NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

PHY 201

CLASSICAL MECAHNICS

NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

PHY 201

CLASSICAL MECAHNICS

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PHY 201 Classical Mechanics I Course Material

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CONTENTS

## MODULE 1:	MOTION IN CENTRAL FORCE FIELDS

The primary application of the theory of central-force motion is in astronomy. The motion of bodies which act under influence of a central force is extremely important physical problem which lies in motion of celestial bodies, such as planets, moons, comets, stars, etc. Artificial satellites orbiting the earth are a familiar part of modern technology; but how do they stay in orbit, and what determines the properties of their orbits? In this module we shall discuss problems of bodies that are encounterd in celestial mechanics.

Unit 1 Vector Analysis

1.0 Introduction

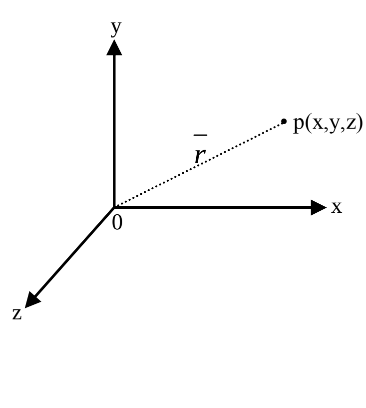
Physics makes use of equations, formulas, and vectors. A *vector* is a special type of quantity in which there are two components: *magnitude* and *direction*. Vectors are used commonly in physics to represent phenomena such as force, velocity, and acceleration. (Vectors are customarily written in boldface, as opposed to variables, constants, and coefficients, which are usually written in talics). A simple example is the motion of an airplane; to describe this motion completely, we must say not only how fast the plane is moving, but also in what direction. The speed of the airplane combined with its direction of motion together constitute a vector quantity called *velocity*. In contrast, real numbers, also called *scalars*, are one-dimensional (they can be depicted on a line); they have only magnitude. Scalars are satisfactory for representing phenomena or quantities such as temperature, time, and mass.

The discusssion of motion in two or three dimensions is vastly simplified only when concept of vector calculus is introduced. In this unit you will need to refresh your knowledge of vector concepts from elementary mathematics before proceeding.

2.0 Aims and Learning objectives

By the end of this unit students should be able to manipulate vectors fluently including use of scalar and vector quantities in physics.

3.1 Vector Position



The point P is in cartesian coordinates and has no direction, such quantity is called *scalar*.

However, $\overline{op}$ is a position vector which defines the position of the point p from the origin.

$$\overline{op} = \bar{r}(position\ \ vector) = f(x, y, z)$$
$$= \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

The letters $\hat{i}, \hat{j}, \hat{k}$ (could also be written in bold **i**, **j**, **k)** are called 'unit vectors' and

$$r^2 = x^2 + y^2 + z^2$$

In general,

$$\text{unit vector} = \frac{\bar{r}}{|r|}$$

When the point p coincides with origin 0, $\bar{r} = 0\,(zero)$, this particular vector $\overline{op} = 0$ is called '*zero vector'*

3.2 Addition and Subtraction of vectors

Vectors can be added or subtracted. Consider two vectors $\bar{a}$ and $\bar{b}$ (could also be denoted using arrows $\vec{a}$ and $\vec{b}$ or using the cap $\hat{a}$ and $\hat{b}$)

$\bar{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $\bar{b} = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$

$\bar{a} + \bar{b} = \bar{b} + \bar{a}$

3.3 Multiplying of a vector with another vector

Consider two vectors $\bar{a}$ and $\bar{b}$,

1. Scalar product (Dot product)

   $\bar{a} \cdot \bar{b} = a\,b\cos\theta$ (scalar result)

   If $\bar{a}$ is perpendicular to $\bar{b}$, $\bar{a} \cdot \bar{b} = 0$

   Note: $\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$ and $\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$

2. Cross product

   $\bar{a}\ \mathrm{x}\ \bar{b} = a\,b\sin\theta$ (vector result)

   $\bar{a}\ \mathrm{x}\ \bar{b} = -\,\bar{b}\ \mathrm{x}\ \bar{a}$

   If $\bar{a}$ is parallel $\bar{b}$, $\bar{a}\ \mathrm{x}\ \bar{b} = 0$

3. Tripple product

   (a) scalar triple product

   $$(\bar{a}\ \mathrm{x}\ \bar{b}).\bar{c} = |\bar{a}\,x\bar{b}|\bar{c}\cos\phi$$

   $$(\bar{a}\ \mathrm{x}\ \bar{b}).\bar{c} = (\bar{b}\ \mathrm{x}\ \bar{c}).\bar{a} = (\bar{c}\,x\,\bar{a}).\bar{b}$$

   (b) vector triple product

   $$(\bar{a}\,x\bar{b})x\bar{c} = \bar{b}(\bar{a}.\bar{c}) - \bar{c}(\bar{a}.\bar{b})$$

3.4 Gradient, Divergence and Curl

Fields are quantities which are function of position in space.

There are two types of fields: *scalar field* , denoted $\phi(x,y,z)$ or $\phi(\bar{r})$ and *vector* field, denoted

$\overline{A}$ (x,y,z) or $\overline{A}(\overline{r})$. Both are functions of position in space

Define a symbol

$$\nabla = del = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(i) $\nabla\phi = (del\phi) = grad(\phi)$ ≡ gradient of the scalar field ϕ

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$ *≡ gradient of a scalar gives a vector*

(ii) Divergence of vector $\overline{A}$ (can also be $\hat{A}$, $\vec{A}$ etc.) is

$$\nabla . \overline{A} = Div\overline{A}$$ (divergence of $\overline{A}$)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ ≡ gives scalar quantity

(iii) $$\nabla\, x\, \overline{A} = Curl\overline{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)x\left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$ ≡ which will give a vector quantity

Self Assessement Exercise

1. For what values of q are the two vectors A = **i** + **j** + **q**k and B = q**i** – 2**j** + 2q**k** parallel to each other?

2. A particle of mass m is subjected to a force F = kx where k is a constant. At x = 0 its velocity is v_0. Show that at x = 2, its velocity will be $v = \sqrt{v_0^2 + \frac{4k}{m}}$

3. The velocity of a moving particle is given by $v = a\hat{i} + bt\hat{j} + ct^{-1}\hat{k}$ in which a, b, and c are constants. Find r

4.0 Conclusion

Vectors can be manipulated in the same manner as done with quantities in mathematics; addition, subtraction, multiplication, etc. These techniques are used in a range of unseen mathematically applied problems in physics.

5.0 Summary

We treated how to add, subtract and multiply vectors, including taking the dot, divergence and cross product of vectors.

6.0 Tutor-Marked Assignment (TMA)

Question 1.1

If $A = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $B = 4\hat{i} + j + 3\hat{k}$

(i) calculate A – B (ii) calculate A curl B (iii) evaluate A dot B and
(iv) verify that $(A \bullet B)^2 + (A \mathbf{X} B)^2 = A^2B^2$

Question 1.2

Find the angle between the two vectors $A = 6\hat{i} + 5\hat{j} + 4\hat{k}$ and $B = \hat{i} + 2\hat{j} + 3\hat{k}$

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

Unit 2 Central – Conservative Forces

1.0 Introduction

All forces are either conservative or nonconservative. A force that offers the oppportunity of conversion between kinetic and potential energies is a conservative force. An essential feature of conservative forces is that their work is always reversible. The work of a conservative force can always be represented by a potential energy function, but work of nonconservative forces cannot. A central force is a conservative field, that is, it can always be expressed as the negative gradient of a potential: A force that can be derived from a potential energy $V(r)$ that only depends on the distance to the source is called *central***.** As a consequence of being conservative, a central force field is irrotational, that is, its curl is zero. Since the central force is *conservative* the *energy* of the planet must be a constant over the orbit. The *gravitational force, electric force* between charges and *spring force* in elevator are examples of *central forces*

Defintions of Central, Conservative and Central-conservative forces are teated in this unit. The physical meaning of gradient of a vector will also be seen. The necessary and sufficient conditions for a force field to be conservative and some quick tests are also touched.

2.0 Aims and Learning objectives:

By the end of the unit students should be able:

- to know what central, conservative and central-conservative forces are.
- mathematically understand the conservative theorems of energy, linear momentum and angular Momentum.

3.0 Definition of Central forces

An external force is said to be central if it is always directed towards or away from a fixed point, called 'the centre of force'.

A force whose line of action is always directed toward a fixed point. The central force may attract or repel. The point toward or from which the force acts is called the center of force. If the central force attracts a material particle, the path of the particle is a curve concave toward the center of force; if the central force repels the particle, its orbit is convex to the center of force. Undisturbed orbital motion under the influence of a central force satisfies Kepler's law of areas.

If we choose the origin to be at this centre, this means that $\overline{F}$ is always parrallel to $\overline{r}$

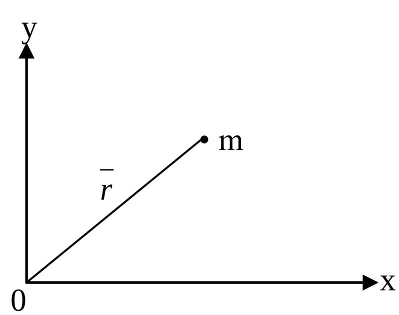


Figure 1.2: A particle in a plane

Mathematically,

$$\overline{F} = f(\overline{r})\overline{r_1}$$

where $\overline{r_1} = \frac{\overline{r}}{|r|} \equiv$ unit vector

If $f(\overline{r}) \langle\ 0$ is attractive

$f(\overline{r}) \rangle\ 0$ is repulsive

3.1 Properties of Central Force Fields

If a particle moves under the action of a central force field;

(a) The angular momentum of the particle is conserved

(b) The path (or orbit) of the particle must be a plane curve.
i.e. the particle moves in a plane curve
proof:

Central force field $\overline{F} = f(r)\overline{r_1}$ where $\overline{r_1}$ is unit vector

$$\overline{r} \; x \; \overline{F} = \overline{r} \; x \; f(r)\overline{r_1} = 0, \qquad F = m\overline{a} = m\frac{d\overline{v}}{dt}$$

$$\overline{r} \; x \; \overline{F} = \overline{r} \; x \; m\frac{d\overline{v}}{dt} = 0 \quad (\text{since } \overline{r} \; x \; \overline{F} = 0)$$

$$\text{So } \; m\frac{d}{dt}\left(\overline{r} \; x \; \overline{v}\right) = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\left(\overline{r} \; x \; \overline{v}\right) = 0$$

$$\Rightarrow \quad \overline{r} \; x \; \overline{v} = constant \quad (h, say)$$

$$\overline{r}.\left(\overline{r} \; x \; \overline{v}\right) = \overline{r}.\overline{h}$$

$$\text{i.e} \quad \left(\overline{r} \; x \; \overline{r}\right).\overline{v} = \overline{r}.\overline{h}$$

$$\Rightarrow \quad 0.\overline{v} = \overline{r}.\overline{h} \qquad (\text{since } \overline{r} \; x \; \overline{r} = 0)$$

$$0 = \overline{r}.\overline{h} \qquad \Rightarrow \quad \overline{r} \text{ is perpendicular to } \overline{h} \quad (\text{i.e } \overline{r} \perp \overline{h})$$

Hence the motion of a particle under the influence of a central force takes place in a plane.

3.2 Work Performed by Conservative force fields

From everyday experience, work is related with the magnitude of the force acting during performing a work and the magnitude of the displacement of the body from one point to another. In general, work does depend on the path taken between any two points.

Let V be a scalar fuction such that

$$\overline{F} = -\nabla V \quad , \qquad \text{where} \quad \overline{F} = ma \equiv \text{force field (a vector space)}$$

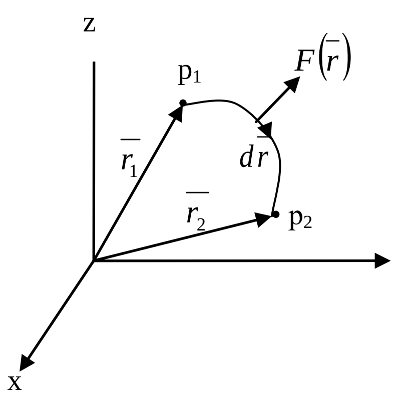


Figure 1.3: Moving particle from point p_1 to p_2

We want to find the work W, done by the force F in moving a particle from point p_1 to p_2

$$W = \int_{p_1}^{p_2} \overline{F} d\overline{r} = -\int_{p_1}^{p_2} \nabla V \, d\overline{r}$$

$$W = -\int_{p_1}^{p_2}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V\,d\overline{r} = -\int_{p_1}^{p_2}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$W = -\int_{p_1}^{p_2} dV \quad = \quad \left[-V\right]_{p_1}^{p_2} \quad = \quad V(p_1) - V(p_2) \quad = \quad V_1 - V_2$$

V_1 and V_2 are potential energies (or simply potentials) at point p_1 and p_2 respectively.

