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Course Developer Mr. Efiong A. Ibanga

Physics Unit

School of Science and Technology

National Open University of Nigeria

Course Coordinator Mr. Ibanga, E. A.

National Open University of Nigeria

Programme Leader Dr. Ajibola S. O.

National Open University of Nigeria

NATIONAL OPEN UNIVERSITY OF NIGERIA

MODULE 1: EINSTIEN'S POSTULATES AND LORENTZ TRANSFORMATIONS

UNIT 1: GALILEAN TRANSFORMATION AND INVARIANCE OF PHYSICAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

In your previous work, especially in Elementary Mechanics, you learned that the motion of an object in space was not absolute but relative an observer. To a passenger in a moving car, a passerby who is at rest by the side of the road appears to be in motion. The question is "who is moving between the two?" This question can have an answer only if the *frame of reference* (a *coordinate system*) from which each observer is describing the motion is clearly defined. In the passerby's frame of reference, he is at rest relative to the ground while the passenger in the car is in motion. On the other hand, in the frame of reference attached to the moving car, the passenger sees himself to be at rest relative to the car and the passerby in motion relative to him. Thus, *both the passenger and the passerby see themselves in motion relative to each other.*

The theory of relativity is a study of the relationships between observations made in different frames of reference which are in relative motion to each other. The mathematical basis of comparing two descriptions is called *transformation*. The subject also deals with effects that are observed when objects are in relative motion to each other at speeds close to the speed of light in a vacuum (*relativistic speeds*). Such effects include length contraction, time dilation, mass increase and others of electromagnetic and optical significance. You will be introduced to some of these in this course. Also in this course, we will be dealing with *special relativity* as against *general relativity*. While the former deals with physical laws expressible in equations having the same form (*invariance)* in reference frames which are in relative motion at constant velocity (*inertial frames of reference*), in the latter, equations expressing the laws of physics have the same form in all frames of reference regardless of their state of motion (*accelerating and rotating frames*).

Relativity is one of the main pillars of *Modern Physics* as against what is known as *Classical Physics.* Classical physics comprises the Newtonian mechanics, sundry phenomena which can be explained in terms of Maxwell's theory of electromagnetic interaction and its applications, thermodynamics and the kinetic theory of gases. Modern physics on the other hand comprises the theory of relativity and its effects, quantum theories and associated phenomena, and, in particular, the application of these theories to the study of the atom and the nucleus.

We begin our study with a look at the description of position of an event in space at a given time. We will use the familiar Cartesian coordinate system to define our frame of reference for this purpose.

1.1 OBJECTIVE

At the end of this unit you will be able to

a. Carry out simple Galilean transformations of some physical equations

b. Demonstrate the invariance of equations of Newtonian mechanics under Galilean transformation

c. Show that physical equations of electromagnetic phenomena are non-invariant under Galilean transformation.

d. Perform simple calculations involving Galilean transformations of physical equations

e. Discuss the idea of the universal frame of reference and the *ether hypothesis*

f Describe the Michelson-Morley Experiment

1.2 MAIN BODY

1.2.1 Frames of Reference

Every *physical event* occurs somewhere in *space* (a portion of the physical universe) in a definite time interval. In three-dimensional Euclidean space it is convenient to specify the position of such an event using a *frame of reference* which comprises a three-dimensional rectangular *coordinate system*. You already know that the position of a point in space in Cartesian coordinate system is specified by an ordered set, (x, y, z) called the Cartesian coordinates of the point. This, of course, is not the only type of coordinate system. You may possibly have done some work with polar, cylindrical and spherical coordinates in other physics or mathematics courses. These will not be used in our analysis here.

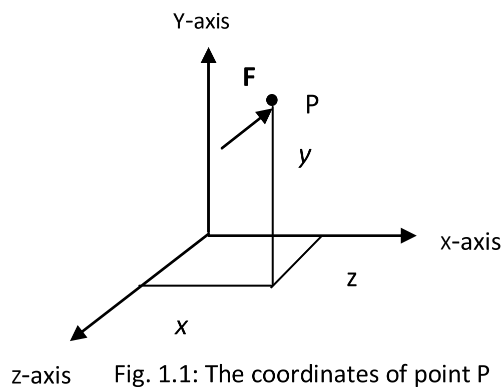


Fig. 1.1: The coordinates of point P

Consider a particle of mass m at a point P acted upon by a force **F** at an instant of time t as shown in figure 1.1 The coordinates in space (spatial coordinates) of the event as measured by an observer at the origin of the coordinate system are (x, y, z, t). Notice that space and time are intricately intertwined. This is true because every event in nature has a definite location (space) as well as duration (time). As such, we speak of space-time rather than space and time separately. So, the set of four numbers (x, y, z, t) tells us that at time t, the coordinates of the particle at point P are x, y, z.

Now, we can write Newton's second law of motion for this event in terms of the components F_x, F_y and F_z of the force **F** as

$$F_x = m\frac{d^2x}{dt^2}$$

$$F_y = m\frac{d^2y}{dt^2}$$

(1.1)

$$F_z = m\frac{d^2 z}{dt^2}$$

It is important for you to note that these equations are valid only if the frame of reference described by the coordinates x, y, z is i*nertial. A frame of reference is said to be inertial if an object in it, which is not under the influence of a force, will remain at rest if it was initially at rest or continue in its motion with constant velocity, if it was initially in motion*.

We now proceed to examine the description of the same event as seen by two observers which are in motion relative to each other at constant velocity in a straight line.

1.2.2 The Galilean Transformation

Consider the two observers O and $O^{'}$at the origins of the frames of reference S and $S^{'}$respectively, which are in relative motion at constant velocity *v* as illustrated in figure 1.2. *Frames of reference which are in translational motion at constant velocity relative to each other are called inertial frames of reference*. Suppose also that the origins O and $O^{'}$as well as the axes of the coordinates of these frames are coincident at an initial time $t = t^{'} = 0$.

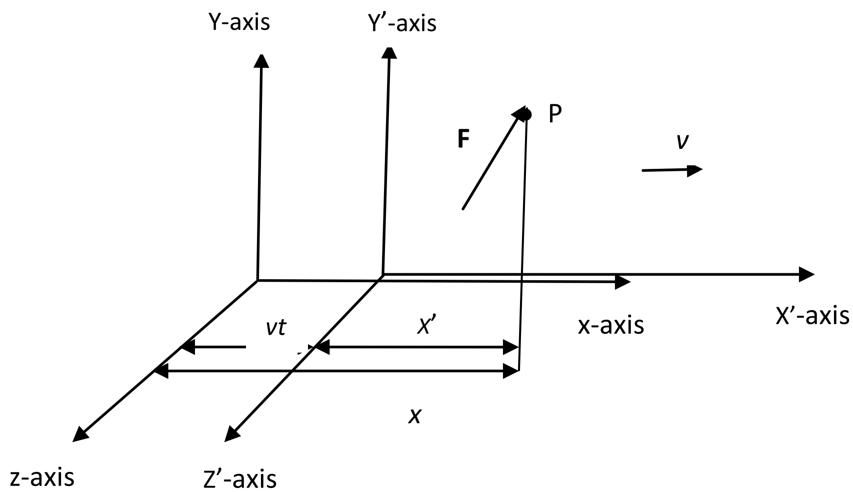


Fig. 1.2: Two frames of reference S and $S^{'}$ in uniform translation. x and $x^{'}$ are supposed to be collinear.

The two observers are equipped with measuring instruments to determine the coordinates of the event at P. Measurements made in the S frame are related to those made in the $S^{'}$ by the *Galilean transformation* as follows:

$x^{'} = x - vt$

$y^{'} = y$

$z^{'} = z$

(1.2)

$t' = t$

Let us write, for example, the equations of Newton's second law of motion for this event using these transformations for each of S and S' frames. The x, y, and z components of the force **F** which acts on the particle at P for the S frame are

To obtain the corresponding equations in the S' frame, we first differentiate (1.1) twice to obtain the components of the acceleration as follows:

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

and

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}$$

Also from (1.1) $t = t'$

Thus, we can replace *t* by t' in the above equation and write

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

Obviously,

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2}$$

and

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2}$$

Therefore, the components of the force in the S' frame can be written in terms of the components of the acceleration as follows:

$$F_{x'} = m\frac{d^2\ x'}{dt'^2}$$

$$F_{y'} = m\frac{d^2y'}{dt'^2}$$

(1.3)

$$F_{z'} = m\frac{d^2z'}{dt'^2}$$

Comparing equations (1.2) and (1.3), you will find out that they have the same form. We say that *the equations are invariant under Galilean transformation.* Now, this is quite significant in the sense that Newton's laws constitute the main platform on which the motions of objects are analyzed in classical physics and govern the behavior of mechanical systems. It follows that *the laws that govern the behavior of mechanical systems will always have the same form in all inertial frames of reference.* As an example of this generalization, consider two laboratory experiments conducted under the same conditions to determine the gravitational constant g. Let one laboratory be situated in a frame of reference at rest with respect to the surface of the earth and the other mounted on a train travelling at constant velocity on a straight horizontal track. If the windows of the moving laboratory are closed, the experimenter in the train will not be able to tell whether or not he is in motion. Interestingly, the two experimenters will obtain the same value for the gravitational constant g.

Before we proceed, let us at the moment illustrate the concepts we have discussed by solving a few practical problems through a few self assessment questions SAQs. Work carefully through all of them and there after compare your answers with those given at of this section.

SAQ 1: A man in a boat moving at constant speed of 60km/h relative to the shore throws an object in the forward direction with a speed of 30km/h. What is the speed of the object as measured by an observer at rest at the shore?

SAQ 2: A passenger in a train moving at 35km/h looks out and sees a man standing on the platform of the station at $t = t' = 0$. Twenty seconds after, the man on the platform determines that a bird flying in the same direction as the train is 800m away. What is the average speed of the bird as determined by the passenger?

SAQ 3: A swimmer can swim with a speed c in the still water of a lake. In a stream in which the speed of the current is v (which, we assume, is less than c), the swimmer can also swim with a speed c relative to the water in the stream. Suppose the swimmer swims upstream a distance L and then returns downstream to the starting point. Find the time taken to make the round trip and compare it with the time taken to swim across the stream a distance L and return.

SAQ 4: An observer at rest with respect to the ground observes a particle of mass $m_1 = 3kg$ moving along the x-axis with a velocity $u_1 = 3\ m/s$. It approaches a second particle of mass $m_2 = 1kg$ moving with velocity a $u_2 = -3\ m/s$ along the same axis. After head-on collision, he finds that the velocity of m_2 is $u_2' = -3\ m/s$ along the x-axis. What are the momenta before and after the collision as seen by a moving observer walking with a velocity of 2 m/s relative to the ground along the x-axis?

1.2.3 Non- Invariance of Electromagnetic Phenomena under Galilean Transformation

You saw in previous section that physical equations that describe the behavior of mechanical systems have the same form in frames of reference that are in uniform translational motion relative to each other (inertial frames). Now, we want to see whether the equations that govern the behavior of electromagnetic phenomena will have the same form under a similar transformation. The behavior of an electromagnetic phenomenon can, in general, be described in terms of Maxwell's equations. It is from these equations that the differential equation of the electromagnetic wave in free space is obtained. *Maxwell's equations express the spatial variation with respect to time of time-dependent electric field intensity **E** and magnetic induction **B***. You can find the derivation of these equations in any standard textbook on electrodynamics. Here, we will simply write down the equation of electromagnetic wave in free space in terms of the coordinates of the electric field and then try to verify its invariance or otherwise under Galilean transformation.

The equation is

$$\nabla^2 \boldsymbol{E} = \varepsilon_o \mu_o \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

, (1.4)

where $\varepsilon_o \, and \, \mu_o$ are the permittivity and permeability of free space respectively. In Cartesian coordinate system the electric field vector **E** can be written in terms of its components as

$\boldsymbol{E} = \boldsymbol{e}_x E_x + \boldsymbol{e}_y E_y + \boldsymbol{e}_z E_z$, where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are the unit vectors in the x-, y-, and z-directions respectively. Equation (1.4) then separates into three equations obeyed by each of the components of **E**. Thus, for the x - component, we can write equation (1.4) as

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

(1.5)

where

$c = \frac{1}{\sqrt{\epsilon_o \mu_o}} = 2.998 \times 10^8 \text{ms}^{-1}$, the speed of all electromagnetic waves in free space.

Now we proceed to transform equation (1.5) to a new set of coordinates (x', y', z', t') of a moving frame of reference at constant velocity ***v*** relative to those of the stationary frame (x, y, z, t). The objective is to find out whether our transformation will give us our equation (1.5) written in terms of x', y', z' and t'. If it does, then we can say that the equation is invariant (has the same form) under the transformation, otherwise it is invariant (has a different form from equation 1.5).

From (1.1), we have

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = \frac{\partial t'}{\partial t} = 1$$

and

$$\frac{\partial x'}{\partial y} = \frac{\partial x'}{\partial z} = \frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial z} = \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = \frac{\partial t'}{\partial x} = \frac{\partial t'}{\partial y} = \frac{\partial t'}{\partial z} = 0$$

Next, we apply the chain rule and the above equations to obtain the second derivatives. We begin by obtaining the first derivative w.r.t. x' as follows:

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E_x}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial E_x}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial E_x}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial E_x}{\partial x'}$$

Then, we differentiate again to obtain the second derivative w.r.t. x'

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial^2 E_x}{\partial x'^2}$$

Similarly, by differentiating with respect to $y'$and z' respectively we obtain

$$\frac{\partial^2 E_x}{\partial y^2} = \frac{\partial^2 E_x}{\partial y'^2}$$

and

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial^2 E_x}{\partial z'}$$

Furthermore,

$$\frac{\partial \mathrm{E_x}}{\partial \mathrm{t}} = \frac{\partial \mathrm{E}_x}{\partial \mathrm{x}'}\frac{\partial \mathrm{x}'}{\partial \mathrm{t}'} = \frac{\partial \mathrm{E}_x}{\partial \mathrm{x}'}\left(\frac{\partial \mathrm{x}}{\partial \mathrm{t}} - v\right) = -v\frac{\partial \mathrm{E}_x}{\partial \mathrm{x}'} + \frac{\partial \mathrm{E}_x}{\partial \mathrm{t}'}$$

and

$$\frac{\partial^2 E_x}{\partial \mathrm{t}^2} = \frac{\partial^2 E_x}{\partial \mathrm{t}'^2} - 2v\frac{\partial^2 E_x}{\partial \mathrm{x}'\partial \mathrm{t}'} + v^2\frac{\partial^2 E_x}{\partial \mathrm{x}'^2}$$

We can now substitute these second derivatives in the wave equation (1.5) and obtain

$$\frac{\partial^2 E_x}{\partial \mathrm{x}'^2} + \frac{\partial^2 E_x}{\partial \mathrm{y}'^2} + \frac{\partial^2 E_x}{\partial \mathrm{z}'^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t'^2} - \frac{1}{c^2}\left(2v\frac{\partial^2 E_x}{\partial \mathrm{x}'\partial \mathrm{t}'} - v^2\frac{\partial^2 E_x}{\partial \mathrm{x}'^2}\right) \tag{1.6}$$

Comparing equation 1.5 and 1.6, it is obvious that the *electromagnetic wave equation (1.5) does not retain the same form (is not invariant) under Galilean transformation*.

Looking back the way we came, we can see that Maxwell's equations, from which the electromagnetic wave equation was obtained, and *all electromagnetic phenomena they describe, are also not invariant under Galilean transformation.* By extension of the argument, the speed of electromagnetic waves should also be seen to be dependent on the speed of its source, since the propagation speed depends on the form of the wave equation. For instance, if the source of light is moving with a speed v relative to a stationary observer in the direction of propagation of light wave, the resultant speed of light as measured by the observer will be $c + v$ and $c - v$ if the source moves in opposite direction. This clearly contradicts the experimental observation that *the speed of light is constant and independent of the speed of the source*. Take, for example, an elementary particle such an electron accelerated to a speed 0.97c in a linear accelerator. If it is caused to emit another particle such as a neutrino whose speed is 0.99c, then the speed of the neutrino will be 1.96c, which is greater than the speed of light. The question then is "is there any particle in nature known to possess the speed greater than the speed of light in free space?" We will discuss this problem before we come to an end of this module.

You will recall from your study of relative velocities in your Elementary Mechanics that the situation we are examining here is analogous to the motion of a boat in a river with a stream speed v. In this case, the speed of the boat relative to a stationary observer is the sum of the speed of the boat relative to the water of the river and the speed of the stream. *In the case of light waves the speed is measured relative to what or light waves are waves in what medium?* Physicists of the time when the theory of special relativity was in its formative stages were very mechanistic and believed that light wave, just like water wave, sound wave and other forms of mechanical waves required a material medium for its propagation. This is how the concept of the *ether* arose. We will now take a brief look at it.

1.2.4 The Ether Hypothesis

We mentioned in the last section that physicists of the time reasoned that light waves required material media for their propagation just like sound waves required air and water waves required water as their propagation media. It was therefore reasonable to assume that material medium called *ether* pervaded all space. But, unlike other mechanical waves, light waves can propagate through the vacuum. Thus, it was also reasonable for the hypothetical ether to be assumed to be massless, although it must possess elastic properties in order to allow the electromagnetic disturbance to propagate through it.

With the assumption of the existence of ether, it was expected that electromagnetic phenomena in the form represented by Maxwell's equations were *valid in the frame of reference which was at rest with respect to ether* – the so-called *ether frame.* A solution of the equations would yield a *fixed propagation velocity* of the electromagnetic disturbance through space. The outcome of this expectation was in perfect agreement with experimentally determined value of the velocity of light c by Fizeau in 1849.

However, in a new frame of reference which is in uniform translation with respect to the ether frame, Maxwell's equations would have a different form and their solutions would yield a different value of the propagation velocity of the electromagnetic disturbance in ether-pervading space. In other words, a Galilean transformation of the equation would yield the non-invariance which we had earlier verified in section 3.2.

Now, the interesting aspect of the ether hypothesis is that if ether exists, then all motions can be described relative to it. In other words, the ether frame could be taken to be the *universal frame.* Furthermore, objects in motion relative to ether should experience "ether wind" and therefore it would be possible to detect it. In the next sub-section, we will examine an experimental attempt at detecting ether, the famous Michelson-Morley experiment.

1.2.5 Michelson-Morley Experiment

The experiment was designed to detect the motion of the earth through the hypothetical ether. If ether exists, the Earth should move through it at, at least, the speed of $3\times 10^4 ms^{-1}$, which is its orbital speed about the Sun. The ether frame was assumed to be at rest with respect to the centre of the solar system or the centre of the universe. If we take the motion of the solar system with respect to the centre of the universe into consideration, the speed of the Earth through ether will be even greater.

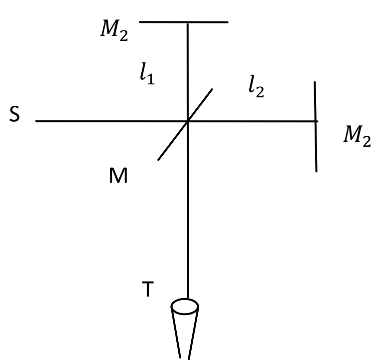


Fig. 1.4 An interferometer

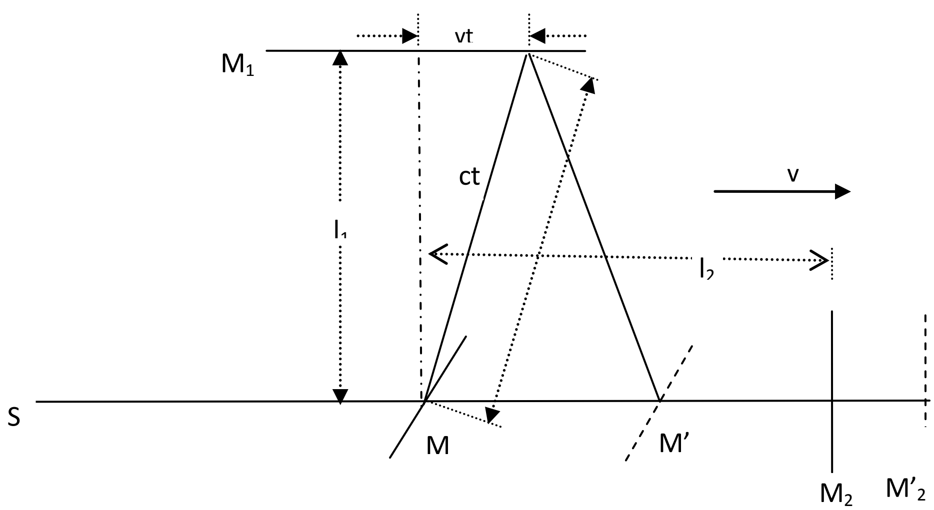


Figure 1.5: An interferometer moving against ether

The idea of the experiment is to find the directional variation of the velocity of light as a result of the motion of the Earth through ether.

The experiment therefore consists of the measurement of the velocity of light, as seen from the frame of reference fixed with respect to the earth, in two perpendicular directions. To achieve this, the experimenters designed a device called *interferometer* invented by Michelson and set up the experiment as shown schematically in figure 1.4.

