking at how conducting and non-conducting media affect the propagation of plane electromagnetic waves.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- qualify the phenomenon of reflection and refraction of plane waves at boundaries
- distinguish between reflection and refraction at boundary between dielectrics and at the surface of conductors
- solve problems involving electromagnetic boundary conditions
- derive the laws of reflection and refraction
- analyse the characteristics of monochromatic plane wave on a boundary
- explain polarisation by reflection on a boundary between two dielectrics

- comfortably work with reflection coefficient.

3.0 MAIN CONTENT

3.1 Reflection and Refraction of Plane Boundary of Plane Waves

In reality, plane electromagnetic waves frequently encounter obstacles along their propagation paths: hills, buildings, metallic antennas aimed at receiving the messages the waves carry, objects from which they are supposed to partly reflect. In such cases, the wave induces conduction currents in the object (if the object is metallic), or polarisation current (if the object is made of an insulator). These current are, of course, sources of a secondary electromagnetic field. This field is known as scattered field, and the process that creates it is known as scattering of electromagnetic waves. The objects or obstacles are called scatters.

When plane waves are incident on a boundary between different media, some energy crosses the boundary, and some is reflected. In other words, when a plane electromagnetic is incident on a planar boundary between two media, one of these waves is radiated back into the half-space of the incident wave: this wave is known as the reflected wave. There is also a wave in the other half-space (except in the case of a perfect conductor), propagating generally in a different direction from the incident wave; it is therefore called the refracted or transmitted wave. We define transmission and reflection coefficients to quantify the transmission and reflection of wave energy. These coefficients are properties of the two media. The transmission and reflection coefficients are determined by matching the electric and magnetic fields in the waves at the boundary between the two media.

In this session, for easy understanding, we shall consider:

- Boundary conditions on electric and magnetic fields.
- Boundary conditions on fields at the surfaces of conductors.
- Monochromatic plane wave on a boundary:
    - Directions of reflected and transmitted waves (laws of reflection and refraction)
    - Amplitudes of reflected and transmitted waves (Fresnel's equations)
    - The special case of a boundary between two dielectrics
    - The special case of the surface of a conductor
- Monochromatic plane wave:
    - Total internal reflection
- Reflection coefficient for a conducting surface.

3.2 Electromagnetic Boundary Conditions 1: Normal Component of $\vec{B}$

We can use Maxwell's equations to derive the boundary conditions on the magnetic field across a surface. Electromagnetic shows that the normal component of current, electric displacement, and magnetic induction should be continuous when cross a material interface or boundary; while the tangential component of the electric field and the magnetic field should be continuous across the material interface. Let us take the magnetic boundary condition as the example to illustrate the calculation. From the Gaussian theorem we have

$$\iiint \vec{\nabla} \cdot \vec{B} dV = \oiint \vec{B} \cdot d\vec{S}.$$

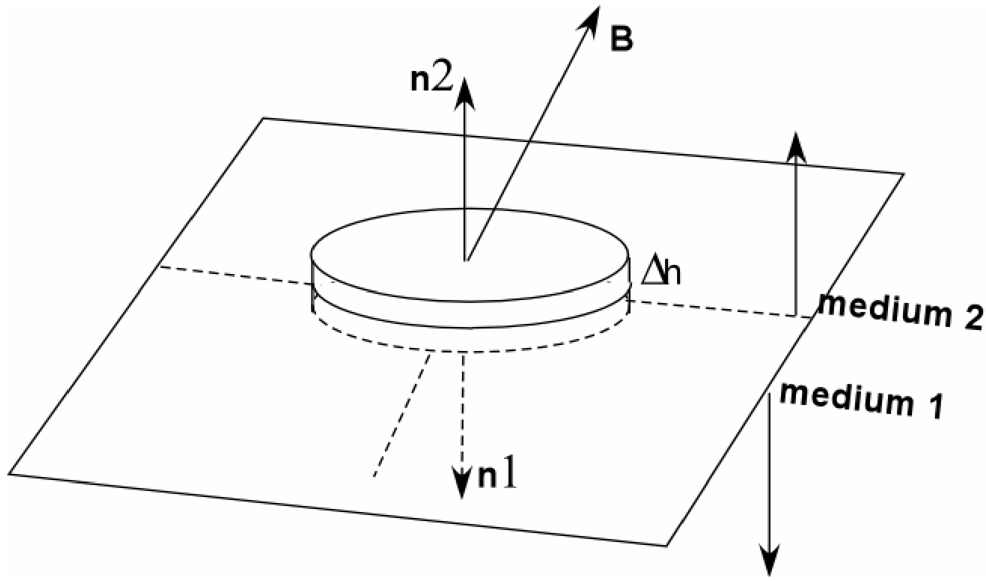

Fig.3.1: Illustration of the Electromagnetic Boundary Conditions

By making a small disc with the thickness of Δh and its central line is coincident with the boundary of two media (Figure 6) we have

$$\iiint \vec{\nabla} \cdot \vec{B} dV = 0.$$

This coincides with the Maxwell's equation (M4):