We see here that the work done is independent of the path transversed by the particle.

If the work done by a force field in moving a particle from one point to another point is independent of the path joining the points, then the force field is said to be conservative.
Precondition:

$$\overline{F} = -\nabla V$$
$$\overline{\nabla}\, x\, \overline{F} = \nabla\, x\, (-\nabla V) \quad = \quad 0$$

Note.

- It can be shown that the force field F(r) is conservative implies that $\oint F(r).dr \quad = \quad 0$
- $\oint F(r).dr \quad = \quad 0$ for any closed path implies 'there exists a function V(r) such that $F(r) = -\nabla V(r)$'
- $\overline{F}$ is conservative if curl F = $\overline{\nabla}\, x\, \overline{F} \quad = \quad 0$

3.3 Definition of Central Conservative Forces*:*
These are forces which exibit both the properties of conservative and central forces. Central forces are always conservative

3.4 Conservative theorems

3.4.1 Energy
The work done W by net force on a particle by displacing it from an initial potential V_1 to a final potential V_2 equals the change in the particle's kinetic energy from T_1 to T_2

$$W = V_1 - V_2$$
$$W = T_2 - T_1$$

Equating the two equations,

$$V_1 - V_2 = T_2 - T_1$$
$$\text{i.e. } V_1 + T_1 = V_2 + T_2 = \text{constant}$$
$$T + V = K.E + P.E = E \text{ (total energy)}$$

This result is known as the work - energy theorem.

3.4.2 Torque and Conservation of Angular momentum
When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).
Torgue, N is the moment of the force

$$\overline{N} = \bar{r}\, x\, \overline{F} = r\, x\, \frac{d}{dt}\left(m\bar{v}\right) = \frac{d}{dt}\left(\bar{r}\, x\, m\bar{v}\right) \qquad \text{using vector identity}$$

$$= \frac{d}{dt}\left(\bar{r}\ x\ \bar{p}\right)$$

$$\bar{r}\ x\ \bar{p} \quad = \quad \bar{L} \quad (angular\ momentum)$$

$\therefore$

$$\bar{N} = \frac{d\bar{L}}{dt}$$

If $\bar{N} = 0$ $\quad \Rightarrow \quad \frac{d\bar{L}}{dt} = 0$

$\Rightarrow \quad \bar{L}$ = constant (implies L is conserved)

$$\bar{N} = \bar{r}\ x\ \bar{F} \quad = \quad \frac{d\bar{L}}{dt}$$

Here $\bar{r}$ is parallel to $\bar{F}$ $\quad \Rightarrow \quad \bar{r}\ x\ \bar{F} = 0$ i.e. $\bar{N} = 0$

$\Rightarrow \quad \frac{d\bar{L}}{dt} = 0 \quad \Rightarrow \quad \bar{L} = constant$ (means is conserved)

3.4.3 Impulse and Conservation of Linear momentum, p

The change in momentum of a particle during a time interval equals the the impulse of the net force that acts on the particle during that interval.

Impulse of the force = $\int_{t_1}^{t_2} \bar{F} dt = \int_{t_1}^{t_2} \frac{d\bar{p}}{dt} dt = t_2 \int_{t_1} dp = [p]_{t_1}^{t_2} = p_2 - p_1$ = rate of change in momentum

$$\bar{F} = \frac{d\bar{p}}{dt}$$

If $\bar{F} = 0$ $\quad \Rightarrow \quad \frac{d\bar{p}}{dt} = 0$

$\Rightarrow \quad \bar{p}$ = constant (implies p is conserved)

This result is known as impulse - momentum theorem

Conclusion

The wok done in moving a particle from one point to another in a conservative force field is independent of the path taken. Energy, linear and angular momentum are conserved during motion in central conservative field.

5.0 Summary

A central force is always directed along the line connecting the centre of two bodies, it is conservative and the potential is derivable from it. The motion of a particle in a central force field always takes place in a plane. Central force motion occurs in celestial bodies and nuclear interaction.

6.0 Tutor-Marked Assignment (TMA)

Question 2.1

Find out if the force given below is conservative and hence determine the potential energy for it: $F_x = 2ax(z^3 + y^3)$, $F_y = 2ay(z^3 + y^3)$, $F_z = 3az^2(x^2 + y^2)$

Question 2.2

Find the component of the force for the potential energy $V = axy^2z^3$

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

Unit 3 Kinematics in Polar coordinates

1.0 Introduction

Mechanics often includes the study of the motion of particles along curves; this is known as kinematics. In problems with particular symmetries, it is often convenient to use non- Cartesian co-ordinates. In particular, in the case of axial or spherical symmetry, we may use cylindrical polar co-ordinates ρ, ϕ, z, or spherical polar co-ordinates r, θ, ϕ. These co-ordinates, though curvilinear, are still orthogonal in the sense that the three coordinate directions at each point are mutually perpendicular.These are related to Cartesian co-ordinates. In this unit, kinematics is considered in terms of polar coordinates.

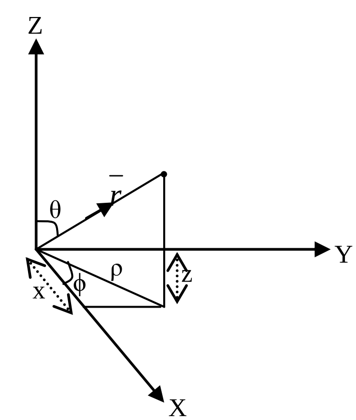
2.0 Aims and Learning objectives:

By the end of the unit students should be able:

- evaluate velocity and acceleration components in terms of polar coordinates.

3.0 Polar Coordinates

One can migrate from one coordinate system to another



There are two types of polar coordinates:
cylindrical and spherical polar coordinates.
ρ comes in only in terms of cylindrical

cartesian (x,y,z)
cylindrical (ρ,ϕ,z)
spherical (r,θ,ϕ)

From pythagorean theorem, we have

$$r = \sqrt{\left(x^2 + y^2 + z^2\right)}$$

Figure 1.4: Polar coordinates

Table 1.1: Relation between cartesian and polar coordinates

Cartesian	*Cylindrical*	*Spherical*
x	$\rho\cos\phi$	$r\sin\theta\cos\phi$
y	$\rho\sin\phi$	$r\sin\theta\sin\phi$
z	z	$r\cos\theta$

In the Newton's second law of motion

$$F = ma, \qquad a = \frac{dv}{dt}$$

This is in the cartesian coordinates.
We wish to change it to spherical polar coordinates
From x, y, z in cartesian, we get the velocity components v_x, v_y and v_z
From r, θ, ϕ in spherical polar, we get the velocity components v_r, v_θ, v_ϕ

3.1 Velocity and acceleration components in polar coordinates

In many practical problems, potential energy function is spherically symmetric; it depends only on the distance from the origin given by:

$$r^2 = \left(x^2 + y^2 + z^2\right)\text{, and}$$

$$v_r^2 = \left(v_x^2 + v_y^2 + v_z^2\right)$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

In polar coordinates,

$$x = r\sin\theta\cos\phi, \qquad \frac{dx}{dt} = \dot{r}\sin\theta\cos\phi \qquad \text{(Note: } \dot{r} = \frac{dr}{dt}\text{)}$$

$$y = r\sin\theta\sin\phi, \qquad \frac{dy}{dt} = \dot{r}\sin\theta\sin\phi$$

$$z = r\cos\theta, \qquad \frac{dz}{dt} = \dot{r}\cos\theta$$

Therefore

$$v_r^2 = \left(\dot{r}\sin\theta\cos\phi\right)^2 + \left(\dot{r}\sin\theta\sin\phi\right)^2 + \left(\dot{r}\cos\theta\right)^2$$

$$= \dot{r}^2\left[\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right]$$

$$= \dot{r}^2\left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right]$$

$$= \dot{r}^2$$

$$v_r^2 = \dot{r}^2 \qquad \Rightarrow \qquad v_r = \dot{r}$$

For v_θ

$$x = r\sin\theta\cos\phi, \qquad \frac{dx}{d\theta} = r\cos\theta\cos\phi$$

$$y = r\sin\theta\sin\phi, \qquad \frac{dy}{d\theta} = -r\cos\theta\sin\phi$$

$$z = r\cos\theta, \qquad \frac{dz}{d\theta} = -r\sin\theta$$

and proceeding as we did for v_r, we get

$$v_\theta = r\dot{\theta} \qquad \text{where } \dot{\theta} = \frac{d\theta}{dt}$$

and for v_ϕ,

$$v_\phi = r\sin\theta\dot{\phi} \qquad \text{where } \dot{\phi} = \frac{d\phi}{dt}$$

So

$$v^2 = \left(v_r^2 + v_\theta^2 + v_\phi^2\right)$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$$

Since the force acting on the particle is a central conservative force, the path selected by the particle would be a two dimensional plane.
Let it be the xy plane then,

$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$
$$v_\phi = 0$$

and

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad \Rightarrow \quad \bar{v} = \bar{v}_r + \bar{v}_\theta$$

$$\bar{v} = \dot{r}\,\bar{r}_1 + r\dot{\theta}\,\bar{\theta}_1$$ where $\bar{r}_1$ and $\bar{\theta}_1$ are unit vectors

acceleration can be derived to be

$$\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\bar{r}_1 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\bar{\theta}_1$$

conservatism and associated potential

4.0 Conclusion

One can transform from one coordinate system to another; in this case from cartesian to polar.

5.0 Summary

The velocity and acceleration components for a two dimensional motion in a central conservative fields were obtained.

6.0 Tutor-Marked Assignment (TMA)

Question 3.1

Given that $\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$ in spherical coordinates, show that the inverse square law of force in three dimensions $F = \left(\frac{-k}{r^2}\right)\hat{e}_r$ is conservative

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

Unit 4 Energy Conservation in Central – Conservative Force Fields

1.0 Introduction

For a particle moving under any central, conservative force, information about the radial motion may be obtained from the radial energy equation., which results from eliminating the time derivative of θ with respect to time from equations of a particle motion in central conservative force field.

In this unit, we shall consider conservatism and associated potential by using polar coordinates to determine the total energy using initial conditions.

2.0 Aims and Learning Objectives

By the end of the unit students should be able:

• mathematically treat conservation of energy and angular momentum in planar motion to get the Radial and conservation energy equations.

3.0 Radial Energy Equation

For a motion of particle in a central conservative force field, equation of motion is:

$$\overline{F} = m\overline{a} = f(r)\overline{r_1} \quad (\textit{central conservative force})$$

$$= \quad m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\overline{r_1} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\overline{\theta_1}\right]$$

⇒

$$f(r) = m\left(\ddot{r} - r\dot{\theta}^2\right) \qquad (1) \qquad \text{and}$$

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad (2)$$

Now,

$$\overline{F} = -\nabla V(r) \quad \text{(conservative force)}$$

$$f(r)\overline{r_1} = -\nabla V(r)$$

$$V(r) = -\int f(r)dr \quad \text{(potential)}$$

K.E + P.E = E (toatal energy)

i.e.

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \int f(r)dr = E \qquad (3)$$

Angular momentum $\overline{L} = \overline{r} \; x \; \overline{p}$ (cartesian coordinates)

From $0 = m\left(r\ddot{\theta} + 2m\dot{r}\dot{\theta}\right)$, we get $L = mr^2\dot{\theta}$ (4) (polar coordinates)

Eliminating $\dot{\theta}$ from equations (3) and (4), $\dot{\theta} = \dfrac{L}{mr^2}$ we have

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \int f(r)dr = E \qquad \text{or} \qquad \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E$$

This is the Radial energy equation.

3.1 Energy Conservation Equation

For a particle which moves in a central force field described by a potential function, it posseses spherical symmetry and angular momentum and total energy are conserved.

$$F = f(r)\overline{r_1} \quad , \quad (\overline{r_1} \equiv \text{unit vector}) \text{ is a central conservative force}$$

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E$$

$$L = mr^2\dot{\theta}$$

Let $\quad u = \frac{1}{r}, \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \quad \Rightarrow \quad \frac{dr}{d\theta} = -r^2\frac{du}{d\theta} \quad \Rightarrow \quad \frac{dr}{d\theta}\dot{\theta} = -r^2\frac{du}{d\theta}\dot{\theta}$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = -r^2\frac{du}{d\theta}\dot{\theta}$$

$$L = mr^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{L}{mr^2}$$

Now

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E$$

Substituting for $\dot{r}$ and multiplying by $\frac{2m}{L^2}$, we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E - V) \qquad \text{---- Energy Conservation Equation}$$

This equation can be integrated to obtain the equation of the orbit in any of the following the forms

$$r = r(\theta)$$
$$r = r(t)$$
$$\theta = \theta(t)$$

3.2 Equation of Orbit

If we eliminate time from the equations of motion, we get the path of the particle.