Consider the interferometer at rest with respect to ether as shown in the figure 1.4. A ray of light from a collimated source S is incident at angle of 45° upon a half-silvered mirror M. The mirror splits the ray into two rays such that half of the light is reflected while the remainder is transmitted. The reflected ray travels to M1 while the transmitted ray travels to M2. These rays are reflected back to M. Half of each of the rays returning to M is directed into the telescope and recombine to give rise to the light seen by the observer O. The intensity of the light seen by O depends on the phase relationship between the two recombining rays. Since the two rays were originally in phase before being split by M, their phase relationship upon recombination depends on the lengths of the two arms l_1 and l_2 of the interferometer as follows:

if

$$l_1 = l_2, l_2 \pm \frac{\lambda}{2}, l_2 \pm \frac{2\lambda}{2}, l_2 \pm \frac{3\lambda}{2}, \ldots,$$

rays are in phase

if

$$l_1 = l_2 \pm \frac{\lambda}{4}, l_2 \pm \frac{3\lambda}{4}, l_2 \pm \frac{5\lambda}{4}, \ldots,$$

rays are out of phase

where λ is the wavelength of light.

Now, consider the interferometer in motion with velocity *v* with respect to ether as shown in figure 1.5. Note that *v* is perpendicular to ray1 (which travels to M_1) and parallel to ray2 (which travels to M_2) respectively. We also assume that $l_1 = l_2 = l$. That is, the lengths of the arms are equal. M' represents the position of M at the instant ray1 reflected from M_1 meets it after travelling a distance $2vt$. M_2' indicates the position of M_2 at the instant it reflects ray2.

To evaluate the phase relationship between the two recombining rays, we calculate the time required for each ray to make a round trip in their respective paths. Notice that, as seen from the ether frame, ray1 travels in oblique path of length ct in time t. During this time, the apparatus moves through a distance equal to vt to the right. Thus, from the diagram, we have

$$c^2t^2 = v^2t^2 + l^2$$

$$t^2 = \frac{l^2}{(c^2 - v^2)}$$

$$t^2 = \frac{l^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

$$t = \frac{l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Now, v is very small when compared with c, that is $\frac{v^2}{c^2} \ll 1$. We can apply the binomial expansion obtain

$$t = \frac{l}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

Thus, the round trip time t_1 for ray1 is

$$t_1 = 2t = \frac{2l}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

. 1.7

Calculating the round trip time for ray 2 is particularly simple. Take a closer look at figure 1.5. The situation is analogous to the swimmer which does a round trip by going downstream and then returning upstream to the starting point. The velocity of light as seen by an observer fixed with respect to the apparatus is $c + v$ while going downstream (to the right) and $c - v$ while going upstream (to the left). The round trip time for this ray is

$$t_2 = \frac{l}{c + v} + \frac{l}{c - v} = \frac{(c - v)l + (c + v)l}{(c + v)(c - v)}$$

$$= \frac{2cl}{(c^2 - v^2)} = \frac{2l}{c}(1 + \frac{v^2}{c^2})^{-1}$$

Applying the binomial expansion as before, we obtain

$$t_2 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right).$$

1.8

Comparing t_1 with t_2 (i.e. equations 1.7 and 1.8) you will find out that they are clearly different. Thus, the recombining rays are out of phase. To find the phase difference of these recombining rays, we must find the difference in time δt of the recombining rays. We have

$$\delta t = t_1 - t_2$$

$$= \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$= \frac{lv^2}{c^3}.$$

1.9

The corresponding path difference δl of the recombining rays is

$$\delta l = c\delta t = \frac{lv^2}{c^2}$$

1.10

The phase difference associated with this is

$$\frac{dl}{\lambda} = \frac{c\delta t}{\lambda} = \frac{l}{\lambda}\frac{v^2}{c^2}$$

1.11

where λ is the wavelength of light used.

This phase difference should give rise to a certain number of fringes which can be seen in the telescope when the apparatus is rotated through an angle. Thus, rotating the apparatus through 90^o, the phase difference in the new position is

$$\frac{dl'}{\lambda} = -\frac{l}{\lambda}\frac{v^2}{c^2}$$

1.12

The change in the phase difference, which corresponds to the number of fringes, ΔN arising from this is

$$\Delta N = \frac{dl}{\lambda} - \frac{dl'}{\lambda} = 2\frac{l}{\lambda}\frac{v^2}{c^2}$$

1.13

In the actual experiment, $l = 10\ m$ and $\lambda = 6 \times 10^{-7} m$ were used. Substituting these values in the above equation gives the value of ΔN approximately as 0.5. Thus, when the apparatus is rotated the bright fringes are replaced by the dark ones and vice versa as seen by an observer looking through the telescope.

To the surprise of everyone, *the fringes were not observed*. Despite the repetition of the experiment with increased accuracy at different locations, times of the day and seasons of the year, no effect was observed. So, how do we interpret this result? The classical theory (as contained in the Galilean transformation) predicts that the velocity of light is not constant but depends on the direction along which it is measured. The null result of the Michelson-Morley experiment therefore shows that *the velocity of light is the same* when measured along two perpendicular directions in a frame of reference that is supposedly moving relative to the ether frame.

Of course, scientist did not just quit the concept of the existence of ether. Spirited attempts were made to explain the null result. Here are some of the viewpoints advanced to retain the ether concept:

1. The ether drag hypothesis: The concept explains that ether was attached to and dragged along by bodies of finite masses as they move. This would certainly validate the null result of the Michelson-Morley experiment. But evidence from the measurement of the velocity of light in rapidly flowing water earlier made by Fizeau and stellar aberration strongly contradicted the hypothesis.
2. Lorentz contraction: The concept assumed that material objects in relative motion to ether contract by a factor of $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ in the direction of motion. Thus, the arm of the interferometer moving parallel to ether will contract and a null result would be obtained. This hypothesis was also ruled out by the results of appropriate experiment which showed that no such effects occurred.
3. Emission theories: In this theory, Maxwell's equations were modified in such a way that the velocity of lights was always dependent on the velocity of its source. This was in conflict with experimental evidence concerning binary stars.

As there was no strong experimental evidence in support of the existence of ether, it was finally abandoned. Furthermore, there was sufficient experimental evidence to show that Maxwell's equations were correct. Therefore the Galilean transformation needed to be replaced by another transformation which ensures the invariance of both the laws of mechanics and electromagnetic theory.

SAQ 5

In a Michelson-Morley experiment, an interferometer with arms of 11m and sodium light of 5900 Å was used. If the velocity of the earth through ether is $3\times 10^4 ms^{-1}$, calculate the expected total fringe shift when the apparatus is rotated through 90°.

SAQ 6

In a Michelson-Morley experiment, an equal-arm interferometer with arms 10m and light of wavelength 600 Å was used. The expected number of fringes was 0.005. Calculate the velocity of the earth relative to ether.

Summary

We summarize all we have studied as follows:

- There is no absolute rest or motion in nature. For the purpose of describing the motion of an object in space, we require a *frame of reference* relative to which the motion must be ascribed.
- Every event in nature occurs somewhere in space at a definite time interval, thus, the coordinates of an event in space are (x, y, z, t). We therefore speak of space-time and not space and time separately.
- If two observers that are at rest with respect to each other observe the same event which occurs at a point in space at an instant of time, they will agree on the physical laws governing that event. In other words the two observers will write equations having the same form to represent the physical law that govern the event in their respective frames of reference.
- Frames of reference in uniform translational motion relative to each other are called inertial frames of reference.
- Galilean transformation provides a way of relating observations of physical phenomena made by observers in inertial frames of reference.
- The laws that govern the behaviour of mechanical systems are invariant under Galilean transformation.
- The laws that govern that govern electromagnetic phenomena are not invariant under Galilean transformation.
- Initially, light waves were thought to propagate through elastic but massless medium called ether, which could be regarded as the universal frame of reference.
- Michelson-Morley experiment failed to detect ether.

Conclusion

We conclude this unit by observing that if ether truly exists, then the Galilean transformation should be valid for all physical phenomena, whether they are mechanical or electromagnetic. The ether frame should then become the universal frame relative to which all motions could be ascribed and the velocity of light in it is *c*. It would then be possible to talk about absolute motion. The null result of the Michelson-Morley experiment leaves us with the option of either modifying the Galilean transformation or Maxwell's equations. Experiments show overwhelmingly that Maxwell's equations are correct and do not need any modification. It implies that another transformation is required in which the laws of physics, whether mechanical or electromagnetic must be valid.

Tutor Marked Assignments (TMA)

TMA 1

A 1-kg ball is constrained to move to the north at 3m/s. It makes a perfectly elastic collision with an identical second ball which is at rest, and both balls move in a north-south axis after the collision.

Calculate the total momentum before and after the collision as measured by an observer moving northwards at 1.5 m/s.

TMA 2

In an experiment similar to the Michelson-Morley experiment the distance d of either mirror from the half-silvered mirror is 25.9 m. The wavelength of the light used is 5890 Å and the apparatus is capable of detecting a time difference equal to one-hundredth of the period of this light. If the null effect is obtained when the apparatus is rotated through 90°, what is the maximum possible value of the apparatus through ether?

TMA 3

In a Michelson-Morley experiment, the optical path of each beam is 11 m and wavelength of light used is 5500Å. Assuming $\frac{v}{c}$ to be 10^{-4}, calculate the expected fringe shift if ether exists.

TMA 4

Assume the orbital speed of the earth 3$\times 10^4$m/s is the speed of the earth through ether. If it takes t_A seconds to travel through an equal-arm interferometer in a direction parallel to this motion, calculate how long it will take light to travel perpendicular to this motion.

TMA 5

A shift of one fringe in the Michelson-Morley experiment corresponds to a change in the round-trip travel time along one arm of the interferometer by one period of light (about$2 \times 10^{-15}s$) when the apparatus is rotated by 90°. If the length of the arms of the interferometer is 11m, what velocity through ether would be deduced from a shift of one fringe.

Solution to SAQ and TMA

SOLUTIONS TO SAQ and TMAs

SAQ1

SOLUTION: Let v_b and v_s be the velocities of the boat and stream respectively. Then, the velocity of the observer v in terms of v_b and v_s is

$v = v_b + v_s$

i.e. $v = 60 + 30\ km/h = 90\ km/h$

The velocity of the object as seen by the observer at the shore is 90km/h

SAQ 2

SOLUTION

The coordinates of the bird as assigned by the man on the platform are

$(x, y, z, t) = (800m, 0,0,20s)$

The distance of the bird as determined by the passenger is

$x' = x - vt = 800 - 35 \times 20m = 100m$

The coordinates of the bird as determined by the passenger are

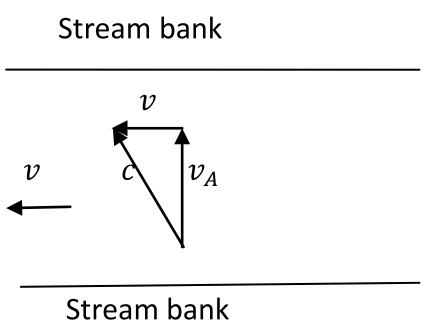
$(x', y', z', t') = (100m, 0,0,20s)$.

Thus, the average speed $\bar{v}$ of the bird as determined by the passenger is

$$\bar{v} = \frac{100m}{20s} = 5ms^{-1}$$

SAQ 3

SOLUTION



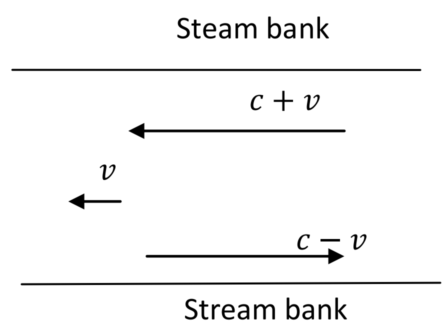


Fig.1.3a: The swimmer’s speed v_A perpendicular to the banks Fig.1.3b. The swimmer's speed v_B parallel to the banks

Figure1.3. The motion of a swimmer as seen by an observer at rest with respect to the banks of the stream.

Let the round trip time to cross the river be T_A and the round trip time downstream and upstream be T_B.

We begin with the swim across the river. Notice that to swim across the river perpendicular to the banks; the swimmer must aim at an angle upstream so that the upstream component of his velocity compensates the stream speed downstream (Figure1.3a).

Thus,
$v_A^2 = c^2 - v^2$
,

where v_A is the velocity of the swimmer perpendicular to the banks of the stream.

$$v_A = c(1-\frac{v^2}{c^2})^{\frac{1}{2}}$$

The time t_A to cross the stream along the straight-across direction is

$$t_A = \frac{L}{v_A} = \frac{L}{c}(1-\frac{v^2}{c^2})^{-\frac{1}{2}}$$

So,
$$T_A = \frac{2L}{c}(1-\frac{v^2}{c^2})^{-\frac{1}{2}}$$

Next, we consider the swim parallel to the banks of the stream. The velocity downstream is $c+v$ while the upstream velocity is $c-v$ (Figure1.3b).

The time for the round trip is simply

$$T_B = \frac{L}{c+v} + \frac{L}{c-v}$$

$$= \frac{L(c+u) + L(c-u)}{c^2 - v^2}$$

$$= \frac{2L}{c}(1-\frac{v^2}{c^2})^{-1}$$

Comparing T_A and T_B we find that the round trip times are different.

SAQ4

SOLUTION

For the observer at rest with respect to the ground, the u_1^* velocity of m_1 after collision is obtained first.

$$initial\ momentum = final\ momentum$$

$$m_1u_1 + m_2u_2 = m_1u_1^* + m_2u_2^*$$

$$(3kg)(4m/s) + (1kg)(-3m/s) = (3kg)u_1' + (1kg)(3m/s)$$

$$u_1^* = 2m/s$$

Now, using the Galilean velocity transformations,

$u_1' = u_1 - v = 4m/s - 2m/s = 2m/s$

$u_2' = u_2 - v = -3m/s - 2m/s = -5m/s$

$u_1^{*\prime} = u_1^* - v = 2m/s - 2m/s = 0$

$u_2^{*\prime} = u_2^* - v = 3m/s - 2m/s = 1m/s$

$(initial\ momentum)' = m_1u_1' + m_2u_2' = (3kg)(2m/s) + (1kg)(-5m/s) = 1kg.m/s$

$(final\ momentum)' = m_1u_1^{*\prime} + m_2u_2^{*\prime} = (3kg)(0) + (1kg)(1m/s) = 1kg.m/s$

We can see that under Galilean transformation, the observer in motion also determines that momentum is conserved though at different values to those of the observer at rest.

SAQ5

Solution

Let *l* be the lengths of the arms of the interferometer, *v* the velocity of the earth through ether, λ the wavelength and c the velocity of light used.

Then, $l = 11m, \lambda = 5900\ \text{Å}, v = 3 \times 10^4 m/s\,, and\ c = 3 \times 10^8 m/s$

Let the expected total fringe shift be ΔN.

Then,

$$\Delta N = 2\frac{l}{\lambda}\frac{v^2}{c^2} = \frac{2 \times 11 \times (3 \times 10^4)^2}{(5900 \times 10^{-10}) \times (3 \times 10^8)^2} = 0.37$$

SAQ6

Solution

Let the number of fringes be ΔN, *l* the length of the arms, *v* the velocity of the earth relative to ether, λ the wavelength of light and *c* the velocity of light.

Then, $\Delta N = 0.005$, $l = 10m$, $\lambda = 600\ \text{Å}$ and $c = 3 \times 10^8 m/s$.

$$\Delta N = 2\frac{l}{\lambda}\frac{v^2}{c^2}$$

$$0.005 = \frac{2 \times 10 \times v^2}{(600 \times 10^{-10})(3 \times 10^8)^2}$$

$v = 1.16 \times 10^3\ m/s$

References

1. Introduction to Special Relativity by Wolfgang Rindler, Oxford University Press, 1990.
2. Special Relativity by A. P. French, Noton, 1968

3. Fundamentals of Modern Physics by Robert Eisberg, John Wiley, 1972

UNIT2: EINSTEIN'S POSTULATES AND LORENTZ TRANSFORMATION

CONTENTS

2.0 Introduction

In unit1, we saw how the null result of Michelson- Morley experiment led to the failure of the concept of the absolute space or universal frame of reference in which both the mechanical and electromagnetic phenomena are valid. Thus, instead of a constant velocity of electromagnetic waves c, an observer in a reference frame moving at a velocity v should measure $c + v$ or $c - v$ depending on the direction of relative motion. The velocity of the electromagnetic waves, therefore, is not invariant under Galilean transformation and Maxwell's equations will change their form on transformation from one system to another. This scenario places before us the following options:

1. If electromagnetic theory is correct, then it must be invariant under Galilean transformation just like Newtonian mechanics. Since this is not the case, Maxwell's equations which form the basis of electromagnetic phenomena are incorrect and need to be modified.

2. Galilean transformation is applicable to mechanics but there is a preferred frame of reference in which the electromagnetic phenomena (Maxwell's equations) are valid. In any other frame of reference, they must be suitably modified.

3. Maxwell's equations are valid and there is a transformation under which both the mechanical and electromagnetic phenomena are valid but it is not Galilean. This implies that Newton's laws need to be modified.

Experimental evidence is overwhelmingly in support of the correctness of Maxwell's equations. Therefore the first option is not tenable.

The second option is ruled out by the null result of Michelson-Morley experiment. One of the important conclusions of this experiment is that the velocity of light is the same in all directions and is independent of the relative motion of the source and the observer.

Albert Einstein was guided by the above considerations to formulate his theory of special relativity which we will deal with in this unit. He reasoned that the velocity of light was isotropic and is independent on the relative motion of the source and the observer. Furthermore, he saw the need to replace the Galilean transformation and modify the laws of mechanics and so he chose the third option.

2.1 Objectives

After completing this unit, you will be able to

a. State the postulates of special relativity
b. Explain the concept of simultaneity.
c. Carry out the process of the derivation of the Lorentz transformation relations
d. Discuss and show proof of some of the properties of Lorentz transformation
e. Discuss the problem of clock synchronization
f. Demonstrate the invariance of Maxwell's equations under the
g. Solve simple problems involving Lorentz transformation Lorentz transformation

2.2 Main Body

2.2.1 Einstein's Postulates

Einstein's special theory of relativity consist of two postulates namely,

I. *Physical laws are the same for all inertial frames of reference.* Consequently, *all inertial frames are equivalent*.

The postulate stresses that it is impossible by means of any physical measurement to find a state of absolute motion or universal frame of reference.

II. *The velocity of light in free space has the same value in all inertial frames of reference and is independent of the motion of the source.*

These postulates, as you can see, appear to be anything but radical. But, as we will soon see, they have far reaching physical consequences. The immediate consequence of the first postulate was the compelling need to modify the Galilean transformation.

The second postulate on the other hand has to do with finite velocity of interaction between particles. You already know that the interaction between two particles is described in terms of their potential energy. That is to say, interaction is a function of position. Thus, when one particle changes position (moves), the other particle is influenced to move relative to it instantaneously. This implies that the signal of interaction propagates at infinite velocity. This assumption is inherent in the Galilean transformation. In practice, a change in one body requires a finite time interval to begin to manifest in another body at a distance. Therefore, *the signal of interaction propagates at a finite velocity* and this velocity is the maximum. Clearly, there cannot be any motion with velocity greater than this in nature. This is exactly what the second postulate states. In other words, the velocity of propagation of the signal of interaction-the velocity of light in free space-is the same in all inertial frames of reference and is the limiting velocity. *No material object can be accelerated to a velocity in excess of c, the velocity of light in free space.* This assertion has been tested and found to agree with experiment. For example, the velocity of the radiation emitted in the decay of π-meson moving at a velocity in excess of 0.99975c was measured by Farley et al to be $(2.9979 \pm 0.0003) \times 10^8$m/s. This answers the question we asked in sub-section 1.2.3. Recently (in October 2011), a group of scientist working at a linear accelerator in Stanford University, England, claimed that they discovered neutrino which travels at speed in excess of *c*. This claim is still being investigated by scientists elsewhere to confirm its veracity or otherwise. Until an experimental confirmation proves otherwise, *the speed of light c remains the limiting speed of all material particles*.

2.2.2 Simultaneity

Before we turn our attention to the Lorentz transformation, let us briefly discuss the idea of the measurement of time. Consider the fourth of the Galilean transformation equations, namely

$$t' = t$$

The equation seems to say that time is absolute or universal for all inertial frames. Is this true? To answer this question, let us investigate the idea of measurement of time. The basic process involved in the measurement of time is the measurement of simultaneity. Thus, if we say that "the plane lands at 4 o'clock," what this means is that the two events are simultaneous at the same location namely, 1.The plane landed and 2. The short hand of the clock is at 4 while the long hand is at 12. What happens if the two events occur at different locations (i.e. if the clock is far removed from the vicinity of the plane)? Of course this will involve the transmission of signal between the two physically separated locations. But, we have seen that interaction signals must have finite velocities. There must be a difference in time between the two events at different locations and so they are no longer simultaneous. This is the point where Galilean transformation is wrong by assuming that the time of occurrence of two events at different locations is the same. In other words, the equation $t' = t$ is not correct and must be modified.