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (2.1)$$

integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \vec{\nabla} \cdot \vec{B} dV = \oint_S \vec{B} . \vec{dS} = 0 \qquad (2.2)$$

where V is the volume of the pillbox, and S is its surface. We can break the integral over the surface into three parts: over the flat ends (S1 and S2) and over the curved wall (S3):

$$\int_{S_1} \vec{B} . \vec{dS} + \int_{S_2} \vec{B} . \vec{dS} + \int_{S_3} \vec{B} . \vec{dS} = 0 \qquad (2.3)$$

In the limit that the length of the pillbox approaches zero, the integral over the curved surface also approaches zero. If each end has a small area A, then equation (2.3) becomes:

$$B_{1n}A + B_{2n}A = 0 \quad (2.4)$$

or

$$B_{1n} = B_{2n} \quad (2.5)$$

In other words, the normal component of the magnetic field $\vec{B}$ must be continuous across the surface. By using similar approaches, the general conditions on electric and magnetic fields at the boundary between two materials can be summarised as follows:

Boundary Condition	Derived from…	Applied to…
$B_{1n} = B_{2n}$	$\vec{\nabla} \cdot \vec{B} = 0$	pillbox
$E_{2t} = E_{1t}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Loop
$D_{2n} - D_{1n} = \rho$	$\vec{\nabla} \cdot \vec{D} = \rho$	Pillbox
$H_{2t} - H_{1t} = -J$	$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{D}$	Loop

Static electric fields cannot persist inside a conductor. This is simply because the free charges within the conductor will re-arrange themselves to cancel any electric field; this can result in a surface charge density, ρ. We have seen that electromagnetic waves can pass into a conductor, but the field amplitudes fall exponentially with decay length given by the skin depth, δ:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \quad (2.6)$$

As the conductivity increases, the skin depth gets smaller. Since both static and oscillating electric fields vanish within a good conductor, we can write the boundary conditions at the surface of such a conductor:

$$E_{1t} \approx 0, E_{2t} \approx 0$$

$$D_{1n} \approx -\rho, D_{2n} \approx 0$$

Lenz's law states that a changing magnetic field will induce currents in a conductor that will act to oppose the change. In other words, currents are induced that will tend to cancel the magnetic field in the conductor. This means that a good conductor will tend to exclude magnetic fields. Thus the boundary conditions on oscillating magnetic fields at the surface of a good conductor can be written:

$$B_{1n} \approx 0, B_{2n} \approx 0$$

$$H_{1t} \approx -J, H_{2t} \approx 0.$$

We can consider an "ideal" conductor as having infinite conductivity. In that case, we would expect the boundary conditions to become:

$$B_{1n} = 0, B_{2n} = 0$$
$$E_{1t} = 0, E_{2t} = 0$$
$$D_{1n} = -\rho, D_{2n} = 0$$
$$H_{1t} = J, H_{2t} = 0.$$

Strictly speaking, the boundary conditions on the magnetic field apply only to oscillating fields, and not to static fields. But it turns out that for superconductors, static magnetic fields are excluded as well as oscillating magnetic fields. This is not expected for classical "ideal" conductors.

3.3 Waves on Boundaries

We now apply the boundary conditions to an electromagnetic wave incident on a boundary between two different materials. We shall use the boundary conditions to derive the properties of the reflected and transmitted waves, for a given incident wave. Consider a monochromatic wave incident at some angle on a boundary. We must consider three waves: the incident wave itself; the reflected wave, and the transmitted wave on the far side of the boundary.

The electric field components for these waves can be written (respectively):

$$\vec{E}_I(\vec{r},t) = \vec{E}_{0I}\, e^{i(\omega_I t - \vec{k}_I . \vec{r})} \qquad (2.7)$$

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R}\, e^{i(\omega_R t - \vec{k}_R . \vec{r})} \qquad (2.8)$$

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T}\, e^{i(\omega_T t - \vec{k}_T . \vec{r})} \qquad (2.9)$$

Let us first consider the time dependence of the waves. The boundary conditions must apply at all times: for example, the tangential component of the electric field, $\vec{E}_t$ must be continuous across the boundary at all points on the boundary at all times. This means that all waves must have the same time dependence, and therefore:

$$\omega_I = \omega_R = \omega_T = \omega \qquad (2.10)$$

Reflection at a boundary cannot change the frequency of an incident monochromatic wave. Some surfaces reflect some wavelengths better than others, which is why they can appear coloured under white light; but the frequency of the light does not change.

3.4 Laws of Reflection and Refraction

Now let us consider the relationships between the directions in which the waves are moving. We shall find that these relationships are just the laws of reflection and refraction that we are familiar with from basic optics. However, our goal is now to derive these laws from Maxwell's equations, by applying the boundary condition waves across boundaries. We start from the fact that the boundary conditions must be satisfied at all points on the boundary. This means that the waves must all change phase in the same way as we move from one point to another on the boundary. Since the phase of each of the waves at a position $\vec{r}$ is given by $\vec{k}.\vec{r}$, where $\vec{k}$ is the appropriate wave vector, we must have:

$$\vec{k}_I.\vec{p} = \vec{k}_R.\vec{p} = \vec{k}_T.\vec{p}, \qquad (2.11)$$

where $\vec{p}$ is any point on the boundary.