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = f(r) \qquad (1)$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \qquad (2)$$

From $\quad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0$, we have

$$\frac{m}{r}\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 0, \quad \Rightarrow \quad \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0, \quad \Rightarrow \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

i.e. $\qquad r^2\dot{\theta} = constant \ (say, h)$

$$r^2\dot{\theta} = h \quad \Rightarrow \quad \dot{\theta} = \frac{h}{r^2} = hu^2$$

Substituting for $\dot{\theta}$ in $m\left(\ddot{r} - r\dot{\theta}^2\right) = f(r)$ we get, $m\left(\ddot{r} - h^2r^{-3}\right) = f(r)$

Now,

$$r = \frac{1}{u} \qquad \Rightarrow \qquad \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(\dot{r}\right) = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}$$

$m\left(\ddot{r} - h^2r^{-3}\right) = f(r)$ becomes,

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = f\left(\frac{1}{u}\right) \quad \text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{-f\left(\frac{1}{u}\right)}{mh^2u^2}$$

This is the equation for the path of the particle.
Note:
This equation can be applied to find path of particle that the force involved is directed towards a centre (that is ellipse, circular, parabolic, hyperbolic, etc.).

For a given central force, increasing the velocity causes the orbit to change from a circle to an ellipse to a parabola to a hyperbola, with the changes occurring at certain critical velocities. For example, if the speed of the Earth (which is in a nearly circular gravitational orbit) were increased by about a factor of 1.4, the orbit would change into a parabola and the Earth would leave the Solar System.

4.0 Conclusion

From the radial energy equation and the energy conservation equation and using polar coordinates, the path or differential equation of orbit can be obtained by integrating.

5.0 Summary

This differential equation of orbit can be applied to find path of particle that the force involved is directed towards a centre (that is ellipse, circular, parabolic, hyperbolic, etc.).

6.0 Tutor-Marked Assignment (TMA)

Question 4.1

Consider the family of orbits in a central potential for which the total energy is constant. Show that if a stable circular orbit exist, then the angular momentum associated with this orbit is larger than that for any other orbit of the family.

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

Unit 5 Central – Conservative Force and Planetary Motion

1.0 Introduction

We now turn to the problem of determining the orbit of a particle moving under a central conservative force. The motion under the influence of a central force is an extremely important problem in the motion of celestial bodies, such as planets, moons, comets ete. Undisturbed orbital motion under the influence of a central force satisfies Kepler's law of areas. When satelite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's laws describe more general case: elliptical orbit of a planet around the sun or a satellite around a planet

Kepler's laws will be reviewed and the orbits of a planet surrounding a heavy sun explained. The differential equation for orbit of motion in an inverse square law force field is derived.

2.0 Aims and Learning objectives

By the end of the unit students should be able:

establish that Kepler's laws are just consequences Newton's laws of gravitation and that of motion.

3.0 Kepler's Laws

Kepler's three laws of planetary motion can be described as follows:

- The paths of planets about the sun are elliptical in shape, with the center of the sun being located at one focus. (The Law of Ellipses)
- An imaginary line drawn from the center of the sun to the center of the planet will sweep out equal areas in equal intervals of time. (The Law of Equal Areas)
- The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the sun. (The Law of Harmonies)

Of course, Kepler's Laws originated from observations of the solar system. It should be known that these laws are consequences of Newton's laws of gravitation and of Motion.

3.1 Motion in an Inverse Square Law Force Field

Let us consider the central conservative force F given by

$$F = f(r)\overline{r_1} \quad , \qquad (\overline{r_1} \equiv \text{unit vector}) \quad f(r) = -\frac{k}{r^2}$$

i.e.

$$f\left(\frac{1}{u}\right) = -ku^2$$

So

$$\frac{d^2u}{d\theta^2} + u = \frac{-f\left(\frac{1}{u}\right)}{mh^2u^2} = \frac{ku^2}{mh^2u^2} = \frac{k}{mh^2}$$

Thus differential equation for orbit of motion in an inverse square law force field is

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}$$

Given that a particle is continuously subjected to a force directed toward a fixed central point, and that the magnitude of the force is inversely proportional to the square of the particle's distance from that central point, the usual way of determining the motion of the particle is by solving the appropriate differential equation. In this way it can easily be shown that the particle's path will be a conic, i.e., an ellipse, hyperbola, or parabola, with the central point located at one focus, and that the line from the central point to the particle sweeps out equal areas in equal times.

SELF ASSESSEMENT EXERCISE

1. What do you understand by a central conservative force field? A particle of mass m moves according to the equations:

$$x = x_0 + at^2$$
$$y = bt^3$$
$$z = ct$$

   (i) Find the angular momentum L at any time t.
   (ii) Find the force F and from it, the torque, N acting on the particle.
   (iii) Verify that the angular momentum theorem $\frac{d\vec{L}}{dt} = \vec{r} \ X \ \vec{F} = \vec{N}$

2. The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as $V(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$ where a and b are positive constants and x is the distance of separation between the two atoms.
   (i) at what values of x is V(x) equal zero
   (ii) at what values of x is V(x) a minimum
   (iii) determine the force between the atoms

3. Calculate the velocity vector in spherical polar coordinates.
   Assuming that a central conservative force is acting on a particle so that the path selected is on xy plane, show that the acceleration componenets are:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\dot{\theta} + 2\dot{r}\dot{\theta}$$

4. Show by means of the substitution $r = \frac{1}{u}$ that the differential equation for the path of the particle in a central field is $\frac{d^2u}{dt^2} + u = \frac{-f\left(\frac{1}{u}\right)}{mh^2u^2}$

4.0 Conclusion

We established mathematically that Kepler's Laws which originated from observations of the solar system are just consequences of Newton's laws of gravitation and of Motion.

5.0 Summary

We saw that Newton's law of graviation is example of a central force field. The orbits of a planet surrounding a heavy sun as described by Kepler's laws, we showed that Kepler's second law followed from conservation of angular momentum and the third law is a consequence of the inverse square force law

6.0 Tutor-Marked Assignment (TMA)

Question 5.1

Show mathematically that Kepler's laws of planetary motion are just consequences of Newton's laws of universal gravitation and motion.

Question 5.2

A particle moves under the influence of a central force given by $f(r) = \frac{-k}{r^n}$. If the particle's orbit is circular and passes through the force centre, show that n = 5

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

MODULE 1 Solutions to TMA

Unit 1

question 1.1

$A = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $B = 4\hat{i} + j + 3\hat{k}$

(i) A – B = (2i – 4i) + (3j – j) + (4k – 3k) = -2i + 2j + k

(ii) $A\,x\,B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 1 & 3 \end{vmatrix} = \hat{i}(3.3 - 4.1) + \hat{j}(4.4 - 2.3) + \hat{k}(2.1 - 4.3) = \hat{i}(9-4) + \hat{j}(16-6) + \hat{k}(2-12)$

= 5i + 10j – 10k

(iii) $(A \bullet B)^2 + (A\mathbf{X}B)^2 = [(2i + 3j + 4k) \text{ x } (4i + j + 3k)]^2 + (5i + 10j - 10k)^2$

$= (2.4 + 3.1 + 4.3)^2 + \{5.5 + 10.10 + (-10).(-10)\}$

$= (23)^2 + (25 + 100 + 100) = 529 + 225 = 754$

$A^2B^2 = (2i + 3j + 4k)^2(4i + j + 3k)^2 = (4 + 9 + 16)(16 + 1 + 9) = 29 \text{ x } 26 = 754$

Question 1.2

$A = 6\hat{i} + 5\hat{j} + 4\hat{k}$ and $B = \hat{i} + 2\hat{j} + 3\hat{k}$ and we know $A \bullet B = |A||B|\cos\theta$ by definition

So $|A| = \sqrt{6^2 + 5^2 + 4^2} = \sqrt{77}$ and $|B| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

and A•B = (6i+ 5j + 4k)(i + 2j + 3k) = 6 + 10 + 12 = 28

$$\cos\theta = \frac{A \bullet B}{|A||B|} = \frac{28}{\sqrt{77x14}} = \sqrt{\frac{8}{11}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\sqrt{\frac{8}{11}}\right) = 31.5^0$$

Unit 2

Question 2.1

A force F is conservative if the curl of F is zero. i.e $\nabla XF = 0$

Now $F_x = 2ax(z^3 + y^3)$, $F_y = 2ay(z^3 + y^3)$, $F_z = 3az^2(x^2 + y^2)$

$$\Rightarrow \quad F = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z = 2ax(z^3 + y^3)\hat{i} + 2ay(z^3 + y^3)\hat{j} + 3az^2(x^2 + y^2)\hat{k}$$

$\therefore$

$$\nabla\, X\ F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2ax(z^3+y^3) & 2ay(z^3+y^3) & 3az^2(x^2+y^2) \end{vmatrix} = i\left\{\frac{\partial}{\partial y}3az^2(x^2+y^2) - \frac{\partial}{\partial z}2ay(z^3+y^3)\right\}$$

$$+ \quad j\left\{\frac{\partial}{\partial z}2ax(z^3+y^3) - \frac{\partial}{\partial x}3az^2(x^2+y^2)\right\}$$

$$+ \quad k\left\{\frac{\partial}{\partial x}2ay(z^3+y^3) - \frac{\partial}{\partial y}2ax(z^3+y^3)\right\}$$

$$\nabla\, X\, F \quad = \quad i(6ayz^2 - 6ayz^2) + j(6axz^2 - 6axz^2) \quad + \quad k(0 - 6axy^2) \quad = \quad -6axy^2\hat{k}$$

$\nabla XF \neq 0$ hence force is not conservative, therefore potential energy cannot be determined in this case

Question 2.2

The force F and potential energy V are related by the expression

$$F = -\nabla V \quad = \quad -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(axy^2z^3)$$

$$= \quad -\left[iay^2z^3 + j2axyz^3 + k3axy^2z^2\right]$$

$$= \quad iF_x + jF_y + kF_z$$

Hence force components are :

$$F_x = iay^2z^3, \quad F_y = j2axyz^3, \quad F_z = k3axy^2z^2$$

and

$$F = \hat{i}ay^2z^3 + \hat{j}2axyz^3 + \hat{k}3axy^2z^2$$

Unit 3

Question 3.1

The force $F = \left(\dfrac{-k}{r^2}\right)\hat{e}_r$ is conservative if $\nabla\ X\ F = 0$ and

So,

$$\nabla\ X\ F \quad = \quad \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ \dfrac{-k}{r^2} & 0 & 0 \end{vmatrix}$$

$$\nabla \times F = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{-k}{r^2} & 0 & 0 \end{vmatrix}$$

Now $$= \frac{1}{r^2 \sin\theta}\left\{\hat{e}_r[0-0] + r\hat{e}_\theta\left[\frac{\partial}{\partial\phi}\left(\frac{-k}{r^2}\right) - o\right] + r\sin\theta\,\hat{e}_\phi\left[0 - \frac{\partial}{\partial\theta}\left(\frac{-k}{r^2}\right)\right]\right\}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\phi}\left(\frac{-k}{r^2}\right)\hat{e}_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{-k}{r^2}\right)\hat{e}_\phi\right]$$

$$= 0$$

We have shown that $F = \left(\frac{-k}{r^2}\right)\hat{e}_r$ is conservative since we evaluated $\nabla \times F = 0$

Unit 4

Question 4.1

The total energy, E is related to the potential, V(r) by the expression

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E$$

From which

$$L^2 = 2mr^2\left[E - V(r) - \frac{1}{2}m\dot{r}^2\right]$$

E and V(r) are the same for all orbits and the different values of angular momentum, L result only from different values of $\frac{1}{2}m\dot{r}^2$.

For stable circular motion, $\dot{r} = 0$ and for all other motions $\dot{r} \neq 0$

Therefore for non-circular motions, $\dot{r} \rangle 0$ and L is smaller than for the circular case. That is, L for circular motion is the largest among the family.

Unit 5

Question 5.1

Kepler's First Law

From conservation of angular momentum $\frac{d\theta}{dt} = \frac{L}{mr^2}$ (1)

Since the central force is conservative the energy of the planet must be a constant over the orbit

$$E = \frac{mv^2}{2} - \frac{GMm}{r} = cons\tan t$$

Substituting for the velocity in polar coordinates $mv^2 = m\left(\frac{dr}{dt}\right)^2 + mr^2\left(\frac{d\theta}{dt}\right)^2$ and using (1),

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{m} + \frac{2GM}{r} - \frac{L^2}{m^2 r^2} \qquad (2)$$

Equations (1) and (2) allow us to find the orbits. We can for example, solve (2) for dr/dt and integrate to find $r(t)$. We can also divide the equations for $\frac{dr}{dt}$, $\frac{d\theta}{dt}$ and integrate to get an equation for the orbit.