A realistic method of determination of simultaneity requires the use of a real signal with the highest known velocity which is an electromagnetic wave such as light. This leads us to Einstein's definition of simultaneity which states that *two instants of time t_1 and t_2, observed at two points $x_1$and x_2 in a particular frame of reference, are simultaneous if light signals simultaneously emitted from the geometrically measured midpoint between $x_1$and x_2 arrive at $x_1$at t_1 and at x_2 at t_2*. The converse is also true i.e. *t_1 and t_2 are simultaneous if light signals emitted at t_1 from x_1 and t_2 from x_2 arrive at the mid-point simultaneously*. The concept is illustrated in figure 2.1. Notice that these definitions intimately mix the times t_1, t_2 and spatial coordinates x_1, x_2. In Einstein's theory simultaneity does not have an absolute meaning, independent of spatial coordinates, as it does in the classical theory.

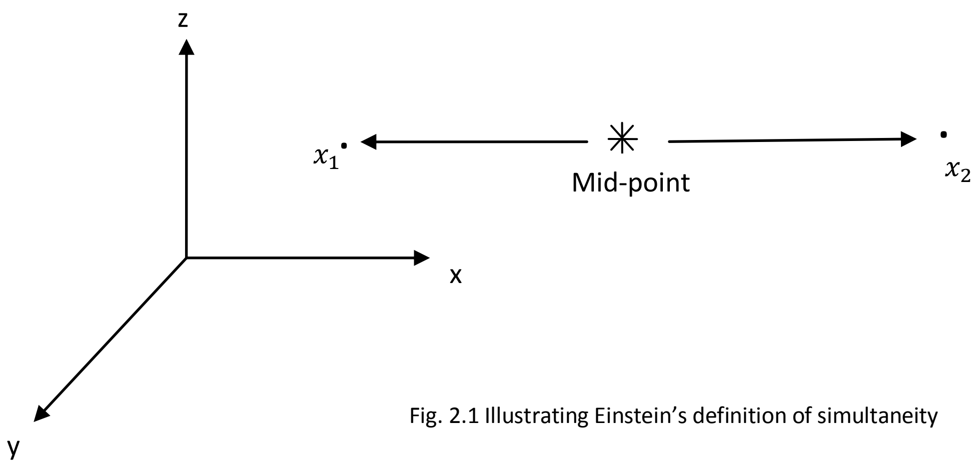


Fig. 2.1 Illustrating Einstein's definition of simultaneity

A consequence of these definitions is that *two events which are simultaneous in an inertial frame of reference are in general not simultaneous in another inertial frame of reference in relative motion to it*. Consider a thought experiment in which we follow the progress of a light pulse as noted by two observers in inertial reference frames. Suppose a box car is travelling to the right at a very high constant velocity ***v*** as illustrated in figure 2.2. A high-speed flashbulb B is placed at the geometrical centre of the car where the observer O' is located. The bulb is equipped with reflectors so that it can send light pulses in opposite directions when it is switched on. Photocells are fitted at opposite ends

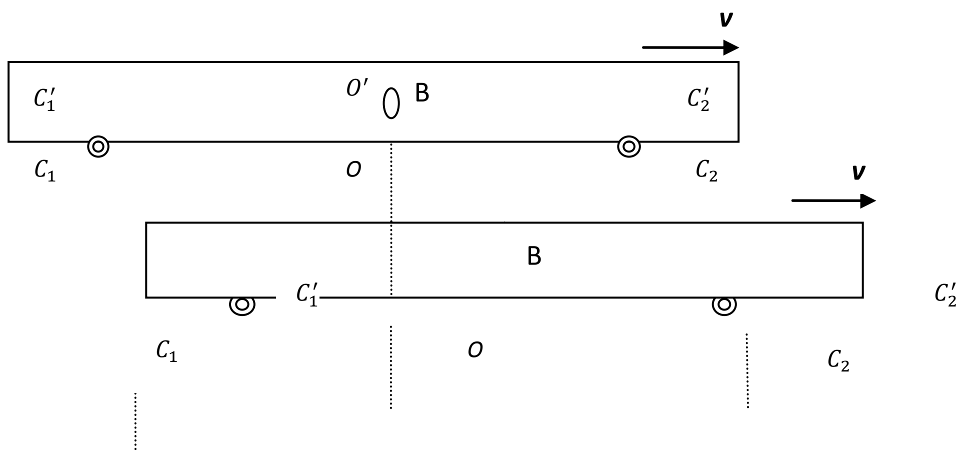


Fig.2.2 For O' the events at C_1' and C_2' occur simultaneously whereas for *O* the event at C_1 occurs before the one at C_2

enable an observer detect when the light pulses strike the ends of the car. At the instant O' is abreast of *O*, an observer at rest with respect to the earth, the flashbulb is switched on. Notice that the two observers are in inertial reference frames (one is the boxcar and the other is the earth), and so the speed of light must be constant to both of them. To the observer O' the flashbulb is at rest relative to his reference frame. When the bulb is switched on, light pulses travel equal distances in equal times to the two ends C_1' and C_2' of the car. For him the experiment is the same whether or not the car is in motion. Notice that the pulses of light will also travel equal distances to C_1 and $C_{2,}$ the ends of the boxcar as measured by O, in equal times if the two observers are abreast and O' is at rest relative to O. So, the light pulses hit the two ends of the car simultaneously.

Now, to the stationary observer O, C_1' approaches while C_2' recedes from him. Thus, for him, the distance traveled by the light pulse to the left end of the car OC_1' is shorter than the distance to the right end of the car OC_2' . Light pulses to C_1' arrive before those traveling to C_2'. Notice that light pulses from O' arrive C_2 ahead of light pulses from O' to C_1. We conclude that the light pulses do not hit the two ends simultaneously.

From this experiment we can conclude that time is not a universal quantity because:

Events which are simultaneous in one inertial system may not be simultaneously in another.

You must understand that this situation exists only if the two events occur at different locations. In our experiment, one event took place at one end of the car, and the other took place at the opposite end. However it must be pointed out that both observers are correct, even though their results differ as a result of their relative motion.

2.2.3 Lorentz Transformation

We now proceed to re-examine the concept of space and time in the light of Einstein's postulates. This is of course what Einstein himself did.

Consider two observers O and O' in two inertial frames of reference S and S', who view the same event, with S' moving in the x-direction relative to S at constant velocity v as in figure 1.2. Relative to their respective coordinate systems O assigns the location and time coordinates of the event as (x, y, z, t) while O' assigns (x', y', z', t') as location and time coordinates for the same events. The desired coordinate transformation consists of a set of algebraic equations which connect the two set of coordinates, that is, a relation that allows us to calculate the primed set of coordinates in terms of the unprimed ones and vice versa. You will recall that Galilean relativity gave these relations as in equation 1.1, namely, $y' = y,\ z' = z,\ \ t' = t$ and $x' = x - vt$. Of course, this conforms with common experience provided $v \ll c$, that is, the relative motion of the coordinate systems occurs at ordinary velocity. However, at relativistic velocity, that is, at velocity close to that of light ($v \approx c$), these transformation relations contradict Einstein's postulates and are incorrect as we will soon see.

Let us assume that at the instant the origins of the two frames are coincident, a flash bulb explodes as shown in figure 2.3 a. A light sphere then expands in all directions at a velocity c. At an instant t after the bulb had exploded, observer O in the S frame observes that the moving frame has travelled a distance $x = vt$ in the x-direction as shown in figure 2.3b, and that the radius of the expanding light sphere is $r = ct$. You will recollect, from your study of coordinate geometry, that $r^2 = x^2 + y^2 + z^2$.

Thus, $x^2 + y^2 + z^2 = c^2t^2$ 2.1

The observer O' in the S' frame observes a light sphere with radius $r' = ct'$.

Thus, $x'^2 + y'^2 + z'^2 = c^2t'^2$ 2.2

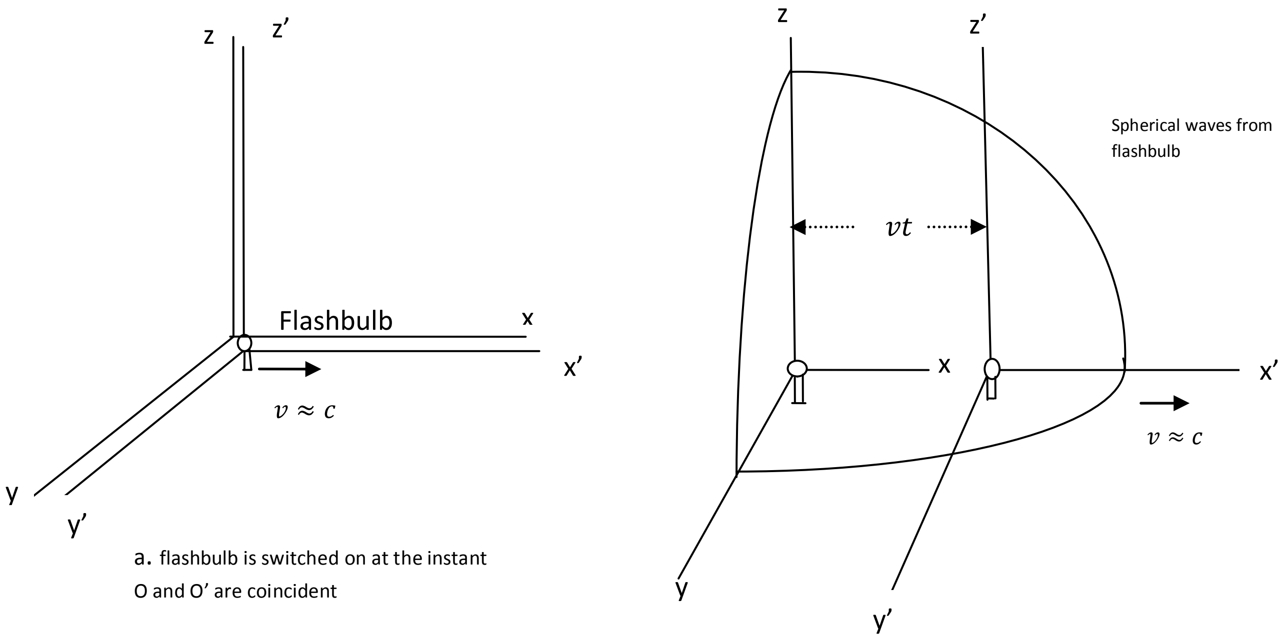
Notice that the two equations have the same form and the velocity c is the same in both frames in compliance with Einstein's postulates.

Now, from Galilean transformation, we have $y' = y$, $z' = z$, $t' = t$ and $x' = x - vt$. Putting these in equation 2.2, we obtain,

$(x - vt)^2 + y^2 + z^2 = ct$

or, $x^2 - 2xvt + v^2t^2 + y^2 + z^2 = ct$ 2.3

Of course, as you can see, equation 2.3 does not have the same form as equation 2.1. Therefore it is quite clear that Galilean transformation fails in the realms of relativistic velocity. We require a new set of transformation relations which will transform equation 2.2 into equation 2.3.



a. flashbulb is switched on at the instant O and O' are coincident

b. t seconds later, radius of sphere is ct

Fig. 2.3 The light sphere expands with velocity c.

The relative velocity of the reference frames $v \approx c$

It is reasonable to assume a linear relationship for the transformation equations for the fact that *a single event in one inertial frame of reference must correspond to a single event in the other in the other inertial frame.* Thus, the most general of such a set (relationship) may be expressed in the form

$$x' = A(x - vt)$$
$$y' = By$$
$$z' = Cz$$
$$t' = Dx + Et \qquad 2.4$$

A, B, C, D and E are constants.

Since the motion is along the x-axis only, then $B = C = 1$ so that $y' = y$ and $z' = z$ as before.

If our assumption is correct, then when we make the substitution of equation 2.4 in equation 2.2, we should obtain equation 2.1. Now, do these substitutions yourself and cross-check with the working given below:

Putting the values of x' and t' from equation 2.4 into equation 2.2, we have

$$A^2(x - vt)^2 + y^2 + z^2 = c^2(Dx + Et)^2$$

Expanding the brackets, we obtain

$$A^2(x^2 - 2xvt + v^2t^2) + y^2 + z^2 = c^2(D^2x^2 + 2EDxt + E^2t^2)$$

Rearranging the equation

$$A^2x^2 - 2A^2xvt + A^2v^2t^2 + y^2 + z^2 - c^2D^2x^2 - 2EDc^2xt - E^2t^2c^2 = 0$$

Collecting like terms,
$(A^2 - c^2D^2)x^2 + (A^2v^2 - E^2c^2)t^2 - 2xt(A^2v + EDc^2) + y^2 + z^2 = 0$

Now, comparing this last equation with equation 2.1 (i.e. equating corresponding coefficients), we find that
$A^2 - D^2c^2 = 1$

$A^2v^2 - E^2c^2 = -c^2$

$A^2v + EDc^2 = 0$ 2.5

Can you solve the set of simultaneous equations 2.5? Try it out. At the end of it all, you should obtain something similar to this:

$$A = D = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

and

$$-E = D\frac{v}{c^2}$$

where $\beta = \frac{v}{c}$

Putting these values in equation 2.4 we obtain

$$x' = \gamma(x - vt) = \frac{(x - vt)}{\sqrt{1-\beta^2}}$$

$$y' = y$$

$$z' = z$$

and $t' = \gamma\left(t - \frac{\beta x}{c}\right)$

$$= \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

2.6

The set of equations 2.6 are called the *Lorentz- Einstein relations* or the so-called *Lorentz transformations*. They tell us how the (primed) coordinates of an event (x', y', z', t') in the S' frame are related to the coordinates of the *same event* (x, y, x, t) as measured in the S frame when both frames are in relative motion at constant velocity.

The *inverse Lorentz transformation equations* are obtained simply by replacing v by $-v$ in the corresponding transformation equations 2.6. If we do this, we will get the following set of equations:

$$x = \gamma(x' + vt')$$

$$= \frac{(x' + vt')}{\sqrt{1 - \beta^2}}$$

$$y = y'$$

$$z = z'$$

and

$$t = \gamma\left(t' + \frac{\beta x'}{c}\right)$$

$$= \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \beta^2}} \qquad 2.7$$

Equation 2.7 allows us to transform from the primed frame to unprimed frame.

Now, examine these transformation equations (2.6 and 2.7) and you will quickly discover that at non-relativistic velocities (at ordinary velocities), i.e. when $v \ll c,\ \beta = \frac{v}{c} \rightarrow 0$ and $\gamma \rightarrow 1$, so that $x' = x - vt$. This justifies our common experience that *at ordinary velocities, the Lorentz transformation reduces to the Galilean transformation.*

2.2.4 Properties of the Lorentz Transformation

We will now proceed to take a closer look at the Lorentz-Einstein transformation relations with a view to finding out the physical contents in them.

1. **The Relativity of Simultaneity**

The most striking feature of the Lorentz – Einstein relations is the *transformation of time,* which exhibits the relativity of simultaneity. We have already studied this in section 2.2.2. Here, we want to show how the concept is the direct consequence of the Lorentz – Einstein transformation. We will again put forward a similar argument as before.

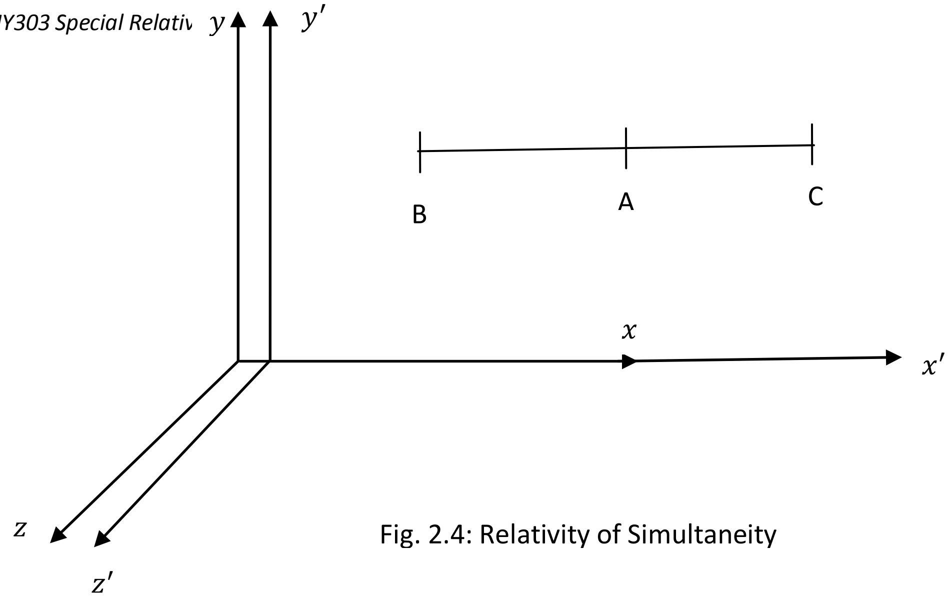


Fig. 2.4: Relativity of Simultaneity

Consider two frames inertial frames of reference S and S' which are in relative motion in the common positive x - direction with constant velocity v. Consider two events, such as the explosion of a pair of flash bulbs that occur simultaneously at points B and C in the S' frame as shown in figure 2.4. For an observer at the mid-point A of the distance between B and C in this frame, the times t_B and t_C of the occurrence of the said events at B and C are equal i.e. $t'_B = t'_C$ or the time interval between the events $t'_B - t'_C = 0$.

However, for an observer in the S frame, the times for the events t_B and t_C are related to those in the S' frame by the Lorentz equations. Of course this makes sense from the point of the discussions we have had.

Thus, $t_B = \gamma\left(t'_B + \beta\frac{x'_B}{c}\right)$

and

$$t_C = \gamma\left(t'_C + \beta\frac{x'_C}{c}\right)$$

.

The time interval between the two events for this observer is

$$t_B - t_C = \gamma\left(t'_B + \beta\frac{x'_B}{c}\right) - \gamma\left(t'_C + \beta\frac{x'_C}{c}\right)$$

$$= \gamma\left\{t'_B - t'_C + \frac{\beta}{c}(x'_B - x'_C)\right\} \neq 0$$

.

You can see how we have arrived at our earlier conclusion that *events which are simultaneous in one frame of reference are not necessarily simultaneous in another frame of reference in relative motion to it*. *Simultaneity has only a relative and not an absolute meaning*.

2. **Symmetry of the Relations**:

Carefully, examine equations 2.6 (and of course 2.7). You will realize that they are symmetric not only in x but also in ct. Let us attempt to verify this assertion. Simply replace t in equation 2.6 by $\frac{\tau}{c}$ and t' by $\frac{\tau'}{c}$ and then divide the second equation by c, that is,

$$x' = \gamma\left(x - v\frac{\tau}{c}\right)$$

and

$$\frac{\tau'}{c} = \gamma\left(\frac{\tau}{c} - \beta\frac{x}{c}\right) \qquad 2.8$$

Therefore

$$\tau' = \gamma(\tau - \beta x)$$

or

$$\tau' = \gamma\left(\tau - \frac{vx}{c}\right) \qquad 2.9$$

By comparison, we can easily see that 2.8 and 2.9 are symmetric if $\tau = ct$ and $\tau' = ct'$.

3. **Significance of the Lorentz factor γ**

For $v \neq 0$, the Lorentz factor γ is always greater than unity. Besides, as $v \to c, \gamma \to \infty$. Also, $v > c$ leads to an imaginary value of γ and for $v \ll c$, Lorentz transformation reduces to Galilean transformation. The physical meaning of all of this is that the relative velocity of the two inertial frames of reference must be less than c, since finite real coordinates in one frame must correspond to finite real coordinates in any other frame.

4. **The Relationship between the Coordinates and the Differentials**

Since the standard Lorentz transformation is linear and homogeneous, the coordinate differences as well as the differentials satisfy the same transformation equations as the coordinates themselves. In other words,

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

$\Delta y' = \Delta y$

$\Delta z' = \Delta z$

$\Delta t' = \gamma\left(\Delta t - \beta \frac{\Delta x}{c}\right)$ 2.10

and

$dx' = \gamma(dx - vdt)$

$dy' = dy$

$dz' = dz$

$dt' = \gamma\left(dt - \beta \frac{dx}{c}\right)$

2.11

2.2.5 Simultaneity and Synchronization of clocks

Let us proceed to apply our discussion of the simultaneity to an interesting problem; synchronization of clocks. Now try to reflect on the process of setting the time on your wrist watch or clock. How do you do that? Well, you will say that is very simple. You simply compare the time on your clock with the time, say of a radio station such the BBC or a nearby clock, and then adjust the hands of your clock as appropriate.

2.2.6 Invariance of Maxwell's Equation

Among the successes of the Lorentz –Einstein transformation, it must be able to demonstrate the invariance of both the mechanical and electromagnetic laws of physics. At the beginning of this unit we pointed out that any transformation that must replace the Galilean transformation must satisfy this requirement. We now proceed to demonstrate the invariance of Maxwell's equations in the light of special relativity and Lorentz –Einstein transformation.

The electromagnetic wave equation which is derived from Maxwell's equations is

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

where $c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = 2.998 \times 10^8$ ms^{-1}, the speed of all electromagnetic waves in free space. As in unit one, we restrict our equation to the x-component of the electric field just for simplicity of the proof.

