NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 423

COURSE TITLE: INTEGRAL EQUATIONS

**COURSE GUIDE**

# MTH 423
# INTEGRAL EQUATIONS

**Course Team** Prof. O. J. Adeniran (Course Writer) - UNAB Abeokuta
Dr. A. Bankole (Course Editor) - UNILAG
Dr. S. O. Ajibola (Programme Leader) - NOUN

**NATIONAL OPEN UNIVERSITY OF NIGERIA**

National Open University of Nigeria
Headquarters
14\16 Ahmadu Bello Way
Victoria lsland, Lagos

Abuja office
5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e- mail: centralinfo@nou.edu.ng
URL: www.noun.edu.ng

Published by
National Open University of Nigeria

Printed 2014

ISBN: 978-058-108-1

**CONTENTS** **PAGE**

## INTRODUCTION

An integral equation is an equation in which an unknown function appears under an integral sign. Integral equation bear a very close kinship with differential equations and quite often, problems may be formulated either in differential form or in integral form.

Very often, integral equations cannot be solved analytically and a numerical approach has to be adopted; particularly for equations over arbitrarily profiles. It is the desire of the author that through this course, you will be encouraged to develop an enquiring attitude towards integral equation and relate the lessons learnt in this course to the world around you. Furthermore; you are encouraged to build upon the lessons learnt in the prerequisite course to strengthen your understanding of the underlying principles at work in the application of integral equation.

This course, **MTH 423: Integral Equations,** comprises a total of four modules and ten units as follows:

Module 1 is composed of 3 Units
Module 2 is composed of 2 Units
Module 3 is composed of 3 Units
Module 4 is composed of 2 Units

In module 1, you will learn the preliminary concepts of linear integral equation; convert ordinary differential equations into integral equation and transformation of Sturm-Lowville problems to integral equation. You shall also learn how to classify linear integral equation and find approximate solutions to integral equation in Unit 3.

In module 2, you shall learn that the Volterra integral equation is integral equation with an integration limit containing one of the variables of integration. You will learn to use the Resolvent Kernel to solve this class of integral equation. Also you shall discover that for many integral equations, you must carry out a Laplace Transformation to arrive at a solution; and that the consequence of this is the inverse transforms which implies Convolution.

Module 3 will discuss the Fredholm Equations with Degenerate Kernels and the general method of finding solutions which will make you conversant with Eigen-functions, as well as Eigen-vectors and Symmetric Kernels. You will also learn how to easily represent a function by a series of orthogonal functions and expand $K$ in Eigen-functions. Of the several definitions and theorems which you will be introduced to, shall be those related to positive kernels and convergence

– a necessary condition for determining a solution for integral equation in deriving a numerical solution.

Module 4 will take you through the processing of $1^{st}$ Eigen-value Integral Transforms via Laplace Transforms, Convolution Theorem and Inverse Laplace Transform. The application of the transform and Fourier integral equations will be the concluding part of your study of MTH 423.

## COURSE AIM

The aim of this course is to hone your understanding of integral equation, whilst acquainting you with the graphical and mathematical significance of integral equation and its relationship with partial differential equations. throughout the course, you shall be learn that for every analytical approach to integral equation solving, there is a numerical method, and indeed, that some intricately irregular multi-variable profiles can only be resolved numerically All these are expected to motivate you towards further enquiry into this very interesting and highly specialised mathematical habitat.

## COURSE OBJECTIVES

You are expected to conscientiously and diligently work through this course. Upon completion you should be able to:

- explain the basic concepts underlying linear integral equation
- investigate the equations which describe the displacement of a loaded elastic sting
- treat the shop stocking problem
- convert ordinary differential equations into integral equations
- transform Sturm Lowville problems to integral equation
- work through a series of examples of transformations and conversions, and their solutions
- classify linear integral equation
- find approximate solutions for integral equation
- recognise Volterra integral equation
- identify the three types of Volterra integral equation
- arrive at the Resolvent kernel of a Volterra equation
- solve convolution type kernels of the Volterra integral using Laplace transform
- comfortably solve Fredholm equations
- identify a Neumann series
- solve Fredholm equations with degenerate kernels
- derive the general method of solution of Fredholm equations

- work with Eigen functions and eigenvectors
- prove that symmetric and continuous kernels that are not identically zero possess at least one Eigen value
- write the Hilbert – Schmidt theorem
- state the convergence theorem
- prove that functions can be represented by series of orthogonal functions
- expand $K$ in a series of Eigen functions
- define positive kernels
- apply the convolution theorem
- calculate the first Eigen value of an integral equation
- use the variational formula
- recognise integral Laplace transforms as transforms
- derive the solution of integral equation using inverse Laplace transform
- apply Laplace transform through worked examples
- understand and solve integral equation by the method of Fourier integral transforms.

## WORKING THROUGH THE COURSE

This course requires you to spend quality time to read. The course content is presented in clear mathematical language that you can easily relate to and the presentation style is adequate and easy to assimilate.
You should take full advantage of the tutorial sessions because this is a veritable forum for you to "rub minds" with your peers – which provides you valuable feedback as you have the opportunity of comparing knowledge and "rubbing minds" with your course mates.

## COURSE MATERIALS

You will be provided course materials prior to commencement of this course, which will comprise your Course Guide as well as your study units. You will receive a list of recommended textbooks which shall be an invaluable asset for your course material. These textbooks are however not compulsory.

## STUDY UNITS

You will find listed below the study units which are contained in this course and you will observe that there are four modules. The first module comprises three units, the second has two units, the third has three units and the last module has two units.

**Module 1**

| | |
|---|---|
| Unit 1 | Linear Integral Equation: Preliminary Concepts |
| Unit 2 | Conversion of Ordinary Differential Equations into Integral Equation |
| Unit 3 | Classification of Linear Integral Equation |

**Module 2**

| | |
|---|---|
| Unit 1 | S2 Volterra Integral Equation |
| Unit 2 | Convolution Type Kernels |

**Module 3**

| | |
|---|---|
| Unit 1 | Fredholm Equations with Degenerate Kernels |
| Unit 2 | Eigenfunctions and Eigenvectors |
| Unit 3 | Representation of a Function by a Series of Orthogonal Functions |

**Module 4**

| | |
|---|---|
| Unit 1 | Calculation of $1^{st}$ Eigenvalue |
| Unit 2 | The Application of the Transform |

## TEXTBOOKS

Kendall, E. A. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equation*. Boca Raton: CRC Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis.* Cambridge Mathematical Library.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971*). Problems and Exercises in Integral Equation*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). "Chapter 19. Integral Equation and Inverse Theory". *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

## ASSESSMENT

Assessment of your performance is partly through Tutor-Marked Assessments which you can refer to as TMAs, and partly through the final examinations.

## TUTOR-MARKED ASSIGNMENT

This is basically a continuous assessment which accounts for 30% of your total score. During this course, you will be given four tutor-marked assignments (TMAs) and you must answer three of them to qualify to sit for the final examinations. Tutor-Marked Assignments are provided by your course facilitator and you must return the answered TMAs back to your course facilitator within the stipulated period.

## FINAL EXAMINATION AND GRADING

You must sit for the final examination which accounts for 70% of your score upon completion of this course. You will be notified in advance of the date, time and the venue for the examinations which may, or may not coincide with National Open University of Nigeria semester examination.

## SUMMARY

Each of the four modules of this course has been designed to stimulate your interest in integral equation through associative conceptual building blocks in the study and application of integral equation to practical problem solving.

By the time you complete this course, you should have acquired the skills and confidence to solve many integral equations. Make sure that you have enough referential and study material available and at your disposal at all times, and – devote sufficient quality time to your study.

I wish you the best in your academic pursuits.

**MAIN COURSE**

**CONTENTS** **PAGE**

# MODULE 1



## UNIT 1 LINEAR INTEGRAL EQUATION: PRELIMINARY CONCEPTS

**CONTENTS**



## 1.0 INTRODUCTION

In integral equations, an unknown function which is the subject seeking a solution always appears under an integral sign. These equations bear a close kinship with differential equations suggesting that a differential equation can be formulated as an integral equation and vice-versa.

The analytical method remains the standard method of solving integral equations, however, where the analytical method fails; the equation can be solved numerically.

Let us commence with two common problems to illustrate the basic concepts of linear integral equations; loaded elastic string and the shop stocking problem.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

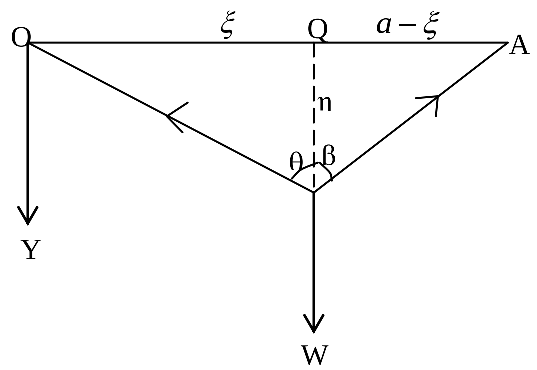
- explain the basic concepts underlying linear integral equations
- investigate the equations which describe the displacement of a loaded elastic sting
- treat the shop stocking problem.

## 3.0 MAIN CONTENT

### 3.1 Linear Integral Equation: Preliminary Concepts

Let us take a look at some problems, the types of which we encounter every day and which give rise to integral equation.

#### 3.1.1 A Loaded Elastic String




Consider a weightless elastic string as shown in the above figure, stretched between two horizontal points O and A and suppose that a weight W is hung from the elastic string and that in equilibrium the position of the weight is at a distance $\xi$ from O and at a depth Y below OA. If W is small compared to the initial tension T in the string, it can be assumed that the tension of the string remains T during the further stretching. The vertical resolution of forces gives the equilibrium equation

$$T\,(\eta/\xi)\ +\ T\,(\eta/(a-\xi)) - W = O$$

Where $AO\ =\ a$ (1.1)

The drop Y due to a weight W situated a distance $\xi$ along the string from O is given by

$$Y\ =\ W(a-\xi)\xi/Ta \qquad (1.2)$$

The drop Y in the string at a distance $x$ from O is given by

$$Y = x\,y/\xi, \qquad 0 \le x \le \xi \qquad (1.3)$$
$$y = (a-x)\eta/(a-\xi), \ \xi \le x \le a \ (1.4)$$

Eliminating y, these two results can be written in the form

$$y = W\,G(x,\xi)/T \quad (1.5)$$

where

$$G(x,\xi) = x(a-\xi)/a, \qquad 0 \le x \le \xi$$
$$= \xi(a-x)/a, \qquad \xi \le x \le a \qquad (1.6)$$

Suppose now that the string is loaded continuously with a weight distribution $W(x)$ per unit length, the elementary displacement at the point distance $x$ from O, due to the weight distribution over $\xi \le x \le \xi + \partial\xi$ is

$$\partial y = W(\xi)\partial\xi \ \ G(x,\xi)/T$$
$$0 \le x, \xi \le a \quad (1.7)$$

On integrating, displacement due to the complete weight distribution is given by

$$y(x) = T^{-1}\int_0^a G(x,\xi)W(\xi)d\xi, \qquad 0 \le x \le a \qquad (1.8)$$

Thus, the displacement of the string is given in terms of the weight distribution. However, if we are given the displacement of the string, what is the weight distribution?

In this case, we can sew site to equation. (1.8) the form

$$y(x) = (Ta)^{-1}\left[x\int_0^x (a-\xi)W(\xi) + (a-x)\int_x^a \xi \ W(\xi)D\xi\right]$$
$$(1.9)$$

Different this twice, we obtain

$$y^{11}(x) = (Ta)^{-1}\,W(x)$$

i.e. $\quad W(x) = Ta\,y^{11}(x) \quad (1.10)$

### 3.1.2 The Shop Stocking Problem

A shop starts selling some goods. It is found that a proportion $K(t)$ remains unsold at time t after the shop has purchased the goods. It is required to find the stock at which the shop should purchase the goods so that the stock of the goods in the shop remains constant (all processes are deemed to be continuous).

Suppose that the shop commences business in the goods by purchasing an amount A of the goods at zero time, and buys at a rate $Q(t)$ subsequently. Over the time interval

$$K(t-\tau)\,Q(\tau)\,d\tau \qquad (1.11)$$

Thus, the amount of goods remaining unsold at time t, and which was bought up to that time, is given by

$$AK(t) + \int_0^t K(t-\tau)\,Q(\tau)\,d\tau \qquad (1.12)$$

This is the total stock of the shop and is to remain constant at its initial value and so

$$AK(t) + \int_0^t K(t-\tau)\,Q(\tau)\,d\tau \qquad (1.13)$$

And the required stocking rate $Q(t)$is the solution of this integral eqn.

## 4.0 CONCLUSION

You have learnt the processes involved in the two illustrative problems. It is easy to formulate similar solutions for a vast array of problems.

## 5.0 SUMMARY

The two problems presented demonstrate how to formulate and derive an integral equation for a suitably structured problem. It also demonstrates the process of solving the integral equation developed.

1. Apart from the Loaded Elastic String and the Shop Stocking Problem, can you make a list of 5 different types of problems which can be solved using integral equation?

2. A transport company distributed workshops within a metropolis which receives and repairs its broken down vehicles. The workshop manager discovers that he must always reroute a *Y (t)*% of his workshop allocation of vehicles to alternative location every day as he cannot accommodate them in his workshop overnight, and he calls you in to tell him the optimum number of requests for repairs he should entertain every day such that the workshop is 100% utilised when all related processes are assumed to be continuous. Formulate an integral equation to help the workshop manager.

## 7.0 REFERENCES/FURTHER READING

Arfken, G. & Weber, H. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Andrei, D. P. & Alexander, V. (1998). *Manzhirov Handbook of Integral Equations.* Boca Raton: CRC Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind. Cambridge Monographs on Applied and Computational Mathematics*.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T & Flannery, B.P. (2007). "Chapter 19: Integral Equations and Inverse Theory". *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. A *Course of Modern Analysis*. Cambridge Mathematical Library.