For simplicity, let us choose our coordinates so that the boundary lies in the plane z = 0. Then any point $\vec{p}$ on the boundary can be written:

$$\vec{p} = (x, y, 0) \qquad (2.12)$$

Now we can (without loss of generality) further specify the coordinate system so that $\vec{k}_I$ lies in the x − z plane, i.e. the y component of $\vec{k}_I$ is zero:

$$\vec{k}_I = (k_I \sin\theta_{I,} 0, k_I \cos\theta_I) \qquad (2.13)$$

where θ_I is the angle between the direction of travel of the incident wave and the boundary.
Now let us apply equation (2.11):

$$\vec{k}_I.\vec{p} = \vec{k}_R.\vec{p} = \vec{k}_T.\vec{p}$$

to points on the boundary with x = 0, i.e. $\vec{p} = (0, y, 0)$. We find:

$$k_{Iy} = k_{Ry} = k_{Ty}. \qquad (2.14)$$

Therefore, the directions of the incident, reflected and transmitted waves all lie in the plane y = 0. Now let us consider points on the boundary with y = 0, i.e. $\vec{p} = (x, 0, 0)$. This time, using equation (2.11) gives:

$$k_{Ix} = k_{Rx} = k_{Tx} = k_I \sin\theta_I \qquad (2.15)$$

which (since the vertical components of the wave vectors are all zero) can be written:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T. \qquad (2.16)$$

But since the incident and reflected waves are travelling in the same material with the same frequency, the magnitudes of the wave vectors must be the same:

$$k_I = k_R \,. \tag{2.17}$$

Combining equations (2.15) and (2.16) we find:

$$\theta_I = \theta_R \qquad \text{(The law of reflection)} \tag{2.18}$$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \qquad \text{The law of refraction (Snell's law)} \tag{2.19}$$

3.5 Reflection and Refraction at a Boundary between Dielectrics

As an example, consider a monochromatic wave incident on a boundary between two dielectrics (e.g. air and glass). Since the conductivity is zero on both sides of the boundary, the wave vectors of all waves must be real.

Also, we have:

$$\frac{\omega}{k_I} = v_1 \,, \ \frac{\omega}{k_T} = v_2 \tag{2.20}$$

Where v_1 is the phase velocity in medium 1, and v_2 is the phase velocity in medium. 2.

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} \tag{2.21}$$

We define the refractive index n of a material as the ratio of the speed of light in a vacuum to the speed of light in the material:

$$n = \frac{c}{v} \tag{2.22}$$

Then equation (2.21) can be written:

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \tag{2.23}$$

This is the familiar form of Snell's law.

3.6 Reflection and Refraction at the Surface of a Conductor

For a wave incident on a conductor, $\vec{k}_T$ will be complex:

$$\vec{k}_T = \vec{\alpha} - i\,\vec{\beta} \tag{2.24}$$

For a good conductor (i.e. $\sigma >> \omega\varepsilon_2$):

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}} \tag{2.25}$$

so:

$$k_T = \sqrt{\vec{\alpha}^2 + \vec{\beta}^2} = \sqrt{\omega\mu_2\sigma_2} \tag{2.26}$$

Applying the law of refraction (2.19):

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \approx \sqrt{\frac{\sigma_2}{\omega\varepsilon_1}} >> 1 \tag{2.27}$$

where we have assumed that $\mu_2 \approx \mu_1$. Since the largest value of $\sin\theta_I$ is 1, equation (2.27) tells us that $\sin\theta_T \approx 0$, so the direction of the transmitted wave in a good conductor must be (close to the) normal to the surface.

SELF-ASSESSMENT EXERCISE

i. Derive (from Maxwell's equations) the boundary conditions on electric and magnetic fields at the interface between two media.
ii. Apply the boundary conditions on electric and magnetic fields to derive the laws of reflection and refraction.
iii. Energy and Momentum.

We shall use Maxwell's macroscopic equations in (M1, M2, M3, M4), on the energy and momentum of the electromagnetic field and its interaction with matter.

4.0 CONCLUSION

We have seen in this unit that at plane boundaries, plane waves can either be reflected or refracted and that the reflection or refraction satisfies electromagnetic boundary conditions specific to the boundary media interface.

The behaviour of electromagnetic plane waves at these boundary conditions is subject to the laws of reflection and refraction while the boundaries may be dielectric or conductive.

5.0 SUMMARY

Plane electromagnetic waves are affected by boundaries in a manner subject to the nature of the boundary material.

6.0 TUTOR-MARKED ASSIGNMENT

1. State any two important laws which govern the propagation of plane light waves at the plane boundary between two media.
2. What does Fresnel's Law state?
3. Derive an expression for Snell's Law.

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MODULE 3

UNIT 1 ENERGY THEOREM IN MAXWELL’S THEORY

CONTENTS

1.0 INTRODUCTION

Electromagnetic waves have associated energy and momentum and we shall be considering both Maxwell’s energy and momentum theorems in explaining these phenomena.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the energy theorem in Maxwell’s electromagnetic theory
- state why the energy theorem is also known as Poynting’s theorem
- quantify the momentum theorem in Maxwell's electromagnetic theory
- explain the term electric volume force
- write an expression for the Maxwell stress tensor
- analyse relative electric permittivity.