Our objective here is to show that the above wave equation retains the same form when it is expressed in terms of the primed coordinates (x', y', z', t').

First of all, find the first partial derivatives of x', y', z' and t' w.r.t. x, y, z and t from the Lorentz equations as follows:

$$\frac{\partial x'}{\partial x} = \gamma \frac{\partial (x - vt)}{\partial x} = \gamma$$

and

$$\frac{\partial x'}{\partial t} = \gamma \frac{\partial (x - vt)}{\partial t}$$

$$= -\gamma v$$

$$\frac{\partial t'}{\partial x} = \gamma \frac{\partial \left(t - \frac{\beta}{c} x\right)}{\partial x}$$

$$= -\gamma \ \frac{\beta}{c}$$

and

$$\frac{\partial t'}{\partial t} = \gamma \frac{\partial \left(t - \frac{\beta}{c} x\right)}{\partial t} = \gamma$$

Also,

$$\frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = 1$$

and

$$\frac{\partial x'}{\partial y} = \frac{\partial x'}{\partial z} = \frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial z} = \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = \cdots = 0$$

Now use the chain rule to write first derivative of the electric field component of the wave equation as

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E_x}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial E_x}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial E_x}{\partial t'}\frac{\partial t'}{\partial x}$$

$$= \gamma \frac{\partial E_x}{\partial x'} - \gamma \ \frac{\beta}{c}\frac{\partial E_x}{\partial t'}$$

Differentiating w.r.t. x we obtain

$$\frac{\partial^2 E_x}{\partial x^2} = \gamma \left(\frac{\partial^2 E_x}{\partial x'^2} + \frac{v^2}{c^2}\frac{\partial^2 E_x}{\partial t'^2}\right) - \frac{2v}{c^2 - v^2}\frac{\partial^2 E_x}{\partial x' \partial t'}$$

Similarly, if we differentiate w.r.t. t, we obtain

$$\frac{\partial E_x}{\partial t} = -\gamma v \frac{\partial E_x}{\partial x'} + \gamma \frac{\partial E_x}{\partial t'}$$

and

$$\frac{\partial^2 E_x}{\partial t^2} = \gamma^2 \left(v^2 \frac{\partial^2 E_x}{\partial x'^2} + \frac{\partial^2 E_x}{\partial t'^2} \right) - 2v\gamma \frac{\partial^2 E_x}{\partial x' \partial t'}$$

Also,

$$\frac{\partial^2 E_x}{\partial y^2} = \frac{\partial^2 E_x}{\partial y'^2}$$

and

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial^2 E_x}{\partial z'^2}$$

Now, substituting these values in the wave equation, we obtain

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x'^2} + \frac{\partial^2 E_x}{\partial y'^2} + \frac{\partial^2 E_x}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t'^2}$$

Thus, the equation of electromagnetic waves and consequently Maxwell's equations of electromagnetic phenomena are invariant (retain the same form) under Lorentz transformation.

We have come to the end of this unit except for a few self assessment questions to reinforce the concepts you have studied. You are now ready and equipped with sufficient information to attempt the next set of SAQs. Be sure you do them entirely on your own. You will be glad you did. None of them is difficult or outside what we have been discussing in this unit. When you have finished, turn to the solutions on page… and compare your results.

SAQ 7

An observer O' in a space ship moving at a speed of 0.7c along the positive x-axis relative to a stationary observer O on earth determines the coordinates of a lightning bolt $(x', y', z', t') = (80, 0, 0, 8 \times 10^{-8} s)$. What are the coordinates of the bolt as determined by the observer$O$?

SAQ 8

Show that if (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are the coordinates of an event in S_1 and the corresponding event in S_2 respectively, then the expression $ds_1^2 = dx_1^2 + dy_1^2 + dz_1^2 - c^2 dt_1^2$ is invariant under the Lorentz transformation of the coordinates.

SAQ 9

Show that (i) $\gamma v = c(\gamma^2 - 1)^{1/2}$ (ii) $c^2 d\gamma = \gamma^3 v dv$

SAQ 10

An observer O in an S frame determines that two events are separated in space and time by 800m and $6 \times 10^{-7}s$. At what speed must an observer O' in the S' frame be moving relative to O in order that the events are simultaneous to O'?

SAQ 11

A particle moves with constant velocity of $c/2$ in the x-y plane relative to an observer O in the S- frame. The trajectory of the particle makes an angle of 30° with the x-axis. Write down the equations of motion of the particles as determined by an observer O' in the S'-frame which is moving at a speed of 0.7c relative to O along their common positive x-axis.

Summary

Here is the summary of the salient points we have discussed in this unit:

- Einstein's special theory of relativity consist of two postulates namely,

Physical laws are the same for all inertial frames of reference. Consequently, *all inertial frames are equivalent*.

The postulate stresses that it is impossible by means of any physical measurement to find a state of absolute motion or universal frame of reference.

The velocity of light in free space has the same value in all inertial frames of reference and is independent of the motion of the source.

The postulate emphasizes the fact that the speed of light c remains the limiting speed of all material particles.

- Galilean transformation is inconsistent with the special theory of relativity at speeds close to the speed of light c.
- Lorentz transformation is consistent with the theory of special relativity and provides a means of relating the coordinates of an event as viewed by observers in inertial frames of reference in relative motion at relativistic speeds.
- At ordinary speeds, Lorentz transformation reduces to the Galilean transformation.
- Events which are simultaneous in one frame of reference are not necessarily simultaneous in another frame of reference in relative motion to it. Simultaneity has only a relative and not an absolute meaning.

Conclusion

We conclude this unit by emphasizing that Lorentz transformation provides a means of ensuring the invariance of physical equations whether they are mechanical or electromagnetic.

Tutor Marked Assignments

1. Evaluate the values of

$$\frac{1}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

for (i) $v = 10^{-2}c$ (ii) $v = 0.9998c$

2. Show that $d(\gamma v) = \gamma^3 dv$ with the symbols having the meaning as defined in this unit.

3. As determined by O'a lightning bolt strikes at $x' = 60m, y' = z' = 0, t' = 8 \times 10^{-8}s$. O' has a velocity of $0.6c$ along the x-axis of O. What are the space-time coordinates of the strike as determined by O?

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UNIT3: KINEMATIC CONSEQUENCES OF LORENTZ TRANSFORMATION

CONTENTS

3.0 Introduction

In the previous unit, we saw how Einstein's postulates required the modification of our consideration of the fundamental nature of space and time.

In this unit you are going to learn about some of the *kinematical consequences of the special theory of relativity*. In other words, we are interested in the physical consequences of replacing the Galilean with the Lorentz – Einstein transformation when motions at relativistic speeds are involved. In particular, we will discuss how these transformations effect the measurements of length and time intervals by observers in different frames of reference.

The transformation equations you have studied in the previous unit will come handy here, so you will do well to take a brief look at them before proceeding with this discussion.

3.1 Objectives

At the end of this unit, you should be able to

a. Discuss and perform simple calculations involving relativistic length contraction
b. Discuss and perform simple calculations involving relativistic time dilation
c. Derive the equations for the transformation of velocity

d. Perform simple calculations involving velocity transformation.
e. Discuss the twin paradox
f. Apply the concept of velocity of addition to discuss relativistic Doppler effect
g. Perform simple calculations involving relativistic Doppler effect

3.2 Main Body

3.2.1 Length contraction

You already know how to measure the length of a rod from your school practical physics. Remember that you simply had to take the metre scale readings against the two ends of the rod and find the difference.

In practice, the length of an object which is at rest with respect to an observer O in the S- frame is simply the difference between the spatial coordinates of its end-points. This is illustrated in figure 3.1

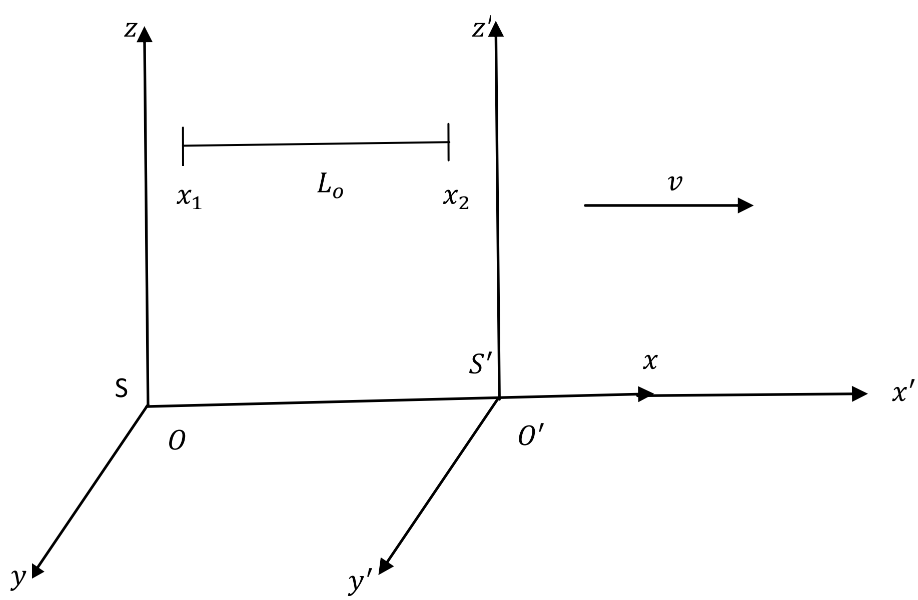

Figure 3.1: L_o is the difference between

The length $L_o = x_1 - x_2$. L_o is called the *rest length,* since the body is at rest relative to the observer in the S-frame.

To an observer O'in the S' - frame, which is in relative motion to S- frame at a relativistic velocity v along the common positive x-direction, the coordinates the ends of the rod are measured as x_1' and x_2'. Now the measurements in the two frames are related through the Lorentz transformation as

$x_1 = \gamma(x_1' + vt')$ and $x_2 = \gamma(x_2' + vt')$

So that,

$$L_o = x_2 - x_1 = \gamma(x_2' - x_1') = \gamma L = \frac{L}{\sqrt{1-\beta^2}} = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$$

where L is the length of the object as measured by O'

Thus,

$$L = L_o\sqrt{1-\beta^2} = L_o\sqrt{1-\frac{v^2}{c^2}} \quad 3.1$$

How do you interpret this equation? Well, notice that L is less than one, since the Lorentz factor is $\sqrt{1-\frac{v^2}{c^2}}$ is less than one. Why? This is because the Lorentz factor i.e. $\sqrt{1-\frac{v^2}{c^2}}$ decreases as v increases. Thus, to the observer in motion, the object appears to be shortened in the direction of motion. You must note that the *rod appears shortened only if the direction of motion is parallel to its length*. This shortening of length is called Lorentz - *FitzGerald contraction.* This effect is reciprocal. In other words,

$$L_o = L\sqrt{1-\beta^2} = L\sqrt{1-\frac{v^2}{c^2}}$$

Thus, *objects appear to be longest when they are at rest relative to the observer.* When it is in motion, it appears to be contracted in the direction of motion by the factor $\sqrt{1-\beta^2}$. However, because $y = y'$ and $z = z'$are perpendicular to the motion, *the apparent dimensions of the object remain unchanged in a direction perpendicular to its motion*. Let us make a generalized statement of the concept we have introduced here*:*

The length of an object is a maximum in a frame of reference relative to which it is at rest and its length appears to be contracted in a frame of reference relative to which it is in motion.

The length of the object measured in the frame of reference relative to which it is at rest is called the *proper length*.

One more point and then we move to the next sub-section. Notice that we have been a bit economical with the use of terms here by saying that "the rod appears to be shortened."In reality, nothing has happened to the rod. The rod appears shorter because, from *velocity perspective*, the eye of the observer can only see what, in the moving rod's frame, could be described as *later* events, from behind. The result is that the rear of the rod appears to be closer to its front. If the eye of the observer was able to move along with the rod at the same relativistic speed, the proper length of the rod would be restored. So we see here that in relativity, there is a world of difference between what we see at an instance of time and what we know has actually taken place. What we see is a composite of events that occurred progressively earlier as they occurred farther and farther away.

3.2.2 Time Dilation

When we say that something is dilated, we mean that it is enlarged. With that in mind, let us consider the problem of measurement of the time interval of two events as measured in two reference frames which are in relative motion at a relativistic velocity.

Suppose we have a clock at a fixed point x'in the S'- frame which is in relative motion to the S-frame at a constant velocity v along the common positive x-axis. This clock is used to measure the interval between two events which occur at the same position x' at different instants of time t_1' and t_2'. The time interval between these events is $t_2' - t_1' = \Delta t'$.

Now to an observer in the S-frame, the two events appear to occur at a fixed position x at different instants of time t_1 and t_2. The time interval is $t_2 - t_1 = \Delta t$.

The relationship between the times of occurrence of the events in the two frames is given in terms of the Lorentz transformation as

$$t_1 = \gamma\left(t_1' + \beta\frac{x'}{c}\right)$$

and

$$t_2 = \gamma\left(t_2' + \beta\frac{x'}{c}\right)$$

$$t_2 - t_1 = \gamma(t_2' - t_1')$$

i.e.

$$\Delta t = \gamma\Delta t'$$

or

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}$$

or

$$t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

3.2

Equation 3.2 summarizes the fact that as v increases, γ and therefore Δt increases. The time measured by observer O in the S-frame is longer. *This is known as time dilation.* Now, the observer O' in the S'-frame is at rest relative to the clock and the occurrences of the events at the same position. The time of the occurrence of the events measured by him ($\Delta t'$) is known as the *proper time.*

We shall generalize by saying that *a clock measures longer time interval of events in a frame of reference relative to which it is in motion than a clock in a frame of reference relative to which it at rest* or *a clock moving with uniform velocity v through an inertial frame S runs slow relative to the standard clock at rest in S.*

The fastest rate (smallest time interval Δt) is measured by the clock in the rest frame and is called the *proper time.*

You have to note that that, like length contraction, time dilation is also a 'velocity perspective" effect. In reality, nothing has happened to the clock in motion. If the eyes of the observer were able to move at the same velocity of the clock, everything would return to normal.

But then, are there any physical manifestations of time dilation (and length contraction) in nature? Or is there any empirical evidence of these concepts? Well, yes. Have you heard of μ *mesons (mu mesons or muons)?* You will certainly discuss the properties of these particles in greater detail in your Modern Physics or Elementary Particle Physics course. They are unstable elementary particles with very short life span (about $2 \times 10^6 s$ on the average). So, they decay shortly after they come into existence. Typical speeds of muons are in the neighbourhood of the speed of light (the speed of muon is about $2.994 \times 10^8 ms^{-1}$). Now, muons are created in the upper atmosphere at an altitude of about an earth radius by incoming cosmic rays, yet a profuse supply of them reaches the earth at sea level. How come? In a time of $2 \times 10^6 s$, a muon travelling at a speed of $2.994 \times 10^8 ms^{-1}$ can only cover $600\ m$ of the altitude. So, if they are found at sea level, then we have an issue to resolve here. Let us look at the problem from the reference frame of the muon. Its lifetime t_m in this frame is unaffected by the motion, right? In the frame of reference of an observer on the ground the time, we call t_o appears to be dilated by the factor

$$t_o = \frac{t_m}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2 \times 10^{-6} s}{\sqrt{1 - \frac{(0.998c)^2}{c^2}}}$$

$$= 3.1639 \times 10^{-5} s.$$

In this time, the muon will travel a

distance
$y = vt = (2.994 \times 10^8\ ms^{-1})(3.1639 \times 10^{-5}\ s) = 9{,}4727.7\ m \cong 9500\ m.$

Thus, despite its brief life span, the muon makes it to the earth from the altitude where it is created.

Now, rework the problem from the perspective of length contraction and satisfy yourself that the results are the same.

There is another striking evidence of time dilation called the *relativistic focusing* of electrically charged particles, which plays a role in operation of high-energy particle accelerators. This and others will be discussed in greater detail when we consider experimental evidences of the theory of special relativity elsewhere in this course.

Now, try out the following SAQs.

SAQ 12

Calculate the minimum speed required for muons produced at a height of 400 km to reach the surface of the surface of the Earth if their life time is 2.2μs

SAQ 13

Obtain the volume of a cube, the proper length of each edge of which is l_o when it is moving with a velocity v along its edges

SAQ 14

Calculate the percentage contraction of a rod moving with a velocity of $0.8\ c$ in a direction inclined at $60°$ to its own length.

SAQ 15

A pi-meson (pion) has a mean life time of $2 \times 10^{-8} s$ when measured at rest. How far does it go before decaying into another particle if its speed $0.99\ c$?

Let us proceed to discuss yet another consequence of the Lorentz coordinate transformation. How can we obtain the velocity of an object in a frame of reference as measured by an observer in another frame of reference in relative motion at constant velocity to it? Turn to the next sub-section for a discussion on this.

3.2.3 Velocity Addition

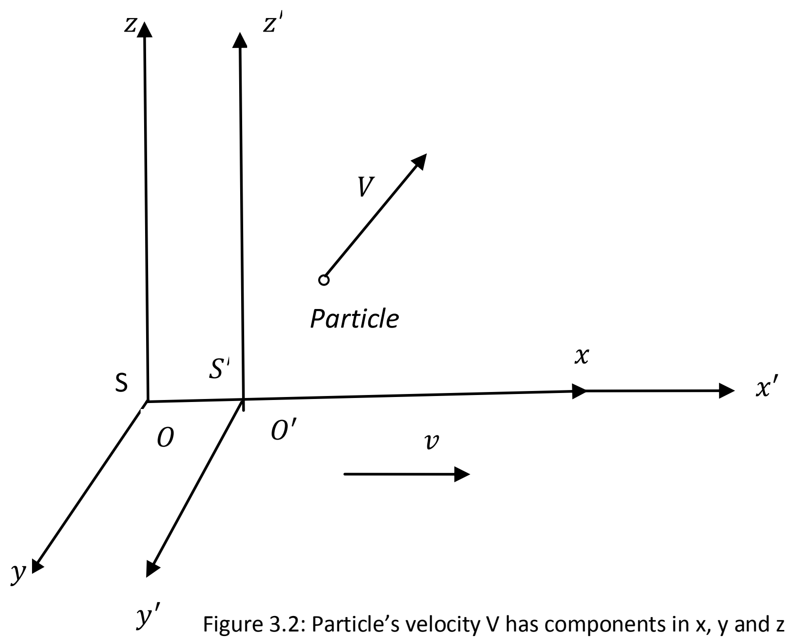

Figure 3.2: Particle's velocity V has components in x, y and z - directions

Now, consider the particle in the frame of reference s' which is in uniform motion with a velocity v relative to the reference S as shown in figure 3.2. The particle is in motion with velocity V' as measured by observer O' who is at rest in the s'- frame. We want to find out the velocity of the particle as measured by observer O in the S-frame. Let us be more specific. Suppose the particle is a photon of light emitted in the s'- frame in the x'- direction. Then, as measured by O in the S-frame, the velocity of the particle will be $c + v$, which clearly contradicts the postulate of special relativity, namely that the speed of light in free space is independent of the speed of its source.

Now, back to the problem at hand, we see that the observers O in the S-frame and O' in the s'- frame will measure the components of the velocity vector V of the particle as

$$V_x = \frac{dx}{dt}$$

,

$$V_y = \frac{dy}{dt}$$

,

$$V_z = \frac{dz}{dt}$$

and

$$V_x' = \frac{dx'}{dt'}$$

$$V_y' = \frac{dy'}{dt'}$$

$$V_z' = \frac{dz'}{dt'}$$

respectively.

To properly transform the velocity of the particle from one frame to another, we have to fall back on Lorentz coordinate transformation, namely

$$x' = \gamma(x - vt)$$

$$= \frac{(x - vt)}{\sqrt{1-\beta^2}}$$

$$y' = y$$

$$z' = z$$

and

$$t' = \gamma\left(t - \frac{\beta x}{c}\right) = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

where

$$\beta = \frac{v}{c}$$

and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Differentiating these equations and noting that v is constant, we obtain

$$dx' = \frac{dx - vdt}{\sqrt{1-\beta}}$$

$$dy' = dy$$

$$dz' = dx$$

and

$$dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1-\beta^2}}$$

3.3

$$V_x' = \frac{dx - vdt}{dt - \frac{vdx}{c^2}}$$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$= \frac{V_x - v}{1 - \frac{v}{c^2}V_x}$$

Thus,

$$V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}V_x}$$

3.4

$$V_y' = \frac{dy}{\frac{dt - \frac{vdx}{c^2}}{\sqrt{1-\beta^2}}}$$

$$= \frac{\frac{dy}{dt}}{\frac{1}{\sqrt{1-\beta^2}}\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{V_y\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}V_x}$$

Thus,

$$V_y' = \frac{V_y\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}V_x}$$

3.5

$$V_z' = \frac{dz}{\frac{dt - \frac{vdx}{c^2}}{\sqrt{1-\beta^2}}}$$

$$= \frac{\frac{dz}{dt}}{\frac{1}{\sqrt{1-\beta^2}}\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{V_z\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}V_x}$$

Thus,

$$V_z' = \frac{V_z\sqrt{1-\beta^2}}{1 - \frac{v}{c^2}V_x} \qquad 3.6$$

Equations 3.3 through 3.5 are the relativistic velocity transformation equations. The corresponding inverse velocity transformation equation are obtained if you simply replace v by $-v$ in the above set of equations, namely

$$V_x = \frac{V_x' + v}{1 + \frac{v}{c^2}V_x'} \qquad 3.7$$

$$V_y = \frac{V_y'\sqrt{1-\beta^2}}{1 + \frac{v}{c^2}V_x'} \qquad 3.8$$

$$V_z = \frac{V_z'\sqrt{1-\beta^2}}{1 + \frac{v}{c^2}V_x'} \qquad 3.9$$

If you now let your $V_x' = c$, i.e. if the particle emitted in the S'-frame travels at the speed of light c in the x-direction as measured by the observer O' in the S' frame, then the observer O in the S-frame will measure the velocity as

$$V_x = \frac{V_x' + v}{1 + \frac{v}{c^2}V_x'}$$

$$= \frac{c+v}{1+\frac{vc}{c^2}}$$

$$= \frac{c(c+v)}{c+v} = c$$

Therefore, both observers measure the same value of the speed of light in agreement with the postulate of the special relativity.