Kepler's Second Law

This law is a consequence of the conservation of angular momentum in a central force field. Recall from the definition of the vector product $\vec{L} = m\vec{r}\ X\ \vec{v}$

and the fact that $\vec{L}$ is constant, $\vec{r}$ and $\vec{v}$ must stay in a plane. Let us use polar coordinates (r, θ) in this plane. We have for the velocity $\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta$

Hence, the magnitude of the angular momentum is $L = r^2\frac{d\theta}{dt}$

In a time interval dt, the area swept by the vector $\vec{r}$ per unit time is just $\frac{1}{2}r^2\frac{d\theta}{dt} = cons\tan t$

Kepler's Third Law

Dividing Newton's second law by the mass of the planet we find the equation of motion

$$\frac{dv}{dt} = -\frac{rGM}{r^2}$$

The period T will depend on G, some parameter describing the orbit (such as the semi-major axis a), and the mass of the sun.

$$T = M^\alpha a^\beta G^\gamma$$

Using the method of dimension, we get

$$T^2 = const.\frac{a^3}{GM}$$

Question 5.2

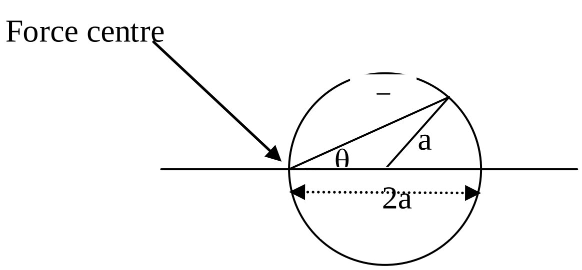


Figure 1.5: Sketch of orbit

For a central force motion, equation of orbit is given by

$$\frac{d^2u}{d\theta^2}+u=\frac{-f\left(\frac{1}{u}\right)}{mh^2u^2}$$

But $u=\frac{1}{r}$ so0

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{-r^2f(r)}{mh^2} \qquad (1)$$

From Figure 1.5, equation of orbit is
r = 2acosθ (2)
Putting eqn (2) in (1) we have,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{2a\cos\theta}\right)+\frac{1}{2a\cos\theta}=\frac{-(2a\cos\theta)^2f(r)}{mh^2}$$

i.e

$$\frac{1}{2a}\frac{d^2}{d\theta^2}\left(\frac{1}{\cos\theta}\right)+\frac{1}{2a\cos\theta}=\frac{-4a^2f(r)\cos^2}{mh^2}$$

But,

$$\frac{d^2}{d\theta^2}(\cos\theta)^{-1}=\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos^2\theta}\right)=\frac{1}{\cos\theta}+\frac{2\sin^2\theta}{\cos^3\theta}$$

Therefore,

$$\frac{1}{2a}\left[\frac{2}{\cos\theta}+\frac{2\sin^2\theta}{\cos^3\theta}\right]=\frac{-4a^2}{mh^2}f(r)\cos^2\theta$$

Form which,

$$f(r)=\frac{-mh^2}{4a^3\cos^3\theta}-\frac{mh^2\sin^2\theta}{4a^3\cos^5\theta}=\frac{-mh^2(\cos^2\theta+\sin^2\theta)}{4a^3\cos^5\theta}=\frac{-mh^2}{4a^3\cos^5\theta}\frac{8a^2}{8a^2}=\frac{-8ma^2h}{(2a\cos\theta)^5}=\frac{-k}{r^5}$$

where $k=8ma^2h$ ≡ constant

Comparing $f(r)=\frac{-k}{r5}$ we just got with $f(r)=\frac{-k}{r^n}$ ⇒ n = 5

MODULE 2: OSCILLATORY MOTION

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by an organ pipe and the back-and-forth motion of the pistons in a car engine. This kind of motion is called periodic motion or oscillation.

A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. There are many oscillatory systems; the vibrations of molecules and interaction between atoms, oscillations of an electrical circuit and springs. There is reciprocal nature of correspodence between mechanical compliance and electrical capacitance.

Unit1 Linear Simple Harmonic Osillator

1.0 Introduction

The linear simple harmonic oscillator (SHO) is the foundation of the theory of oscillations. We discuss equilibria in physical systems and how small oscillations about equilibria can in most cases be described by the SHO equation.This unit deals with a range of simple harmonic oscillatory phenomena, explaining the general techniques for analysing and predicting them.

2.0 Aims and Learning objectives

By the end of this unit students should be able to;

find the linear approximation to any dynamical system near equilibrium and also know how to derive and solve the wave equation for small oscillations.

3.0 Simple Harmonic Motion (SHM)

If one displaces a system from a position of stable equilibriun, the system will move back and forth, that is, it will oscillate about the equilibrium position. The maximum displacement is called the amplitude, A.

The time, T, to go through one complete cycle is called the period of oscillation and its inverse is called the frequency, f.

$$f = \frac{1}{T}$$

For many systems, if the amplitude is small enough, the restoring force F is directly proportional to the displacement from equilibrium x and satisfies Hook's law, given by:

$$F = - kx$$

where k is a positive constant known as the force constant and has units of N/m (or kg/s^2).

The motion of such system is called simple harmonic motiom (SHM). We can compute the motion using Newton's second law (F = ma) to have

$$-kx = m\frac{d^2x}{dt^2} = m\ddot{x}$$

The solution of this equation gives the displacement, x as a function of time, t.

The general solution is of the form

$$x = A\cos(\omega t + \phi)$$

where ϕ is called the phase and it defines the initial displacement $x = A\cos\phi$

$$\frac{dx}{dt} = \dot{x} = -\omega A\sin(\omega t + \phi) \text{ and } \frac{d^2x}{dt^2} = \ddot{x} - \omega^2 A\cos(\omega t + \phi)$$

$$-kx = m\frac{d^2x}{dt^2} = m\ddot{x} \text{ becomes}$$

$$-kA\cos(\omega t + \phi) = m\left(-\omega^2 A\cos(\omega t + \phi)\right)$$

$$k = m\omega^2 \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{\omega}}$$

The equation of a simple harmonic motion is thus

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \text{or} \qquad \ddot{x} + \omega^2 x = 0$$

By definition, after a period T later the motion repeats itself, therefore:

$$x = A\cos\omega t = A\cos(\omega t + \omega T) = A\cos\omega t\cos\omega t - A\sin\omega t\sin\omega t \backslash$$

This equation can be solved if we set

$$\omega T = 2\pi \quad \Rightarrow \quad \omega = \frac{2\pi}{T} = 2\pi f \quad \equiv \textit{angular frequency}$$

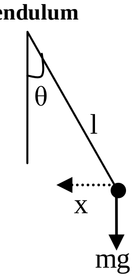
Simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

The projection of the motion of a particle along y-axis implies that the particle also exhibits simple harmonic motion. Therefore, the uniform circular motion can be considered as a combination of two simple harmonic motions, one along the x-axis and the other along y-axis, with the two differing in phase 90^o.

3.1 Examples of SHM

Several examples of SHM or SHO (simple harmonic motion or simple harmonic oscillation) exist, three, each with its equation of motion, are presented below in Figures 2.1 (a), (b) and (c)

a. A Simple Pendulum

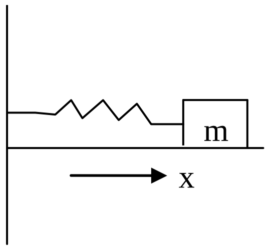


The equation of motion is

$$m\ddot{x} + mg\frac{x}{l} = 0 \quad or \quad ml\ddot{\theta} + mg\theta = 0$$

$$\omega^2 = \frac{g}{l}$$

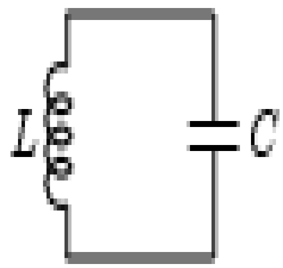
b. A maas m on a frictionless plane connected by a spring to a wall



$$m\ddot{x} + kx = 0$$

$$\omega^2 = \frac{k}{m}$$

c. An electrical LC resonant circuit



$$L\ddot{q} + \frac{q}{C} = 0$$

$$\omega^2 = \frac{1}{LC}$$

4.0 Conclusion

Linear Simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle. Several oscillatory systems exist that exibit and their equations of motion can easily be established.

5.0 Summary

To prove a simple harmonic motion, the following steps are very important:

- Draw a free body diagram, showing all the forces acting on it at any instant.
- Denote the direction of the acceleration $a \quad or \quad \frac{d^2x}{dt^2}$ in that of increasing x.
- Apply Newton's secon law, F = ma
- Evaluate an expression for the acceleration a, and compare with with a = - ωx^2

6.0 Tutor-Marked Assignment (TMA)

Question 1.1

Show that the equation

$$x(t) = a\cos(\omega t - \phi)$$

where a > 0, $\omega > 0$ and ϕ are constants, is indeed a solution to equation of a simple harmonic motion. Sketch the graph of this equation and explain in details how you could obtain the amplitude, period and frequency.

7.0 References/Further Readings

Pain, H. J. (1999) *The Physics of Vibrations and Waves*, 5th Edition, John Wiley & Sons, Chichester UK.

Crawford Jr, F. S. (1968) *Waves, Berkeley Physics Course, Vol. 3, McGraw-Hill, New York NY.*

Unit 2 Conservation of Energy in SHM

1.0 Introduction

A number of energy conserving physical systems that exhibit *simple harmonic oscillation* about a stable equilibrium state exist. One of the main features of such oscillation is that, once excited, it never dies away. However, the majority of the oscillatory systems which we generally encounter in everyday life suffer some sort of irreversible energy loss due; for instance, to frictional or viscous heat generation whilst they are oscillating. We shall examine the conservation of the mechanical energy involved in a linear SHM.

2.0 Aims and Learning objectives:

By the end of this unit students should be able to;

find the expressions for the kinetic and potential energies of a SHM and also establish that total energy remains constant during a SHM.

3.0 Energy of Simple Harmonic Motion

Energies in a simple harmonic motion are:

kinetic energy, K.E = $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$ and

potential energy, P.E = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$

So total energy is

E = K.E + P.E = $\frac{1}{2}mv^2 + kx^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$

$= \frac{1}{2}kA^2$ (For a spring mass: $m\omega^2 = k$)

The time average kinetic energy and time average potential energy are expressed as:

$$\langle K.E \rangle = \frac{1}{T}\int_t^{t+T} \frac{1}{2}m\dot{x}^2 \, dt \quad \text{and}$$

$$\langle P.E \rangle = \frac{1}{2}\int_t^{t+T} \frac{1}{2}kx^2 dt \quad \text{respectively.}$$

where T is the period of oscillation.

4.0 Conclusion

In the absence of non-conservative forces, the total mechanical energy of a SHM is constant.

5.0 Summary

The energy oscillates back and forth between K.E and P.E, in such a way that the sum remains constant. In reality, however, most systems are affected by non-conservative forces.

6.0 Tutor-Marked Assignment (TMA)

Question 2.1

A simple harmonic oscillator consists of a 100g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced a distance of 3cm and released from rest. Calculate the following quantities:

(a) the natural frequency

(b) the period

(c) the total energy

(d) the maximum velocity

Question 2.2

The oscillator in the above problem is set into motion by giving it an initial velocity of 10^{-2} ms^{-1} at its equilibrium position. Calculate;

(a) the maximum displacement

(b) the maximum potential energy

Question 2.3

Considering a simple harmonic oscillator, calculate the time averages of the kinetic and potential energies over one cycle and show that the two quantities are equal.

7.0 References/Further Readings

Pain, H. J. (1999) *The Physics of Vibrations and Waves*, 5th Edition, John Wiley & Sons, Chichester UK.

Crawford Jr, F. S. (1968) *Waves, Berkeley Physics Course, Vol. 3, McGraw-Hill, New York NY.*

Unit 3 Damped Oscillatory Motion

1.0 Introduction

Most simple harmonic oscillators in the real world are damped – mechanical oscillators, electrical oscillators, etc. We assume that a damping force linear in velocity is applied to the harmonic oscillator. For a mechanical oscillator, this could be a frictional force. For an electrical oscillator, this could be a resistive element. We investigate the effect of damping on the SHO

2.0 Aims and Learning objectives:

By the end of this unit students should be able to;
derive and solve the wave equation for damped oscillatory motions which is due to dissipative non- conservative forces.