# UNIT 2 CONVERSIONS OF ORDINARY DIFFERENTIAL EQUATIONS INTO INTEGRAL EQUATIONS

## CONTENTS



## 1.0 INTRODUCTION

There are many ordinary differential equations which can be converted into corresponding integral equations and we shall proceed to study how these transformations can be carried out; particularly in the classical case of the Sturm Lowville problems and a host of others illustrative of this transformation process.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- convert ordinary differential equations into integral equations
- transform Sturm Lowville problems to integral equations
- work through a series of examples of transformations and conversions, and their solutions.

## 3.0 MAIN CONTENT

### 3.1 Conversion of Ordinary Differential Equations into Integral Equations

$$y^{11}(x) + a_1(x)\, y^1(x) + a_2(x)\, y(x) = f(x) \qquad (1.14)$$

with the initial condition,

$$y(0) = y_0, \quad y^1(0) = y_1 \qquad (1.15)$$

Let $\psi(x) = y^{11}(x)$ (1.16)

Then, $y^1(x) = \int_0^x \psi(u)\, du + y_1$ (1.17)

$$y(x) = \int_0^x (x-u)\, \psi(u) du + y_1 x + y_0 \quad (1.18)$$

Substituting the relations 1.16 to 1.18 into the differential equation, it follows that

$$\psi(x) + \int_0^x [a_1(x) + a_2(x)(x-u)]\psi(u)\, du$$
$$= f(x) - y_1 a_1(x) - y_1 x\, a_2(x) - y_0 a_2(x) \qquad (1.19)$$

Equation (1.19) can be written in the form

$$\psi(x) + \int_0^x K(x,u)\psi(u)\, du = g(x) \qquad (1.20)$$

Which is an integral equation for $\psi(x)$

**Example 1.1**

Form the integral equation corresponding to

$$y^{11} + 2xy^1 + y = 0, \quad y(0) = 1, \quad y^1(0) = 0$$

**Solution**

Let $y_x^{11} = \psi(x), \quad y^1 = \int_0^x \psi(u) du$

$$y = \int_0^x (x-u)\psi(u) du + 1$$

Thus, $\psi(x) + 2x\int_0^x \psi(u) du + \int_0^x (x-u)\psi(u)\, du + 1 = 0$

i.e. $\psi(x) + \int_0^x (3x-u)\psi(u) + 1 = 0$

### 3.2 Transformation of Sturm - Linville Problems to Integral Equation

A problem which is associated with an expression of the form

$$Ly = \frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) - q(x)y, \quad x_1 \le x \le x_2 \quad (1.21)$$

and boundary condition of the form

$$a_1 y(x_1) + b_1 y^1(x_1) = 0 \quad (1.22)$$
$$a_2 y(x_2) + b_2 y^1(x_2) = 0$$

is said to be of Sturm-Lowville type.

There are two problems which are of interest here, namely:

$$Ly = f(x) \qquad x_1 \le x \le x_2 \ (1.23)$$

and

$$Ly + \lambda r(x) y = 0 \qquad x_1 \le x \le x_2 \ (1.24)$$

are continuous in the interval $x_1 \le x \le x_2$, and in addition $P(x)$ has a continuous derivative and does not vanish.

The differential equation (1.23) corresponds to a displacement $y$ caused by some forcing function $f$, and the differential equation (1.24) forms together with the boundary condition, an Eigenvalue problem.

Suppose that $Q_1, Q_2$ are solutions of the equation $Ly = 0$

with $a_1 Q_1(x_1) + b_1 Q_1(x_1) = 0$

$a_2 Q_2(x_2) + b_2 Q_2^1(x_2) = 0 \qquad (1.25)$

then,

$$\begin{aligned} 0 &= Q_2 LQ_1 - Q_1 LQ_2 \\ &= Q_2 \frac{d}{dx}\left(P\frac{d\varepsilon_1}{dx}\right) - Q_1 \frac{d}{dx}\left(P\frac{dd_2}{dx}\right) \\ &= \frac{d}{dx}\left(P\left(Q_2\frac{dQ_1}{dx} - Q_1\frac{dQ_2}{dx}\right)\right) \end{aligned}$$

Thus,

$$P\left(Q_2 \frac{dQ_1}{dx} - Q_1 \frac{dd_2}{dx}\right) = \text{constant} \quad (1.26)$$

Using the method of variation of parameters, look for a solution of the form

$$y(x) = z_1(x)\,Q_1(x) + z_2(x)\,Q_2(x) \qquad (1.27)$$

where $z_1$ and $z_2$ are to be determined.

Thus,

$$y^1 = z^1{}_1\, Q_1 + z_2^1\, Q_2 + z_1 Q_1^1 + z_2 Q_2^1 \qquad (1.28)$$

Let $\quad z_1{}^1\, Q_1 + z_2^1 Q_2 = 0,$ so that

$$Ly = \frac{d}{dx}\left[P(x)\left(z_1(x)\,Q_1^1\,(x) + z_1(x)\,Q_2^1(x)\right)\right]$$
$$- q(x)\,(z)\left(z_1(x)\,Q_1(x) + z_2(x)\,Q_2(x)\right)$$
$$= \quad P\left(z_1^1\, Q_1^1 + z_2^1\, Q_2^1\right) \qquad (1.29)$$

Since $LQ_1 = LQ_2 = 0$

Thus, $z_1$ and $z_2$ are given by the solutions of equations

$$z_1^1\, Q_1 \;+ z_2^1\, Q_2 \;=\; 0 \qquad (1.30)$$
$$P\left(z_1^1\, Q_1^1 + z_2^1\, Q_2^1\right) = \; f(x) \quad (1.31)$$

Whence, $z_1{}^1 = \dfrac{fQ_2}{P\left(Q_2\, Q_1{}^1 - Q_1 Q_2^1\right)} \qquad z_2{}^1 \;=\; \dfrac{-\,fQ_1}{P\left(Q_2 Q_1{}^1 - Q_1 Q_2^1\right)} \qquad (1.32)$

The denominator in these two expressions is constant by (1.26) and by a suitable scaling of $\phi_1$ and $\phi_2$ may be taken as $-1$.

Thus,

$$z_1^1 = -\, fQ_2, \qquad z_2^1 \;=\; fQ_1 \qquad (1.33)$$

It follows that

$$z_1(x) \;=\; -\int_x^x Q_2(u)\, f(u)\, du \qquad (1.34)$$
$$z_2(x) \;=\; \int_p^x Q_1(u)\, f(u)\, du \quad (1.35)$$

where the unspecified limits of integration are the equivalent of the arbitrary constants of integration and are determined by the necessity of $y$ satisfying the boundary condition.

Now,

$$a_1 y + b_1 y^1 = a_1\left(z_1 Q_1 + z_2 Q_2\right) + b_1\left(z_1 Q_1{}^1 + z_2 Q_2^1\right) \qquad (1.36)$$

Since $z_1^1\, Q + z_2{}^1\, Q_2 \;=\; 0$

Also $a_1 Q_1(x_1) \; + b_1\, Q_1^1(x_1) = 0 \qquad (1.37)$

Hence,

$$0 = a_1 y(x_1) + b_1 y^1(x_1) = z_2(x_1)\left(a_1 Q_2(x_1) + b_1 Q_2{}^1(x_1)\right)$$
(1.38)

First let us assume that neither $Q_1$ nor $Q_2$ satisfies both boundary condition, hence, it follows that $z_2(x_1) = 0$ and so

$$z_2(x) = \int_{x_1}^{x} Q_1(u)\, f(u)\, du \qquad (1.39)$$

Similarly,

$$\begin{aligned} a_2 y + b_2 y^1 &= a_2\left(z_1 Q_1\ z_2 Q_2\right) + b_2\left(z_1^1 Q_1 + z_1 Q_1^1 + z_2^1 Q_2 + z_2 Q_2{}^1\right) \\ &= a_2\left(z_1 Q_1 + z_2 Q_2\right) + b_2\left(z_1 Q_1^1 + z_2 Q_2^1\right) \\ &= z_2\left(a_2 Q_2 + b_2 Q_2^1\right) + z_1\left(a_2 Q_1 + b_2 Q_1{}^1\right) \end{aligned}$$

Since $a_2\ Q_2(x_2) + b_2 Q^1(x_2) = 0,$ we have

$$0 = a_2 y(x_2) + b_2 y^1(x_2) = z_1(x_2)\left(a_2 Q_1(x_2) + b_2 Q_1{}^1(x_2)\right)$$

Thus, it follows that $z_1(x_2) = 0$ and so

$$\begin{aligned} z_1(x) &= -\int_{x_2}^{x} Q_2(u)\, f(u) du \\ &= \int_{x}^{x_2} Q_2(u)\, f(u) du \quad (1.39) \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= z_1(x) Q_1(x) + z_2(x) Q_2(x) \\ &= Q_1(x)\int_{x}^{x_2} Q_2(u) f(u) du + Q_2(x)\int_{x_1}^{x} Q_1(u) f(u) du \end{aligned}$$

$$y(x) = \int_{x_1}^{x_2} G(x,u)\, f(u) du \qquad (1.40)$$

where

$$G(x,u) = Q_1(u) Q_2(x) \quad x_1 \le u \le x \qquad (1.41)$$

The quantity $G(x,u)$ is termed the Green's fin associated with the operate L and the boundary condition specified.

We would see that the Eigenvalue problem (1.24) defined and the boundary condition (1.25) can be reformulated as the integral equation

$$y(x) + \lambda \int_{x_1}^{x_2} G(x,u)\, r(u)\, y(u) du = 0 \qquad (1.42)$$

by just replacing $f(x)$ by $\lambda r(x)\, y(x)$.

Let us now consider the case where one of the solutions $\phi_1$ and $Q_2$ of $Ly = 0$ do satisfy both boundary condition while the other will not satisfy either boundary condition. Then, following the provided argument, if follows that

$$y(x) = Q(x)\int_x^x \psi(u) + (u)du \quad + \psi(x)\int_x^\beta Q(u)f(u)du \qquad (1.43)$$

where $x$ and $\beta$ are arbitrary and here $\psi(x)$ does not satisfy either boundary conditions.

Since both $y$ and $Q$ satisfy the boundary condition, if follows that

$$0 = a_1 y(x_1) + b_1 y^1(x_1) = \left(a_1\psi_1(x_1) + b_1\psi_1{}^1(x_1)\right)\int_{x_1}^\beta Q(u)f(u)du \qquad (1.44)$$

$$0 = a_2 y(x_2) + b_2 y^1(x_2) = (a_2\psi_2(x_2))\int_{x_2}^\beta Q(u)f(u)du \qquad (1.45)$$

$\psi(x)$ does not satisfy either boundary condition and so if follows that from (1.44) $\beta = x,$ and from (1.45) we have

$$\int_{x_1}^{x_2} Q(u)f(u)du = 0 \quad (1.46)$$

and the solution is only possible when this relation exists between $f$ and $Q.$ Thus, the integral equation formulation becomes

$$y = A\,Q(x) + \int_{x_1}^{x_2} G(x,u)\,f(u)\,du \quad (1.47)$$

Where +e $A = \int_x^{x_1} \psi(u)\,f(u)\,du$ is an arbitrary constant and

$$G(x,u) = Q(u)\psi(x) \qquad x_1 \le u \le x$$

$$= Q(x)\psi(u) \quad x \le u \le x_2 \qquad (1.48)$$

**Example 1.2**

Find an integral equation formulation for the problem defined by

$$\frac{d^2y}{dx^2} + 4y = f(x) \qquad 0 \le x \le \pi/4, \qquad y = 0 \text{ at } x = 0, \text{ and } y = 0 \text{ at } x = \pi/4$$

**Solution**

The solutions of $\frac{d^2y}{dx^2} + 4y = 0$ which satisfy the boundary condition at $x = 0$ and $x = \pi/4$ are $Sin2x$ and $Cos2x$ respectively.

Neither satisfies both boundary conditions.

Let, $\quad y \;=\; w\sin 2x + z\cos 2x$

$\therefore \quad y^1 \;=\; w^1\sin 2x + z^1\cos 2x + 2w\cos 2x \;2z\sin 2x$

$\therefore \quad\quad \;=\; 2w\cos x - 2z\sin 2x \;$ if $\; w^1\sin x + z^1\cos 2x = 0$

$\therefore \quad y^{11} \;=\; 2w^1\cos 2x - 2z^1\sin 2x - 4w\sin 2x - 4z\cos 2x$

Thus,

$$y^{11} + 4y = f$$

becomes

$$2w^1\cos 2x - 2z\sin 2x = f$$

whence,

$$z^1 = -\frac{1}{2} f\ \sin 2x, \qquad w^1 = \frac{1}{2} f\ \cos 2x$$

Thus,

$$z(x) \;=\; -\frac{1}{2}\int_\alpha^x f(u)\sin 2u\,du \;\text{ and }\; w(x) \;=\; \frac{1}{2}\int_\alpha^x f(u)\cos 2u\,du$$

$$\therefore \quad y \;=\; \frac{\sin 2x}{2}\int_\beta^x f(u)\cos 2u\,du \;-\; \frac{\cos 2x}{2}\int_\alpha^x f(u)s2u\,du$$

Now $\quad y = 0\;$ at $\;x = 0,$ so that

$$0 \;= 0 - \frac{1}{2}\int_\alpha^0 f(u)\sin 2u\,du.$$

$\therefore \quad \alpha \;=\; 0$

Also, $\;y = 0\;$ at $\;x = \pi/4,\;$ so that

$$0 \;= \frac{1}{2}\int_\beta^{\pi/4} f(u)\cos 2u\,du \;- 0$$

$\therefore \;\; \beta = \pi/4.$

Thus,

$$y \;= -\frac{1}{2}\sin 2x\int_x^{\pi/4} f(u)\cos 2u\,du - \frac{1}{2}\cos 2x\int_0^x f(u)\sin 2u\,du$$

$$=\; \int_0^{\pi/4} G(x,u)\ f(u)\,du$$

where $\quad G(x,u) \;= \dfrac{-1}{2}\cos 2x\sin 2u \quad 0 \le u \le x$

$$= \frac{-1}{2}\sin 2x \cos 2u \quad x \le u \le \pi/4.$$

**Example 1.3**

Transform the problem defined by

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

when $y = 0$ at $x = 0$ and $y^1 = 0$ at $x = 1$ into integral equation form.