You must remember, in all of this discussion, that velocity addition formulae are applicable only when the two relative velocities are parallel to each other.

One final note; if v is very small in comparison with c, i.e. $v \ll c$, we obtain back our classical, Galilean transformation, namely

$$V_x' = V_x - c$$

$$V_y' = V_y$$

and

$$V_z' = V_z$$

The concept you have just studied will come in handy when we will be discussing the concept of relativistic Doppler Effect. So, be sure you are quite conversant with it. For now, we will make do with a few self assessment questions to test your understanding of all we have been discussing here.

SAQ 16

A spaceship moving away from the earth at speed of 0.80c fires a missile parallel to its direction of motion. The missile moves at a speed of 0.60c relative to the spaceship. What is the speed of the missile as measured by an observer on Earth?

SAQ 17

In an experiment, a radioactive nucleus moving at a velocity of 0.40c relative to the laboratory frame emits a β-particle with a velocity of 0.90c in the direction of and relative to the nucleus. What is the velocity of the β-particle as measured by an experiment in the laboratory?

SAQ 18

Two spaceships approach the Earth from opposite directions. According to an observer on the Earth, ship A is moving at a speed of 0.753c and ship B at a speed of 0.851c. What is the speed of ship A as observed from ship B?

When you have finished turn to the answer page and follow through the suggested solutions to effect corrections where you have gone wrong.

We want to discuss something you are going to find absolutely fascinating. It is called the *twin paradox* or the paradox of Langevin and is also the consequence of the transformations we have been discussing so far. So, move on to the next sub-section.

3.2.4 Twin Paradox

Consider two twin brothers, Taiwo and Kehinde, born at the same time. Later on in life when they are grown up, Kehinde decides to embark on a round trip to space, flying off at a constant relativistic velocity (a velocity close to that of light), leaving Taiwo behind on Earth. Now, based on their knowledge of relativity, Taiwo sees his brother's clock as running slowly and expects him to be younger when he returns. Of course, Kehinde sees his own clock which has remained stationary relative to him in his moving frame of reference as running normally.

On the other hand, Kehinde sees his brother Taiwo as moving relative to him in opposite direction with the same velocity, his clock also running slowly and therefore expects him to be younger on his return. Again, relative to Taiwo, his earth clock also runs normally.

Yet, when Kehinde finally arrives, both brothers find and agree that Kehinde is younger. The *paradox* is that each twin expects the other to be younger and this arises from the fact that motion is considered to be relative and not absolute.

So, how do we resolve the paradox? Well, the disagreement between the twins has to do with the fact that Kehinde considers his twin brother Taiwo to be in motion relative to him and by symmetry of the situation his brother should be younger. But, to achieve a constant velocity, Kehinde has to *start out* on his journey; he is momentarily accelerated and feels a sudden jerk backwards as he takes off. In his round trip, Kehinde also has to *return*; he is swerved to the sides of the ship as it makes a turn toward his brother or momentarily reverses its direction. Finally, Kehinde is decelerated to rest on his return to his brother and is jerked forward as he does so. All these effects of the stages of Kehinde's motion are not felt by Taiwo. While Taiwo has been observing his brother's motion from a single inertial frame of reference, Kehinde has been jumping from one inertial frame to another. The asymmetry in their ages arises from this fact. Even if the outward acceleration and final deceleration were seen to be symmetrical by the two observers, the acceleration or deceleration involve in returning is not. So, there is no doubt that it is Kehinde and not Taiwo that was in motion all along and so all observers agree to the fact that it is Kehinde's clock that ran slowly and so he is younger.

3.2.5 Relativistic Doppler Effect

At this point in your study of Physics, you are familiar with the concept of Doppler effect. You have experienced this phenomenon before. As a car or an airplane approaches you, the pitch (frequency) of its sound rises and as it recedes (moves away from you) the pitch of its sound falls. So we can say that, for mechanical waves (waves that require elastic material media for their propagation), *when the source of wave and an observer are in relative motion with respect to the material medium in which the wave propagates, the frequency of the waves observed is different from the frequency of the source*. You will recall that the relationship between the frequencies f observed by the observer and f' emitted by source is given by the Doppler equation as

$$f' = f\left(\frac{v - v_o}{v - v_s}\right)$$

or

$$f' = f\left(\frac{1 - v_o/v}{1 - v_s/v}\right) \qquad 3.10$$

where v is the velocity of the wave relative to the medium, v_s the velocity of the source relative to the medium and v_o the velocity of the observer relative to the medium. Note here that the source and the observer are assumed to move along the same straight line. In the more general case where the source and the observer move at an angle θ relative to each other, it could be shown that the frequencies are related by the equation

$$f' = f\left(\frac{1 - v_{os}/vcos\theta}{1 - v_s/v}\right) \qquad 3.11$$

where $v_{os} = v_o - v_s$ is relative the velocity between the observer and the source.

If $v_{os} < 0$ i.e. negative, then the observer is approaching the source and if $v_{os} > 0$, i.e., then the observer is receding from the source.

Armed with these reminders, you are ready to for the task at hand, namely Doppler effect in the case where the source or the observer is moving at relativistic speed. Doppler effect can be observed with all kinds of waves whether mechanical or electromagnetic. In classical Doppler effect, it makes the difference whether the source, the observer or neither were at rest relative to the material medium. In other words, the Doppler shift for the motion of the source is different from that of the motion of the observer. For instance, suppose a source moving at a constant velocity of $30\ m/s$ toward and observer at rest emits sound waves having the frequency of $f = 1000\ Hz$. The medium in this case is still air in which we assume the velocity of propagation of sound waves to be $340\ m/s$. The frequency f' of the sound detected by the observer will be $1097\ Hz$. If, on the other hand, the source is at rest and the observer is approaching it with a velocity of $30\ m/s$ the frequency f' of the sound detected by the observer is $1088\ Hz$. Still, if the source and the observer approach each other with a velocity of $15\ m/s$ each, i.e. their relative velocity of approach is $30\ m/s$, then the frequency f' of the sound detected by the observer is $912\ Hz$. It is clear that it is not the relative velocity of the observer and the source that determines the Doppler shift; rather it is the velocity of each with respect to the medium.

In the case of electromagnetic waves, the points you must note are as follows:

1. Electromagnetic waves (light in our discussion) do not consist of matter in motion, that is, they do not require any material media for their propagation. Therefore, the velocity of the source relative to the medium does not come into the discussion.

2. The velocity of propagation c is constant and independent of the motion of the source or the observers as stipulated by the postulate of special relativity.

3. Doppler effect for electromagnetic waves must be analyzed by means of the principle of relativity in which *only he relative motion between the source and the observer is involved*.

Now, let us consider a plane harmonic electromagnetic wave emitted in an inertial frame of reference S' which is in motion with a constant velocity v in the positive $x-$direction relative to another inertial reference frame S. An observer O' in S' frame will describe the wave by $sin(k'x' - \omega't')$ multiplied by suitable amplitude. Similarly, observer O in the S frame describes it by $sin(kx - \omega t)$. In compliance with the theory of relativity, the phase of the wave must of necessity be invariant. Thus, we write

$$kx - \omega t = k'x' - \omega't'$$

i.e.

$$kx - \omega t = k'\frac{(x - vt)}{\sqrt{1-\frac{v^2}{c^2}}} - \omega'\frac{\left(t - \frac{vx}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{k' + \omega v}{\sqrt{1-\frac{v^2}{c^2}}}x - \frac{k'v + \omega'}{\sqrt{1-\frac{v^2}{c^2}}}t$$

$$\therefore k = \frac{k' + \omega' v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

and $\omega = \frac{k'v+\omega'}{\sqrt{1-\frac{v^2}{c^2}}}$ 3.12

Now, $\omega = kc$.

Can you show this? Try it out. Just remember that $\omega = 2\pi f$, $c = f\lambda$ and $k = \frac{2\pi}{\lambda}$

Therefore, multiplying both sides of the first of equations 3.12 by c, we have

$$kc = \frac{k'c + \omega' v/c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\omega = \frac{\omega' + \ \omega' v/c}{\sqrt{1 - \frac{v^2}{c^2}}} = \omega' \frac{(1 + v/c)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \omega' = \frac{\omega\sqrt{1 - v^2/c^2}}{1 + v/c}$$

$$\text{or } f' = \frac{f\sqrt{1 - v^2/c^2}}{1 + v/c} \qquad 3.13$$

Equation 3.13 is the formula for the *relativistic Doppler shift* in the case in which the waves are observed in a direction parallel to v. In the general case in which the waves are observed at an angle θ to v, we can modify the formula to read

$$f' = \frac{f\sqrt{1 - v^2/c^2}}{1 + v\cos\theta/c} \qquad 3.14$$

Note:

1. The formula, unlike its classical counterpart, does not make any distinction between the source motion and the observer motion. *Doppler effect depends only on the relative motion between the source and the observer.*

2. In equations 3.13 and 3.14, it is assumed that the source and the observer are receding from each other. In this case $\theta = 180°$ and we can write the equation in the form

$$f' = \frac{f\sqrt{1 - v^2/c^2}}{1 + v/c} = f\sqrt{\frac{1 - v/c}{1 + v/c}} = f\sqrt{\frac{c - v}{c + v}}$$

3. If the source and the observer are approaching each other, then $\theta = 0°$ and we use the equation in the form

$$f' = \frac{f\sqrt{1 + v^2/c^2}}{1 - v/c} = f\sqrt{\frac{1 + v/c}{1 - v/c}} = f\sqrt{\frac{c + v}{c - v}}$$

4. If the wave is observed transverse to the direction of motion of the source, then $\theta = 90°$ and we use the equation in the form $f' = f\sqrt{1 - v^2/c^2}$

With these hints at the back of your mind, attempt the following SAQs

SAQ 1

One of the strongest emission lines observed from distant galaxies comes from hydrogen and has a wavelength of 122 nm (in the ultraviolet region).

a. How fast must a galaxy be moving away for us in order for the line to be observed in the visible region at 366 nm?

b. What would be the wavelength of the line if that galaxy were moving toward us at the same speed?

SAQ 2

A physics professor claims in court that the reason he went through the red light ($\lambda = 650\ nm$) was that, due to his motion, the red colour was Doppler shifted to green ($\lambda = 550\ nm$). How fast was he going?

SAQ 3

FRSC speed trap radar operating at a frequency of $2.0 \times 10^{20}\ Hz$ detects an oncoming car. The patrol officer observes a frequency shift of $5 \times 10^{-3} Hz$. What is the speed of the car?

We have come to the end of this unit. The TMAs are exactly of the kind you have just done in the SAQs. No tricks. Finish them off.

Summary

The summary of all we have learnt in this unit here.

The kinematic consequences of the theory of special relatively are:

- Length contraction: - To the observer in motion, objects appear to be shortened in the direction of motion. Objects appear shortened only if the direction of motion is parallel to its length. Objects appear to be longest when they are at rest relative to the observer.
- Time dilation: - A clock measures longer time interval of events in a frame of reference relative to which it is in motion than a clock in a frame of reference relative to which it at rest. The fastest rate (smallest time interval) is measured by the clock in the rest frame and is called the proper time.
- Velocity Addition: - Relativistic velocity addition relations are obtained by taking the differentials of the corresponding Lorentz coordinate transformation equations.

- Twin paradox is a consequence of time dilation or length contraction and is resolved by noting that that one of the twins that embarked on the space trip at relativistic velocity has to return to make comparison of clocks.
- Relativistic Doppler effect:- Doppler effect for electromagnetic waves must be analyzed by means of the principle of relativity in which only he relative motion between the source and the observer is involved.

Conclusion

Length contraction, time dilation velocity transformation and relativistic Doppler effect are the direct kinematical consequences and application of the Lorentz-FitzGerald coordinate transformation.

Tutor Marked Assignments

1. Calculate the percentage contraction of a rod moving with a velocity $0.8c$ in a direction inclined at $60°$ to its own length.

Ans. 9%

2. How fast would a rock ship have to go relative to an observer for its length to be contracted to 90% of its length when at rest?

Ans.$3.25 \times 10^5 km$

3. Determine the time (as measured by a clock at rest on the rocket) taken by a rocket to reach a distant star and return to the earth with constant velocity v equal to $\sqrt{0.9999}\,c$ if the distance of the star is 4 light years.

Ans. 0.08 year

4. How fast would a rocket ship have to go for each year on the ship to correspond to two years on the earth?

Ans.$2.6 \times 10^8\ m/s$

5. A 40-year-old astronaut marries a 20-year-old girl just before setting out on a space voyage. When he returns to earth, she is 35 and he is 42. How long was he gone according to earth clocks and what was his average speed during the trip?

6. According to an observer on the earth, a spaceship is going east with a speed $0.60\ c$ and is going to collide head on in $5\ s$ with a comet going west at $0.80\ c$ (a) How fast does the spacecraft see the

comet to be approaching? (b) According to their clocks, hoe much time do they have to get out of the way?

Ans.(a) $0.946\ c$ (b) $4.0\ s$

7. A distant galaxy is moving away from Earth at such high speed that the blue hydrogen line at a wavelength of $434\ nm$ is recorded at $600\ nm$ in the red range of the spectrum. What is the speed of the galaxy relative to the Earth?

Ans. $9.4 \times 10^7 m/s$

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MODULE 2: CONSEQUENCES OF THE TRANSFORMATIONS OF MOMNETUM AND ENERGY

UNIT 1: Relativity of Mass

UNIT2: Relativistic Energy

UNIT3: Experimental Verification of Special Relativity

UNIT 1: Relativity of Mass

CONTENTS

1.0 Introduction

Up to this moment, you have studied only the kinematic consequences of the relativistic motion. In mechanics, a complete discussion of motion must include the description of the forces which bring about the observed change of states of motion. Newton's laws together with the laws of conservation of momentum and energy provide the platform on which classical mechanics rests. You will remember, from your study of elementary mechanics, that at ordinary speeds, it is assumed that the mass of an object in motion does vary with its speed. To that extent, Newton's laws of motion are correct. In the realms of relativistic speeds, can we still depend on this assumption? In this unit, we look into this matter. You will discover that, in fact, at relativistic speeds the mass of an object in motion is a function of its speed.

1.1 Objective

At the end of this unit, you should be able to

- Derive the formula for the variation of mass with velocity
- Perform simple calculations involving the use of the formula for the variation of mass with velocity
- Sketch the curve of mass as a function of relativistic speed

- Discuss the concept of force and Newton's second law in relativistic mechanics.

1.3.0 Variation of Mass with Velocity

Our intention is to consider the effect of very high speeds on the mass and consequently momentum and force acting on an object. Talking about momentum, reminds you of Newton's second law of motion. This law explains the relationship between the two quantities. Do you remember the statement of this law? It says *the rate of change of momentum is directly proportional to the applied force.* In mathematical terms we write this as $\vec{F} = \frac{d(m\vec{v})}{dt}$. For constant mass m, we obtain $\vec{F} = m\vec{a}$ or $\vec{a} = \frac{\vec{F}}{m}$. But we know that $v = u + at$ so that as $t \to \infty$, $v \to \infty$. So, classically, if a constant force is applied on an object for a sufficiently long time, its velocity increases infinitely. This clearly contradicts the theory of special relativity as the velocity of an object is prevented from reaching or exceeding the velocity of light in relativistic mechanics. So at very high velocities, something is wrong with $\vec{F} = m\vec{a}$. How do get around this difficulty? How about making mass a function of velocity? That is $m = m(v)$, provided $m(v) \to 0$ as $v \to 0$. With this assumption, let us investigate one of the basic laws of Newtonian mechanics that we would like to preserve in the relativistic range, namely, the law of conservation of linear momentum.

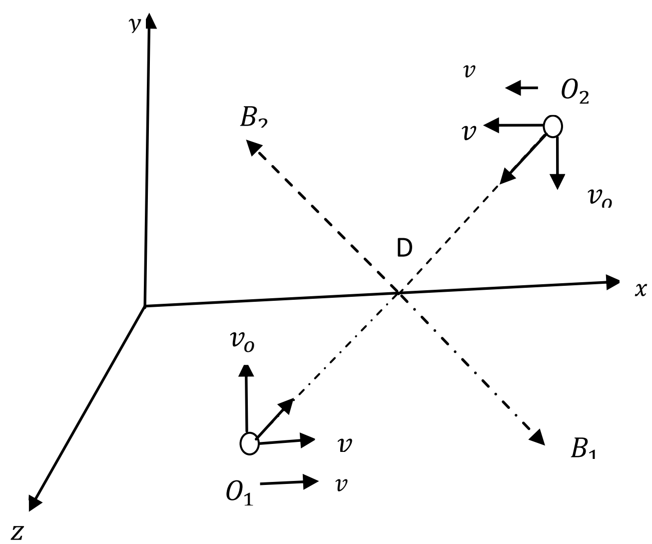

Figure 1.1 Glancing collision of B_1 and

Consider an elastic collision of two identical balls B_1 and B_2 each of mass m_o measured in a frame of reference in which they were at rest. Two observers O_1 and O_2 are in motion at a very high (relativistic) velocity v in opposite direction parallel to the x-axis. The situation is illustrated in figure 1.1. Observer O_1carries with him ball B_1while observer O_2carries with him ball B_2. While

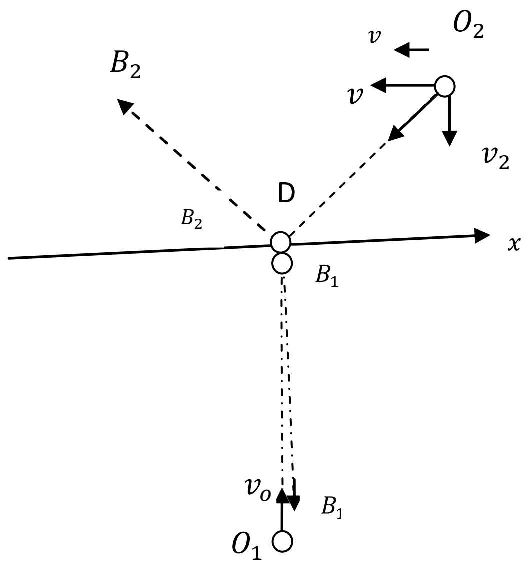

Figure 1.2: Glancing collision as seen by O_1

passing each other, each observer throws his ball *perpendicular to his direction of motion* with a velocity v_o (where $v_o \ll v$) such that a glancing collision of the balls takes place at point D, which is mid-way between the two observers and rebounds with the same velocity and at equal angles to the x-axis (not shown). As seen by each observer, the path of motion of his ball is strictly $\pm y$. Thus, as seen by an observer at rest at D in the *xyz* frame, the paths of the balls are as shown in the figure. For him,

$$\left[\sum p_{yi}\right]_{before\ collision} \Rightarrow m_o v_o + m_o(-v_o) = 0$$

$$\left[\sum p_{yi}\right]_{after\ collision} \Rightarrow m_o(-v_o) + m_o v_o = 0.$$

$$\therefore \left[\sum p_{yi}\right]_{before\ collision} = \left[\sum p_{yi}\right]_{after\ collision} = 0$$

In the x-direction, he sees the balls approaching each other with equal velocity v before collision and recede from each other after collision. Thus he writes

$$[\sum p_{xi}]_{initial} \Rightarrow m_o v + m_o(-v) = 0$$
$$[\sum p_{xi}]_{final} \Rightarrow m_o v + m_o(-v) = 0$$

$$\therefore [\sum p_{xi}]_{initial} = [\sum p_{xi}]_{final} = 0$$

Therefore for this observer, momentum is conserved.