3.0 MAIN CONTENT

3.1 The Energy Theorem in Maxwell's Theory

Scalar multiplying (M1) by $\vec{H}$, (M2) by $\vec{E}$ and subtracting, we obtain

$$\vec{H}\cdot(\nabla\times\vec{E})-\vec{E}\cdot(\nabla\times\vec{E})=\nabla\cdot(\vec{E}\times\vec{H})$$

$$=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{E}\cdot\vec{j}-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \qquad (6.24)(3.1)$$

$$=-\frac{1}{2}\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)-\vec{j}\cdot\vec{E}$$

Integration over the entire volume V and using Gauss' theorem (the divergence theorem), we obtain

$$-\frac{\partial}{\partial t}\int_V\frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)d^3x'=\int_V\vec{j}\cdot\vec{E}d^3x'+\int_A(\vec{E}\times\vec{H})\cdot\vec{n}\,d^2x' \qquad (6.25)(3.2)$$

But, according to Ohm's law in the presence of an electromotive force field, the linear relationship between the current and the electric field is

$$\vec{j}=\sigma\left(\vec{E}+\vec{E}^{EMF}\right) \qquad (6.26)(3.3)$$

which means that

$$\int_V\vec{j}\cdot\vec{E}d^3x'=\int_V\frac{j^2}{\sigma}d^3x'-\int_V\left(\vec{j}\cdot\vec{E}^{EMF}\right)d^3x' \qquad (6.27)(3.4)$$

Inserting this into equation (3.2)

$$\int_V\vec{j}\cdot\vec{E}d^3x'=\int_V\frac{j^2}{\sigma}d^3x'+\frac{\partial}{\partial t}\int_V\frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)d^3x'+\int_A(\vec{E}\times\vec{H})\cdot\vec{n}\,d^2x'$$

(6.28)(3.5)

i.e.

Applied electric power = Joule heat + Field energy + Radiated power

Which is the **energy theorem in Maxwell's theory** also known as **Poynting's theorem**?

It is convenient to introduce the following quantities:

$$U_e = \frac{1}{2}\int_V \vec{E}\cdot\vec{D}\,d^3x'$$

$$U_m = \frac{1}{2}\int_V \vec{H}\cdot\vec{B}\,d^3x' \qquad (3.6)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

where Ue is the **electric field energy**, Um is the **magnetic field energy**, both measured in J, and $\vec{S}$ is the **Poynting vector (power flux)**, measured in W/m^2.

3.2 The Momentum Theorem in Maxwell's Theory

We now investigate the momentum balance (force actions) in the case that a field interacts with matter in a non-relativistic way. For this purpose we consider the force density given by the Lorentz force per unit volume $\rho\vec{E}+\vec{j}\times\vec{B}$

Using Maxwell's equations (M1-M4) and symmetrising, we obtain

$$\rho\vec{E}+\vec{j}\times\vec{B} = (\nabla\cdot\vec{D})\vec{E}+\left(\nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}\right)\times\vec{B}$$

$$= (\nabla\cdot\vec{D})\vec{E}+(\nabla\times\vec{H})\times\vec{B}-\frac{\partial\vec{D}}{\partial t}\times\vec{B}$$

$$= (\nabla\cdot\vec{D})\vec{E}-\vec{B}\times(\nabla\times\vec{H})-\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)+\vec{D}\times\frac{\partial\vec{B}}{\partial t} \qquad (3.7)$$

$$= (\nabla\cdot\vec{D})\vec{E}-\vec{B}\times(\nabla\times\vec{H})-\frac{\partial}{\partial t}(\vec{D}\times\vec{B})-\vec{D}\times(\nabla\times\vec{E})+\vec{H}(\nabla\cdot\vec{B})$$

$$= \left[(\nabla\cdot\vec{D})\vec{E}-\vec{D}\times(\nabla\times\vec{E})\right]+\left[(\nabla\cdot\vec{B})\vec{H}-\vec{B}\times(\nabla\times\vec{H})\right]$$

$$= -\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)$$

One verifies easily that the ith vector components of the two terms in square brackets in the right hand member of (3.7) can be expressed as

$$\left[(\nabla\cdot\vec{D})\vec{E}-\vec{D}\times(\nabla\times\vec{E})\right]_i = \frac{1}{2}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial x_i}-\vec{D}\cdot\frac{\partial\vec{E}}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(E_iD_i-\frac{1}{2}\vec{E}\cdot\vec{D}\delta_{ij}\right) \qquad (3.8)$$

and

$$\left[(\nabla\cdot\vec{B})\vec{H}-\vec{B}\times(\nabla\times\vec{H})\right]_i=\frac{1}{2}\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial x_i}-\vec{B}\cdot\frac{\partial\vec{H}}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(H_iB_i-\frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij}\right)\quad(3.9)$$

respectively.