**Solution**

The solution to this problem is

$$y = \sin\frac{(2n-1)\pi x}{2}, \ \lambda = \left[\frac{(2n-1)\pi}{2}\right]^2 \qquad n = 1, 2, 3, \cdots$$

The two solutions $\frac{d^2y}{dx^2} = 0$ which satisfy the boundary conditions are respectively $y = x$ and $y = 1$. (neither satisfies both b.c)

Following through the usual process, it follows that the solution of

$\frac{d^2y}{dx^2} = f(x)$ under the boundary condition specified is

$$y = x\int_1^x f(u)\,du + \int_x^0 u f(u)\,du$$

and so the integral formulation is

$$y(x) = \lambda\int_0^1 K(x,u)\,y(u)\,du$$

where

$$\begin{aligned} K(x,u) &= x \qquad 0 \le x \le u \\ &= u \qquad u \le x \le 1 \end{aligned} \qquad \begin{cases} 1 \le u \le x \\ x \le u \le 0 \end{cases}$$

**Example 1.4**

Transform the problem by $\frac{d^2y}{dx^2} + y = f(x)$

and the boundary condition $y = 0$ at $x = 0$ and $x = \pi$ into integral equation form and indicate what condition must be satisfied by $f(x)$.

**Solution**

Now $\sin x$ satisfies the equation $\frac{d^2y}{dx^2} + y = 0$ and both boundary condition

The second solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ is $\cos x$, and this satisfies neither boundary conditions
Let $y = z \sin x + w \cos x$

Following the same process, it follows that

$$y = \sin x \int^x \cos u \; f(u)du + \cos x \int_x \sin u \; f(u)du$$

Now $y$ is to vanish at $x = 0$, and so the limit of integration on the second integral is zero $y$ must also vanish $x = \pi$ and it follows therefore, that

$$y(\pi) \quad \cos \pi \int_\pi^0 \sin u \; f(u)\, du \quad = 0$$

Thus, for a solution to be possible

$$\int_0^\pi \sin u \; f(u)du \; = 0$$

and $y(x) = A \sin x + \int_0^\pi G(x, u) \, f(u)du$

where $A$ is arbitrary and

$$G(x,u) = -\sin u \cos x \qquad 0 \le u \le u$$
$$= -\sin x \cos u \qquad x \le u \le \pi.$$

## 4.0 CONCLUSION

A Sturm–Lowville differential equation with boundary conditions may be solved by a variety of numerical methods on most occasions; however, there are situations where it becomes necessary to carry out intermediate calculations.

## 5.0 SUMMARY

Ordinary differential equations can be transformed into integral equations.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Transform the problem defined by *y'' – Ky = 0* when y=0 at x=2 and y'=0 at x=4 into integral equation?

2. A Sturm-Lowville type problem can be associated with an expression of the form

$$Ly = \frac{d}{dx}\left( P(x)\frac{dy}{dx}\right) - q(x)y, \quad x_1 \le x \le x_2 \quad (1.21)$$

Write down the form of the second boundary condition when the first is of this form.

$a_1 y(x_1) + b_1 y^1(x_1) = 0$?

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M, Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory": Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis.* Cambridge Mathematical Library.

# UNIT 3 CLASSIFICATION OF LINEAR INTEGRAL EQUATION

## CONTENTS



## 1.0 INTRODUCTION

Integral equations are classified according to Limits of integration, placement of unknown function and nature of known function. These result in Fredholm and Volterra equations on the one hand and integral equations on the other hand. Finally, the homogeneous and non-homogeneous fall into the last class, making a total of eight distinct classes of integral equations.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- classify linear integral equations
- find approximate solutions for integral equations.

## 3.0 MAIN CONTENT

### 3.1 Classification of Linear Integral Equation

Let $K(x, y)$ be a function of two variables $x$ and $y$ defined and let $f(x)$ and $Q(x)$ be two functions of the variable $x$ continuous in the interval $a \le x \le b,$ which are connected by the functional equation

$$f(x) = Q(x) - \lambda \int K(x, y) Q(y) dy \quad (1.49)$$

The functional equation (1.49) is called a linear integral equation of the 2nd kind with the kernel $K(x, y)$. In this equation, every continuous find $Q(x)$ is transformed into another continuous find $f(x)$; the transformation is linear, since to $c_1 Q_1 + c_2 Q_2$, there corresponds to the analogous combination $c_1 f_1 + c_2 \ f_2$.

If the find $f(x)$ vanishes identically, we are dealing with a homogenous integral equation. If a homogenous equation possesses a solution other than the trivial solution $Q = 0$, the solution may be multiplied by an arbitrary constant factor and may therefore, be assumed normalised.
If $Q_1, Q_2, \cdots, Q_n$ are solutions of the homogenous equation, then, all linear combination $C_1 \ Q_1 + \cdots + C_n \ Q_n$ are solutions.

It can also be proved that linearly independent solutions of the same homogenous internal equation are orthornormal. A value $\lambda$ for which the homogenous equation possesses non-vanishing solutions is called an Eigenfunction of the kernel for the Eigenvalue $\lambda$. Their number is finite for each Eigenvalue.

The integral equation $\int_a^b K(x, y) Q(y) dy = f(x)$ (1.50)
1st kind. The integral equation

$$Q(x) = \lambda \int_a^b K(x, y) Q(y) dy + f(x), \quad a \le x \le b \qquad (1.51)$$

is termed a Fredholm equation of the 2nd kind.
If $K(x, y) = 0 \quad y > x,$ (1.52)
the kernel is said to be of Volterra type.

The integral equation

$$\int_a^x K(x, y) Q(y) dy = f(x) \quad a \le x \qquad (1.5)$$

is termed a Volterra integral equation of the 1st Kind.
If $K(x, y) = K(y - x)$, the kernel is said to be of convolution form.

The integral equation

$$Q(x) = \lambda \int_0^x K(x, y) Q(y) dy + f(x) \quad a \le x \qquad (1.54)$$

is termed a Volterra integral equation of the 2nd kind.
In general, it is a Volterra integral equation of the integral equation of the 2nd kind.

If we differentiate equation (1.53) w.r.t $x$, it follows that

$$K(x, x) \ Q(x) + \int_a^x \frac{\partial K(x, y)}{\partial x} Q(y) \, dy = f^1(x) \qquad (1.55)$$

If $K(x, x)$ is non-zero, it is possible to divide through by it, and it is clear that it is an associated Volterra integral equation of the 2nd kind.

The kernel is said to be symmetric.

if $K(x, y) \;=\; -K(y, x)$

The kernel is said to be anti-symmetric

if $K(x, y) \;=\; \overline{K}(y, x)$

The kernel is said to be Hermitian

if $K(x, y) \;=\; \overline{K}(y, x)$

## 3.2 Approximate Solutions

We split the interval into n equal sub-interval, and suppose that we may write approximately

$$K(x, y) \;=\; K_{rs}\left(\frac{r-1}{n} \le x \le \frac{r}{n}, \frac{s-1}{n} \le y \le \frac{s}{n}\right)$$

where $Krs$ are constants.

Similarly, when we write

$$f(x) \;=\; f_r \quad \left(\frac{r-1}{n} \le x\,\frac{r}{n}\right)$$

the equation (1.54) becomes

$$Q(x) \;=\; f_r + \lambda \sum_{s=1}^{n} K_{rs} \int_{\frac{s-1}{n}}^{s/n} Q(y)dy \quad (1.56)$$

$$\frac{r-1}{n} \le \; x \le \frac{r}{m}$$

This shows that $Q$ also will be a step find taking the values $Q_r$, say.

Equation (1.56) becomes

$$Q_r - \frac{\lambda}{n}\sum_{s=1}^{n} K_{rs}\,Q_s \;=\; f_r \qquad (1.57)$$

Let $K$ be the $n \times n$ matrix with elements $\frac{K_{rs}}{n}$ and let $Q$ be $n$ vectors, then, we have W4

$$(I - \lambda K)Q \;=\; f \qquad (1.58)$$

The system has thus been reduced approximately to a set of linear algebraic equations. For these, the theory is well-known and a computational solution is straight forward.

In a sense, the solution of (1.54) may be regarded as the limit of (1.57) as $n \to \infty$.

**Exercises**

Solve the equations:

(i) $Q(x) \; - 2\int_0^1 xy\, Q(y)dy \; = \; x^2$

(ii) $Q(x) \; + \; \int_0^1 e^{x-t} Q(t)dt \; = \; x \qquad Q(0) = 0$

approximately at the parts. $x = 0, \frac{1}{2}, 1$ and $y = 0, \frac{1}{2}, 1$
Compare your results with the exact solution in case (ii).

## 4.0 CONCLUSION

Linear integral equations can be classified into several groups and sub-groups such as: Fredholm, Hermitian, Volterra integral equation and those integral equations which are either symmetric or anti-symmetric.

## 5.0 SUMMARY

Linear integral equations can be classified according to their common characteristics.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. In how many ways can integral equations be classified?
2. What type of integral equation has a fixed (constant) limit of integration?
3. Distinguish a Volterra type of integral equation from a Fredholm integral equation.
4. A homogeneous equation is identically non-zero. True or False?

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M, Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing.* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# MODULE 2

Unit 1 S2 Volterra Integral Equations
Unit 2 Convolution Type Kernels

## UNIT 1 S2 VOLTERRA INTEGRAL EQUATIONS

**CONTENTS**



## 1.0 INTRODUCTION

Volterra integral equations have integration limit which include the variable as opposed to the Fredholm integral in which the integration limits are constants.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recognise Volterra integral equations
- comprehend that there are the three types of Volterra integral equations
- arrive at the Resolvent kernel of a Volterra equation.

## 3.0 MAIN CONTENT

Volterra integrals are characterised by the limit of integration being one variable and of which there are three types. A common solution to Volterra integrals is to employ the formalism known as the Resolvent.

### 3.1 Volterra Integral Equations

A kernel $K(x, y)$is said to be of Volterra type if $K(x, y) = 0, \quad y > x$ (2.1)

There are three types of Volterra integral equations.

These are:

(i) The equation of the first type.

$$f(x) = \int_0^x K(x, y)\, Q(y)\, dy \qquad (2.2)$$

(ii) The equation of the second type.

$$Q(x) = \lambda \int_0^x K(x, y)\, Q(y)\, dy + f(x) \qquad (2.3)$$

(iii) The homogenous equation of the second type.

$$Q(x) = \lambda \int_0^x K(x, y)\, Q(y)\, dy \qquad (2.4)$$

The following properties arise:

(i) It is necessary for consistency in the equation of the first kind i.e. $f(0) = 0$

(ii) Any solution to the equation of the second kind cannot be correct unless $Q(0) = f(0)$

(iii) If $K$ is non-singular, there are no Eigenvalue and Eigenfunctions associated with the homogenous equation (2.4)

(iv) The equation of the first type can be differentiated to give the equivalent equation

$$K(x, x)\, Q(x) + \int_0^x \frac{\partial K(x, y)}{\partial x} Q(y) = f^1(x) \qquad (2.5)$$

**Example 2.1**

Solve the integral equation

$$Q(x) = 3\int_0^x \cos(x - y)\, Q(y)\, du + e^x$$

**Solution**

Here $Q(0) = f(0) = 1$

Differentiating w.r.t $x$, it follows that

$$Q^1(x) = 3\, Q(x) - 3\int_c^x \sin(x - y)\, Q(y)\, dy + e^x$$

Thus, $Q^1(0) = 3Q(0) + 1 \quad 4$

Differentiating w.r.t $x$ again, we have

$$Q^{11}(x) = 3\ Q^1(x) - 3\int_0^x \cos(x - y)\, Q(y)\, dy + e^x$$

$$= 3\, Q^1(x) - Q(x) + 2e^x$$

This equation can simply be solved thus:

$$Q^{11} \ -3\ Q^1(x)\ +\ Q(x)\ =\ 2e^x$$

Consider the homogenous equation

$$Q^{11} - 3\ Q^1 + Q = 0$$

Let $Q\ =\ e^{mx} \Rightarrow m^2 - 3m + 1 = 0, \quad m = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

**Example 2.2**

Solve the integral equation

$$Q(x)\ =\ x + 1 +\ \int_0^x (1 + 2\,(x - y))\, d(y)\, dy$$

**Solution**

Differentiating once, it follows that $(Q(0) = f\,(0) = 1)$

$$Q^1(x)\ \ = + Q(x) + 2\int_0^x Q(y)dy$$

$$Q^1(0)\ =\ 1 +\ Q\,(0)\ = 2$$

Differentiating again, we have

$$Q^{11}(x)\ \ =\ \ Q^1(x)\ \ + 2\,Q(x)$$

i.e. $Q^{11} - Q^1 - 2\,Q\ =\ 0$

Let $Q = e^{mx}, m^2 - m - 2\ = 0 \qquad (m+1)(m+2) = 0$

$\Rightarrow\ \ m\ \ = -\ \ \text{or}\ 2$

$$Q\ \ =\ \ A\,e^{-x} + B\,e^{2x}$$

## 3.2 Resolvent Kernel of Volterra Equation

Let us consider the equation:

$$Q(x)\ - \lambda\int_0^x K\,(x, y)\,Q(y)\,dy\ = f\,(x) \qquad (2.6)$$

We can set about the solution by guessing that at least for small $x$ the integral term will be small. First approximation is then

$$Q_0\,(x)\ \ =\ \ f(x) \qquad (2.7)$$

So that $\int_0^x K(x, y)\,Q(y) \div \int_0^x K\,(x, y)\,Q_0\,(y)dy$

$$=\ \int_0^x K\,(x, y)\,f\,(y)\,dy \qquad (2.8)$$

The second approximation $Q_1(x)$, is then

$$Q_1(x)\ \ =\ \ f(x)\ \ +\ \lambda\int_0^x K(x, y)\,f_0(y)dy$$

$$=\ \ f(x)\ \ +\ \lambda\ \int_0^x K\,(x, y)\,f_0(y)\,dy \ \ (2.9)$$

Repeating the argument, we obtain a sequence of approximations.