3.0 Damped Harmonic Motion:

In real systems, the dissipative forces (non- conservative forces) retard the oscillatory motion by causing amplitude to decrease. Consequently, the mechanical energy of the system diminishes with time. Thus, the oscillatory motion of the system is damped.
The non – conservative force (called damping force) is approximately equal

$$- r\,v = -r\dot{x}$$

r is a constant giving the damping strength and v is the velocity.
The equation of a damped harmonic oscillatory motion is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \qquad \text{or} \qquad m\ddot{x} + r\dot{x} + = 0$$

The solution of the differential equation

$$m\ddot{x} + r\dot{x} + = 0$$

is of the form

$$x(t) = Ae^{-\frac{t}{\tau}}\cos(\omega t + \phi)$$

For simplicity, let's take x = A at t = 0, then ϕ = 0
If we plug the solution

$$x(t) = Ae^{-\frac{t}{\tau}}\cos(\omega t)$$

into Newton's second law, we get the damping time, τ as

$$\tau = \frac{2m}{r}$$

and the angular frequency, ω as

$$\omega = \omega_0\sqrt{1 - \frac{1}{(\omega_0\tau)^2}}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the un-damped angular frequency

The larger the damping constant r the shorter the damping time τ. There are three damping regimes as indicated in Figure 2.2:
(a) underdamped (b) critically damped (c) overdamped

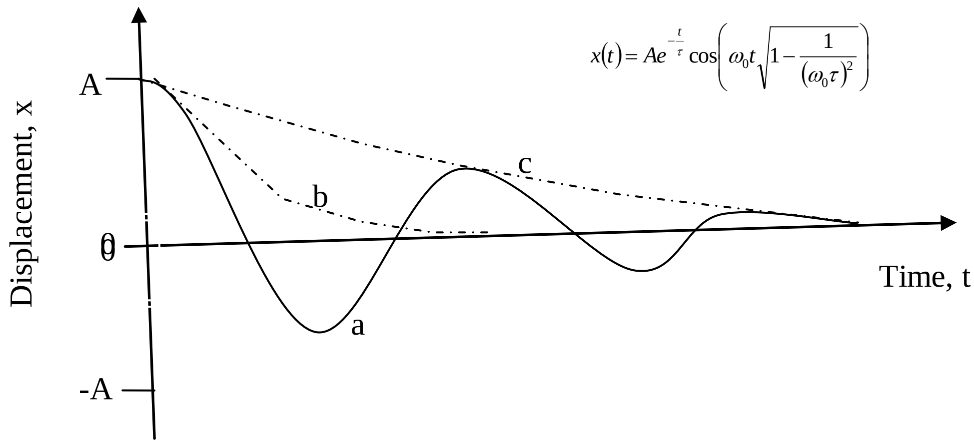


Figure 2.2: Sketch of damping wave function

Underdamping, the system oscillates with steady decreasing amplitude when it is displaced and released. In critical damping, the system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released. For overdamping, there is no oscillation but the system returns to equilibrium more slowly than with critical damping.

4.0 Conclusion

Critical damping is of practical importance in recording instruments such as ballistic galvanometer which experiences sudden impulses and are required to return to zero displacement in minimum time.

The degree of damping a mechanical system is important for example, a good suspension system of a car should be slightly undercritically damped to ensure comfortable ride.

5.0 Summary

In a damped oscillatory motion the following points must be remembered:

- When the retarding force is much smaller than the restoring force, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases. Any system that behaves in this manner is called a damped oscillator.
- During the damped oscillatory motion, the amplitude decays exponentially with time.
- In the absence of retarding force, the system oscillates with its natural frequency.
- If the system is so viscous that the retarding force is greater than the restoring force then the system is over damped.
- Irrespective of the case whether the system is over damped or under damped, the friction is present and the energy of the oscillator eventually falls to zero. The lost mechanical energy dissipates into internal energy in the retarding medium.

6.0 Tutor-Marked Assignment (TMA)

Question 3.1

Consider an electrical circuit consisting of an inductor, of inductance L, connected in series with a capacitor, of capacitance C, and a resistor, of resistance R. Such a circuit is known as an LCR circuit, for obvious reasons. Show that the current in the circuit execute damped harmonic oscillations.

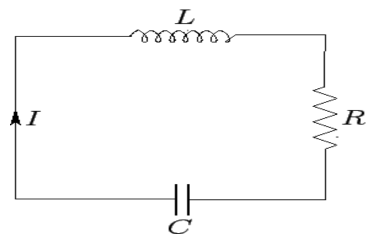


Figure 2.3: LRC circuit

7.0 References/Further Readings

Pain, H. J. (1999) *The Physics of Vibrations and Waves*, 5th Edition, John Wiley & Sons, Chichester UK.

Crawford Jr, F. S. (1968) *Waves, Berkeley Physics Course, Vol. 3, McGraw-Hill, New York NY.*

Unit 4 Forced Oscillatory Motion

1.0 Introduction

So far we have considered an oscillator in isolation: initial conditions are imposed and then the system evolves, perhaps with damping. Frequently in reality we happen upon driven oscillators, oscillators that are exposed to a continuous driving force – e.g., a driven LRC circuit. The oscillator's evolution is modified by the existence of the additional driving force. There are many situations in which a system may be driven by a regular or irregular external force. For example, machinery may vibrate its local enviroment; an electromagnet may vibrate the cone of a loudspeaker, an electrical current may drive the oscillator in a watch. Sometimes we may wish to suppress the vibrations, while in other situations we may wish to enhance them. We investigate the effect of driving forces on the SHO.

2.0 Aims and Learning objectives

By the end of this unit students should be able to;

•use an external oscillatory force in Newton's second law to get an equation of a driven damped oscillation.

• to know that when the system executing forced oscillation behaves in such a way that its natural frequency becomes equal to the frequency of oscillation, the system is said to be in resonance.

3.0 Forced Oscillations and Resonance:

When an oscillatory system is acted upon by an external force we say that the system is driven (or forced).To compensate the energy loss in the system in a damped system due to the retarding forces, an external force is applied. This force acts in the direction of motion of the oscillator and does a positive work on the system. As a result of which, the amplitude of motion remains constant when the energy input per cycle exactly equals the energy lost due to damping. The system which oscillates in this manner experiences forced oscillation.

Consider an external oscillatory force $F = F_0 \cos(\omega_d t)$

where F is varying force with time t, F_0 driving force and ω_d driving angular frequency.

Newtos's second law for the system becomes

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F_0 \cos\omega_d t \quad \text{or} \quad m\ddot{x} + r\dot{x} + kx = F_0 \cos\omega_d t$$

Again, if we try a solution of the form

$$x(t) = A\cos(\omega_d t)$$

and plug into the Newton's second law, we get the amplitude that has a resonance form

$$A(\omega_d) = \frac{F_0}{m\sqrt{(\omega_d - \omega_0)^2 + \frac{r^2\omega_d^2}{m^2}}}$$

When the system executing forced oscillation behaves in such a way that its natural frequency becomes equal to the frequency of oscillation, the system is said to be in resonance. At resonance, the applied force remains in phase with the velocity so that the power transferred to the system is of maximum value.

The soldiers marching on a bridge are asked to break the steps to avoid the resonance condition or else it collapses after hitting its resonant frequency.
Resonance not only occurs in mechanical systems but also in electrical circuits. Microwaves with frequency similar to the natural frequency of vibration of water molecules are used in microwave oven. When food is placed in the oven, the water molecules of the food resonant, absorbing energy from the microwaves and consequently get heated up.
Magnetic resonance imaging systems have greatly improved medical diagnoses in present day technology.

4.0 Conclusion

Sometimes we may wish to suppress the vibrations, while in other situations we may wish to enhance them, all these we employ driving forces on the SHO and mathematically investigate the effects to get the equation for the motion for forced oscillations.

5.0 Summary

When the system executing forced oscillation behaves in such a way that its natural frequency becomes equal to the frequency of oscillation, the system is said to be in resonance. Many mechanical systems and electrical circuits experience resonance phenomena.

6.0 Tutor-Marked Assignment (TMA)

Question 4.1

Consider an LCR circuit consisting of an inductor, L, a capacitor, C, and a resistor, R, connected in series with an emf of voltage V(t). Show that the current in this arrangement will execute forced or driven damped harmonic oscillations. Name one application of an LRC circuit.

7.0 References/Further Readings

Pain, H. J. (1999) *The Physics of Vibrations and Waves*, 5th Edition, John Wiley & Sons, Chichester UK.

Crawford Jr, F. S. (1968) *Waves, Berkeley Physics Course, Vol. 3, McGraw-Hill, New York NY.*

Unit 5 Coupled Oscillation

1.0 Introduction

Coupled simple harmonic oscillators are physically important and seen everywhere in nature. The electromagnetic field at any point in space can be considered a simple harmonic oscillator, but Maxwell's Equations couple the oscillator at one point to the oscillators everywhere else in space. Coupled oscillations are critical in particle physics and also use when discussing rigid-body etc

2.0 Aims and Learning objectives

By the end of this unit students should be able to;

•to establish equations of motion for two coupled oscillatory systems (normal modes) and other similar physical systems.

• solve the differential equations to obtain the normal frequencies.

3.0 Normal Frequencies and Normal Mode of Vibration: Two Body Oscillations

A vibration involving only one independent variable, say x (or y), is called a normal mode of vibration and has its own normal frequency.

Consider the coupled oscillations, when two masses are connected with each other by strings and oscillating together, Figure 2.4

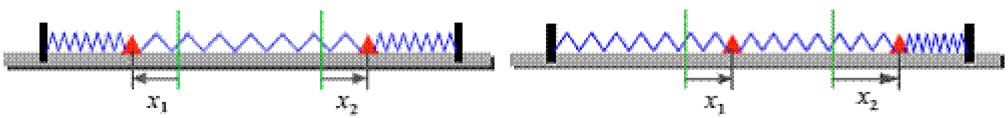


Figure 2.4: coupled system

The masses are placed on a frictionless track and joined up by ideal strings as shown in the figure. There are two kinds of motion which distinguish themselves by being very simple and the two differential equations of motion are:

$$k(x_2 - 2x_1)\hat{i} = m\ddot{x}_1\,\hat{i}$$

$$k(x_1 - 2x_2)\hat{i} = m\ddot{x}_2\,\hat{i}$$

If we solve the two equations we will obtain two frequencies each corresponding to the kind of motion.

In one kind of motion x_1 and x_2 remain equal and the whole system oscillates back and forth without the stretching of the middle string. In this kind of motion the frequency of oscillation is given by,

$$\omega_0 = \sqrt{\frac{k}{m}}$$

In the other kind of motion, x_1 and x_2 remain exactly opposite and the motion is like "in and out" type. In this case, the net restoring force on the body is three times as compared to the previous case and thus the frequency of oscillation of the system is given by,

$$\omega = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_0$$

Self Assessement Exercise

1. A simple harmonic oscillator consists of a 100g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced a distance of 3cm and released from rest. Calculate the following quantities:
   (a) the maximum displacement
   (b) the maximum potential energy

2. (a) Define Normal frequencies and Normal modes
   (b) What do you understand by forced frequency and resonance?
   (c) Show that the function $y = A\cos\omega t + A\sin\omega t$ can be written as $y = C\cos(\omega t - \delta)$
   where $C = \sqrt{A^2 + B^2}$ and $\delta = \tan^{-1}\left(\frac{B}{A}\right)$

3. A mass of 2 kg is attached to a spring of elasticity constant 8 Nm^{-1}. At time t = 0, the mass is displaced to a position x = 0.2m and released from rest. Find the position x of the mass as a function of time t.

4.0 Conclusion

In two coupled body system, there are two kinds of motion which distinguish themselves by being very simple each with its differential equation of motion. Each of the equations could be solved to get the normal frequencies.

5.0 Summary

A vibration involving only one independent variable, say x (or y), is called a normal mode of vibration and has its own normal frequency. The simple motion of the two-body coupled oscillator is called normal modes. They have the property that when the system starts motion in one of these modes then it will continue in that mode.

6.0 Tutor-Marked Assignment (TMA)

Question 5.1

Consider the LC circuit pictured in Figure 2.4 below. How many normal modes of oscillation will be there?

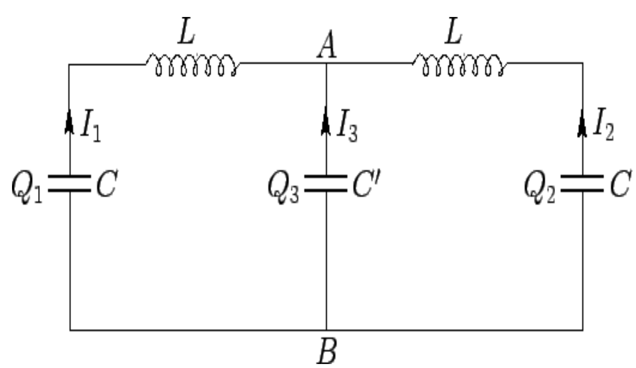


Figure 2.5: LC coupled circuit

7.0 References/Further Readings

Pain, H. J. (1999) *The Physics of Vibrations and Waves*, 5th Edition, John Wiley & Sons, Chichester UK.