Now, let us take a look at the collision from the view-point of observer O_1. He sees himself to be at rest in his frame of reference and observer O_2approaching with velocity v. Remember that v is very large. The collision as seen by observer O_1 is shown in figure 1.3. For him, his ball moves strictly in the $\pm y$ direction with velocity v_o so that $v_{1x} = 0$ and $v_{1y} = v_o$, v_{1x} and v_{1y} being the x and y components of the velocity v_1 of the ball B_1. On the other hand, he sees ball B_2 move with velocity v_2 whose x and y components are v_{2x} and v_{2y} respectively, with v_{2y} reversed after collision. He writes the law of conservation of the y- directed momentum as

$$\left[\sum p_{yi}\right]_{before\ collision} = \left[\sum p_{yi}\right]_{after\ collision}$$

$$\Rightarrow m_o v_o + m(-v_{2y}) = m_o(-v_o) + m v_{2y}$$

$$\Rightarrow 2m_o v_o = 2m v_{2y}$$

$$\Rightarrow m = \frac{m_o v_o}{v_{2y}}$$

Notice that although he started with equal masses m_o for both balls, he reasons that the mass of ball B_2 may have changed and writes m for it since it is moving with relativistic velocity v. He however retains m_o for the ball B_1 because it is moving slowly with velocity v_o so that its mass is essentially constant. Now, remember that observer O_2 threw his ball perpendicular to his direction of motion with velocity v_o as measured by him. But observer O_1 measures this as v_{2y} because O_2 is moving at relativistic speed relative to him. So, we will be justified to use the velocity transformation equation 3.8, namely, $V_y = \frac{V_y'\sqrt{1-\beta^2}}{1+\frac{v}{c^2}V_x'}$. In this case, $V_y = v_{2y}$, $V_y' = v_o$ and $V_x' = v_{1x} = 0$. Making these substitutions, we obtain

$$v_{2y} = \frac{v_o\sqrt{1-\beta^2}}{1+\frac{v}{c^2}v_{1x}} = v_o\sqrt{1-\beta^2}$$

i.e. $v_{2y} = v_o\sqrt{1-\beta^2}$

Substituting this for m, we have,

$$m = \frac{m_o}{\sqrt{1-\beta^2}}$$

or

$$m = \frac{m_o}{\sqrt{1-v^2/c^2}} \qquad 2.1$$

The mass m is called the *relativistic mass.* m_o is *called the rest mass*. Let us examine its physical significance of equation 2.1.

1. First of all we note that by the symmetry of the problem, observer O_2 see a similar situation and concludes that the mass of the ball B_1 is also given by equation 2.1.
2. Secondly, at very high (relativistic) velocity, an object's mass as seen by an observer increases with increase in velocity. As $v \to c, m \to \infty$. What this means is that the inertia of the object increases with increase in velocity at relativistic speeds. Because the velocity of

an object is prevented from reaching the velocity of light, the mass and therefore momentum of an object cannot increases infinitely. Also, as $v \rightarrow 0, m \rightarrow m_o$.
It is instructive to examine the variation of m/m_o with v/c graphically as illustrated in figure 1.3.

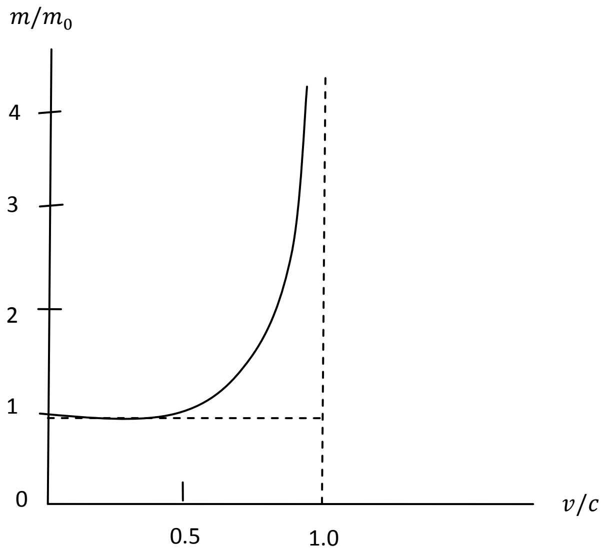

Figure 1.3: Variation of mass with velocity

As you can see from the graph, the mass is constant at $m_o$when v is small. For a velocity $v = 0.1c$, the relativistic mass is only 0.5% greater than the rest mass m_o. However, as v increases the rest mass increases rapidly. At a velocity of $0.9\ c$, the relativistic mass is over 100% greater than the rest mass. It is obvious that velocity of the object cannot be up to or exceed c.

1.3.1 Momentum and Force in Relativistic Mechanics

We may ask. What is the correct way of representing momentum and force in the relativistic range? Well, first of all you will have to note carefully that at very high velocities force could no longer be represented by Newton's second law as $F = ma$ since m is not constant but varies with velocity. We emphasize here that $F = ma$ *is not correct in the relativistic range*. The best we can do is to leave the relation in the form $\vec{F} = \frac{d(m\vec{v})}{dt}$. This implies $\vec{F} = m\frac{dv}{dt} + v\frac{dm}{dt}$ and $\frac{dm}{dt}$ does not vanish if the velocity of the body varies with time. The resultant force is always equal to the rate of change of momentum.

Then, provided m is given by our equation 2.1, momentum can be written as $\vec{p} = m\vec{v}$. Thus, for very high (relativistic velocity), momentum is correctly represented by

$$\vec{p} = \frac{m_o\vec{v}}{\sqrt{1-v^2/c^2}} \qquad 2.2$$

In terms of components, we can write equation 2.2 in the form

$$p_x = \frac{m_o v_x}{\sqrt{1-v^2/c^2}}\,,\; p_y = \frac{m_o v_y}{\sqrt{1-v^2/c^2}} \text{ and } p_z = \frac{m_o v_z}{\sqrt{1-v^2/c^2}} \qquad 2.3$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$

Here is another care you have to take. The velocity v that appears in the denominator of these expressions is always the velocity of the object as measured from an inertial frame of reference. It is not the velocity of an inertial frame itself. The velocity in the numerator can be any of the components of the velocity vector.

Newton's second law can then be written in the form

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\frac{m_o\vec{v}}{\sqrt{1-v^2/c^2}}\right] = \frac{d}{dt}(m\vec{v})$$

As you can easily verify from our analysis so far, if the net force acting on a system of particles is zero, momentum in relativistic range is conserved provided the mass of a particle is given by equation 2.2.

SAQ 1

How fast must a particle be accelerated by a linear accelerator for its mass to be 50% larger than its rest mass?

SAQ 2

Calculate the momentum of a proton moving at a speed of 0.90c.

Summary

We summarize all we have studied in this unit as follows:

- At ordinary speeds Newton's second law in the form $F = ma$ is correct but at very high (relativistic) speeds this form is not correct.
- The mass of a body moving at a speed v relative to an observer is larger than its mass when it is at rest relative to the observer (rest mass) by the factor $1/\sqrt{1-v^2/c^2}$
- The phenomenon of the increase in mass with speed is reciprocal. Two observers in inertial frames of reference in relative motion at constant velocity will observe the same effect.
- Relativistic mass increases are significant only at speeds close to the speed of light.
- At relativistic speeds Newton's second in the form $\vec{F} = \frac{d}{dt}(m\vec{v})$ provided $m = \frac{m_o}{\sqrt{1-v^2/c^2}}$
- The momentum of a particle in motion at relativistic velocity is given by $\vec{p} = \frac{m_o\vec{v}}{\sqrt{1-v^2/c^2}}$

- In the relativistic range, momentum is conserved provided the mass of the object is given by the relativistic mass.

Conclusion

We conclude this unit by reiterating that at speed close to the speed of light, the mass of an object in motion is a function of its velocity and Newtonian mechanics in the relativistic range is correct only if we take into account the phenomenon of the relativistic mass.

Tutor Marked Assignments

1. How fast must an electron move if it mass must be equal to the rest mass of an alpha particle?
2. A man whose mass at rest on the ground equals 100 kg is in flight in a rocket ship moving at a speed of $4.2 \times 10^7\ m/s$. Determine his mass while in flight as measured by an observer on earth.
3. What is the momentum of an electron moving at a speed of $0.87c$?

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UNIT2: Relativistic Energy

CONTENTS

2.0 Introduction

In the previous unit we looked into the definition of momentum and force in the realms of relativistic velocity. We saw there that the mass of an object in motion at relativistic velocity was a function of the velocity. We spelt out the condition under which the principle of conservation of linear momentum could be preserved in the relativistic range. You know that work is done on an object when it is displaced in the direction of the force acting on it. This work is retained in the system as the capacity to do an equal amount of work by the body and this is what you refer to as the energy of the system. We now proceed in this unit to examine the concept of energy and the principle of conservation of energy for objects that are in motion at relativistic speeds.

2.2 Objectives

At the end of this unit, you should be able to

- Derive the equation for the relativistic work and kinetic energy
- Discuss the mass-energy relationship
- Carry out calculations involving the relativistic mass-energy conversion
- Derive the relationship total energy, momentum and the rest energy of a particle
- Carry out momentum and energy transformation

2.3.0 Relativistic Work and Energy

You are familiar with the definition of energy as the capacity to work. Work is said to be done on an object when a forces displaces it in its (force's) direction. Consider an element of work dW done on a particle when a force $\vec{F}$ moves it through an element of displacement $d\vec{s}$. We can write this as

$$dW = \vec{F}.d\vec{s} = Fds$$

In the relativistic range, the force is given as

$$F = \frac{d}{dt}\left(\frac{m_o v}{\sqrt{1 - v^2/c^2}}\right)$$

Using your product rule you can write this as

$$F = \frac{m_o}{\sqrt{1 - v^2/c^2}}\frac{dv}{dt} + v\frac{d}{dt}\left(\frac{m_o}{\sqrt{1 - v^2/c^2}}\right)$$

$$= \frac{m_o}{\sqrt{1 - v^2/c^2}}\frac{dv}{dt} + m_o v\frac{v/c^2}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}$$

$$= \frac{m_o}{(1 - v^2/c^2)^{3/2}}\left[(1 - v^2/c^2) + \frac{v^2}{c^2}\right]\frac{dv}{dt}$$

$$= \frac{m_o}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}$$

Putting this result in the work equation, we obtain

$$W = \int_0^s Fds = \int_0^s \frac{m_o}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt}ds$$

Now, using the chain rule, you can write

$$\frac{dv}{dt}ds = \frac{dv}{ds}\frac{ds}{dt}ds = v\frac{dv}{ds}ds = vdv$$

$\therefore$

$$W = \int_0^v \frac{m_o v}{(1 - v^2/c^2)^{3/2}}dv$$

Integrating by parts, we have

$$W = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} - m_o c^2$$

⇒

$$W = mc^2 - m_o c^2 \tag{2.1}$$

2.1 is the equation of relativistic work. It tells us that we cannot accelerate an object to speeds up to or greater than the speed of light. If $v \to c$, then $m \to \infty$. The amount of work required to attain this status becomes infinite and impossible to furnish.

If, we ignore dissipative (frictional) forces and assume constant potential energy for the body, then the work-energy theorem (a form conservation of energy principle) tells us that the work done on the body appears as its kinetic energy. Thus, if we write T for the kinetic energy, we obtain

$$T = mc^2 - m_o c^2 \tag{2.2}$$

As always, you should demand that equation 2.2 be consistent with the definition of kinetic energy as we all know it at ordinary velocities. Thus, in the limit as $v/c \ll 1$, we can use the series expansion formula

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Here $x = v^2/c^2$. Our relativistic mass formula could then be approximated as

$$m = \frac{m_o}{\sqrt{\sqrt{1 - v^2/c^2}}} \cong m_o\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right] = m_o + \frac{1}{2}m_o\left(\frac{v}{c}\right)^2$$

Putting this approximate value in equation 2.2, we obtain

$$T = \left[m_o + \frac{1}{2}m_o\left(\frac{v}{c}\right)^2\right]c^2 - m_o c^2 = m_o c^2\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1\right] = \frac{1}{2}m_o v^2$$

for

$v/c \ll 1$

Therefore, equation 2.2 correctly represents kinetic energy at all velocities.

2.3.1 Mass-Energy Equivalence

If we write equation 2.2 out in the form:

$$T = mc^2 - m_o c^2 = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} - m_o c^2$$

we easily observe that kinetic energy T is a function of velocity v. Also, since T is energy, the other two terms on the right hand side of the equation are also energy terms and we can write

$T(v) = E(v) - E(0)$,

or $E(v) = T(v) + E(0)$ 2.3

where $E(v) = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} = mc^2$ and $E(0) = m_o c^2$ which the value of $E(v)$ for $v = 0$.

$E(v)$ is the total energy of the particle moving at a relativistic speed v, $E(0)$ is the energy of the particle when it is at rest i.e. $v = 0$. Further more if we consider each term individually, we write

$E(0) = m_o c^2$ 2.4

$E(v) = mc^2$ 2.5

These are the famous Einstein's mass-energy equations. It is clear from these equations that mass and energy are related by the factor c^2 and are equivalent. It implies that *the principle of conservation of mass or energy no longer makes sense as the conservation of mass-energy does*.

Better still, if we subtract 2.4 from 2.5, we obtain

$E(v) - E(0) = mc^2 - m_o c^2$

or

$\Delta E = \Delta mc^2$

2.6

where $\Delta m = m - m_o$

Equation 2.6 states that *a change in energy leads to a corresponding change in mass*. More generally, it says that when mass is destroyed, it appears as energy and also if energy disappears, it emerges as mass.

It is an irony of intellectual feat that Einstein, a pacifist, was the one who wrote the equation that was later to give an insight into the design of atomic and nuclear bombs, which are undoubtedly the world's most potent weapons of mass destruction. What equation 2.6 suggests is that if a heavy particle splits into smaller parts, it will release a tremendous amount of energy. This is the basis of nuclear fission as a source of energy. On the other hand, when lighter particles fuse together to form a heavier one, energy is also released. This is the basis of fusion reaction that is responsible for the tremendous energy radiated by the sun.

If the particle has potential energy V, we could write a more general equation of the form

$mc^2 = T + V + m_o c^2$ 2.7

Equation 2.7 emphasizes the fact that the relativistic mass is a direct measure of the total energy of the particle.

Another important information we could obtain from equation 2.2 is that momentum is conserved. As such it is useful to express the total energy in terms of momentum rather than energy. Recall that

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}}$$

Squaring both sides of the equation, we have

$$m^2 = \frac{{m_o}^2}{(1 - v^2/c^2)}$$

$$m^2(1 - v^2/c^2) = {m_o}^2$$

Multiplying both sides by c^2, we obtain

$$m^2c^4 - m^2v^2c^2 = {m_o}^2c^4$$

Now, we use that fact that $E = mc^2$, $E_o = m_oc^2$ and $p = mv$ write this equation as

$$E^2 = (pc)^2 + E_o^2 \qquad 2.8$$

Equation 2.8 is the relationship between the momentum and energy of a particle. It explains why, in relativistic theory, we must replace the conservation of total energy. We can then state that, *as viewed from a specified frame of reference, the total relativistic energy of an isolated system remains constant.*

2.3.2 Transformation of Momentum and Energy

We want to conclude this unit with a look at the concepts of momentum and relativistic energy as described by two observers in relative motion at a relativistic velocity. As you may have observed, what we may be describing as the rest mass energy of a particle at rest in one frame of reference may turn out to be the energy due to the motion relative to an observer in another frame of reference. It is reasonable that we work out a way of transforming these quantities from one inertial frame to another.

As before, consider two observers O and O'in inertial frames of reference S and S' which are in relative motion along the x-axis at a relativistic velocity v. For observer O, the components of the momentum and the relativistic energy of a particle of rest m_0 with velocity V along the positive x-axis are

$$p_x = \frac{m_0 V}{\sqrt{1 - V^2/c^2}}$$

$p_y = 0$
$p_z = 0$
and

$$E = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}}$$

Observer O' assigns to this particle the components of the momentum and relativistic energy as

$$p'_x = \frac{m_0 V'}{\sqrt{1 - V'^2/c^2}}$$

$p'_y = 0$

$p'_z = 0$

and

$$E' = \frac{m_0 c^2}{\sqrt{1 - V'^2/c^2}}$$

where V' is the velocity of the particle along the positive x-axis as measured by this observer. Notice that O' assigns to the particle the same rest mass, m_0. Why?

We have to find the primed quantities in terms of the unprimed ones. We have to first of all transform the velocity terms. That is, we must evaluate the quantities

$V'/\sqrt{1 - V'^2/c^2}$ and $c^2/\sqrt{1 - V'^2/c^2}$

in terms of V using the velocity transformation equation,

i.e.

$$V' = \frac{V - v}{1 - \frac{v}{c^2} V}$$

You can begin by squaring both sides and then divide the result by c^2 to obtain

$$V'^2 = \frac{V^2 - 2V + v^2}{1 - \frac{2Vv}{c^2} + \frac{V^2v^2}{c^4}}$$

$$\frac{V'^2}{c^2} = \frac{\frac{V^2}{c^2} - \frac{2Vv}{c^2} + \frac{v^2}{c^2}}{1 - \frac{2Vv}{c^2} + \frac{V^2v^2}{c^4}}$$

$$1 - \frac{V'^2}{c^2} = 1 - \frac{\frac{V^2}{c^2} - \frac{2Vv}{c^2} + \frac{v^2}{c^2}}{1 - \frac{2Vv}{c^2} + \frac{V^2v^2}{c^4}}$$

⇒

$$1 - \frac{V'^2}{c^2} = \frac{1 - \frac{2v^2}{c^2} + \frac{V^2v^2}{c^4}}{1 - \frac{2Vv}{c^2} + \frac{V^2v^2}{c^4}}$$

Now, factorizing the numerator and the denominator of the right hand side

⇒

$$1 - \frac{V'^2}{c^2} = \frac{(1 - v^2/c^2)(1 - V^2/c^2)}{(1 - Vv/c^2)^2}$$

We can now take the reciprocal and then the square root of both sides.

$$\frac{1}{\sqrt{1 - V'^2/c^2}} = \frac{1 - Vv/c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - V^{2/c^2}}}$$

2.9

We only need to multiply both sides of 2.9 by m_0c^2 in order to obtain our relativistic energy equation as assigned by observer O'. Doing just that, we obtain

$$\frac{m_0c^2}{\sqrt{1 - V'^2/c^2}} = \frac{m_oc^2 - m_oVv}{\sqrt{1 - v^2/c^2}\sqrt{1 - V^2/c^2}}$$

⇒

$$\frac{m_0c^2}{\sqrt{1 - V'^2/c^2}} = \frac{m_oc^2/\sqrt{1 - v^2/c^2} - \left(m_{ov}/\sqrt{1 - v^2/c^2}\right)V}{\sqrt{1 - V^2/c^2}}$$

∴

$$E' = \frac{E - vp_x}{\sqrt{1 - v^2/c^2}}$$

2.10

Also, the momentum assigned by O' is

$$p'_x = \frac{m_0 V'}{\sqrt{1 - V'^2/c^2}} = \frac{m_0 c^2 V'}{c^2\sqrt{1 - V'^2/c^2}} = \frac{E'V'}{c^2}$$

$$= \frac{\dfrac{m_o c^2/\sqrt{1 - v^2/c^2} - \left(m_{ov}/\sqrt{1 - v^2/c^2}\right)V}{\sqrt{1 - V^2/c^2}} . \dfrac{V - v}{1 - vV/c^2}}{c^2}$$

⇒

$$p'_x = \frac{m_o(c^2 - vV)}{c^2\sqrt{1 - \frac{V^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}} . \frac{V - v}{c^2 - vV} c^2$$

$$= \frac{m_o V - m_o v}{\sqrt{1 - \frac{V^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{p_x - vE/c^2}{\sqrt{1 - v^2/c^2}}$$

∴

$$p'_x = \frac{p_x - vE/c^2}{\sqrt{1 - v^2/c^2}}$$

2.11

Remember that we have agreed, while studying the glancing collision, that momentum in the directions which are perpendicular to the direction of motion is not changed. With this in mind, we collect our transformation equations for the relativistic energy and the components of the momentum together to obtain the following set:

$$p'_x = \frac{p_x - vE/c^2}{\sqrt{1 - v^2/c^2}}$$

$$p'_y = p_y$$

2.12

$$p'_z = p_z$$

$$E' = \frac{E - vp_x}{\sqrt{1 - v^2/c^2}}$$

If we examine the equations 2.12, we recognize that the relativistic energy and the momentum equations we have obtained are of the same form as the Lorentz transformation equations 2.6. Thus, we conclude that momentum and energy transform exactly as the space-time quantities x, y, z and t.

We have come to the end of this unit. You can now handle work out the following SAQs which are the direct application of the relations we have established in this unit.

SAQ 1

How much energy in joule and electron volt is required to give an electron a speed of 0.9 c starting from rest?

SAQ 2 What is the change in mass of copper when the mass of 1 g from 0 to 100$°C$? The specific heat of copper is $0.40J/gK$.

SAQ 3

Find the kinetic energy of an electron moving at (a) $v = 1.00 \times 10^{-4}c$ (b) $v = 0.999c$

SAQ 4

Calculate the momentum of the a proton whose kinetic energy is $200\ Mev$

SAQ 5

Calculate the amount of energy required to accelerate an electron to a speed of 0.9c, starting from rest.

Summary

We summarize this unit as follows:

- As observed from a given reference frame, the total relativistic energy of an isolated system remains constant.
- A change in energy leads to a corresponding change in mass.
- Total relativistic energy is the sum of the kinetic and the rest energy
- Momentum and energy transform exactly as the space-time quantities x, y, z and t.

Conclusion

We can conclude here that mass and energy are equivalent and that the a change one of these leads to a change in the other.