$$Q_n(x) = f(x) + \lambda \int_0^x K(x, y)\, Q_{n-1}(y)\, dy \qquad (2.10)$$

Write equation (2.10) in the form

$$Q_n = f + \int K\, Q_{n-1} \qquad (2.11)$$

So that $Q_{n-1} = f + \lambda \int K\, Q_{n-2}$ (2.12)

Therefore, $Q_n - Q_{n-1} = \lambda \int K\,(Q_{n-1} - Q_{n-2})$ (2.13)

Now set $\psi_0 = f = Q_0$ and $\lambda^n \psi_n = Q_n - Q_{n-1}$ (2.14)

then,

$$\lambda^n\, \psi_n = \lambda \int K\, \lambda^{n-1} \psi_{n-1}$$

i.e.

$$\psi_n = \int K \psi_{n-1} \qquad n \geq 1 \qquad (2.15)$$

Now, $Q_0(x) = f(x)$

$$\psi_1(x) = \int_0^x K(x, y)\, f(y)\, dy$$

$$\psi_2(x) = \int_0^x K(x, y)\ \psi_1(y) dy = \int_0^x K(x, z) \psi_1(z) dz$$

$$= \int_0^x K(x, z)\, dz \int_0^z K(z, y)\, f(y)\, dy$$

$$= \int_0^x f(y) dy \int_0^x K(x, z)\, K(z, y)\, dz$$

$$= \int_0^x f(y)\, K_2(x, y) \qquad (2.16)$$

where $K_2 = \int_y^x (x, z)\, K(z, y)\, dz$

By repetition of the argument, we have

$$\psi_n(x) = \int_0^x K_n(x, y)\, f(y) dy \qquad (2.17)$$

Where $K_1(x, y) = K(x, y)$ and

$$K_{n+1}(x, y) = \int_y^x K(x, z)\, K_n(z, y)\, dz \qquad (2.18)$$

Also, from equation $\lambda^n \psi_n = Q_n - Q_{n-1}$, so that

$$\sum_{r=0}^{n} \lambda^r \psi_r = (Q_n - Q_{n-1}) + (e_{n-1}\ Q_{n-2}) + \cdots + (Q_1 - Q_0) + Q_0$$

$$= Q_n \qquad (2.19)$$

$$\therefore \quad Q_n \quad \sum_{r=0}^{n} \lambda^r \psi_r \qquad (2.20)$$

By considering equation (2.20), (2.17), we have

$$Q_n = f(x) + \int_0^x \left( \sum_{r=1}^{n} \lambda^r\ K_r\ (x, y) \right) + (y)\, dy$$

$(\Psi_0 = Q_0)$ (2.21)

Thus, it is plausible to suppose that

$$Q(x) = \lim_{n\to\infty} Q_n(x)$$

$$= \quad f(x) \quad + \quad \int_0^x \left[\sum_{r=1}^{\infty} K_r(x, u)\right] + f(y)dy \qquad (2.22)$$

$$= \quad f(x) \quad -\lambda \int_0^x R(x, y, \lambda)\, f(y)dy \quad (2.23)$$

where $R(r, y) = -\sum_{r=0}^{\infty} \lambda^r K_{r+1}(x, y).$ (2.24)

The function $R$ is called the Resolvent kernel.

Let us now determine the conditions under which the power series on the right hand side of equation is convergent.

Suppose that over $0 \le x, \quad y \le l, \left|K(x, y)\right| \le K$

Then,

$$\left|K_2(x, y)\right| = \left| \int_y^x K(x, z)\, K(z, y)\, dz \right|$$

$$\le \quad K^2(x-y) \quad = \quad (x-y)K^2 \quad x \ge y \qquad (2.25)$$

Also $K_2(x, y) = 0, \qquad y \ge x$

Similarly,

$$\left|K_3(u, y)\right| \quad = \quad \left|\int_y^x K_2(x, z)\, K(z, y)\, dz\right|$$

$$\le K^3 \int_y^x (x-z)dz = \frac{1}{2} K^3 (x-y)^2 \quad x \ge y \quad (2.26)$$

and $K_3(x, y) = 0, \qquad x \le y$

Proceeding in this way, it follows that:

$$\left| K_n(x, y)\right| \le \frac{1}{(n-1)}1 \;\; K^n (x-y)^{n-1} \quad x \ge y \quad (2.27)$$

$$= \quad 0 \qquad x \le y$$

Thus, the series $\lambda^n K_n(x, y)$ is dominated by the series with the n^th term

$$\frac{\lambda^n}{(n-1)!} \; K^n (x-y)^{n-1} \quad (2.28)$$

now $\left|x-y\right| \le 2l,$ and so the later series is dominated by the series with n^th term

$$\frac{\lambda^n K}{(n-1)!}(2lK)^{n-1} \qquad (2.29)$$

This is the typical term of an exponential series and so it follows that the series 2.23 for $R(x, y, \lambda)$ always converge.

The uniqueness of the solution follows easily because, if $Q_A(x)$, $Q_B(x)$ are both solution, then,

$$Q_A(x) \quad - Q_A(x), \quad = \quad \int_0^x K(x, y)(Q_A(y) Q_B(y)dy) \qquad (2.30)$$

Since the resolvent kernel series converges for all values of $\lambda$ is the original kernel $n$ bounded.

This is equivalent to saying that there is no Eigenvalue. Thus, $Q_n(x) = Q_B(x)$

**Example 2.3**

Solve the integral equation

$$Q(x, y) \quad = \quad f(x, y) + \int_0^x \int_0^y \exp(x - u + y + v) Q(u, v) \; d_u d_v$$

$$K_1(x, y; u, v) = \exp(x - u + y - v)$$

$$K_2(x, y; u, v) = \int_u^x \int_v^u K(x, y, x^1, y^1) \quad K(x^1, y^1, u, v) dx^1 dy^1$$

$$= \quad \exp(x - u + y - v) \quad \int_u^x dx^1 \int_v^y dy^1 = (x-u)(y-v) \exp \quad (x - u + y - v)$$

Similarly,

$$K_3(x, y; u, v) = \quad \int_u^x \int_v^y \exp(x - u + y - v)\left[(x - x^1)(y - y^1)\right] dx^1 dy^1$$

$$= \quad \exp(x - u + y - v) \int_v^y \left(xx^1 - \tfrac{x^{12}}{2}\right) \Big|_u^x \left(y - y^1\right) dy^1$$

$$= \exp(x - u + y - v)\left[\left(x^2 - \frac{x^2}{2} - xu + \frac{u^2}{2}\right)\left(y^2 - \frac{y^2}{2} - vy + \frac{v^2}{2}\right)\right]$$

$$= \quad \frac{1}{2^2}(x-u)^2 \; (y-v)^2 \quad \exp(x - u + y - v)$$

Hence,

$$K_n(x, y; u, v) \quad = \quad \frac{(x-u)^{n-1}(y-v)^{n-1}}{[(u-1)!]^2} \exp(x - u + y - v)$$

and so

$$R(x, y; u, v,) \quad = \quad -\sum_{u-1}^{\infty} K_n(x, y; u, v)$$

$$= \quad -\exp(x - u + y - v) \sum_{u=0}^{\infty} \frac{(x-u)^u \; (y-v)^x}{(u!)^2}$$

The solution is therefore, given as

$$\mathrm{Q}(\mathrm{x}, \mathrm{y}) \quad = \quad f(x, y) - \int_o^x \int_o^y R(x, y; u, v) \quad f(u, v) dudv$$

## 4.0 CONCLUSION

Certain properties arise as a consequence of the three types of Volterra integrals.

## 5.0 SUMMARY

There are three types of Volterra integral equations, and can be solved using the Resolvent kernel.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. How many different type of Volterra Integrals are there. 1, 2, 3 or 4?
2. Which of these three is a Volterra integral equation of the first type?

$$f(x) = \int_0^x K(x, y)\,Q(y)\,dy$$

$$Q(x) = \lambda \int_0^x K(x, y)\,Q(y)\,dy$$

$$Q(x) = \lambda \int_0^x K(x, y)\,Q(y)\,dy + f(x)$$

3. And which is a Volterra integral of the third type?

## 7.0 REFERENCES/FURTHER READING


Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# UNIT 2 CONVOLUTION TYPE KERNELS

## CONTENTS



## 1.0 INTRODUCTION

Laplace transformation serves as a powerful tool in the solving of integral equations. Convolution, the inverse of Laplace transformation, is necessary to transform the solution back to the originating domain.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe how convolution type kernels of the Volterra integral can be solved using Laplace transform
- solve Fredholm equations
- identify a Neumann series.

## 3.0 MAIN CONTENT

## 3.1 Convolution Type Kernels

If the kernel of the Volterra integral is of the form $K(x-y)$, the equation is said to be of convolution type and may be solved by using the Laplace transform. The method of solution depends upon the well known result in Laplace transform that:

$$\int_o^\infty e^{-px} \int_o^x F(x-y)\,G(y)dy\,dx$$

$$= \int_o^\infty e^{-px} F(x)dx \int_o^\infty e^{-px} G(x)\,dx \quad (2.36)$$

The term $\int_o^x F(x-y)\,G(y)\,du = \int_o^x F(y)\,G(x-y)\,dy \qquad (2.37)$

is the convolution, (faltung) of the two functions $F(x)$ and $G(x)$.

Let us denote $\int_o^\infty e^{-px}\, G(x)\,dx$, the Laplace transform of $G(x)$ by $\overline{G}$ .

Consider the integral equation of the first kind.

$$f(x) = \int_0^x K(x-y)\, d(y) dy \qquad (2.38)$$

On taking the Laplace transform, it follows that,

$$\overline{F} = \overline{K}\ \overline{Q} \qquad (2.39)$$

Thus $\overline{Q} = \bar{f}/\bar{k}$ , (2.40) provided the transforms exist.

The solution is found by finding the inverse transform of $\overline{Q}$. It is also possible to solve the inhomogeneous Volterra equation of the 2[nd] kind with the convolution kernels in exactly the same way.

The equation

$$Q(x) = f(x) + \int_o^x k(x-y)\, Q(y)\, dy \text{ transforms into}$$

$$\overline{Q} = \overline{F} + \overline{K}\ \overline{Q}$$

where $\overline{Q} = \left(1-\overline{K}\right)^{-1} \bar{f} \qquad (2.41)$

and $Q(x)$ may be found.

**Example 2.5**

Solve the integral equation

$$\int_o^x \sin x\,(x-y)\, d(y)\, dy = 1-\cos\beta x$$

Note that the equation in self-consistent

Taking the Laplace transform, we have

$$\frac{\alpha}{p^2+\alpha^2}\ \overline{Q} = \frac{1}{p} - \frac{p}{p^2+p^2} = \frac{p^2+\beta^2-p^2}{p\left(p^2+\beta^2\right)}$$

Thus, $\overline{Q} = \dfrac{\beta^2\left(p^2+\alpha^2\right)}{\alpha\, p\left(p^2+\beta^2\right)} = \dfrac{\alpha}{p} + \left(\dfrac{\beta^2-\alpha^2}{\alpha}\right)\dfrac{p}{p^2+\alpha^2}$

Therefore, $Q = \alpha + \left(\dfrac{\beta^2-\alpha^2}{\alpha}\right)\cos\beta x$

**Example 2.6**

Solve the integral equation

$$\int_o^x Q(x-y)\left[Q(y)-2\sin ay\right] dy = x\cos\, ax$$

**Solution**

Taking the transform if follows that

$$\overline{Q}\left\{\overline{Q} - \frac{2a}{p^2+a^2}\right\} = \frac{p^2-a^2}{\left(p^2+a^2\right)^2}$$

$$(\mathbf{NB})\int\left(x^2 \; f(x)\right) = (-1)^u \frac{d^u}{dp^n}\,|f\,(f))$$

Thus, $\overline{Q} = \sin ax \pm \cos ax$ are the two possible solutions

**Example 2.7**

Solve the integral equation

$$Q(x) = x^3 + \int_o^x e^{3(x-y)}\, Q\,(y)\, dy$$

**Solution**

It follows that

$$\overline{Q} = \frac{3!}{p4} + \frac{1}{p-3}\,\overline{Q}$$

$$\therefore \quad \overline{Q} = \left[1 - \frac{1}{p-3}\right]^{-1} \frac{3!}{p4}$$

Hence, $\overline{Q} = \dfrac{p-3}{p-4} \cdot \dfrac{3!}{p4} = \dfrac{3!}{p4}\left(1 + \dfrac{1}{p-4}\right)$

$$= \frac{3\,!}{p4} + \frac{3!}{p4(p-4)}$$

$$\therefore \quad Q = x^3 + \int_0^x e^{4(x-y)}\, y^3\, dy$$

**Example 2.8**

Solve the integer-differential equation

$$Q^{11}(x) + \int_o^x e^{2(x-y)}\, Q^1(y) dy = 1$$

where $Q(o) = o, \quad Q^1(o) = o$

**Solution**

Taking the Laplace transforms, it follows that

$$P^2\overline{Q} + \frac{P\,\overline{Q}}{P-2} = \frac{1}{P} \qquad \left(\overline{Q} = \frac{p-2}{p^2(p-1)^2}\right)$$

and $\overline{Q} = \dfrac{1}{p^2(p-1)^2} = \dfrac{1}{(p-1)^2} - \dfrac{2}{p-1} + \dfrac{1}{p^2} + \dfrac{2}{p}$

Hence,

$$Q(x) = xe^x - 2e^x + x + 2$$

Cossection

$$\overline{Q}\left[p^2 + \frac{p}{p-2}\right] = \frac{1}{p}$$

i.e. $$\overline{Q}\left[\frac{p(p-1)^2}{p-2}\right] = \frac{1}{p}$$

$$\Rightarrow \quad \overline{Q} = \frac{p-2}{p^2(p-1)^2} = \frac{-3}{p} - \frac{2}{p^2} + \frac{3}{p-1} - \frac{-1}{(p-1)^2}$$

$$\therefore \quad Q(x) = -3 - 2x + 3e^x - x\ e^x$$

[Reuse partial fraction]

**NB:** $$\frac{p-2}{p^2(p-1)^2} \equiv \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2}$$

$$\Rightarrow \quad p-2 \equiv Ap^3 - 2p^2A + Ap + Bp^2 - 2Bp + B + cp^3 - cp^2 + Dp^2$$

## 3.2 Fredholm Equations

The Volterra equations considered are a special case of the equation.

$$Q(x) - \lambda \int_0^1 k(x,y)\,Q(y)\,dy = f(x) \quad (o \le x \le)(3.1)$$

Evidently, the special case is where $k(x, y) = o$ for $y \ge x$

We shall take the interval $(0, 1)$ as standard and for simplicity write the integrals without the limit.