Crawford Jr, F. S. (1968) *Waves, Berkeley Physics Course, Vol. 3, McGraw-Hill, New York NY.*

MODULE 2 Solutions to TMA

Unit 1
Question 1.1

The equation of a SHM is given by $m\ddot{x} = -kx$ (1)

Now $x(t) = a\cos(\omega t - \phi)$ (2)

Therefore $\dot{x} = -\omega a \sin(\omega t - \phi)$ so that $\ddot{x} = -\omega^2 a \cos(\omega t - \phi)$

Substituting we have,

$$-m\omega^2 a\cos(\omega t - \phi) = -ka\cos(\omega t - \phi) \qquad (3)$$

It follows that equation (3) is indeed a solution to equation (1) provided $\omega = \sqrt{\frac{k}{m}}$

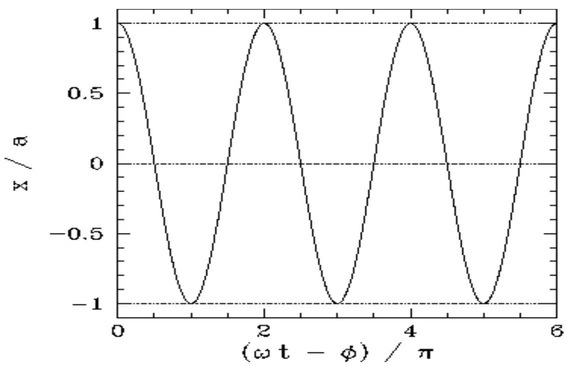


Figure 2.6: Wave sketch

Figure 2.6 shows a graph of x versus t obtained from Equation (2). The type of behavior shown here is called *simple harmonic oscillation*. It can be seen that the displacement x *oscillates* between x = -a and x = +a . Here, a is termed the *amplitude* of the oscillation. Moreover, the motion is *repetitive* in time (*i.e.*, it repeats exactly after a certain time period has elapsed). In fact, the *repetition period* is $T = \frac{2\pi}{w}$

This result is easily obtained from Equation (1) by noting that $\cos\theta$ is a periodic function of θ with period 2π: *i.e.*, cos(θ + 2π) = cosθ. It follows that the motion repeats every time ωt increases by 2π: *i.e.*, every time t increases by $\frac{2\pi}{\omega}$. The *frequency* of the motion (*i.e.*, the number of oscillations completed per second) is $f = \frac{1}{T} = \frac{\omega}{2\pi}$

It can be seen that ω is the motion's *angular frequency*; *i.e.*, the frequency f converted into radians per second. Of course, f is measured in *Hertz*--otherwise known as *cycles per second*. Finally, the *phase angle*, ϕ, determines the times at which the oscillation attains its maximum displacement, $x = a$. In fact, since the maxima of $\cos\theta$ occur at $\theta = n\,2\pi$, where n is an arbitrary integer, the times of maximum displacement are

$$t_{\max} = T\left(n + \frac{\phi}{2\pi}\right)$$

Clearly, varying the phase angle simply shifts the pattern of oscillation backward and forward in time.

Unit 2
Question 2.1

The cgs unit $dyne/cm = \dfrac{\left(\dfrac{gm-cm}{s^2}\right)}{cm} = \dfrac{gm-cm}{s^2} x \dfrac{1}{cm} = gm/s^2$

(a) Use $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{10^4(dyne/cm)}{100gm}} = \dfrac{1}{2\pi}\sqrt{\dfrac{10^4((gm-cm/s^2)/cm)}{10^2 gm}} = \dfrac{10}{2\pi}\left(s^{-1}\right) = 1.6Hz$

(b) Period $T = \dfrac{1}{f} = \dfrac{2\pi}{10} = .63\sec$

(c) $E = \dfrac{1}{2}kA^2 = \dfrac{1}{2}x10^4 x3^2 = 4.5x10^4 (gm-cm)/s^2 = 4.5x10^4 erg = 4.5x10^4 x10^{-7}\, Joules = 4.5x10^{-3} J$

(d) The max. velocity is attained when the total energy of oscillator equals the kinetic energy.

$$\frac{1}{2}mv_{\max}^2 = \frac{1}{2}kA^2 \qquad \Rightarrow \qquad v_{\max} = \sqrt{\frac{kA^2}{m}}$$

Question 2.2

(a) Maximum displacement x_0 is achieved when the total energy equals the potential energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_0^2 \qquad \Rightarrow \qquad x_0 = \sqrt{\frac{m}{k}}v_0 \quad \text{where } v_0 \text{ is initial velocity}$$

(b) $$(P.E)_{\max} = \frac{1}{2}kx_0^2$$

Question 2.3

The position and velocity for a SHO are given by

$x = A\sin\omega_0 t$ and $\dot{x} = \omega_0 A\cos\omega_0 t$ where $\omega_0 = \sqrt{\dfrac{k}{m}}$

The time averages of the kinetic and potential energies are:

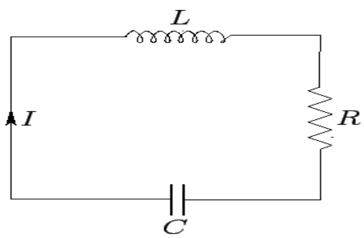
$$(K.E) = \frac{1}{T}\int_t^{t+T}\frac{1}{2}m\dot{x}^2\,dt = \frac{1}{2T}mA^2\omega_0^2\int_t^{t+T}\cos^2\omega_o t dt = \frac{mA^2\omega_0^2}{4}$$

where $T = \dfrac{2\pi}{\omega_0}$ and

$$(P.E) = \frac{1}{T}\int_t^{t+T}\frac{1}{2}kx^2 dt = \frac{1}{2T}\int_t^{t+T}\sin^2\omega_0 t dt = \frac{kA^2}{4} = \frac{mA^2\omega_0^2}{4} \quad \text{since} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$\therefore$ $(K.E) = (P.E)$

Unit 3
Question 3.1



Suppose that I(t) is the instantaneous current flowing around the circuit. The potential differences across the inductor and the capacitor are $L\dot{I}$ and $\frac{Q}{C}$, respectively. Here, Q is the charge on the capacitor's positive plate, and $I = \dot{Q}$. Moreover, from *Ohm's law*, the potential difference across the resistor is V = IR. Now, *Kichhoff's second circuital law* states that the sum of the potential differences across the various components of a closed circuit loop is zero. It follows that

$$L\dot{I} + RI + \frac{Q}{C} = 0$$

Dividing by L, and differentiating with respect to time, we obtain

$$\ddot{I} + \upsilon\dot{I} + \omega_0^2 I = 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\upsilon = \frac{R}{L}$

Comparison with the Equation of a Damped motion, $m\ddot{x} + r\dot{x} + = 0$, reveals that $\ddot{I} + \upsilon\dot{I} + \omega_0^2 I = 0$ is a damped harmonic oscillator equation. Thus, provided that the resistance is not too high (*i.e.*, provided that $\upsilon < \frac{\omega_0}{2}$, which is equivalent to $R < 2\sqrt{\frac{L}{C}}$, the current in the circuit executes damped harmonic oscillations of the form [cf, $x(t) = Ae^{-\frac{t}{\tau}}\cos(\omega t + \phi)$]

$$I(t) = I_0 e^{-\frac{\upsilon t}{2}}\cos(\omega t - \phi)$$

where I_0 and ϕ are constants and $\omega = \sqrt{\omega_0^2 - \frac{\upsilon^2}{4}}$.

We conclude that when a small amount of resistance is introduced into an LC circuit the characteristic oscillations in the current vary exponentially at a rate proportional to the resistance.

Multiplying Equation $L\dot{I} + RI + \frac{Q}{C} = 0$ by I, and making use of the fact that $I = \dot{Q}$, we obtain

$$L\dot{I}I + RI^2 + \dot{Q}\frac{Q}{C} = 0$$

which can be rearranged to give

$$\frac{dE}{dt} = -RI^2$$

where $E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}$

Clearly, E is the circuit energy: *i.e.*, the sum of the energies stored in the inductor and the capacitor. Moreover, the circuit energy decays in time due to the power RI^2 dissipated via *Joule heating* in the resistor. Note that the dissipated power is always positive: *i.e.*, the circuit never gains energy from the resistor.

Unit 4

Question 4.1

Let I(t) be the instantaneous current flowing around the circuit. Now, according to *Kichhoff's second circuital law*, the sum of the potential drops across the various components of a closed circuit loop is equal to zero. Thus, since the potential drop across an emf is *minus* the associated voltage, we obtain

$$L\dot{I} + RI + Q/C = V,$$

where $\dot{Q} = I$.

Suppose that the emf is such that its voltage oscillates *sinusoidally* at the angular frequency ω > 0, with the peak value $V_0 > 0$, so that

$$V(t) = V_0 \sin(\omega t).$$

Dividing by L, and differentiating with respect to time, we obtain

$$\ddot{I} + \nu\dot{I} + \omega_0^2 I = \frac{\omega V_0}{L}\cos(\omega t),$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\upsilon = \frac{R}{L}$

Comparison with $x(t) = x_0 \cos(\omega t - \varphi)$ reveals that this is a driven damped harmonic oscillator equation. It follows, by comparison with the analysis contained in the previous section, that the current driven in the circuit by the oscillating emf is of the form

$$I(t) = I_0 \cos(\omega t - \varphi),$$

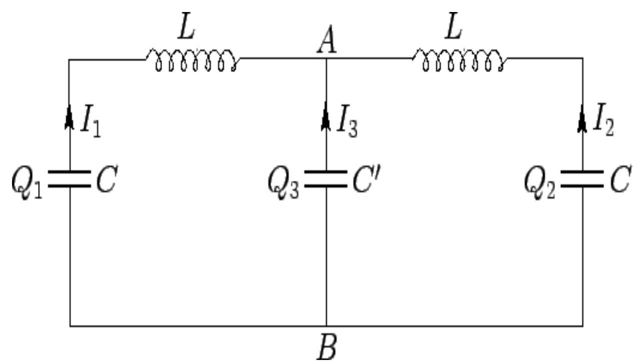
where

$$I_0 = \frac{\frac{\omega V_0}{L}}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \upsilon^2\omega^2\right]^{\frac{1}{2}}}$$

$$\varphi = \tan^{-1}\left(\frac{\upsilon\omega}{\omega_0^2 - \omega^2}\right)$$

LRC circuits are often employed as analogue *radio tuners*. In practice, the values of L and R are fixed, whilst the value of C is varied (by turning a knob which adjusts the degree of overlap between two sets of parallel semicircular conducting plates) until the signal from the desired radio station is found.

Unit 5
Question 5.1



According to *Kichhoff's first circuital law*, the net current flowing into each junction is zero. It follows that $I_3 = (I_1 + I_2)$. Hence, this is a *two degree of freedom* system whose instantaneous configuration is specified by the two independent variables $I_1(t)$ and $I_2(t)$. It follows that there are *two* independent normal modes of oscillation.

MODULE 3: LAGRANGE AND HAMILTONIAN MEACHANICS

It may be difficult or even impossible to obtain explicit expressions for forces of constraint by applying Newtonian procedure. In order to circumvent some of the practical difficulties which may arise in attempts to apply Newtonian mechanics to particular problems, alternative procedures are necessary. This alternative methods are contained in Hamilton's princiciple and the equations of motion which result from the application of this principle; the Lagrange's equations of motion. The Lagrange's equations can be obtained in a variety of ways. We are accustomed to thinking of mechanical systems in terms of vector quantities such as force, velocity, angular momentum, torque, etc., but in Lagrangian formulation, the equations of motion are obtained entirely in terms of scalar operations.

Unit 1 Frame of Reference and Constraints of Motion

1.0 Introduction

Constraints are restrictions imposed in the free motion of a particle (or a system of particles). Imposing constraints on a system is simply another way of stating that there are forces present in the problem that cannot be specified directly, but are known in term of their effect on the motion of the system. In order for Newton's laws of motion to have meaning, a reference frame (coordinate system) which is fixed in space with respect to the distant fixed stars must be chosen with respect to which the motions of bodies can be measured

2.0 Aim and Objectives

By the end of this study unit students will be able to:

• distinquish between '*inertia frame of reference*' and '*non-inertal frame of reference*'

• know how to impose contraints on a system in order to simplify the methods to be used in solving physics problems

3.0 Frames of References

A frame of reference may refer to a *coordinate system* or set of axes within which to measure the position, orientation, and other properties of objects in it, or it may refer to an *observational reference frame* tied to the state of motion of an observer. It may also refer to both an observational reference frame and an attached coordinate system, as a unit.

Newton realized that in order for the laws of motion to have meaning, a reference frame (coordinate system) which is fixed in space with respect to the distant fixed stars must be chosen with respect to which the motions of bodies can be measured. A reference frame is called an *inertial frame of reference* if Newton's laws indeed hold in that frame.