Using these two expressions in the *i*th component of equation (3.7) on the preceding page and re-shuffling terms, we get

$$\rho\vec{E}+\vec{j}\times\vec{B}=\frac{1}{2}\left[\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial x_i}-\vec{D}\cdot\frac{\partial\vec{E}}{\partial x_i}\right)+\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial x_i}-\vec{B}\cdot\frac{\partial\vec{H}}{\partial x_i}\right)\right]+\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)$$

(3.10)

$$=\frac{\partial}{\partial x_i}\left(E_iD_i-\frac{1}{2}\vec{E}\cdot\vec{D}\delta_{ij}+H_iB_i-\frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij}\right)$$

Introducing the electric volume force F_{ev} via its *i*th component

$$(F_{ev})_i=\left(\rho\vec{E}+\vec{j}\times\vec{B}\right)_i-\frac{1}{2}\left[\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial x_i}-\vec{D}\cdot\frac{\partial\vec{E}}{\partial x_i}\right)+\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial x_i}-\vec{B}\cdot\frac{\partial\vec{H}}{\partial x_i}\right)\right]\quad(3.11)$$

And the *Maxwell* stress tensor T with components

$$T_{ij}=E_iD_i-\frac{1}{2}\vec{E}\cdot\vec{D}\delta_{ij}+H_iB_i-\frac{1}{2}\vec{B}\cdot\vec{H}\,\delta_{ij}\quad(3.12)$$

We finally obtain the force equation

$$\left[F_{ev}+\frac{\partial}{\partial t}\left(\vec{D}\times\vec{B}\right)\right]_i=\frac{\partial T_{ij}}{\partial x_i}=\left(\nabla\cdot\vec{T}\right)_i\quad(3.13)$$

If we introduce the relative electric permittivity k and the *relative* magnetic permeability k_m as

$$\vec{D}=k\varepsilon_0\vec{E}=\varepsilon\vec{E}\quad(3.14)$$

$$\vec{B}=k_m\mu_0\vec{H}=\mu\vec{H}\quad(3.15)$$

We can rewrite (3.13) as

$$\frac{\partial T_{ij}}{\partial x_i}=\left(F_{ev}+\frac{kk_m}{c^2}\frac{\partial\vec{S}}{\partial t}\right)_i\quad(3.16)$$

Where $\vec{S}$ is the Poynting vector defined in equation (3.6). Integration over the entire volume *V* yields

$$\int_V \rho\left(\vec{E}+\vec{v}\times\vec{B}\right)d^3x' + \frac{1}{c^2}\frac{d}{dt}\int_V \vec{S}d^3x' = \int_S \vec{T}_{\hat{n}}\, d^2x', \tag{3.17}$$

$\text{Force on the matter} + \text{Field momentum} = \textit{Maxwell}\ \text{Stress}$

This expresses the balance between the force on the matter as the rate of change of the electromagnetic field momentum and the Maxwell stress. This equation is called the **momentum theorem in Maxwell's theory.** In vacuum (3.17) becomes

$$\int_V \rho\left(\vec{E}+\vec{v}\times\vec{B}\right)d^3x' + \frac{1}{c^2}\frac{d}{dt}\int_V \vec{S}d^3x' = \int_S \vec{T}_{\hat{n}}\, d^2x', \tag{3.18}$$

$\text{Force on the matter} + \text{Field momentum} = \textit{Maxwell}\ \text{Stress}$

or

$$\frac{d}{dt}P^{\text{Mech}} + \frac{d}{dt}P^{\text{Field}} = \int_S \vec{T}_{\hat{n}}\, d^2x' \tag{3.19}$$

4.0 CONCLUSION

Here, the energy theorem in Maxwell's electromagnetic theory has been established through the Poynting's theorem and the momentum theorem. We discovered the term electric volume force and derived an expression for the Maxwell stress tensor while at the same time qualified relative electric permittivity.

5.0 SUMMARY

Electromagnetic waves convey energy and possess momentum.

6.0 TUTOR-MARKED ASSIGNMENT

1. State the energy theorem and list five practical applications of this theorem.
2. Describe Poynting's Vector and relate it to solar radiation.
3. Does light wave have momentum? If it does, quantify it mathematically.

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UNIT 2 RADIATION FROM EXTENDED SOURCES

CONTENTS

1.0 INTRODUCTION

Electromagnetic propagation in material medium can be radically different from propagation through vacuum as electromagnetism is affected by media and it is reasonable to simplify the propagation properties of electromagnetic waves by considering the geometry of the radiating source.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- appreciate that there are radiation from extended sources
- write down the macroscopic Maxwell equations
- analyse collisional interaction between charge carriers
- describe anisotropic and birefringent medium
- explain how charges moving in matter radiate electromagnetic waves
- derive the Lorentz transformation
- state why the combination of two Lorentz transformations must be a Lorentz transformation.

3.0 MAIN CONTENT

3.1 Radiation from Extended Sources

Certain radiation systems have a geometry which is one-dimensional, symmetric or in any other way simple enough that a direct calculation of the radiated fields and energy is possible. This is for instance the case when the current flows in one direction in space only and is limited in extent. An example of this is the linear antenna.