Put $$Q(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \qquad (3.2)$$

where $$\psi_1(x) = \int k(x, y)\ f(y)\,dy$$

$$\psi_2(x) = \int k(x, y)\,\psi_1(y)\,dy$$

$$= \int k(x, y)\,dy \int k(y, z)\,f(z)\,dz$$

$$= \int k_2(x, y)\,f(y)\,dy$$

and $$K_2(x, y) = \int K(x, z)\ K(z, y)dz$$

$$\psi_u(x) = \int K_u(x, y)\,f(y)\,dy$$

where

$$K_u(x, y) = \int K_{uy}(x, z)\ k(z, y)\,dz$$

The series (3.2) is called the Neumann series just as we consider the series for the Resolvent kernel.

$$-R(x, y; \lambda) = K(x, y) + \lambda\ k_2(x, y) + \qquad (3.3)$$

This series may be proved convergent for a certain sample of values of $\lambda$ under a variety of conditions. We consider one set of these conditions.

## 3.3 Lemma 3.1

Suppose $K(x, y)$ is continuous and

$$\sup_{\substack{o \le x \le 1 \\ o \le y \le 1}} \left| K(x, y) \right| = M$$

Then, the series (3.3) is uniformly convergent for $|\lambda| < M^{-1}$. It is continuous and the series may be integrated term by term. Also, $R(x, y; \lambda)$ is for each $(x, y)$ an analytic function of the complex variable $\lambda$ inside $|\lambda| < M^{-1}$.

**Proof**

We have

$$\begin{aligned} |K_2 (x, y)| &= \left| \int_o^2 K(x, x) K(z, y)\, dz \right| \\ &\le \sup | K(x, z) K(z, y)| \\ &\le \sup|K(x, z)| \sup |K(z, y)| \le M^2 \end{aligned}$$

By repeating this, we get

$$|K_n (x, y)| \le M^n \quad (3.4)$$

Then, the series (3.3) is dominated by $\sum \lambda^n M^n$. The result follows as before by Weierstrass $M$. test in region $|\lambda M| < 1$. The analyticity is obvious since we are considering each $(x, y)$ the given series $\sum a_n \lambda_n$ where $a_n = K_n (x, y)$. The radius of convergence is not less than $M^{-1}$. Note that in this case we have only proved convergence for $|\lambda| < M^{-1}$, whereas the Volterra equations are true for all $\lambda$ finite.

**Example 3.1**

Consider the integral equation:

$$Q(x) - \lambda \int_o^1 Q(y) dy = f(x)$$

In this case, $K(x, y) = 1$ and $K_n (x, y) = 1$

Thus, $R(x, y; \lambda) = -\sum_{r=o}^{\infty} \lambda^r = \dfrac{1}{\lambda - 1}$

Also, $\sup|K(x, y)| = 1$. Since $\frac{1}{\lambda - 1}$ has a pole at 1, the result may not in general be extended to smaller $M$

If $\quad A = \int_o^1 Q(x)\,dx$, and integrate over $(0, 1)$, the equation

$$Q(x) - \lambda\, Q(y)\,dy = f(x)$$

$$\therefore \quad A(1-\lambda) = \int_0^1 f(x)\,dx \quad \Rightarrow A = \frac{1}{1-\lambda}\int_0^1 f(x)\,dx$$

Suppose first that $\lambda \neq 1$. Then,

$$Q(x) = f(x) + \lambda A = f(x) + \frac{\lambda}{1-\lambda}\int_o^1 f(x)dx$$

The equation had thus a unique solution.
Suppose on the other hand that $\lambda = 1$.
Then, from the equation $A(1-\lambda) = \int_o^1 f(x)\,dx$ the original equation will only have a solution if $\int_o^1 f(x)\,dx = 0$.

If $f$ does not satisfy this condition and $\lambda = 1$, the equation has an infinite number of solutions $Q(x) = f(x) + c$ where $c$ is a constant and $\lambda = 1$ if an Eigenvalue with corresponding Eigenfunction $Q =$ constant.

**Theorem 3.1**

Suppose $K$ is continuous in the square $\begin{array}{l} S : o \le x \le 1 \\ \quad o \le y \le 1 \end{array}$ and set $\sup_s |K| = M$.

The Resolvent kernel R is given by

$$-R(x, y; \lambda) = \sum_{r=0}^{\infty} \lambda^r K_{r+1}(x, y) \quad (3.5)$$

Where the series is uniformly convergent for $|\lambda| < m^{-1}$

$R$ is continuous and the series may be integrated term by term. In the domain $\oiint$ where $\oiint$ is analytic. The following relation holds

$$K(x, y) + R(x, y; \lambda) = \lambda \int K(x, z)\,R(z, y; \lambda)\,dz$$

$$= \lambda \int R(x, z; \lambda)\; K(z, y)\,dz$$

Suppose that $f$ is integrable, then, the unique solution for $\lambda \in \oiint$ of (3.1) is

$$Q(x) = f(x) - \lambda \int_0^1 R(x, y; \lambda)\, f(y)dy$$

## 4.0 CONCLUSION

Convolution type integrals may be solved by the use of Laplace transform provided the transform exists.

## 5.0 SUMMARY

It is possible to determine if a Volterra integral is of the convolution type and then solve it using the method of Laplace where the final solution is found by finding the inverse transform. This applies also to the inhomogeneous Volterra equation of the 2nd kind which convolution kernels can be solved in exactly the same way.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. State the name of the integral equation in which the integration limits are constants and do not include the variable?
2. What is the relationship between F(x), G(x) and this term?

$$\int_o^x F(x-y)\,G(y)\,du = \int_o^x F(y)\,G(x-y)\,dy$$

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing.* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# MODULE 3



## UNIT 1 FREDHOLM EQUATIONS WITH DEGENERATE KERNELS

**CONTENTS**



## 1.0 INTRODUCTION

Fredholm integral equations are integral equations in which the integration limits are constants which do not include the variable; and whose solution gives rise to Fredholm theory, the study of Fredholm kernels and Fredholm operators.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Solve Fredholm equations with degenerate kernels
- Derive the general method of solution of Fredholm equations.

## 3.0 MAIN CONTENT

### 3.1 Fredholm Equations with Degenerate Kernels

Consider the Kernel of the form:

$$K(x, y) = \sum_{p=1}^{n} a_p(x) b_p(y) \quad (3.6)$$

where $x$ is finite, and the $a_r$ and $b_r$ form linearly independent sets. A kernel of this character is termed a degenerate kernel.
Also, consider the integral equation of the first kind

$$f(x) = \int K(x, y)\,Q(y)\,dy$$

$$= \sum_{p=1}^{n} a_p(x) \int b_p(y)\,Q(y)\,dy \quad (3.7)$$

1. We note that no solution exist unless $f(x)$ can be written in the form $\sum_{p=1}^{n} f_p\, a_p(x)$ (3.8)

   This is essential for the equation to be self-consistent.
2. The solution is indefinite by any function $\psi(y)$ which is orthogonal to all the $b_p(y)$ over the range of integration.

**Example 3.2**

The integral equation
$\exp(2x) = \int_{\pi}^{\pi} \sin(x+y)\,\phi(y)dy \quad o \le x \le \pi$ is not self-consistent and so does not have a solution.

This is because

$$\int_{\pi}^{\pi} \sin(x+y)\ \phi(y)\,dy = \sin x \int_{o}^{\pi} \cos y\, \phi(y)dy$$

$$+ \cos x \int_{\pi}^{\pi} \sin y\, \phi(y)dy$$

which is a of form $A\sin x + B\cos x$

## 3.2 The General Method of Solution

Look for a solution of the form

$$\phi(y) = \sum_{q=1}^{n} \phi_q\, b_q(y) \qquad (3.9)$$

If it exists, it will be a solution and if it is possible to add $\psi(y)$ to it.

The solution proceeds as follows in the integral equation.

$$f(x) = \lambda \int K(x, y)\,\phi(y)\,dy \qquad (3.10)$$

$$\sum_{p=1}^{n} f_p\, a_p(x) = \lambda \sum_{p=1}^{n} a_p(x) \int b_p(y) \sum \phi_q b_q(y)dy \qquad (3.11)$$

$$= \sum_{p=1}^{n} a_p(x) \sum_{q=1}^{n} \beta_{pq}\, \phi_1 \qquad (3.12)$$

Where $\beta_{pq} = \lambda \int b_p(y)\, b_q(y)\,dy$ (3.13) and so the $\phi_s$ are defined by

$$f_p \quad = \quad \sum_{q=1}^{n} \beta_{pq}\, \phi_q \qquad 1 \le p \le n \qquad (3.14)$$

Since the $b_p$ are linearly independent, the determinant $|\beta_{pq}|$ does not vanish and the $\phi_q$ can be found uniquely. Also, $\psi(y)$ in such that

$$\int \psi(y)\, K(x, y) dy \quad = \quad 0 \qquad (3.15)$$

**Example 3.3**

Consider the solution of the integral equation

$$3\sin x + 2\cos x \quad = \quad \int_{-\pi}^{\pi} \sin\,(x+y)\,\phi(y)dy \quad -\pi \le x \le \pi$$

Now $\sin\,(x+y) \;=\; \sin x \cos y + \sin y \cos x$
and so there is consistency

Note also that $\int_{-\pi}^{\pi} \cos y \cos my\; dy \quad = \begin{cases} o \text{ if } & m \ne 1 \\ \pi \text{ if } & m = 1 \end{cases}$

$$\int_{-\pi}^{\pi} \cos y \sin my\; dy \quad = \quad \begin{cases} o \;\text{ if }\; m \ne 1 \\ \pi \text{ if }\; m = 1 \end{cases}$$

$$\int_{\pi}^{\pi} \sin y \cos my\; dy \quad = \quad \begin{cases} 0 \;\text{ if } \; m \ne 1 \\ \pi \;\text{ if }\; m = 1 \end{cases} \quad \int_{-\pi}^{\pi} \sin y \sin my\; dy = \begin{cases} o \;\text{ if } \; m \ne 1 \\ \pi \text{ if }\; m = 1 \end{cases}$$

Hence, the integral equation in indefinite by a quantity of the form

$$\psi(y) \quad = \quad C_o \quad + \quad \sum_{n=2}^{\infty} \left[C_n \cos ny + dn \sin ny\right]$$

Since $\int_{-\pi}^{\pi} \psi(y)\; \sin\,(x+y)\, dy = \; o$

Now, look for a solution of the form

$$\phi(y) \quad = \quad A\cos y \quad + \quad B\sin y$$

$$\begin{aligned} \therefore \quad \int_{-\pi}^{\pi} \sin(x+y)\,\phi(y)dy \quad &= \sin x \int_{-\pi}^{\pi} \cos y\,(A\cos y + B\sin y)dy \\ &+ \quad \cos x \int_{-\pi}^{\pi} \sin y \;\; (A\cos y + B\sin y)dy \\ &= \quad \prod A \;\; \sin x + \pi B \cos x \\ &\equiv \quad 3\sin x + 2\cos x \end{aligned}$$

Thus, $A \;=\; {}^{3}\!/_{\pi}$ and $B \;=\; {}^{2}\!/_{\pi}$

$\therefore \quad d(y) \quad = \; (3\cos y \; + \; 2\sin y)\,/\pi$

Note that the process is similar to the idea of finding the particular integral and complementary function in differential equation theory.

The solution

$$\phi(y) = (3\cos y + 2\sin y)/\pi$$

May be termed a particular solution while the $\psi(y)$ a complementary function.

## 4.0 CONCLUSION

Fredholm equations can be solved by applying the method of degenerate kernel.

## 5.0 SUMMARY

Fredholm integral equations have limits which are constants and not the variable as in the Volterra integral equations.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. What kind of kernel is of the form $K(x, y) = \sum_{p=1}^{n} a_p(x) b_p(y)$ where $x$ is finite, and $a_r$ and $b_r$ form linearly independent sets?
2. Why does $\exp(2x) = \int_{\pi}^{\pi} \sin(x + y)\phi(y)dy \quad o \le x \le \pi$ not have a solution

## 7.0 REFERENCES/FURTHER READING


Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing*. (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# UNIT 2 EIGENFUNCTIONS AND EIGENVECTORS

## CONTENTS



## 1.0 INTRODUCTION

Many homogeneous linear integral equations may be viewed as the continuum limit of Eigenvalue equation.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- work with Eigenfunctions and Eigenvectors
- prove that symmetric and continuous Kernels that are not identically zero possess at least one Eigenvalue.