If Newton's laws hold in one reference frame then they also hold in any other reference that is in uniform motion (i.e., it is not accelerating) with respect to the first system.

*Non-inertial frame of reference***,** which is not fixed in space, is a moving coodinate system such as the one attached to a falling body or one that is rotating and therefore accelerating.

3.1 Constraints of Motion

Constraints are restrictions imposed in the free motion of a particle (or a system of particles).

e.g. A system of particles, inter particle distance is constant. The motion may be restricted geometrically in a sense that it must stay on a certain definite surface or curve or to be along a specified path and the motion is said to be constrained.

The total force acting on a particle moving under constraint is

$$m\frac{dv}{dt} = F + R$$

where v is velocity, F is external force, R is force of constrained which is the reaction of the constraining agent.

There are two types of constraints:

1. *Holonomic constraints* are those that can be represented as a functions of position vector and time example, $\phi(r_1, r_2, r_3, \text{-------} r_n, t)$
2. *Non – Holonomic constraints* are those that cannot be represented as functions of position vector and time example, x

4.0 Conclusion

If Newton's laws hold in one reference frame then they also hold in any other reference that is in uniform motion (i.e., it is not accelerating) with respect to the first system.

5.0 Summary

A frame of reference may refer to a *coordinate system* or set of axes within which to measure the position, orientation, and other properties of objects in it. Constraints are restrictions imposed in the free motion of a particle (or a system of particles).

6.0 Tutor-Marked Assignment (TMA)

Question 1.1

A particle of mass 2 units moves along space curve whose position vector is given as a function of time t by

$$r = (2t^3 + t)\mathbf{i} + (3t^4 - t^2)\mathbf{j} - 12t^2\mathbf{k}$$

Find a) the velocity
b) the momentum
c) the acceleration
d) the force acting on it at any time t

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

http://www.scienceaid.co.uk/physics/forces.html (Sept. 2009)

http://www.academics.hamilton.edu/physics/smajor/Courses (Sept. 2009)

Unit 2 Generalized Coordinates

1.0 Introduction

The use of generalized coordinates may considerably simplify a system's analysis. They reduce the total number of degrees of freedom available to the system.The choice of generalized coordonates eliminates the need for the constraint force to enter into the resultant system of equations. When you describe a system in terms of generalized coordinates, you pick the coordinates with the goal of completely describing the motion of the system in the fewest number of coordinates.

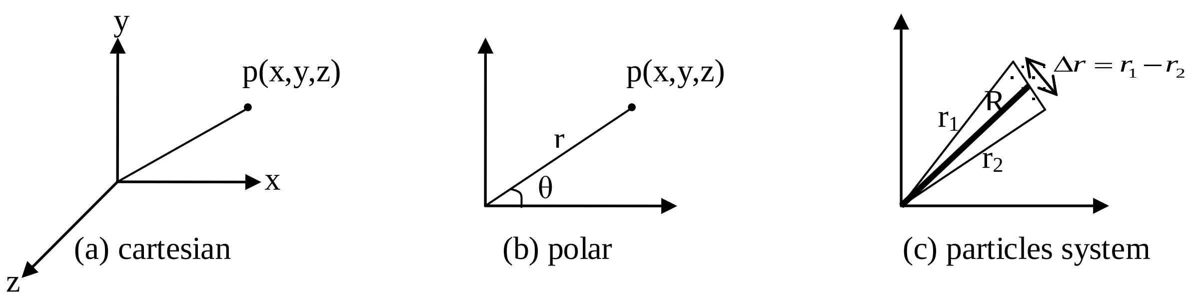
2.0 Aim and Objectives

In this unit students will be able to:

- know that coordinate systems such as cartesian, polar or any other in physics are each, a special case of what is called ''Generalized Coordinates'
- derive other generalized quantities like generalized velocity, generalized momentum, generalized force, etc.

3.0 Generalized Coordinates and Degrees od Freedom

Consider the position of a particle p in cartesian or polar coordinates or consider the coordinates of a system of particles as shown below;



(a) cartesian (b) polar (c) particles system

All these three coordinate systems discussed above and many others are only special cases of what is called generalized coordinate systems.

Transformation from one coordinate system to another coordinate is very possible; example

from cartesian (x,y,z) to polar (r,θ) ⟶ $x = r\cos\theta$, $y = r\sin\theta$ and $\theta = \tan^{-1}\dfrac{y}{x}$

3.1 Definitions:

A minimum number n (designated $q_1, q_2, q_3, \ldots\ldots, q_n$) of coordinates is required to specify the configuration of a given system. These coordinates are known as *Generalized Coordinates.*

Each independent way by which a system may acquire energy is called *degrees of freedom* and the number of coordinates n is known as the number of degrees of freedom of the system.

A particle describe by (x,y,z) has 3 degrees of freedom. A systen of particles (having N number of particles) will have 3N degrees of freedom (for holonomic systems).

Thus the generalized coordinates in

(a) cartesian (q_1,q_2,q_3) are normally represented as (x_1, y_1,z_1)

(b) polar (q_1,q_2) ⟶ (r,θ)
(c) system of particles (q_1,q_2) ⟶ $(R, \Delta r)$ (Note $\Delta r = r_1 - r_2$)

Note:
$q_1 = q_1(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3,\text{-------------},x_N,y_N,z_N)$ and $x_1 = x_1(q_1,q_2,q_3,\text{---------},q_{3N})$
$q_2 = q_2(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3,\text{-------------},x_N,y_N,z_N)$ and $x_2 = x_2(q_1,q_2,q_3,\text{---------},q_{3N})$
⋮ ⋮
$q_{3N} = q_{3N}(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3,\text{-------------},x_N,y_N,z_N)$ and $z_N = z_N(q_1,q_2,q_3,\text{---------},q_{3N})$

3. 2 Other Generalized Quantities

Generalized coordinate ⟶ q_k

Generalized velocity ⟶ $\dot{q}_k$

Generalized momentum ⟶ $p_k = \dfrac{\partial T}{\partial \dot{q}_k}$ where T is kinetic energy

Generalized force ⟶ $Q_k = \sum_{i=1}^{N}\left[F_{ix}\dfrac{\partial x_i}{\partial q_k} + F_{iy}\dfrac{\partial y_i}{\partial q_k} + F_{iz}\dfrac{\partial z_i}{\partial q_k}\right]$

4.0 Conclusion

The knowledge of concepts of generalized coordinates and generalized quantities is very important in formulating the Lagrange's and Hamilton's equations of motion.

5.0 Summary

Coordinate systems such as cartesian, polar or any other in physics are each, a special case of what is called ''Generalized Coordinates'

6.0 Tutor-Marked Assignment (TMA)

Question 2.1

Suppose the position vector of a particle is given by $\bar{r} = \hat{i}b\sin\omega t + \hat{j}b\cos\omega t + \hat{k}c$

(i) Show that the distance from the origin remains a constant given by $r = \left(b^2 + c^2\right)^{\frac{1}{2}}$
(ii) Show that the particle transverse its path with constant speed $v = b\omega$
(iii) Find the acceleration, a
(iv) Show that the acceleration, a is perpendicular to the velocity, v.

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

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http://www.academics.hamilton.edu/physics/smajor/Courses (Sept. 2009)

Unit 3 Lagrange's Mechanics

1.0 Introduction

2.0 Aim and Objectives

- know the importance of concepts such as generalized coordinates and constrained motion.
- derive the Lagrange's equations of motion
- test the elegance and power of the Lagrange method in problem solving as being done using Newton method.

3.0 Lagrange's equations of motion

To obtain a more general form of Lagrange's equations,

Kinetic energy, $T = \frac{1}{2} m_i \dot{x}_i^2$

In generalized coordinates,

$$\frac{\partial T}{\partial \dot{x}_i} \rightarrow \frac{\partial T}{\dot{q}_k}$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k}, \qquad \dot{x}_i = \frac{dx_i}{dt} = \sum \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

$$\Rightarrow \qquad \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j} \quad \text{By cancellation of dots}$$

Hence,

$$\frac{\partial T}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left\{\sum m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}\right\}$$

$$= \sum \left\{ m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k} + m_i \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right)\right\}$$

$$= \left\{ F_i \frac{\partial x_i}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i \dot{x}_i\right)\right\}$$

But

$$Q_k = F_i \frac{\partial x_i}{\partial q_k} \quad \equiv \textit{Generalized force} \quad \text{and} \quad T = \frac{1}{2} m_i \dot{x}_i \quad \equiv \textit{Kinetic energy}$$

$\therefore$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

If again, the system is a conservative one, $Q_k = -\frac{\partial V}{\partial q_k}$

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k}$$

The definition of the Lagrangian is

$$L = T - V \quad \Rightarrow \quad T = L + V$$

$$\frac{d}{dt}\left(\frac{\partial (L+V)}{\partial \dot{q}_k}\right) = \frac{\partial (T - V)}{\partial q_k} \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial V}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

The kinetic energy, T is a function of both q_k and $\dot{q}_k$ but the potential energy is a function of only position q_k, not velocity $\dot{q}_k$, therefore $\frac{\partial V}{\dot{q}_k} = 0$

and we finally have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

These are the Lagrange's equations of motion, also known as the Euler – Langrange equations.

There is one Lagrange equation for each generalized coordinate q_i. When $q_i = r_i$ (i.e. the generalized coordinates are simply the cartesian coordinates), it is straightforward to see that the Lagrange's equations reduce to Newton's second law.
The above derivation can be generalized to a system of N particles. There will be 6N generalized coordinates, related to the position coordinates by 3N transformation equations. In each of the 3N Lagrange's equations, T is the total energy of the system, and V the total potential energy.

Note:
In practice, it is easier to solve a problem using the Euler – Lagrange equations than Newton's laws. This is because not only may more appropriate generalized coordinates q_i be chosen to exploit symmetrics in the system, but constraint forces are replaced with simpler equations.

4.0 Conclusion

In practice, it is easier to solve a problem using the Euler – Lagrange equations than Newton's laws. This is because not only may more appropriate generalized coordinates q_i be chosen to exploit symmetrics in the system, but constraint forces are replaced with simpler equations.

5.0 Summary

To set up an equation of motion:
(i) find T and V
(ii) L = T + V

(iii) substitute in the Lagrange's equation
(iv) solve the equation and you are done.

6.0 Tutor-Marked Assignment (TMA)

Question 3.1

Consider a point mass m falling freely from rest. Derive the equation of motion through the Lagrange formulation.

Question 3.2

Obtain the Lagrangian equation of motion for a 1-D harmonic oscillator, supposing that there is a damping force which is proportional to the velocity and that the system is non-conservative.

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

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http://www.academics.hamilton.edu/physics/smajor/Courses (Sept. 2009)

Unit 4 Hamilton's Mechanics

1.0 Introduction

The Hamiltonian formulation, just like the Lagrangian, is reformulations of Newtonian mechanics and also provide simple techniques for deriving equations of motion using energy relations.

2.0 Aim and Objectives

At the end of this unit students will be able to:

- know the importance of concepts such as generalized coordinates and constrained motion.
- derive the Hamilton's equations of motion also known as 'canonical equations of Hamilton'
- test the elegance and power of the Hamilton method in problem solving as being done using Newton method.

3. 0 Hamilton's Equation of Motion

For a system of particles each having mass m_α described by a set of generalized coordinates q_α, the classical Hamiltonian function is defined by

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L\left(q_\alpha, \dot{q}_\alpha, t\right)$$

where $L\left(q_\alpha, \dot{q}_\alpha\right)$ is Lagrangian.

Now taking the total differential

$$dH = \sum_\alpha \left(p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha - \frac{\partial L}{\partial q_\alpha} dq_\alpha - \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right) - \frac{\partial L}{\partial t} dt$$

$\frac{\partial L}{\partial \dot{q}_\alpha}$ is the definition of the generalized momentum p_α and from Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \quad \Rightarrow \quad \frac{d}{dt} p_\alpha - \frac{\partial L}{\partial q_\alpha} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$$

So we have

$$\frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha \qquad \text{and} \qquad \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$$

Therefore the total differential of the classical Hamiltonian becomes

$$dH = \sum_\alpha \left(p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha - \dot{p}_\alpha dq_\alpha - p_\alpha d\dot{q}_\alpha \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\alpha \left(\dot{q}_\alpha dp_\alpha - \dot{p}_\alpha dq_\alpha \right) - \frac{\partial L}{\partial t} dt$$

From which we have the following equations

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$$

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

and $$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

These equations are called the Hamilton's equations of motion.They are also known as *canonical equations* of Hamilton

4.0 Conclusion

Hamiltoian, just like the Lagrangian, is reformulations of Newtonian mechanics and also provide simple techniques for deriving equations of motion using energy relations.

5.0 Summary

The Hamiltonian method differs from the Lagrangian method in that instead of expressing second-order differential constraints on an n-dimensional coordinate space (where n is the number of degrees of freedom of the system), it expresses first-order constraints on a $2n$-dimensional phase space.