Tutor Marked Assignments

1. Calculate the mass and speed of an electron which has kinetic energy of $1 \times 10^5 eV\ (1.6 \times 10^{14} J)$
   Ans. $1.09 \times 10^{-30} kg$, $1.64 \times 10^8 m/s$
2. What is the minimum energy required to accelerate a rocket ship to a speed of $0.8c$ if its final payload rest mass is $5\ 000\ kg$?
   Ans.$3 \times 10^{20} J$
3. How much mass does an electron gain when it is accelerated to a kinetic energy of $500\ MeV$?
   Ans. $8.9 \times 10^{-28} kg$.
4. A particle with rest mass m_o and kinetic energy $3m_o c^2$ makes a completely inelastic collision with a stationary particle of rest mass $2m_o$. What are the velocity and the rest mass of the composite particle? Ans.$0.645c$; $4.58m_o$.

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UNIT3: Experimental Verification of Special Relativity

CONTENTS

3.0 Introduction

As you have seen from the discussion so far, the theory special relativity is a radical departure from classical physics. Most, if not all, of the conclusions were arrived at through thought experiments. A number of new phenomena such as time dilation, length contraction, relativistic mass increase and mass-energy equivalence were also predicted. Besides, the scientific community of the era of the advent of special relativity was hesitant and accepted the new theory rather slowly and with reservation. This is evidenced by the fact that Einstein, who together with Lorentz was jointly proposed for the 1912 Nobel Prize, was never awarded Nobel Prize for his work on relativity. How reliable is a theory with such an unenviable background? For any scientific theory to be taken seriously, empirical evidence must back it up. The Nobel Committee was cautious and waited for experimental confirmation of the theory.

In this unit, we collect together experimental evidence and tests of the special theory of relativity, some of which have been mentioned in passing in previous units. These evidence and tests are necessary, not only to give a measure of confidence in the correctness of the theory but also to point out the points of departure from classical physics.

3.2 Objectives

At the end of this unit you should be able to.

- discuss experimental tests of special relativity
- solve simple conceptual problems arising from experimental tests of special relativity

3.3.1 Experimental Evidence of the Universality of the Velocity of Light

The main ingredient of the special relativity is, undoubtedly, the one which has to do with the speed of light. Let us revisit it here. It states that *the velocity of light in free space has the same value in all inertial frames of reference and is independent of the motion of the source.* This statement calls for two types of investigation namely:

(1) is the velocity of light isotropic or does it change with direction of motion?

(2) Does the velocity of light change with the relative motion of the source and the observer?

As we saw in section 1.2.5 of unit 1, Michelson - Morley experiment provides the answer to the first question. The experiment compared the upstream-downstream with the cross-stream velocity of light. Within the limit of experimental error, it was shown that the velocity in both directions is the same. Therefore, the velocity of light is isotropic. This experiment also established the fact that there is no preferred frame of reference, the so-called the ether frame.

Despite repetition of the experiment at different times and season, the result was always the same. In 1932, Kennedy and Thorndike repeated the experiment using an interferometer with different length of arms to test the hypothesis of Lorentz contraction of the arm of the interferometer parallel to the direction of motion. This also resulted in the same conclusion that that velocity in the two orthogonal directions was the same.

In recent times, the experiment has been repeated with increasing precision using laser as the source. Yet the results to the best of experimental errors remain the same.

The second question 'does the velocity of light change with the relative motion of the source and the observer?' can be tested by measuring the velocity of light emitted in the same direction as that of a moving source. If an observer at rest in the frame of the moving source assigns c as the velocity of light, then an observer relative to whom the source is moving with a velocity v measures the velocity of light as $c' = v + c$. If v is close to the speed of light, then c' is certainly greater than c. This is an obvious violation of special relativity. So, what special relativity is telling us here is that $c' = c$ i.e. $v = 0$. We know that $v \neq 0$. So, the value obtained for c' is the consequence of relativistic velocity transformation which we have studied earlier. The assertion $c' = c$ has been tested and found to agree with experiment. For example, the velocity of gamma radiation (another electromagnetic radiation propagating at the speed of light c) emitted in the decay of π-meson moving at a velocity in excess of 0.99975*c* was measured in 1964 by Farley et al to be $(2.9979 \pm 0.0003) \times 10^8$m/s. Recently (in October 2011), a group of scientist working at a linear accelerator in Stanford University, England, claimed that they discovered neutrino which travels at speed in excess of *c*. This claim is still being investigated by scientists elsewhere to confirm its veracity or otherwise. At the moment, it accepted that no material object travels at a speed in excess of the speed of light c, which is the limiting speed.

Another piece of evidence comes from the observation of binary stars. A binary star is a pair of stars which are rotating about their common centre of mass. At a given moment, one of the stars is moving toward the earth while the other is receding from it. In that case, the velocity of light with respect to the earth from one star would be larger than that of light from the other star. One experiment that could be performed based on this view is the study of X-rays emitted by a binary pulsar. A binary pulsar is a pair of binary stars in which of them acts as a rapidly pulsating source of X-rays as it revolves about the other star. The pulsar will then be eclipsed from view on the earth when it is in a straight line and above the other star. If the velocity of light (X-rays) were to change as the pulsar moves toward and later away from the earth on its orbit, then the beginning and the end of the eclipse would not be equally spaced in time from the mid-point of the eclipse. The result of this experiment was reported in 1977 by K. Brecher and no such effect was observed. From experiments like these, it is concluded that the velocity of light is independent of the velocity of its source in agreement with special relativity.

3.3.2 Experimental Evidence of Time Dilation and Length Contraction

Time dilation as well as length contraction is real and can be confirmed experimentally. The production of mu meson (muons) in the upper atmosphere by cosmic rays has already been discussed. For emphasis, we recapitulate the conclusion of the discussion here. Muons are unstable elementary particles with very short life span (about $2 \times 10^6 s$ on the average). So, they decay shortly after they come into existence. With typical speeds of about $2.994 \times 10^8 ms^{-1}$, muons which are created in the upper atmosphere at an altitude of about $6\ 000\ km$ by incoming cosmic rays reach the earth at sea level in such profuse supply that one wonders how they manage to achieve this feat. In a time of $2 \times 10^6 s$, a muon travelling at a speed of $2.994 \times 10^8 ms^{-1}$ can only cover $600\ m$ of the altitude. You will recall that we had to rely on the concepts of time dilation and length contraction to explain this.

Production of muons in the laboratory can be achieved through in high-energy accelerators. Their decay is observed by tracking the reaction products which are electrons. The muons can then be trapped and their decay is studied at rest or placed in the beam and their decay studied in motion. When muons are studied at rest their lifetime is 2.198×10^{-6}s. This is the proper time. In a particular experiment published by J. Bailey et al in 1977, muons were trapped in a ring and circulated at very speed and momentum of about $p = 3094\ MeV/c$ and the lifetime came to about $6.438 \times 10^{-5} s$. This confirms that time is dilated while muons were in flight.

Another striking experimental evidence of the time dilation is supplied by ‘electrostatic focusing’. In high energy particle accelerators, it is observed that a stationary cluster of electrons or protons expanding under electrostatic repulsion do so at a certain characteristic rate. When they are in fast moving beams, they expand at a much slower rate.

3.3.3 Experimental Evidence of the Twin Paradox

Direct experiment to test this phenomenon cannot be carried out but an equivalent experiment can be performed. Two atomic clocks are carefully synchronized. Then one of them is flown around the earth in an airplane. On return and comparison with the earth clock, it is found the clock that was in flight is a few seconds behind. Although due the rotation of the earth, the earth clock cannot be said to be in a inertial frame of reference, the observation is in agreement with the prediction of special relativity.

Similar experiments involve cesium atomic clocks, one of which is placed in a space shuttle. Again results show that the clock in flight ran slower on comparison with the earth clock.

3.3.4 Experimental Evidence of the Relativistic Doppler Effect

The experiment of Ives and Stilwel performed in 1938 provides evidence of the agreement with theoretical prediction.

A beam of hydrogen atom, generated in a gas discharge was made to travel down the tube at a relativistic speed. Light emitted by two atoms and in the direction parallel and opposite to the relativistic velocity was simultaneously observed. A spectrograph was used to photograph the characteristic spectral lines from the two atoms and also on the same photographic plate, from atoms at rest. If the classical Doppler formula were valid, the wavelengths of the lines from the two atoms 1 and 2would be place at symmetric intervals $\Delta\lambda_1 = \pm\lambda_0(u/c)^2$ on either side of the line from the atoms at rest (wavelength λ_0). The relativistic formula gives a small additional symmetric shift $\Delta\lambda_2 = +\frac{1}{2}\lambda_0(u/c)^2$.

3.3.5 Experimental Evidence of the Relativistic momentum and Energy

Direct evidence of relativistic momentum and energy comes in handy from experiments involving high-energy accelerators in the study of high-energy and particle physics. The results are in excellent agreement with theoretical predictions. We have discussed this earlier on in section 1.3.0.

Perhaps you have read about the atomic bomb dropped in Japan in 1945 during World War II. This came as a result of the realization of the mass-energy equivalence we have discussed earlier.

Summary

Experiments discussed in this unit are not exhaustive. But their results are in agreement with theoretical predictions.

Conclusion

The phenomena of special relativity are so alien to common experience that there is the need to provide concrete experimental evidence of them. Within the limits of experimental errors, it is found that the practical observations are in excellent agreement with theoretical predictions.

Tutor Marked Assignments

Discuss any two experiments to show evidence of special relativity.

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MODULE 3: ELECTROMAGNETIC FOUR-VECTOR

UNIT 1: Four-Vector

UNIT 2: Magnetism as a Relativistic Phenomenon

UNIT3: Transformation of the Electric and Magnetic Fields

UNIT 1: Four-Vector

CONTENTS

Introduction

From our discussion so far in special relativity, you have seen that *there is no absolute or preferred frame of reference*. All inertial frames are equally valid and the laws of physics retain the same form in all of them. How can we test and ascertain that this is true? We have to have a way of finding out whether or not a physical law satisfies this requirement. One of the ways of doing this is somewhat familiar to you from your study of vectors. This is called *orthogonal transformation*. In this unit, we will briefly recapitulate this procedure and carry it a bit forward to learn something new.

What we will do is simply to consider two fixed coordinate systems (reference frames if you like) which have the same origin but whose axes point in different directions (say by rotating one with respect to the other). A vector (say position vector of the a point in space) is then examined to see whether or not it transforms properly, that is, whether or not it retains the same form in the two coordinate systems under the operation of rotation. A transformation which ensures invariance of such a vector is called an orthogonal transformation. Then we will proceed to show that Lorentz transformation (which is a linear transformation) is an orthogonal transformation using a similar argument.

It will be worthwhile for you to bear in mind that while in an Euclidean space a three – dimensional vector, such the position vector describes a position of a point in space; an event requires a four-dimensional vector or *four-vector* in space-time (Minkowski space) to completely specify it. Our objective then will be to apply an analogous argument of the Euclidean geometry in three dimensions to obtain similar results for the more general space-time in four dimensions.

1.2 Objectives

At the end of the discussion in this unit you should be able to

- carry out simple orthogonal transformation of the coordinates of a position vector, force, momentum etc.
- demonstrate that Lorentz transformation is orthogonal
- perform simple four-vector algebra
- Discuss four-velocity, four-force, four-momentum etc

1.3.0 Orthogonal Transformation

The statement that the laws of physics have the same form in all inertial frames of reference is an expression of symmetry which is a characteristic property of nature. But what is symmetry? Or rather, when do we say that something is symmetrical? Well, we can illustrate the concept with something familiar to all of us. Suppose you rotate a playing card through angle of 180^o i.e. turn it upside down, and after this operation the picture in the card looks the same as it was before it was rotated, then it is symmetrical. Thus, an entity is said to be symmetrical if after an operation on it, it remains the same as it was before the operation. A combination of transformation operations which leaves an entity the same as it was before the operations is called a group. An example of a group is rotation followed by reflection about an axis. Other examples include the Lorentz group which is the translational motion at constant velocity followed by rotation and the Poincaré group.

A transformation which leaves a vector invariant after the rotation of the axes of the coordinate system is said to be orthogonal. Now, let us see how the vector quantities transform when a coordinate system changes. Our specific objective here is to find a specific geometrical transformation law for converting the components of a vector from one frame to another.

Consider two frames of reference (coordinate systems) S and S' with a common origin O show in figure 1.1.

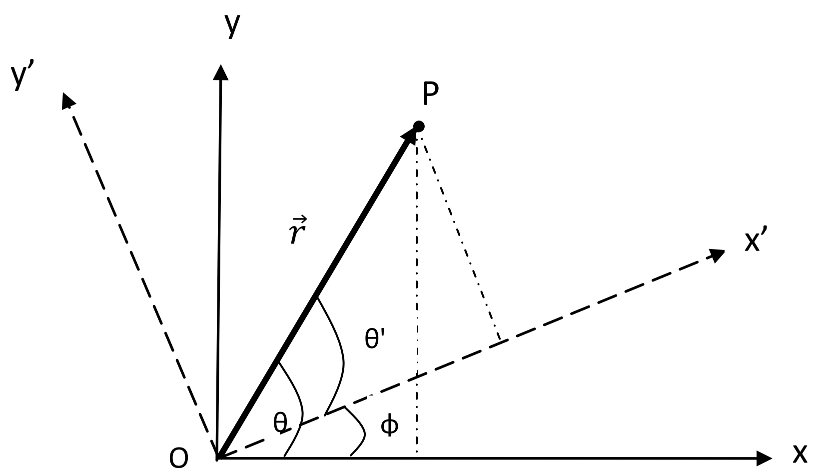

Fig. 1.1: Axes of the frame of reference S are rotated through angle ϕ with respect to those of the S' frame

S' is rotated through an angle ϕ in the counter clockwise direction. Let θ and θ' be the angles which the position vector $\vec{r}$ of a point P in a three-dimensional Euclidean space makes with the x-axis and the x'-axis respectively. Note that the common z-axis (not shown) is perpendicular to the page and points toward you. From the figure we obtain

$r_x = rcos\theta, \quad r_y = rsin\theta$

Also, $r_x' = rcos\theta' = rcos(\theta - \emptyset) = r(cos\theta cos\emptyset + sin\theta sin\emptyset) = r_x cos\emptyset + r_y sin\emptyset$

Similarly from the diagram, we have
$r_y' = rsin\theta' = rsin(\theta - \emptyset) = r(sin\theta cos\emptyset - cos\theta sin\emptyset) = -r_x sin\emptyset + r_y cos\emptyset$

Let us summarize these results.
$r_x' = r_x cos\emptyset + r_y sin\emptyset$

$r_y' = -r_x sin\emptyset + r_y cos\emptyset$ 1.1

Equations 1 are the transformation equations of the components of $\vec{r}$.

Notice that the position P and its position vector $\vec{r}$ have not changed, but the coordinates of P and the components of $\vec{r}$ have. Thus, the transformation of the coordinates of P could also be written as
$x' = xcos\emptyset + ysin\emptyset$
$y' = -xsin\emptyset + ycos\emptyset$
$z' = z$ 1.2

Furthermore, it can easily be seen that the new coordinates (x, y, z) are linear combinations of the old ones (x', y', z'). A coordinate transformation is said to be *linear* if the coordinates are expressed as a linear combination of the old ones.

In matrix notation, our linear transformation of the coordinates of P can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\emptyset & sin\emptyset \\ -sin\emptyset & cos\emptyset \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad 1.3$$

Generally, for a linear transformation of the coordinates due to rotation about an arbitrary axis in three dimensions, we can write

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \qquad 1.4$$

And in matrix notation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad 1.5$$

More compactly, we can write

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j$$

The index 1 stands for x, 2 for y and 3 for z.

$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the transformation matrix.

Before we proceed further, we want to show that our coordinate transformation due to rotation from S'- frame relative to S-frame is indeed orthogonal. This will be true if the length of the position vector of point P, which is $|\overrightarrow{r}|$, is invariant under the coordinate transformation. Now, $|\overrightarrow{r}|$ in the S frame is given as

$$|\overrightarrow{r}|^2 = x^2 + y^2 + z^2$$

$$|\overrightarrow{r'}|^2 = x'^2 + y'^2 + z'^2$$

$$= (xcos\emptyset + ysin\emptyset)^2 + (-xsin\emptyset + ycos\emptyset)^2 + z^2$$

$$= x^2cos^2\emptyset + 2xysin\emptyset cos\emptyset + y^2sin^2\emptyset + x^2sin^2\emptyset - 2xysin\emptyset cos\emptyset + y^2cos^2\emptyset + z^2$$

$$= x^2(sin^2\emptyset + cos^2\emptyset) + y^2(sin^2\emptyset + cos^2\emptyset + 2xysin\emptyset cos\emptyset - 2yxsin\emptyset cos\emptyset) + z^2$$

$$= x^2 + y^2 + z^2$$

Thus, $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$

Therefore, the coordinate transformation is orthogonal under rotation.

We can write this conclusion in a more compact form as follows:

$$\sum_{i=1}^{3}(x_i')^2 = \sum_{k=1}^{3}(x_k)^2 \qquad 1.6$$

where the indexes $i, k = 1,2,3$ denotex, y z respectively.

Now, if you note that $|\overrightarrow{r}|^2$ is a scalar product, namely, $\vec{r}.\vec{r} = |\overrightarrow{r}|^2 = x^2 + y^2 + z^2$, then you will realize that the invariance of the coordinate transformation under rotation as summarized in equation 6 could be written in a more general and compact form as follows:

$$\sum_j \sum_k \sum_l a_{ij}a_{ik}x_jx_k = \sum_k (x_k)^2$$

subject to the condition that

$$\sum_i a_{ij}a_{ik} = \begin{cases} 0 \; if \; j \neq k \\ 1 \; if \; j = k \end{cases}$$

In terms of Dirac delta notation, we write this as

$$\sum_i a_{ij}a_{ik} = \delta_{jk} \qquad 1.\,7$$

where $\delta_{jk} = \begin{cases} 0 \; if \; j \neq k \\ 1 \; if \; j = k \end{cases}$

Therefore, if

$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the matrix of the coordinate transformation under rotation, then for the transformation to be orthogonal, the elements of a_{ij} must satisfy the condition given by equation 1.7.

Three important points are worth noting in this analysis:

1. The components of other vectors transform in exactly the same way as the displacement vector when axes of frames of reference are rotated with respect to each other.

For instance, if p_x, p_y and p_z are components of the momentum vector $\vec{p}$ in the S-frame, then its components in the S'-frame are

$$p_x' = p_x cos\emptyset + p_y sin\emptyset$$

$$p_y' = -p_x sin\emptyset + p_y cos\emptyset$$

$$p_z' = p_z$$

If the S'-frame is rotated through an angle $\emptyset$ with respect the S-frame.

2. The new coordinates are mixtures of the old ones. In other words, the primed quantities are mixtures or combinations of the unprimed ones. The length of an object in the primed coordinate system is a combination of length and width in the unprimed system. As an example of this, if we set $\emptyset = 90°$ in equation 2, then $x' = y$ and $y' = -x$. This is a result of the fact that we are able to move around and look objects at different angles.

3. Since the transformation of the kind we are dealing with is linear, the sum of any two vectors will transform in the same way when the axes of the reference frames are rotated relative to each other. For example, the Lorentz force given as $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ transforms in the same way as $\vec{r}$. If the equation is true for a set of axes, it is also true for axes at any other orientation.

1.3.1 Lorentz Transformation as Orthogonal Transformation

We now proceed to apply the analysis of orthogonal transformation of a vector in a three dimensional Euclidean space to a four dimensional space-time, the so-called the Minkowski space. We have seen that in relativity, space and time are intricately intertwined. Besides, Lorentz transformation is a linear combination of space and time coordinates. An *event* in space-time is therefore completely specified by a vector in a four-dimensional space with coordinates (x, y, z, ict). This type of vector is called a *four-vector.* A point in the space-time is referred to as a *world point*. The distance from the origin to the world point is called an *interval.*

In the previous section, we showed that the scalar product of the three-dimensional vector (displacement vector) $\vec{r}$ is invariant under the rotational transformation. That is, $\vec{r}.\vec{r} = |\vec{r}|^2 = x^2 + y^2 + z^2$ retains the same form for all orientation of the axes. Similarly, the scalar product or the square of the length of the four-vector s^2 is invariant under Lorentz transformation. That is $s^2 = x^2 + y^2 + z^2 - c^2t^2$ is Lorentz invariant. Note here that s^2 will not always be positive compared with to the corresponding r^2 in three dimensions because of the term $-c^2t^2$. So, in order that s^2 might be analogous to r^2 in a three-dimensional vector, we have to replace t by ict in our four-vector analysis, where ict must have the dimensions of length.

SAQ 1

Show that under Lorentz coordinates transformation, the expression $x^2 + y^2 + z^2 - c^2t^2$ is invariant.