## 3.0 MAIN CONTENT

### 3.1 Eigenfunctions and Eigenvectors

Eigenfunction and Eigenvectors associated with the equation:

$$\phi(x) \;=\; \lambda\int\sum_{p=1}^{n} a_p(x)\, b_p(y)\, d(y)dy \quad (3.16)$$

can be found as follows
Rewrite (3.16) in the form

$$\mu\,\phi(y) \;=\; \int\sum_{p=1}^{n} a_p(x)\, b_p(y)\, \phi(y)\, dy \quad (3.17)$$

This equation satisfied by any function $\phi(y)$ such that

$$\int b_p(y)\,\phi(y)\,dy \;=\; o \qquad (3.18)$$

and $\mu = o$, but in general, we shall ignore such functions, any Eigenfunction must be of the form

$$\Phi(x) \;=\; \sum_{p=1}^{n} \phi_p\; a_p(x) \qquad (3.19)$$

Thus $\sum_{p-1}^{n} \phi_p \; a_p(x) \;\; = \;\; \lambda \sum_{p=1}^{n} a_p(x) \int b_p(y) \sum_{q=1}^{n} \phi_q \; a_q(y) dy$ (3.20)

Whence $\qquad \phi_p \; = \; \sum_{q=1}^{n} \phi_q \; K_{pq} \qquad$ (3.21)

$\qquad K_{pq} \; = \; \lambda \int b_p(y)\, a_q(y)\, dy \qquad$ (3.22)

**Example 3.4**

Find the Eigenvalue and Eigenfunction of the system defined by:

$\phi(x) \;\; = \;\; \lambda \int_o^1 (1 + xt)\, \phi(t)\, dt \qquad o \leq x \leq 1$

**Solution**

Let $\Phi(x) \; = \; \phi_o + \phi_1 x \; = \lambda \int_o^1 (1 + xt)(\phi_o + \phi_1 t) dt$

$$= \;\; \lambda\left(\phi_o + \frac{\phi_1}{2}\right) + \lambda\left(\frac{\phi_o}{2} \; + \; \frac{\phi_1}{3}\right) \; x$$

Whence (equating coefficients)

$$(\lambda - 1)\,\phi_o + \lambda\,\frac{\phi_1}{2} \;\; = \;\; o$$

$$\frac{\lambda\,\phi_o}{2} \; + \; \left(\frac{\lambda}{3} - 1\right)\phi_1 \;\; = \;\; o$$

Thus, $\;(\lambda - 1)\left(\lambda/3 - 1\right) \;\; = \;\; \lambda^2/4$

$$\lambda \;\; = \;\; 8 \pm \sqrt{52}$$

and $\qquad \phi_1 \, , \phi_o \;\; = \;\; -\left(7 \pm \sqrt{52}\right) : \;\; \left(4 \pm \sqrt{13}\right)$

Consider now the solution of the integral equation

$$\phi(x) \;\; = \;\; \lambda \int K(x, y) \;\; \phi(y)\, dy \; + \; f(x) \qquad (3.23)$$

Where in this case $K$ in degenerate, any solution will be of the form:

$$\phi(x) \;\; = \;\; \lambda \sum_{p=1}^{n} a_p(x)\,\phi_p \; + \; f(x)$$

(Substituting, we have)

$$= \;\; \lambda \int \sum_{p=1}^{n} a_p(x)\, b_p(y)\, \phi(y) dy \; + \; f(x)$$

$$= \;\; \lambda \sum_{p=1}^{n} a_p(x)\, b_p(y)\left[\lambda \sum_{q=1}^{n} \phi_q a_q(y) + f(y)\right] dy \; + \; f(x)$$

$$= \;\; \lambda \sum_{p=1}^{n} a_p(x)\, \lambda \sum_{q=1}^{n} \int b_p(y)\, a_q\, (y)\, \phi_q \; dy$$

$$+ \quad \sum_{p=1}^{n} a_p(x) \int b_p(y)\, f(y)\, dy + f(x)$$

Set $\quad k_{pq} \quad = \quad \lambda \int b_p(y)\, a_q(y)\, dy$

and $\quad f_p \quad = \quad \int b_p(y)\, f(y)\, dy$

Thus, $\phi(x) \quad = \quad \lambda \sum_{p=1}^{n} a_p(x)\, \phi_p \quad + \quad f(x)$

$$= \quad \lambda \sum_{p=1}^{n} a_p(x) \left[ \lambda \sum_{q=1}^{n} k_{pq}\, \phi_q \, + \, f_p \right] + f(x)$$

$\therefore \quad \phi_p \quad = \quad \lambda \sum_{q=1}^{n} K_{pq}\, \phi_q + f_p \qquad (3.24)$

i.e. $\phi_p \; - \; \lambda \; \sum_{q=1}^{n} K_{pq}\, \phi_q \quad = \quad f_p \qquad (3.25)$

The above equations is a finite system of linear algebraic equations with matrix $A = (k_{pq})$.

The solution depends on whether or not det $(\mathrm{I} - \lambda A)$ is zero.

Set $\quad \wp(\lambda) \quad = \quad \det(\mathrm{I} - \lambda A)$

Then, $\wp(\lambda)$ is a polynomial of degree $n$. If $\lambda$ is not a roof of $\wp(\lambda)$, then, (3.23) has a unique solution.

If you write $d_{pq}$ for the cofactors you will have:

$$\phi_q \quad = \quad \frac{1}{\wp(\lambda)} \sum_{p=1}^{n} d_{pq}\, f_p \qquad (3.26)$$

The solution of equation (3.23) is then,

$$\phi(x) \;=\; f(x) \;+\; \lambda[\wp(\lambda)]^{-1} \sum_{p-1}^{n} a_p(x)\, d_{pq}(\lambda) \int \sum_{q=1}^{n} b_q(y)\, f(y)\, dy$$

$$= \quad f(x) \; - \; \lambda[\wp(\lambda)]^{-1} \int \wp(x, y; \lambda)\, f(y)\, dy \qquad (3.27)$$

Where

$$\wp(x, y; \lambda) \quad = \quad -\sum_{p=1}^{n} \sum_{q=1}^{n} d_{pq}\, a_p(x)\, b_q(y) \qquad (3.28)$$

and $R(x, y; \lambda) \quad = \quad + \dfrac{\wp(x, y; \lambda)}{\wp(\lambda)} \qquad (3.29)$

is the Resolvent kernel

i.e. $\phi(x) \quad = \quad f(x) - \lambda \int R(x, y; \lambda)\, f(y)\, dy \quad (3.30)$

**Examples 3.5**

Solve the integral equation:

$$\phi(x) = \lambda\int_o^1 (1+xt)\,\phi(t)\,dt + f(x)$$

Let $\phi(x) = \phi_o + \phi_1 x + f(x)$

$$= \lambda\int_o^1 (1+xt)\left[\phi_o + \phi_1 t + f(t)\right]dt + f(x)$$

$$= \lambda\left(\phi_o + \frac{\phi_1}{2} + f_o\right) + \lambda x\left(\frac{\phi_o}{2} + \frac{\phi_1}{3} + f_1\right) + f(x)$$

Where $f_r = \int_o^1 t^r f(t)\,dt$

Equating powers of $x$ and solving for $\phi_o$ and $\phi_1$ if follows that:

$$\phi_o\left(\lambda^2 - 16\lambda + 12\right) = \left[-4\lambda(\lambda-3)\,f_o + 6\lambda^2\,f_1\right]$$

$$\phi_1\left(\lambda^2 - 16\lambda + 12\right) = \left[6\lambda^2\,f_o - 12\lambda(\lambda-1)\,f_1\right]$$

The Eigenvalue, are given by the roots of the equation

$$\lambda^2 - 16\lambda + 12 = 0$$

If $\lambda$ is one of the Eigenvalue, say $8+\sqrt{52}$, a solution is possible only if

$$O = \int_o^1 f(x)\left[\frac{8+\sqrt{52}}{2} - \left(7+\sqrt{52}\right)x\right]dx$$

and the solution is indefinite by an arbitrary multiple of $4+\sqrt{13}-\left(7+\sqrt{52}\right)x$

## 3.2 Symmetric Kernels

$K(x, y) = k(y, x)$ and $K$ is continuous

**Theorem 3.2**

Let $K(x, y)$ be symmetric and continuous (and not identically zero). Then, $K$ has at least one Eigenvalue.

**Proof**

We note that the iterated kernels $k_u(x, y)$ are also symmetric and not identically zero.

Suppose result is not true.

Let us assume that $R(x, y;\lambda)$, the Resolvent kernel is an integral function and the series is also convergent for all, $\lambda$. may be integrated term by term.

Now set, $U_n \quad = \quad \int K_n\,(x, x)\,dx$

Then, $U_2\lambda^2 \quad + \quad U_4\lambda^4 \quad + .......$ is absolutely convergent.

Now, we have $\quad U_{n+m} \quad = \quad \iint K_n\,(x, z)\;\; K_m\,(z, x)\,dxdx$

and $\quad U_{2n} \quad = \quad \iint K_n^2\,(x, z)\,dxdz$

(3.40)

Now, $\iint [\alpha\; K_{n+1}\,(x, z) - \beta\; K_{n-1}\,(x, z)]^2 \quad dxdz \geq o$

i.e.

$\alpha^2\; U_{2n+2} - 2\alpha\beta\, U_{2n} + \beta^2 U_{2n-2} \geq o$ (3.41)

for all real $\alpha, \beta$

$\therefore \quad U_{2n}^2 \quad \leq \quad U_{2n+2}\, U_{2n-2}$ (3.4.2)

Form equation (3.40) none of $U_{2n}$ is zero as $K_n$ is not identically zero.

$$\therefore \quad \frac{U_{2n+2}}{U_{2n}} \quad \geq \quad \frac{U_{2n}}{U_{2n-2}}$$

Now, consider series $\sum \lambda^{2n} U_{2n}$ assumed convergent.

The ratio of term is

$$\frac{U_{2n+2}\lambda^{2n+2}}{U_{2n}\kappa^2} \quad = \quad \frac{U_{2n+2}}{U_{2n}}\,\lambda^2$$

This ration is $\geq \quad \frac{U_4}{U_2}\,\lambda^2$ from (3.43)

Thus, for $\frac{U_4}{U_2}\,\lambda^2 \geq 1,$ the forms in the (3.44) series are non-increasing, so as it is a series of positive terms, the series is divergent. This is a contradiction.

Thus, we have seen that poles of $R(x, y; \lambda)$ correspond to Eigenvalue and so K has at least one Eigenvalue.

From the equation (3.44) $\frac{U_4}{U_2}\,\lambda^2 \geq 1,$ the smallest Eigenvalue $\lambda_1$ is such that

$$\lambda_1 \quad \leq \quad \sqrt{\frac{U_4}{U_2}} \quad (3.45)$$

**Theorem 3.3**

If $k(x, y)$ is symmetric and continuous, the:

- number of Eigenfunctions corresponding to each Eigenvalue is finite
- Eigenfunction corresponding to different Eigenvalue are orthogonal
- Eigenvalue is real.

## 4.0 CONCLUSION

Eigenvalues and Eigenfunctions can be found for integral equations of the form $\phi(x) \;=\; \lambda\int\sum_{p=1}^{n} a_p(x)b_p(y)\,d(y)dy$.

## 5.0 SUMMARY

Many homogeneous equations can be solved by determination of their Eigenvalue.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Under what Sturm-Lowville problem assumptions are Eigenfunction corresponding to different Eigenvalue orthogonal?
**2.** Solve?

$$\phi(x) = \lambda\int_0^1 (1 + xt)\phi(t)dt + mf(x); \;\; 0 \leq x \leq 1$$

## 7.0 REFERENCES/FURTHER READING


Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# UNIT 3 REPRESENTATION OF A FUNCTION BY A SERIES OF ORTHOGONAL FUNCTIONS

## CONTENTS



## 1.0 INTRODUCTION

In this unit, we shall take a look at orthogonality of systems and show that Fourier coefficients exist for continuous orthogonal systems and that orthogonal system can be represented by a series of orthogonal functions.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the Hilbert-Schmidt theorem
- state the Convergence theorem
- prove that functions can be represented by series of orthogonal functions
- expand $K$ in a series of Eigenfunctions
- define positive kernels.

# 3.0 MAIN CONTENT

## 3.1 Representation of a Function by a Series of Orthogonal Functions

### 3.1.1 Lemma 3.4

Let $\{\phi_n\}$ be an orthogonal system, and let $f$ be continuous.

Set $\alpha_n = \int_I f(x)\phi_n(x)\,dx$ (3.46)

Then, $\sum \alpha_n^2 \leq \int_I f^2(x)\,dx$ (3.4.7)

and $\alpha_n^{1s}$ are known as the Fourier's coefficient.

**Proof:**

Take any $N$. consider

$$\int_I \left[ f(x) - \sum_1^N \alpha_n \phi_n(x) \right]^2 dx \geq O \qquad (3.48)$$

i.e. $\int \left[ f^2(x) - 2 \sum_1^n \alpha_n \int f(x)\phi_n(x) + \sum_1^n \alpha_n^2 \right] dx \geq o$

i.e. $\int_I f^2(x)dx \geq \sum_1^N \alpha_n^2$ (3.49)

On noting that N is arbitrary

$$f(x) = \sum_1^\infty \alpha_n \phi_n(x) \quad (3.50)$$

We now consider that coefficients, $\alpha_n$, give the best fir in the sense that

$$\int [f(x) - \sum C_n \phi_n(x)]^2 dx \quad (3.51)$$

is a minimum.

The answer is that $C_n = \alpha_n$, the Fourier coefficients.