3.2 Radiation from Charges Moving in Matter

When electromagnetic radiation is propagating through matter, new phenomena may appear which are (at least classically) not present in vacuum. As mentioned earlier, one can under certain simplifying assumptions include, to some extent, the influence from matter on the electromagnetic fields by introducing new, derived field quantities $\vec{D}$ and $\vec{H}$ according to

$$\vec{D} = \varepsilon(t, \vec{x})\vec{E} = k\varepsilon_0 \vec{E} \tag{4.1}$$

$$\vec{B} = \mu(t, \vec{x})\vec{H} = k_m\mu_0 \vec{H} \ . \tag{4.2}$$

Expressed in terms of these derived field quantities, the Maxwell equations, often called **macroscopic Maxwell equations**, take the form as shown previously[M1-M4] Assuming for simplicity that the electric permittivity ε and the magnetic permeability μ, and hence the relative permittivity k and the relative permeability k_m all have fixed values, independent on time and space, for each type of material we consider, we can derive the general telegrapher's equation

$$\frac{\partial^2 \vec{E}}{\partial \varsigma^2} - \sigma\mu \frac{\partial \vec{E}}{\partial t} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = (0,0,0) \tag{4.3}$$

In describing (1D) wave propagation in a material medium, it is known that the existence of a finite conductivity, manifesting itself in a collisional interaction between the charge carriers, causes the waves to decay exponentially with time and space.

Let us therefore assume that in our medium $\sigma = 0$ so that the wave equation simplifies to

$$\frac{\partial^2 \vec{E}}{\partial \varsigma^2} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = (0,0,0) \tag{4.4}$$

If we introduce the *phase velocity* in the medium as

$$v_\varphi = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{kk_m\varepsilon_0\mu_0}} = \frac{c}{\sqrt{kk_m}} \tag{4.5}$$

Where, according to Equation (1.29), $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ is the speed of light, *i.e.*, the phase speed of electromagnetic waves in vacuum, then the general solution to each component of equation (4.4) on the previous page

$$E_i = f(\varsigma - v_\varphi t) + g(\varsigma + v_\varphi t), \quad i = 1,2,3. \tag{4.6}$$

The ratio of the phase speed in vacuum and in the medium

$$\frac{c}{v_\varphi} = \sqrt{kk_m} = c\sqrt{\varepsilon\mu} \overset{\text{def}}{\equiv} n \tag{4.7}$$

is called the **refractive index** of the medium. In general n is a function of both time and space as are the quantities ε, μ, k and k_m. If, in addition, the medium is **anisotropic or birefringent**, all these quantities are rank-two tensor fields. Under our simplifying assumptions, in each medium we consider n = *Const* for each frequency component of the fields. Associated with the phase speed of a medium for a wave of a given frequency ω we have a *wave vector*, defined as

$$\vec{k} \overset{\text{def}}{\equiv} k\hat{\vec{k}} = kv_\varphi = \frac{\omega}{v_\varphi}\frac{\vec{v}_\varphi}{v_\varphi} \tag{4.8}$$

Consider the case of the vacuum where we assume that $\vec{E}$ is time-harmonic, *i.e.*, can be represented by a Fourier component proportional to $\exp\{i\omega t\}$, the solution of equation (4.4) can be written

$$\vec{\mathrm{E}} = \mathrm{E}_0 \exp\left\{i(\vec{\mathrm{k}}\cdot\vec{\mathrm{x}} - \omega t)\right\} \tag{4.9}$$

Where now $\vec{\mathrm{k}}$ is the wave vector *in the medium* given by equation (4.8). With these definitions, the vacuum formula for the associated magnetic field,

$$\vec{\mathrm{B}} = \sqrt{\varepsilon\mu}\,\hat{\vec{k}}\times\vec{\mathrm{E}} = \frac{1}{v_\varphi}\hat{\vec{k}}\times\vec{\mathrm{E}} = \frac{1}{\omega}\vec{k}\times\vec{\mathrm{E}} \tag{4.10}$$

is valid also in a material medium (assuming, as mentioned, that n has a fixed constant scalar value). A consequence of a $k \neq 1$ is that the electric field will, in general, have a longitudinal component. It is important to notice that depending on the electric and magnetic properties of a medium, and, hence, on the value of the refractive index n, the phase speed in the medium can be smaller or larger than the speed of light:

$$v_\varphi = \frac{c}{n} = \frac{\omega}{k} \tag{4.11}$$

Where, in the last step, we have used eqn. (4.8). If the medium has a refractive index which, as is usually the case, dependent on frequency ω, we say that the medium is *dispersive*. Because in this the *group velocity*

$$v_g = \frac{\partial\omega}{\partial k} \tag{4.12}$$

Has a unique value for each frequency component, and is different from *v*'. Except in regions of **anomalous dispersion**, *v*' is always smaller than *c*. In a gas of free charges, such as a *plasma*, the refractive index is given by the expression

$$n^2(\omega) = 1 - \frac{{\omega_p}^2}{\omega^2} \qquad (4.13)$$

Where

$$\omega_p{}^2 = \sum_\sigma \frac{N_\sigma {q_\sigma}^2}{\varepsilon_0 m_\sigma} \qquad (4.14)$$

is the *plasma frequency*. Here m_σ and N_σ denote the mass and number density, respectively, of charged particle species σ. In inhomogeneous plasma, $N_\sigma = N_\sigma(\vec{x})$ so that the refractive index and also the phase and group velocities are space dependent. As can be easily seen, for each given frequency, the phase and group velocities in plasma are different from each other. If the frequency ω is such that it coincides with ω_p at some point in the medium, then at that point $v_\varphi \to \infty$ while $v_g \to 0$ and the wave Fourier component at ω is reflected there.