6.0 Tutor-Marked Assignment (TMA)

Question 4.1

A particle moves in the x-y plane under the influence of a central force depending only on its distance from the origin. Set up the Hamiltonian and get the equations of motion.

Question 4.2

A particle of mass m moves in one dimension under the influence of a force

$$F(x,t) = \frac{k}{x^2} e^{-\left(\frac{t}{\tau}\right)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and total energy, and discuss the conservation of energy for the system.

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

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Unit 5 Between Newtonian, Lagrangian and Hamiltonian Mechanics

1.0 Introduction

Lagrangian mechanics and Hamiltonian mechanics are two important and more abstract alternative formulations of classical mechanics. They bypassed the concept of "force", instead referring to other physical quantities, such as energy, for describing mechanical systems. Lagrange's and Hamilton's equations provide a new and equivalent way of looking at classical mechanics.

2.0 Aim and Objectives

By the end of this study unit, students will be able to:

- see how concept of 'force' in Newtonian mechanics is transformed into physical quantity of energy for use in Lagrangian and Hamiltonian mechanics
- see that the Newtonian, Lagrangian and Hamiltonian mechanics provide equivalent looks into classical mechanics.

3.0 Transformation of Newton's Law from Vector to Scalar Notation

Newton's law, which is in vector notation can be transformed to scalar. The force F on a particle is

$$F = ma = \frac{dp}{dT} = m\dot{v}$$

$$F_i = \frac{d}{dt}\left(m_i \dot{x}\right) = \frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right)$$

Kinetic energy

$$T = \frac{1}{2} m_i \dot{x}_i^2 \qquad \Rightarrow \qquad \frac{dT}{d\dot{x}} = m_i \dot{x}_i = p_x$$

If forces are derivable from potential energy, V then,

$$F = -\nabla V \quad = \quad -\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}\right) \quad = \quad F_i\,, \qquad V = V(x_i)$$

So

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) \quad = \quad -\frac{\partial V}{\partial x_i}$$

This is just Newton's law transformed from vector notation to scalar in cartesian coordinates with T = kinetic energy and V = potential energy

3.1 Between Newtonian, Lagrangian and Hamiltonian Mechanics

Classical mechanics is concerned with the set of physical laws governing and mathematically describing the motions of bodies and aggregates of bodies geometrically distributed within a certain boundary under the action of a system of forces.

The initial stage in the development of classical mechanics is often referred to as Newtonian mechanics, and is associated with the physical concepts employed by and the mathematical methods invented by Newton himself.

Lagrangian mechanics is a re-formulation of classical mechanics that combines conservation of momentum with conservation of energy.

Hamiltonian mechanics is a reformulation of classical mechanics that arose from Lagrangian mechanics, a previous reformulation of classical mechanics.

Self Assessement Exercise

1. Write down expressions for the following quantities and explained the meaning of each symbol involved: (i) generalized velocity (ii) generalized force (iii) generalized kinetic energy

2. Explain the terms 'Generalized coordinates' and 'Degrees of freedom '

3. Distinquish between Holonomic and Non-holonomic constraints.
. Give the generalized coordinates which are applicable to the motion of each of the following:
   (i) A particle moving in a plane under the influence of a force directed towards the origin.
   (ii) A disk rolling on the horizontal xy plane constrained to move so that the plane of the disk is always vertical.

4. Write down the Lagrange's equation of motion for a
   (i) conservative system.
   (ii) non-conservative system.

5. Obtain the Hamiltonian equation of motion for a 1-D harmonic oscillator, supposing that there is a damping force which is proportional to the velocity and that the system is non-conservative.

6. (a) set up the Lagragian for a simple pendulum;
   (b) solve the resulting equation to find the motion of the pendulum.

7. Write three sentences on what you understand on Newtonian mechanics, Lagrangian mechanics and Hamiltonian mechanis; bringing the siimilarities and differences.

4.0 Conclusion

Each of the 3 mechanics of Newton, Lagrange and Hamilton can be preferred to the other to describe mechanical system, depending on convenience.

5.0 Summary

Hamiltonian mechanics is a reformulation of Newtonian mechanics that arose from Lagrangian mechanics, a previous reformulation of Newtonian mechanics.

6.0 Tutor-Marked Assignment (TMA)

Question 5.1

Consider a particle of mass m which moves freely in a conservative force field whose potential energy function is V. Find the Hamiltonian function and show that the canonical equations of motion reduce to Newton's equations (use rectangular coordinate)

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

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MODULE 3 Solutions to TMA

Unit 1

Question 3.1

a) the velocity, $v = \frac{dr}{dt} = (6t^2 + 1)\mathbf{i} + (12t^3 - 2t)\mathbf{j} - 24t\mathbf{k}$

b) the momentum, $p = mv = 2 \times v = 2 \times [(6t^2 + 1)\mathbf{i} + (12t^3 - 2t)\mathbf{j} - 24t\mathbf{k}]$

$= (12t^2 + 2)\mathbf{i} + (24t^3 - 4t)\mathbf{j} - 48t\mathbf{k}$

c) the acceleration, $a = \frac{dv}{dt} = 12t\mathbf{i} + (36t^2 - 2)\mathbf{j} - 24\mathbf{k}$

d) the force acting on it at any time t, $F = ma = 2 \times [12t\mathbf{i} + (36t^2 - 2)\mathbf{j} - 24\mathbf{k}]$

$= 24t\mathbf{i} + (72t^2 - 4)\mathbf{j} - 48\mathbf{k}$

Unit 2

Question 2.1

(i) distance, $r = |\bar{r}| = \sqrt{\hat{i}^2 b^2 \sin^2 \omega t + \hat{j}^2 b^2 \cos^2 \omega t + \hat{k}^2 c^2} = \sqrt{b^2(\sin^2 \omega t + \cos^2 \omega t) + c^2}$

$= \sqrt{b^2 + c^2}$

(ii) velocity vector, $\bar{v} = \frac{d\bar{r}}{dt} = \hat{i} b\omega \cos \omega t - \hat{j} b\omega \sin \omega t$

$\therefore \ velocity, v = |\bar{v}| = \sqrt{\hat{i}^2 b^2 \omega^2 \cos^2 \omega t + (-\hat{j})^2 b^2 \omega^2 \sin^2 \omega t} = \sqrt{b^2 \omega^2} = b\omega$

(iii) acceleration vector, $\bar{a} = \frac{d\bar{v}}{dt} = -\hat{i} b\omega^2 \sin \omega t - j b\omega^2 \cos \omega t$

$\therefore \ acceleration, a = |\bar{a}| = \sqrt{(-\hat{i})^2 (b\omega^2)^2 \sin^2 \omega t + (-j)^2 (b\omega^2)^2 \cos^2 \omega t} = \sqrt{(b\omega^2)^2} = b\omega^2$

(iv) To show that a is perpendicular to v, we only verfy if $\bar{v} \bullet \bar{a} = 0$

$\therefore$

$$\bar{v} \bullet \bar{a} = (\hat{i} b\omega \cos \omega t - \hat{j} b\omega \sin \omega t) \bullet (-\hat{i} b\omega^2 \sin \omega t - j b\omega^2 \cos \omega t)$$

$$= (\hat{i} b\omega \cos \omega t)(-\hat{i} b\omega^2 \sin \omega t) + 0 + 0 + (-\hat{j} b\omega \sin \omega t)(-j b\omega^2 \cos \omega t)$$

$$= -b^2\omega^3 \cos \omega t \sin \omega t + b^2 \omega^3 \sin \omega t \cos \omega t$$

$$= 0$$

Hence , we have shown that the acceleration is perpendicular to the velocity.

Unit 3

Question 3.1

Take x to be the coordinate, which is 0 at starting point.

The kinetic energy is $T \quad = \quad \frac{1}{2}mv^2$ and potential energy is $V = -mgx$

The Lagrangian is

$$L = \quad T - V \quad = \quad \frac{1}{2}m\dot{x}^2 \ + mgx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \quad = \quad \frac{\partial L}{\partial q_k} \qquad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \quad - \frac{\partial L}{\partial q_k} = \quad 0$$

i.e. $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 \ + mgx\right) \quad - \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 \ + mgx\right) = m\ddot{x} - mg = 0$

$\Rightarrow \qquad \ddot{x} = g$ which is the same result we know when filling in the force in Newton's law

Question 3.2

For a 1 –D harmonic oscillator, the Lagrange's equation for conservative systen is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}\right)$$

But if system is non-conservative , it is modified to be $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) == \ Q_k' \ + \frac{\partial L}{\partial q_k}$

$$L = T - V = \frac{1}{2}m\dot{x}^2 \ - \frac{1}{2}mx^2$$

and we have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = -kx$$

Since non-conservative force is present, then there is $Q_k' = -c\dot{x}$

and Lagrange's equation becomes

$$\frac{d}{dt}\left(m\dot{x}\right) = -c\dot{x} + \left(-kx\right) \qquad \Rightarrow \qquad m\ddot{x} + c\dot{x} + kx = 0$$

which is the familiar equation of the damped harmonic oscillator that we know.

Unit 4

Question 4.1

In polar coordinates we choose (r,θ) as the generalized coordinates since it is a 2-D problem

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right) \quad \text{and} \quad V=V(r)$$

$$p_r=\frac{\partial T}{\partial \dot{r}}=m\dot{r} \qquad \Rightarrow \qquad \dot{r}=\frac{p_r}{m}$$

$$p_\theta=\frac{\partial T}{\partial \dot{\theta}}=mr^2\dot{\theta} \qquad \Rightarrow \qquad \dot{\theta}=\frac{p_\theta}{mr^2}$$

$$H=T+V=\frac{1}{2}m\left(\frac{p_r^2}{m^2}+\frac{r^2p_\theta^2}{m^2r^4}\right)+V(r)=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)+V(r)$$

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m} \qquad (1)$$

$$\dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mr^2} \qquad (2)$$

$$-\dot{p}_r=\frac{\partial H}{\partial r}=m\dot{\theta}^2 r \qquad (3)$$

$$-\dot{p}_\theta=\frac{\partial H}{\partial \theta}=0 \qquad (4)$$

Question 4.2

The potential energy V which gives the force $F(x,t)= \dfrac{k}{x^2}e^{-\left(\frac{t}{\tau}\right)}$ must satisfies the relation $F=-\dfrac{\partial V}{\partial x}$

$$V=-\int Fdx=-\int\frac{k}{x^2}e^{-\left(\frac{t}{\tau}\right)}dx=-\left[-\frac{ke^{-\frac{t}{\tau}}}{x}\right]=\frac{k}{x}e^{-\frac{t}{\tau}}$$

The Lagrangian is thus

$$L=T-V=\frac{1}{2}m\dot{x}^2-\frac{k}{x}e^{-\frac{t}{\tau}}$$

The Hamiltonian is given by

$$H=p_x\dot{x}-L=\frac{\partial L}{\partial \dot{x}}-L$$

So that

$$H=\frac{1}{2}m\dot{x}^2+\frac{k}{x}e^{-\frac{t}{\tau}}$$

The Hamiltonian is equal to the total energy, T + V, because the potential does not depend on velocity, but the total energy of the sysytem is not conserved because H contains the time explicitly.

Unit 5

Question 5.1

The Hamiltonian equation can be written as

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L\left(q_\alpha, \dot{q}_\alpha, t\right) \qquad (1)$$

The Lagrangian (using rectangular coordinate) is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V \qquad (2)$$

Linear momentum in rectangular coordinates

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad \textit{and} \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad (3)$$

Hence using (1) and (3)

$$H = \left(m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2\right) - \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V\right] = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V \qquad (4)$$

Equation (4) is the total energy of the particle.

From the canonical equations

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$$

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

$$p_x = \frac{\partial H}{\partial \dot{x}} = m\dot{x}, \quad \dot{p}_x = m\ddot{x}, \qquad \frac{\partial H}{\partial x} = \frac{\partial V}{\partial x}$$

$$p_y = \frac{\partial H}{\partial \dot{y}} = m\dot{y}, \quad \dot{p}_y = m\ddot{y}, \qquad \frac{\partial H}{\partial y} = \frac{\partial V}{\partial y}$$

$$p_z = \frac{\partial H}{\partial \dot{z}} = m\dot{z}, \quad \dot{p}_z = m\ddot{z}, \qquad \frac{\partial H}{\partial z} = \frac{\partial V}{\partial z}$$

$$\dot{p}_x = m\ddot{x} = -\frac{\partial V}{\partial x} = F_x, \qquad \dot{p}_y = m\ddot{y} = -\frac{\partial V}{\partial y} = F_y \ \textit{and} \qquad \dot{p}_z = m\ddot{z} = -\frac{\partial V}{\partial z} = F_z$$

These are just Newton’s equations