To see this, set

$$I_n^* = \int [f(x) - \sum C_n \phi_n]^2 dx \qquad (3.52)$$

$$I_n = \int [f(x) - \sum \alpha_n \phi_n]^2 dx \qquad (3.53)$$

Then, we show that $I_n^* \geq I_n$. For we have,

$$I_n^* = \int f^2(x)\,dx - 2\sum C_n \int f \phi_n \, dx + \sum C_n^2$$

$$\int f^2(x)\,dx - 2\sum \alpha_n C_n + \sum C_n^2$$

$$I_n = \int f^2(x)\,dx - 2\sum \alpha_n^2 + \sum \alpha_n^2$$

$$\therefore \quad I_n^* - I_n = \sum \left[ \left(C_n^2 - \alpha_n^2\right) - 2\alpha_n (C_n - \alpha_n) \right]$$

$$= \sum (C_n - \alpha_n)^2 \geq o \qquad (3.54)$$

As asserted.

**Definitions 3.5**

The set of orthogonal system $\{\phi_n\}$ is said to be complete if

$$\lim_{n\to\infty}\int_I\left[f(x)-\sum\alpha_n\ \phi_n\right]^2 dx \ = \ o \qquad (3.55)$$

for every continuous function $f(x)$. In this case, the Bessel's inequality becomes an equality

i.e. $\int f^2(x)\,dx \ = \ \sum\alpha_n^2$

If $\{\phi_n\}$ is complete, we can then roughly represent any function as a sum

$$f(x) \ = \ \sum\alpha_n\ \phi_n$$

the convergence being in the sense of (3.55)

## 3.2 Expansion of $K$ in Eigenfunctions

We examine the possibility of expanding $K$ as a series of Eigenfunctions.

Consider

$$K(x, y) \ = \ \sum\alpha_n\ \phi_n(y)$$

where $\int K(x, y)\,\phi_n(y)dy \ = \ \alpha_n$

i.e. $\alpha_n \ = \ \lambda_n^{-1}\ \phi_n(x)$

Then, we will have

$$K(x, y) \ = \ \sum\frac{\phi_n(x)\,\phi_n(y)}{\lambda_n} \quad (3.56)$$

This is valid independent of completeness of $\{\phi_n\}$

### 3.2.1 Definitions 3.6 (Positive Kernels)

A kernel is said to be positive if

$$T(\phi, \phi) \ = \ \iint K(x, y)\ \phi(x)\,\phi(y)\,dxdy \ > \ o \qquad (3.57)$$

for all $\phi$ such that $\int \phi^2(x)\,dx \neq o$ (3.58)

It is easily to see that Eigenvalue are strictly positive.

### 3.2.2 Theorem 3.7: Convergence

If $K(x, y)$ is positive, then

$$K(x, y) \ = \ \sum_1^\infty \frac{\phi_n(x)\,\phi_n(y)}{\lambda_n}$$

the series being absolutely and uniformly convergent.

### 3.2.3 Theorem 3.8: Hilbert – Schmidt Theorem

Suppose $f(x)$ can be written in the form

$$f(x) = \int K(x, y)\phi(y)\,dy \quad (3.59)$$

where $K$ is symmetric and continuous,
Then $f = \sum \alpha_n \phi_n$ where the series is absolutely and uniformly convergent and

$$\alpha_p = \int_a^b f(x)\phi_p(x)\,dx \qquad (3.60)$$

A convergence of the above theorem is another formula for the Resolvent $R$.
Consider the equation:

$$\phi(x) - \lambda\int K(x, y)\ \phi(y)\,dy = f(x) \qquad (3.61)$$

Then,

$$\phi - f = \lambda\int k(x, y)\phi(y)\,dy$$

Thus, $\phi - f$ satisfies the condition of theorem 3.8 and we can write

$$\phi(x) - f(x) = \sum \alpha_n \phi_n$$

where $\alpha_n = \int [\phi(x) - f(x)]\phi_n(x)\,dx$

$$= \beta_n - \gamma_n = \int f(x)\phi_n(x)\,dx$$

Multiply 3.61 by $\phi_n(x)$ and integrate and change order of integration. This given

$$\int \phi(x)\phi_n(x)\,dx - \lambda \int \phi(x)dx \int k(x, y)\phi_n(y)\,dy$$

$$= \int f(x)\phi_n(x)\,dx \quad (3.62)$$

Thus, $\beta_n - \dfrac{\lambda\,\beta_n}{\lambda_n} = \gamma_n$

i.e. $\beta_n = \dfrac{\lambda_n}{\lambda_n - \lambda}\gamma_n$, Hence, $\alpha_n = \dfrac{\lambda}{\lambda_n - \lambda}\gamma_n$

$$\therefore\ \phi(x) - f(x) = \lambda \sum \frac{\gamma_n}{\lambda_n - \lambda}\phi_n(x)$$

i.e. $\phi(x) = f(x) + \lambda \sum \dfrac{\lambda_n}{\lambda_n - \lambda}\ \phi_n(x) \quad (3.63)$

This gives the solution of 3.61 in terms of the Eigenfunctions of $\phi_n(x)$.
From 3.63, we have

$$\phi(x) = f(x) - \lambda \sum \frac{\phi_n}{\lambda - \lambda_n} \int \phi_n(y) f(y)\, dy$$

$$= f(x) - \lambda \int \left[ \sum \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n} \right] f(y)\, dy$$

$$= f(x) - \lambda \int R(x, y; \pi)\, f(y)\, dy$$

Where $R(x, y; \lambda) = \sum \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n}$ (3.64)

## 4.0 CONCLUSION

Fourier's coefficients exist for orthogonal systems which are continuous.

## 5.0 SUMMARY

The Hilbert-Schmidt theorem states that when a kernel is positive, a series can be derived which is absolutely and uniformly convergent, and kernel can be represented by a series of Eigenfunctions.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Derive an expression for the Fourier coefficient of the continuous orthogonal system $\{\phi_n\}$?
2. Show that the orthogonal system $\{\phi_n\}$ is complete if

   $\lim_{n\to\infty} \int_I \left[ f(x) - \sum \alpha_n\, \phi_n \right]^2 dx = o$?

   That is, Bessel's inequality becomes an equality.
3. If $T(\phi, \phi) = \iint K(x, y)\, \phi(x)\phi(y)\, dxdy > o$, can we deduce if the associated kernel is positive or negative? Prove it?

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# MODULE 4



## UNIT 1 CALCULATION OF $1^{ST}$ EIGENVALUE

### CONTENTS



## 1.0 INTRODUCTION

Transforms are used to solve equations for which transforms exist while the inverse transform is a convolution. Suitable conditions exist for the transform of a convolution to become the point-wise product of transforms which means that convolution in one domain is the point-wise multiplication in another domain.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply the convolution theorem
- calculate the first Eigenvalue of an integral equation
- use the Variational Formula
- recognise Integral Laplace Transforms as Transforms
- derive the solution of integral equations using inverse Laplace Transform.

## 3.0 MAIN CONTENT

### 3.1 Calculation of 1st Eigenvalue

The modes of vibration in systems are often of great importance. A powerful and simple method for finding them is provided by variational formula.

Let $\phi_1, \phi_2, \ldots$ be Eigenfunctions and $|\lambda_1| < |\lambda_2| < \ldots)$ be the corresponding Eigenvalue.

Set

$$J(\phi, \phi) = \iint K(x, y)\,\phi(x)\,\phi(y)\,dxdy$$

Suppose now that $\phi$ is arbitrary. Then, by the linear formula

$$K(x, y) = \frac{\sum \phi_n(x)\,\phi_n(y)}{\lambda_n}$$

We have:

$$J(\phi, \phi) = \iint \frac{\sum \phi_n(x)\,\phi_n(y)}{\lambda_n}\,\phi(x)\,\phi(y)\,dxdy$$

$$= \sum \frac{\beta_n^2}{\lambda_n} \leq \sum \left|\frac{\beta_n^2}{\lambda_n}\right|$$

Then, $J(\phi, \phi) \leq \sum \frac{\beta_n^2}{|\lambda_n|} \leq \frac{1}{|\lambda_1|}\sum \beta_n^2$

$$\leq \frac{1}{|\lambda_1|}\int \phi^2(x)dx \quad \text{(Bessel equation)}$$

$$\therefore \quad |\lambda_1| \leq \frac{\int {}^2(x)dx}{J(\phi, \phi)} \quad (3.65)$$

where $\lambda_1$ is the smallest Eigenvalue and $\phi$ is arbitrary. Similar results may be obtained for the higher Eigenvalues. However, the first is usually, the most important. $\phi$ is chosen to make $J(\phi, \quad \phi)$ a maximum and a normal function.

This given an estimate of a bound for $\lambda_1$ which usually is fairly accurate.

**Example 3.7**

Consider the kernel T in the square

$o \leq x \leq 1, \quad o \leq y \leq 1$ where

$$T(x, y) = \begin{cases} (1-x)y & o \leq y \leq x \leq 1 \\ (1-y)x & o \leq x \leq y \leq 1 \end{cases}$$

By differentiating the equation

$$\phi(x) - \lambda \int_o^1 \mathrm{T}(x, y)\,\phi(y)\,dy = 0$$

If is easy to see that if reduces to

$$\phi'' + \lambda\,\phi(x) = 0, \quad \phi(o) = \phi(1) = o$$

The Eigenfunction are $\sqrt{2}\sin n\pi x$ (normalized and Eigenvalue are $\lambda_n = (n\pi)^2$

The linear formula given

$$\mathrm{T}(x, y) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi y}{n^2\pi^2}$$

We shall now consider the application of 3.65 to the determination of 1st Eigenvalue

$$(\lambda_1 = \pi^2 = 9.869)$$

First guess $\phi = 1$

$$J(\phi, \phi) = \int_o^1 \left[(1-x)\int_o^x y\,dy + x\int_x^1 (1-y)\,dy\right]dx$$

$$= \frac{1}{12}$$

We get $\lambda_1 = 1\Big/\frac{1}{12} = 12$

Second guess $(R_i\ t_3)$

Choose $\phi$ to be a step function $\phi = 0$ except for $o < x < 1-\alpha$ where $\phi = \beta$.

Choose $\phi$ normalised. Then, $\beta = (1-2\alpha)^{-\frac{1}{2}}$ one find that

$$J(\phi, \phi) = \frac{1}{12}\left(1 + 2\alpha - 8\alpha^2\right)$$

This has a maximum at $\alpha = \frac{1}{8}$, when

$$J(\phi, \phi) = 3/32$$

Then, $\lambda_1 \le \dfrac{1}{3/32} = \dfrac{32}{3} = 10.67$

The estimate is considerably improved, and the choice of a nose complicated $\phi$ will lead to a nose accurate estimate.

## 3.2 Integral Transforms: Laplace Transforms

If $f(t)$ is throughout piecewise, continuous, bounded variation and of exponential order, i.e. $\exists\ M_o$, so, э

$$f(t) \le M_o\, e^{sot}$$

and if we define $F(s) = \int_o^\infty e^{-st} f(t)\, dt$ (4.1)

$S$ may be complex, then, $F(s)$ is known as the Laplace transform of $f$ and is defined when the integral is absolutely convergent for some so, then, it is also for $S$ such that Res > Redo

The largest half-plane in which the integral is absolutely convergent is called the half-plane of convergence. The following hold in this half-plane:

i. $£\{af + bg\} = a£\{f\} + b£\{g\}$ (4.2)

ii. $£\{f^2(t)\} = S^n F(s) - S^{n-1} f(o^+)$ . ... $- F^{n-1} (o^+)$ (4.3)

iii. $£\{e^{at} f(t)\} = F(s-a)$ (4.4)

iv. $£\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$ (4.5)

$F(o^t)$ denotes limit from right

## 3.3 Convolution Theorem

We define a new function $h(t)$ by

$h(t) = \int_o^t g(u) f(t-u)\, du = f * g$ (4.6)

$h(t)$ is called the convolution product of $f$ and $g$ and is written $f * g$ so that we have

$$\int_o^\infty e^{-st} h(t)\, dt = \int_o^\infty e^{-st} (f * g)\, dt \qquad (4.7)$$

$$= \int_o^\infty e^{-st} f(t)\, dt \int_o^\infty e^{-su} g(u) du = F(s) C_1(s)$$

## 3.4 Inverse Laplace Transform

$$f(t) = £^{-1}\{F(s)\} = \frac{1}{2\pi_i} \int_{c-i\infty}^{c+i\infty} F(s)\, ds \quad (4.8)$$

$$= \sum \{\text{residues of } F(x) e^{st} \text{ at poles of } F(s)\}$$

Where C is some real number which is greater than the real part of all the poles of $F(s)$. We can however use any other alternative method to obtain $f(t)$.

## 4.0 CONCLUSION

Kernel can be solved by applying Laplace transform if the transform exists.

## 5.0 SUMMARY

Laplace transform is defined only when an integral is absolutely convergent and the largest half-plane in which the integral is absolutely convergent is called the half-plane of convergence.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Write an expression for the kernel T in the square defined below and find its first Eigenvalue?
   $-1 \leq x \leq 2, \quad 0 \leq y \leq 3$
2. Do you recognise the transform below? Which transform is it?
   $F(s) = \int_o^\infty e^{-st} f(t)\,dt$
3. What is the relationship between an inverse Laplace transform and a convolution?

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing*. (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.

# UNIT 2 THE APPLICATION OF THE TRANSFORM

**CONTENTS**



## 1.0 INTRODUCTION

Laplace and Fourier integral transforms are used to solve integral equations for which the transform exists and this is demonstrated in this unit via worked examples.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply Laplace transform through worked examples
- solve integral equations by the method of Fourier integral transforms.