3.3 Derivation of the Lorentz Transformation

In most cases, the Lorentz transformation is derived from the two postulates: the equivalence of all inertial reference frames and the invariance of the speed of light. However, the most general transformation of space and time coordinates can be derived using only the equivalence of all inertial reference frames and the symmetries of space and time.

The general transformation depends on one free parameter with the dimensionality of speed, which can be then identified with the speed of light *c*. This derivation uses the group property of the Lorentz transformations, which means that a combination of two Lorentz transformations also belongs to the class Lorentz transformations.

The derivation can be compactly written in matrix form. However, for those not familiar with matrix notation, we may also write it without matrices.

3.4 Further Assignments

1) Let us consider two inertial reference frames *O* and *O'*. The reference frame *O'* moves relative to *O* with velocity *v* along the *x* axis. We know that the coordinates *y* and *z* perpendicular to the velocity are the same in both reference frames: *y = y'* and *z = z'*. So, it is sufficient to consider only transformation of the coordinates *x* and *t* from the reference frame *O* to *x' = fx*(*x; t*) and *t' = ft*(*x; t*) in the reference frame *O'*. From translational symmetry of space and time, we conclude that the functions *fx*(*x;*

t) and $ft(x; t)$ must be linear functions. Indeed, the relative distances between two events in one reference frame must depend only on the relative distances in another frame:

$$x_1' - x_2' = f_x(x_1 - x_2, t_1 - t_2),$$
$$t_1' - t_2' = f_t(x_1 - x_2, t_1 - t_2) \qquad (5.1)$$

Because equation (5.1) must be valid for any two events, the functions $fx(x; t)$ and $ft(x; t)$ must be linear functions. Thus

$$x' = Ax + Bt,$$
$$t' = Cx + Dt \qquad (5.2)$$

Where A, B, C and D are some coefficients that depend on v. In matrix form equations (5.2) are written as

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} \qquad (5.3)$$

With four unknown functions A, B, C and D of v.

2) The origin of the reference frame O' has the coordinate $x' = 0$ and moves with velocity v relative to the reference frame O, so that $x = vt$. Substituting these values into equation (5.2), we find $B = -vA$. Thus, the first equation of equations (5.2) has the form

$$x' = A(x - vt), \qquad (5.4)$$

So we need to find only three unknown functions A,C and D of v.

3) The origin of the reference frame O has the coordinate $x = 0$ and moves with velocity $-v$ relative to the reference frame O', so that $x' = -vt'$. Substituting these values in equations (5.2), we find $D = A$. Thus, the second part of equations (2) has the form

$$t' = Cx + At = A(Ex + t) \qquad (5.5)$$

Where we introduced the new variable E=C/A.

Let us change to the more common notation $A = \gamma$. Then equations (5.4) and (5.5) have the form

$$x' = \gamma(x - vt), \qquad (5.6)$$
$$t' = \gamma(Ex + t)\ , \qquad (5.7)$$

or in matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma\begin{pmatrix} 1 & -v \\ E & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}. \qquad (5.8)$$

Now we need to find only two unknown functions γ_v and *Ev* of *v*.

4) A combination of two Lorentz transformations also must be a Lorentz transformation.

Let us consider a reference frame *O'* moving relative to *O* with velocity *v*1 and a reference frame *O''* moving relative to *O'* with velocity *v*2. Then

$$x'' = \gamma_{v2}(x' - v_2 t'), \quad x' = \gamma_{v1}(x - v_1 t), \qquad t'' = \gamma_{v2}(E_{v2}x' + t'), \quad t' = \gamma_{v1}(E_{v1}x + t), \tag{5.9}$$

This can also be put in the matrix form as done earlier.

For a general Lorentz transformation, the coefficients in front of *x* in equation (5.6) and in front of *t* in equation (5.7) are equal, i.e. the diagonal matrix elements in equation (8) are equal.

If we substitute for x' and t' in the first equation of equation (5.9), we obtain

$$x'' = \gamma_{v2}\gamma_{v1}[(1 - E_{v1}v_2)x - (v_1 + v_2)t] \qquad t'' = \gamma_{v2}\gamma_{v1}[(E_{v1} + E_{v1})x + (1 - E_{v1}v_2)t]. \tag{5.10}$$

Similarly, equation (5.10) must also satisfy this requirement:

$$1 - E_{v1}v_2 = 1 - E_{v2}v_1 \quad \Rightarrow \quad \frac{v_2}{E_{v2}} = \frac{v_1}{E_{v1}} \tag{5.11}$$

In the second equation (5.14), the left-land side depends only on *v*2, and the right-hand side only on *v*1. This equation can be satisfied only if the ratio *v/Ev* is a constant *a* independent of velocity *v*, i.e.

$$E_v = \frac{v}{a}, \tag{5.12}$$

Substituting equation (5.12) into equations (5.6) and (5.7), as well as (5.8), we find

$$x' = \gamma_v(x - vt), \quad t' = \gamma_v\left(v/_a\, x + t\right) \tag{5.13}$$

or in the matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & -v \\ \frac{v}{a} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{5.14}$$

Now we need to find only one unknown function γ_v, whereas the coefficient *a* is a fundamental constant independent on *v*.