## 3.0 MAIN CONTENT

## 3.1 The Application of the Transform

**Example 4.1**

Solve the equation:

$$\phi^{11} + 5\ \phi^{1} + 6\phi = e^{-t} \quad t \geq o$$

$$\phi(o) = 2, \quad \phi^{1}(o) = 1$$

Now, let $£(\phi) = \overline{\phi}(s)$ so that

$$£\{\phi^{11}\} = s^2\overline{\phi} - s\,\phi(o) - \phi^{1}(o) = s^2\overline{\phi}(s) - 2s - 1$$

and $£\{e^{-t}\} = \dfrac{1}{s+1}$

$$\therefore \quad s^2\,\overline{\phi} - 2s - 1 + 5\left(s\overline{\phi} - 2\right) + 6\overline{\phi} = \frac{1}{s+1}$$

i.e. $\left(s^2 + 5s + 6\right)\overline{\phi} = 25 + 11 + \dfrac{1}{s+1}$

i.e. $\bar{\phi}(s) = \dfrac{2s^2 + 13s + 12}{(s+1)(s+2)(s+3)}$

Hence, the poles are not $-1, -2, -3$

The residue at $s = -1$ is $\operatorname{Re} s_{s=\alpha} = \bar{\phi}^{(s)(s-\alpha)} 1_{s=\alpha}$

$$\frac{2-113+12}{1\times 2} e^{-t} = \frac{1}{2} e^{-t}$$

That is $-2$ is $6\,e^{-2t}$

and $-3$ is $-\dfrac{9}{2} e^{-3t}$

Thus, $\phi(t) = \dfrac{1}{2}\left\{e^{-t} + 12e^{-2t} - 9e^{-3t}\right\}$

**Example 4.2**

Consider the Volterra equation:

$$f(x) - \int_o^x k(x-y)\, f(y) dy = g(x)$$

We want to use Laplace transform to get a solution.

The equation can be written in the form of

$$f - k * f = g$$

Take the Laplace transform of both sides to give

$$\bar{f} - \bar{k}\,\bar{f} = \bar{g}$$

i.e. $\bar{f}\left(1-\bar{k}\right) = \bar{g} \qquad \therefore \quad \bar{f} = \bar{g} \big/ 1-\bar{k}$

Thus,

$$f = £^{-1}\left\{\frac{\bar{g}}{1-\bar{k}}\right\} = \frac{1}{2\pi_i}\int_\Gamma \frac{\bar{g}}{1-\bar{k}} e^{st}\, ds$$

Take $k(t) = \lambda e^t$ for example, then

$$\bar{k}(s) = \frac{\lambda}{s-1}$$

But $\dfrac{1}{1-\bar{k}} = 1 + \dfrac{\bar{k}}{1-\bar{k}}$

$$f = £^{-1}\left\{\frac{\bar{g}}{1-\bar{k}}\right\} = £^{-1}\left\{\bar{g} + \frac{\bar{k}\,\bar{g}}{1-\bar{k}}\right\}$$

$$= g + £^{-1}\left\{\frac{\lambda\,\bar{g}}{s-1-\lambda}\right\} = g + £^{-1}\left\{\bar{h}\,\bar{g}\right\}$$

Where $\bar{h} = \dfrac{\lambda}{s-1-\lambda}$ and $h = \lambda\, e^{(1+\lambda)t}$

Hence,

$$f(x) = g(x) + \lambda \int_o^x e^{(1+\lambda)(t+u)}\, g(u)\, du.$$

**Example 4.3**

Solve the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = o \qquad (o \le x \le L, \quad t \ge o)$$

$$\phi(x, o) = o \qquad o \le x \le \ell$$

$$\frac{\partial \phi}{\partial t}(x, o) = o \qquad o \le x \le l$$

$$\phi(o, t) = o \qquad t \ge o$$

$$\frac{\partial \phi}{\partial x}(l, t) = a \qquad t \ge o$$

Here we want a solution for $t \ge o$ and for a finite range of re. Take Laplace transform w.r.e $t$ (since the $t-$interval is semi-infinite)

Write $\bar{\phi}(x, s) = \int_0^\infty e^{-st} \phi(x, t)\, dt$

Take the Laplace transform to give

$$\int_0^\infty e^{-st} \frac{\partial^2 \phi}{\partial x^2}(x, t)\, dt - \int_0^\infty e^{-st} \frac{\partial^2 \phi}{\partial t^2}(x, t)\, dt = 0$$

But $\displaystyle\int_0^\infty e^{-st} \frac{\partial^2 \phi}{\partial x^2}(x,t)\, dt = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} \phi(x,t)\, dt$

$$= \frac{\partial^2 \bar{\phi}}{\partial x^2}(x, s)$$

and $\displaystyle\int_0^\infty e^{-st} \frac{\partial^2 \phi}{\partial x^2}(x,t)\, dt = s^2\, \bar{\phi}(x,s) - S\phi(x, 0) + \frac{\partial \phi}{\partial t}(x,0)$

$$= S^2 \bar{\phi}(x, s)$$

from the boundary condition

Hence, we get

$$\frac{\partial^2 \bar{\phi}}{\partial x^2}(x, s) - S^2\, \bar{\phi}(x, s) = 0$$

This is now an ordinary differential equation for $\bar{\phi}$ and to solve it, we need two boundary conditions, we have

$$\bar{\phi}(0, s) \;=\; 0$$

and $\dfrac{\partial \bar{\phi}}{\partial x}(\ell, s) \;=\; \int_0^\infty e^{-st}\,\dfrac{\partial \phi}{\partial x}(\ell, t)\,dt$

$$=\; \int_0^\infty a e^{-st}\,dt = \frac{a}{s}$$

We this, solve the following system

$$\frac{\partial^2 \bar{\phi}}{\partial x^2}(x, s) - S^2\,\bar{\phi}(x, s) \;= 0$$

$$\bar{\phi}(0, s) = 0 \qquad \text{and} \quad \frac{d\bar{\phi}(\ell, s)}{dx} = \frac{a}{s}$$

The solution is

$$\bar{\phi} = A(s)\sinh sx \;+\; B(s)\cosh sx$$

From the first boundary condition $B(s) = 0$ and from the second condition, we have

$$A(s)\,S\cosh sl \;=\; \frac{a}{s}$$

$$A(s) \;=\; \frac{a}{s^2\cosh sl}$$

Here, $\bar{\phi}(x, s) \;=\; \dfrac{a\sinh sx}{s^2\cosh sl}$

and $\phi(x, t) \;=\; \dfrac{a}{2\pi i}\displaystyle\int_{r^1} \dfrac{\sinh sx}{s^2\cosh sl}\,e^{st}\,ds$

where $\tau$ lies to the sight of the poles. The integral has poles at $s = 0$ and at the zeros of $\cosh sl$. Consider first $s = 0$

$$\bar{\phi}(x, s)\,e^{st} = \frac{1}{s^2}\left[sx + \frac{(sx)^3}{6} + \cdots\right]\left[1 - \frac{(sl)^2}{2}\cdots\right][1 + st + \cdots]$$

$$=\; \frac{1}{s^2}\left[sx + 0\left(s^2\right)\right]$$

Simple pole at $s = 0$ with reside $x$.

Now, consider points whose $\cosh ls \;= 0$

$$S = \frac{\pm i\,(2n+1)\pi}{2l}$$

The poles are simple once. We may use the formula:

$$\operatorname*{Res}_{s=a} \;= \frac{f(a)}{g^1(a)}$$

Thus,

$$\operatorname*{Re}_{s=\frac{(2n+1)i\pi}{2l}} s = \frac{Sinh\left[\frac{(2n+1)i\pi x}{2l}\right] e^{i\frac{(2n+1)\pi t}{2l}}}{\ell\left[\frac{(2n+i)i\pi}{2l}\right]^2 Sinh\left[\frac{(2n+1)\pi i}{2}\right]}$$

$$= -\frac{4\ell(-1)^n}{\pi^2}\frac{Sinh\left[\frac{(2n+1)\pi ix}{2l}\right] e^{i\frac{(2n+1)\pi t}{2l}}}{(2n+1)^2 i}$$

Evidently, poles are complex conjugates, so we require twice the seal part. Hence,

$$\phi(x, t) = ax - \frac{8al}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} Sin\left[\frac{(2n+1)\pi x}{2l}\right]$$

The Laplace transform is suitable for problems with a semi-infinite domain for the independent variable. It is also necessary that the (differential) equation should have constant coefficients.

## 3.2 Fourier Integral Equations

If $f(x)$ is a continuous function, then,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{iwx}F(w)dw \quad (4.10)$$

where $$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwu} f(u)du \quad (4.11)$$

Equation 4.11 gives the solution of the integral 4.10 for F and vice versa. If $f(x)$ is seal, using the odd property, of sin $\sin wre$ and the even property of $\cos wre,$ we have, if

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\cos wre\,\phi(w)dw, \quad 0 \le x \quad (4.12)$$

Then,

$$\phi(w) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\cos wx\, f(x)dx, \; 0 \le w \quad (4.13)$$

$\phi(w)$ and $f(x)$ are the $\cos ine$ transforms of one another. If

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\sin wx\,\phi(w)dw, \quad 0 \le x \quad (4.14)$$

Then,

$$\phi(w) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty}\sin wx\, f(x)dx, \; 0 \le w \quad (4.15)$$

$\phi(w)$ and $f(x)$ are the sine transforms of one another.

**Example 4.1**

Solve the integral equation

$$\frac{a}{a^2+x^2} = \int_0^\infty \cos wx\, \phi(w)dw, \quad a>0$$

$$\phi(w) = \left(\frac{2}{\pi}\right)\int_0^\infty \frac{a\cos wx}{a^2+x^2}dx = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{2ia\,e^{iwx}}{a^2+x^2}dx$$

because $\sin wre$ is odd in $x$

Evaluation of the integral by the methods of the complex integral calculus given

$$\phi(w) = e^{-wa}, \quad w>0$$

**Example 4.2**

Solve the integral equation

$$\phi(x) = \lambda\int_0^\infty \cos wx \;\; \phi(w)dw$$

$\phi(x)$ is an even function of $x$

Because the inverse of a $\cos ine$ transform is another $\cos ine$ transformation, we look for a solution of the form

$$\phi(x) = U(x) \pm V(x)$$

where $V(x) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \cos wx\, U(w)dw$

Thus,

$$\phi(x) = U(x) \pm \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \cos wx\, U(w)dw$$

$$= \lambda\int_0^\infty \cos wx\left[U(w) \pm \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \cos wt\, U(t)dt\right]dw$$

$$= \lambda\int_0^\infty \cos wx\, U(w)dw \pm \left(\frac{\pi}{2}\right)^{1/2}\lambda\, U(x)$$

This is true if $\lambda = \pm\left(\frac{2}{\pi}\right)^{1/2}$ Thus, to $\lambda = \left(\frac{2}{\pi}\right)^{1/2}$, there corresponds a solution $U(x)+V(x)$ and to $\lambda = -\left(\frac{\pi}{2}\right)^{1/2}$, there corresponds a solution $U(x)-V(x)$.

This solution will be valid, provided all the integrals exist; $U$ is arbitrary. In this case, the two Eigenvalues $\lambda = \pm\left(\frac{2}{\pi}\right)^{1/2}$, there exist an infinite of Eigenfunctions.

**Example 4.3**

Solve the integral equation

$$\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, \phi(y)dy$$

If $\lambda = \pm 1$, there will not in generally be any solution.
This follows example 4.2

Take the transform of the equation to give

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, \phi(y)dy = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy + \lambda\, \phi(x)$$

It follows that

$$\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy + \lambda^2\phi(x)$$

i.e.

$$\left(1-\lambda^2\right)\phi(x) = f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)\,dy$$

and this solution is valid provided that the integral converge. Now, if $1-\lambda^2 = 0$ and $f(x)$ is a function such that

$$f(x) + \lambda\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \cos xy\, f(y)dy = 0$$

It follows that $\phi(x)$ can be any function for which the integral converge.

**SELF-ASSESSMENT EXERCISE**

(1) Solve the integral equation.

$$\frac{x}{x^2+a^2} = \int_0^\infty \sin wx\phi(w)dw \qquad a > 0$$

(2) Find the Eigenvalues and Eigenfunctions of the integral equation.

$$\phi(x) = \lambda\int_0^\infty \sin xy\, \phi(y)dy$$

(3) Find the solution of the integral equation.

$$\phi(x) = e^{-ax} + \lambda\int_0^\infty \sin xy\, \phi(y)dy, \qquad a > 0$$

$$\pi\lambda^2 \neq 2$$

(4) Find the integral equation:

$$\frac{P}{P^2+a^2} = \int_0^\infty e^{-pt}\, f(t)dt \qquad a > 0$$

(5) Solve the integral equation:

$$\phi(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x-y)\,\phi(y)\,dy$$

(6) Solve the integral equation:

$$\frac{1}{(x+a)^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(y)dy}{x-y}$$

## 4.0 CONCLUSION

Transforms are a useful mathematical tool for solving integral equations for which the applicable transforms exist.

## 5.0 SUMMARY

A Laplace transformation is applicable for problems with a semi-infinite domain for the independent variable.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Solve $\frac{b^2+x^2}{b}\int_0^{\infty} \sin wx\, \emptyset(w)\,dx,\ 0 < b < \pi$

2. Solve $\phi(t) = \lambda \int_0^{\infty} \sin wt\, \phi(f)\,df$

## 7.0 REFERENCES/FURTHER READING

Andrei, D. P. & Alexander, V. M. (1998). *Handbook of Integral Equations*. Boca Raton: CRC Press.

Arfken, G. & Hans, W. (2000). *Mathematical Methods for Physicists*. Port Harcourt: Academic Press.

Kendall, E. A. (1997). *The Numerical Solution of integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics.

Krasnov, M., Kiselev, A. & Makarenko, G. (1971). *Problems and Exercises in Integral Equations*. Moscow: Mir Publishers.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. & Flannery, B.P. (2007). *"Chapter 19. Integral Equations and Inverse Theory". Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press.

Whittaker, E. T. & Watson, G. N. (nd). *A Course of Modern Analysis*. Cambridge Mathematical Library.