5) Let us make the Lorentz transformation from the reference frame *O* to *O'* and then from *O'* back to *O*. The first transformation is

performed with velocity *v*, and the second transformation with velocity -*v*. The equations are similar to equations (5.10):

$$x = \gamma_{-v}\left(x' + v\,t'\right), \quad x' = \gamma_v\left(x - v_1 t\right),$$
$$t = \gamma_{-v}\left(\frac{v}{a}x' + t'\right), \quad t' = \gamma_v\left(\frac{v}{a}x + t\right), \qquad (5.15)$$

Substituting *x'* and *t'* from the first equation (5.15) into the second one, we find

$$x = \gamma_{-v}\gamma_v\left(1 + v^2/a\right)x, \quad t = \gamma_{-v}\gamma_v\left(1 + v^2/a\right). \qquad (5.16)$$

Equation (5.15) must be valid for any x and t, so

$$\gamma_{-v}\gamma_v = \frac{1}{1 + \frac{v^2}{a}} \qquad (5.17)$$

Because of the space symmetry, the function γ_v must depend only on the absolute value of velocity *v*, but not on its direction, so $\gamma_{-v} = \gamma_v$. Thus we find

$$\gamma_v = \frac{1}{\sqrt{1 + \frac{v^2}{a}}} \qquad (5.18)$$

6) Substituting equation (5.18) into equations (5.13) and (5.14) , we find the final expressions for the transformation as

$$x' = \frac{x - vt}{\sqrt{1 + \frac{v^2}{a}}}, \qquad t' = \frac{v/a\, x + t}{\sqrt{1 + \frac{v^2}{a}}}, \qquad (5.19)$$

Which, can also be put in the matrix form.

Equations (5.19) and its matrix equivalent have one fundamental parameter *a*, which has the dimensionality of velocity squared. If $a < 0$, we can write it as

$$a = -c^2 \qquad (5.20)$$

Then equations (5.19) and its matrix equivalent become the standard Lorentz transformation:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{a}}}, \qquad t' = \frac{v/a\, x + t}{\sqrt{1 - \frac{v^2}{a}}}, \qquad (5.21)$$

It is easy to check from equation.(5.21) that, if a particle moves with velocity *c* in one reference frame, it also moves with velocity *c* in any other reference frame, i.e. if $x = ct$ then $x' = ct'$. Thus the parameter *c* is the invariant speed.

Knowing about the Maxwell's equations and electromagnetic waves, we can identify this parameter with the speed of light. It is straightforward to check that the Lorentz transformation (5.21) and matrix equivalent preserves the space -time interval

$$\left(ct'\right)^2 - x'^2 = \left(ct\right)^2 - x^2, \qquad (5.22)$$

or it has the Minkowski metric.

If $a = \infty$, then equations (5.22) and (5.21) produce the non-relativistic Galileo transformation:

$$x' = \gamma\left(x - vt\right), \qquad t' = t \qquad (5.23)$$

or in matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}.$$

If a > 0, we can write it as $a = \sigma^2$. Then equations (5.19) describe a Euclidean space-time and preserve the space-time distance:

$$\left(\sigma t'\right)^2 + x'^2 = \left(\sigma t\right)^2 + x^2 \qquad (5.24).$$

Problem 1:

(A) By examination of (23) show that

$$\omega = \frac{2\pi}{T} \qquad (26)$$

Where T is the time for the electric field to complete one cycle at a fixed z. That is, where T is the temporal period.

(B) By examination of (23) show that

$$k == \frac{2\pi}{\lambda} \qquad (27)$$

Where λ is the wavelength (that is, the spatial period) of the wave.

(C) By examination of (23) show that the phase velocity of this wave is indeed ω / k.

Problem 2:

Use equation (17) above to show that the magnetic field $\vec{b}$ corresponding to the electric field (23) has the form,

$$\vec{b} = \frac{1}{c} E_0 \cos(\omega t - kz + \phi_0)\hat{y} \qquad (28)$$

To solve this problem, substitute (23) into the right hand side of (17) and then integrate to find the magnetic field. (**Don't just show that (28) depends upon substitution alone.)** In integrating, be careful to keep track of the limits.

4.0 CONCLUSION

When electric charge moves through matter, their electromagnetic properties differ from that of the same charges were they to move through a vacuum.

5.0 SUMMARY

Electromagnetic property is a function of medium and propagation is subject to radiation geometry.

6.0 TUTOR-MARKED ASSIGNMENT

1. Is the propagation of electromagnetic waves affected by vacuum?
2. Describe how matter influences electromagnetic waves.
3. When are macroscopic Maxwell's equations most applicable?
4. Explain the phenomenon known as collisional interaction between the charge carriers.
5. From which postulates is the Lorentz transformation derived? State them.

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