Now, let us formally show that the Lorentz transformation is an orthogonal transformation. We begin by writing down the coordinates as seen by an observer in the moving frame S', namely

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Replacing t by ict, we obtain

$$x' = \frac{x + \frac{iv}{c}(ict)}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}} = x\left(\frac{1}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}}\right) + ict\frac{\left(\frac{iv}{c}\right)}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}} = xcos\alpha + (ict)sin\alpha$$

$$y' = y$$

$$z' = z$$

$$ict' = \frac{ict - \left(\frac{iv}{c}\right)x}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}} = ict\left(\frac{1}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}}\right) - x\frac{\left(\frac{iv}{c}\right)}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}} = (ict)cos\alpha - xsin\alpha$$

1.8

where,

$$sin\alpha = \frac{\left(\frac{iv}{c}\right)}{\sqrt{1 + \left(\frac{iv}{c}\right)^2}}$$

$$cos\alpha = \frac{1}{\sqrt{1+\left(\frac{iv}{c}\right)^2}}$$

and

$$\alpha = tan^{-1}\left(\frac{iv}{c}\right)$$

Comparing your equations 1.8 with equations 1.2 you can readily verify that they are analogous, with ict replacing y in 1.8. Thus, the Lorentz coordinate transformation is orthogonal. You can go ahead and show that equations 1.8 satisfy the condition specified in equation 1.7.

If we introduce the notation x_μ for $\mu = 1, 2, 3, 4$ such that $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$, then in terms of the transformation matrix we can write our Lorentz transformation equations 1.8 in the form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad 1.9$$

where,

$$\gamma \frac{1}{\sqrt{1+\left(\frac{iv}{c}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{v}{c}\right)^2}}$$

and

$$\beta = \frac{v}{c}$$

1.3.2 Algebra of Four-Vector

We will now proceed to give the formal definition of a four-vector. A four vector is, simply put, a vector in a four dimensional real space called the Minkowski's space. By analogy to the three-dimensional vector in the Euclidean space, a four-vector can be defined in terms of its components in four possible directions as a set of four quantities denoted by a_μ for $\mu = 1, 2, 3, 4$, where 1, 2, 3 and 4 respectively refer to x, y, z and t. For instance, the momentum four-vector could be written in terms is components in four-dimensional space as p_x, p_y, p_z, p_t or simply p_μ where μ has the values we have already defined and p_t is the energy.

With this notation in mind, we can see that the coordinates of an event in a four-dimensional space could be represented by x_μ.

Just like vectors in three dimensions, addition of four-vectors involves adding the corresponding components. Besides, a four-vector equation is also true for its components.

The quantity $a_x^2 + a_y^2 + a_z^2 - a_t^2$ is invariant under the complete Lorentz group. A four-vector transforms under the complete Lorentz group, that is, for translational motion at constant velocity and rotation. Also, for two four-vectors a_μ and b_μ, their corresponding components transform in the same manner. So,

$$a_x b_x + a_y b_y + a_z b_z - a_t b_t = \vec{a}.\vec{b} - a_t b_t = a_\mu b_\mu$$

is invariant under the complete Lorentz group, provided t is replaced by ict.

Thus, we can write

$$a_\mu b_\mu = a'_\mu b'_\mu \qquad 1.10$$

The above is just an example of a four-vector algebraic operation of the scalar product.

The algebra of four-vectors is very much similar to that of the three-vector and are here tabulated for easy reference

OPERATION	**THREE-VECTOR**	**FOUR-VECTOR**
Vector	$\vec{a} = (a_x, a_{y,} a_z)$	$a_\mu = (\vec{a}, a_t)$
Scalar product	$\vec{a}.\vec{a} = a_x b_x + a_y b_y + a_z b_z$	$a_\mu b_\mu = \vec{a}.\vec{b} - a_t b_t$
Vector operator	$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$\nabla_{\boldsymbol{\mu}} = \left(\nabla, -\frac{\partial}{\partial t}\right)$
Gradient	$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$	$\nabla_{\boldsymbol{\mu}}\varphi = \left(\nabla\varphi, \frac{\partial\varphi}{\partial t}\right)$
Divergence	$\nabla.\vec{a} = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)$	$\nabla_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}} = \nabla.\vec{a} + \frac{\partial \vec{a}}{\partial t}$
Laplacian and D'Alembertian	$\nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	$\nabla_{\boldsymbol{\mu}}\nabla_{\boldsymbol{\mu}} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2$

Table 1: Comparison of the three-vector and four-vector algebra

Notice that the four vectors are made up, in all cases, of the space-like and the time-like parts. The space-like is exactly the equivalent of the three-dimensional counterpart.

1.3.3 Examples of four-vectors in relativistic mechanics

Let us proceed to discuss specific examples of four-vectors. Subsequently, you will learn more as we proceed to discuss the application of this notation to the formulation of the electromagnetic theory.

1.3.3.1 The Position Four-Vector

An event is represented by a point in a four-dimensional real vector space by a *world point.* A world point is the coordinates of a point taken from the origin of a four-dimensional real space. The distance from the origin to a world point or the between two world points is called the i*nterval* and is denoted by S. The quantity $S^2 = x^2 + y^2 + z^2 - c^2t^2$ is invariant under the complete Lorentz group (rotation inversion and translation at constant velocity). We must note that the main difference between the three-vector and the four-vector is that while the three-vector transforms under rotation (orthogonal transformation), a four-vector transforms under the complete Lorentz group.

The position four-vector or four-position is denoted by x_μ for $\mu = 1,2,3,4$.

In terms matrix notation, we can represent the four-position as

$$x_\mu = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

Transformation to the primed coordinates in terms of Lorentz transformation is given as

$$x_\mu{}' = \begin{pmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ x_4{}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

1.11

or

$x' = \gamma v_1 + i\beta\gamma x_4$

$x_2{}' = x_2$

$x_3{}' = x_3$

$x_4{}' = -i\beta\gamma v_1 + \gamma x_4$ 1.12

1.3.3.2 The Velocity Four-Vector

The components of the velocity four-vector are obtained by differentiating the components of the four-position with respect to proper time. We can denote four-velocity by v_μ.

Thus, we have

$$v_\mu = \frac{dx_\mu}{d\tau}$$

Accordingly, the components are as follows:

$$v_1 = \frac{dx_1}{d\tau} = \frac{dx_1}{dt}\frac{dt}{d\tau} = \gamma\frac{dx_1}{dt} = \gamma x = {}^{\cdot}\gamma v_1$$

$$v_2 = \frac{dx_2}{d\tau} = \frac{dx_2}{dt}\frac{dt}{d\tau} = \gamma\frac{dx_2}{dt} = \gamma\dot{y} = \gamma v_2$$

$$v_3 = \frac{dx_3}{d\tau} = \frac{dx_3}{dt}\frac{dt}{d\tau} = \gamma\frac{dx_3}{dt} = \gamma\dot{z} = {}^{\cdot}\gamma v_3$$

$$v_4 = \frac{dx_4}{d\tau} = \frac{dx_4}{dt}\frac{dt}{d\tau} = \gamma\frac{dx_4}{dt} = \gamma\frac{d(ict)}{dt} = \gamma ic$$

1.13

In matrix notation we can write this as

$$v_\mu = \frac{dx_\mu}{d\tau} = \begin{pmatrix} \gamma v_1 \\ \gamma v_2 \\ \gamma v_3 \\ \gamma ic \end{pmatrix}$$

1.14

Transformation to the primed coordinates will give in matrix notation

$$v_\mu{}' = \begin{pmatrix} v_1{}' \\ v_2{}' \\ v_3{}' \\ v_4{}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

1.15

This implies that

$$v_1{}' = \gamma v_1 + i\beta\gamma v_4$$

$$v_2{}' = v_2$$

$$v_3{}' = v_3$$

$$v_4{}' = -i\beta\gamma v_1 + \gamma v_4$$

1.16

Notice the similarity of the transformation matrix of the four-velocity to that of the four-position.

1.3.3.3 The Momentum Four-Vector

You already know how to find the momentum of a moving object, which is simply the product of mass and velocity.

We define the momentum four-vector as product of the rest mass and velocity four-vector and denote it by the symbol p_μ. So we write

$$p_\mu = m_o v_\mu$$

Obviously, this implies

$$p_\mu = \begin{pmatrix} \gamma m_0 v_1 \\ \gamma m_0 v_2 \\ \gamma m_0 v_3 \\ \gamma m_0 ic \end{pmatrix} = \begin{pmatrix} mv_1 \\ mv_2 \\ mv_3 \\ imc \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \frac{iE}{c} \end{pmatrix}$$

1.17

which we may write as

$$p_\mu = \left(\vec{p}, \frac{iE}{c}\right)$$

where $\vec{p}$ is the spatial or space-like component and $\frac{iE}{c}$ is the time-like component. E is the total relativistic energy given as $E = mc^2$

We see here that momentum and energy are four vector conjugates, that is, they are that spatial and temporal parts of a four-vector. Now, any four-vector must transform properly, that is, it transform in the same way as the position four-vector, so we can write

$$p'_x = \gamma\left(p_x - \beta\frac{E}{c}\right)$$

$$p'_y = p_y$$

$$p'_z = p_z$$

$$E' = \gamma(E - \beta p_x c)$$

1.18

as expected

Also, the dot product of a four-momentum with itself must be invariant. Thus,

$$p_\mu\, p_\mu = \sum_i^4 p_i^2 \;= -\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2$$

$$= \gamma^2 m_0^2({v_1}^2 + {v_2}^2 + {v_3}^2 - c^2)$$

$$= \gamma^2 m_0^2(v^2 - c^2)$$

$$= m_0^2 \frac{(v^2 - c^2)}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$= -m_0^2 c^2$$

But,

$$\sum_i^4 p_i^2 \;= -\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2$$

is Lorentz invariant, that is, it a Lorentz scalar. Thus we can write

$$-\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2 = -\frac{E^2}{c^2} + p^2 = -m_0^2 c^2$$

Multiplying both sides by c^2, we obtain

$$E^2 = p^2 c^2 + m_0^2 c^4$$

1.19

which is the basic energy equation of special relativity.

We can write this as

$$E^2 = \sqrt{(m_0 c^2)^2 + (pc)^2}$$

where

$$E_0 = m_0 c^2$$

1.20

is the rest energy

Finally in this section, we take a look at the force four-vector.

1.3.3.4 The Force Four-Vector

Recall that from Newton's second law, a force is the time rate of change of the momentum. So, to obtain the components of the force four vector or four-force, we simply differentiate the four-momentum with respect to proper time. Thus, if we denote the four force by F_μ, we have

$$F_\mu = \frac{dp_\mu}{d\tau} = \gamma \frac{dp_\mu}{dt}$$

1.21

Thus, as before, we obtain the components as

$$F_1 = \gamma \frac{dp_1}{dt} = \gamma F_1$$

$$F_2 = \gamma \frac{dp_2}{dt} = \gamma F_2$$

$$F_3 = \gamma \frac{dp_3}{dt} = \gamma F_3$$

$$F_4 = \gamma \frac{dp_4}{dt} = \gamma \frac{i}{c}\frac{dE}{dt} = \gamma \frac{i}{c}\vec{F}.\vec{v}$$

1.22

Therefore, we have

$$F_\mu = \frac{dp_\mu}{d\tau} = \gamma \frac{dp_\mu}{dt} = \gamma \left(\frac{d\vec{p}}{dt}, \frac{i}{c}\frac{dE}{dt}\right)$$

We can write the above equations in matrix notation as

$$F_\mu = \begin{pmatrix} \gamma F_1 \\ \gamma F_2 \\ \gamma F_3 \\ \gamma F_4 \end{pmatrix} = \begin{pmatrix} \gamma F_x \\ \gamma F_y \\ \gamma F_z \\ \gamma \frac{i}{c}\vec{F}.\vec{v} \end{pmatrix}$$

1.23

1.3.4 Space-Time and its Geometry

1.3.4.1 Space-Time

If you take a look at the equations of Lorentz coordinate transformation, you will find that the space and time coordinates are mixed. So, in relativity, we talk space-time as opposed to space only. Space-time is a four dimensional space. A point in the space-time is called the *world point* and defines an event. Of course, you can reason that an event occurs in space at some specified time. So, space-time is an imaginary concept of space which has four dimensions namely x, y, z and t.

The distance between two world points is called an *interval* and is denoted by S_{12}, which is given as

$$S_{12} = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}$$

1.24

If the interval is infinitesimally close, then we represent it as

$$ds = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \sqrt{c^2d\tau^2 - dl^2}$$

1.25

where,

$$dl^2 = dx^2 + dy^2 + dz^2$$

and

$$\tau = ict$$

If you take a close look at equation 1.24, you will realize that an interval can be real or imaginary. For instance, if $t_2 - t_1 = 0$, i.e., if the event took place at a time interval equal to zero, then the *interval is imaginary* and is referred to as *space-like interval* . The condition to satisfy here is that

$${S_{12}}^2 < 0.$$

For any space-like interval, there is an inertial frame in which the two events are simultaneous, but it is impossible to find an inertial frame in which the two events occur at the same place.

On the other hand, if

$$x_2 = x_1, \quad y_2 = y_1, \quad z_2 = z_1,$$

then, the two events took place at the same point. This implies that

${S_{12}}^2 > 0.$

The *interval is real* and is referred to as *time-like interval*

1.3.4.12 Space-Time Diagrams

In space-time continuum, we plot the space coordinates on the horizontal axis while we plot the time coordinate on the vertical axis as shown in figure 1.2

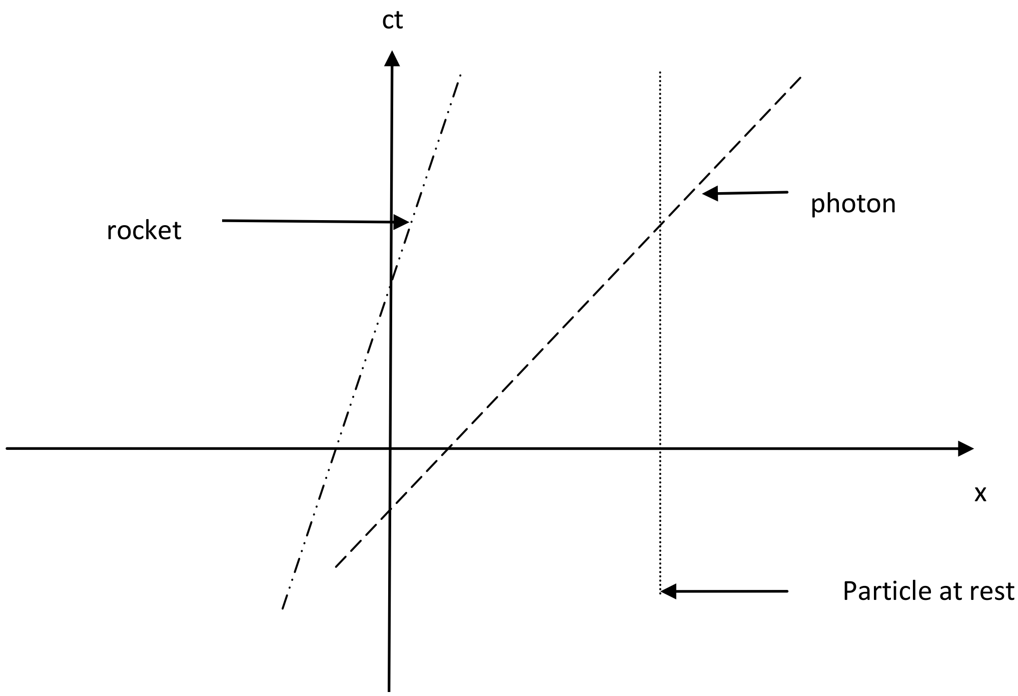

Figure 1.2 Space-time diagrams

As you can see, the vertical line represents a particle at rest. A particle travelling at the speed of light (photon) is represented by a line inclined at an angle of 45^0 to the positive x-axis. A particle going at an ordinary speed (fraction of the speed of light) such as a rocket is represented by line whose slope $\beta = \frac{c}{v}$

The path or trajectory of a particle in a space-time continuum is called *world line*.

Now, suppose a moving observer (an object, particle or signal) starts from the origin $x = 0$ at $t = 0$. His world line must be restricted any point within the light cone bounded by 45^0 lines because no material object can travel at a speed greater than that of light. The situation is illustrated in figure 1.3. The forward light cone represents the observer's future since this is the locus of all points accessible to him from the start, the origin, which represents his present. The backward light cone represents the observer's past. Elsewhere, that is, outside the light cones, is inaccessible to the observer as he can not travel faster than a photon of light.

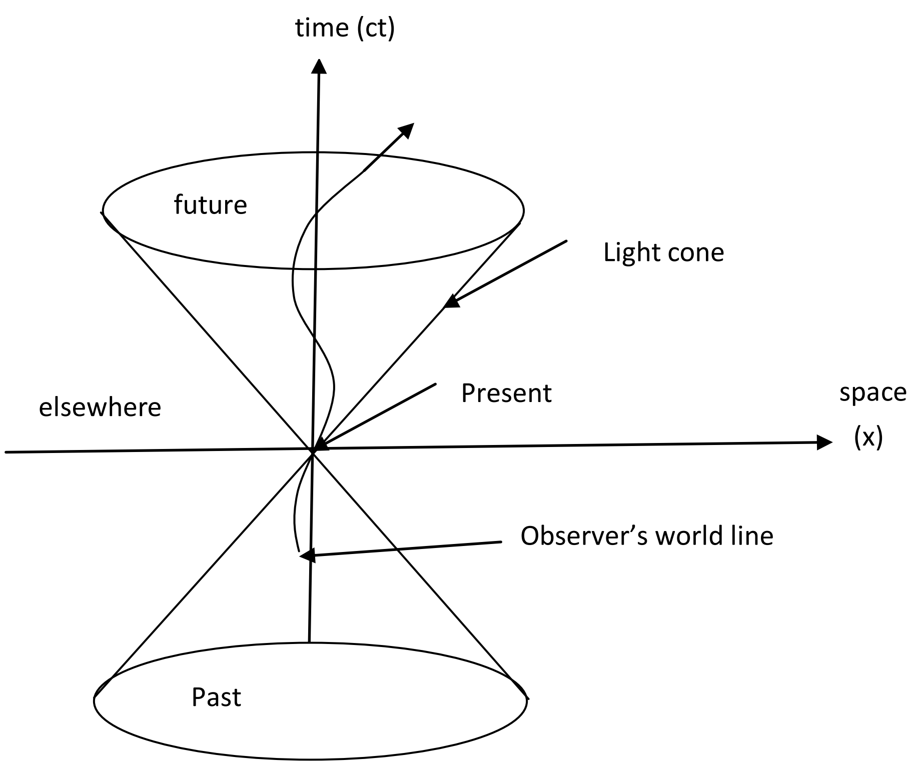

Fig.ure 1.3 Light cones

The implication of the above is that within the regions of the light cone, signals can be sent at speeds less than the speed of light from present to influence an event in the future. That, is the present can influence the future. Similarly, the past can influence the present

Summary

- A combination of transformation operations which leaves an entity the same as it was before the operations is called a group.
- A transformation which leaves a vector invariant after the rotation of the axes of the coordinate system is said to be orthogonal.
- Lorentz coordinate transformation is an orthogonal transformation.
- An *event* in space-time is completely specified by a vector in a four-dimensional space with coordinates (x, y, z, ict). This type of vector is called a *four-vector.* A point in the space-time is referred to as a *world point*. The distance from the origin to the world point is called an *interval.*
- An event is a point on the locus of a space time diagram

Conclusion

Space and time are interwoven to form space-time continuum.

Tutor Marked Assignments

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UNIT 2: Magnetism as a Relativistic Phenomenon

CONTENTS

1.0 Introduction

We have seen that Newtonian mechanics is consistent with the Galilean transformation, but electrodynamics is not. We also saw that classical electrodynamics is consistent with special relativity. That is, Maxwell's equations and Lorentz force both of which constitute the basic formulation of electrodynamics, are Lorentz invariant

You will also remember from your basic electromagnetism courses that magnetism arises from the motion of electric charges (in close loop). Thus, as far as relativity is concerned, what one observer sees as an electrical phenomenon in his frame of reference might appear as a magnetic phenomenon to another observer in a frame of reference in relative motion to his.

In this unit we proceed to show how the electric and the magnetic fields arise as a result of relativistic motion as seen in different inertial frames.

1.1 Objectives

At the end of this unit, you will be able to

- Show that net charge of any macroscopic body is invariant under Lorentz transformation.
- Show that charge density is invariant
- Show that given electrostatics and relativity, we obtain magnetism.

1.3 Electric Charges And Charge Density As Seen In Different Frames of Reference

1.3.2 Electric Charges As Seen By Different Observers In Different Frames of Reference

You already know that the total charge of an isolated system is constant. This, of course, is the statement of the conservation of charge. It demands that the quantity of charge that flows into an isolated body must be exactly equal to the amount that flows out of it.

You will also remember that for a neutral atom, the number of protons in the nucleus equals the number of electrons in the electron cloud. Now, in an atom, these protons and electrons are in motion, yet their charges are exactly balanced out. So, the atom strictly maintains its electrical neutrality. The basic charge of a proton and an electron is the electronic charge; $\pm e$. The net charge on a macroscopic object is an integral multiple of the electronic charge. Thus, different observers in different frames of reference will count the same number of electronic charges in any macroscopic object. In other words, if N electronic charges, Ne, constitute the net charge Q on an object as seen by an observer in the S frame, the observer in the S' frame will also see N electronic charges and the net charge Q' of magnitude Ne. Therefore , the *net charge of any macroscopic body is invariant under Lorentz transformation*. The magnitude of an electric charge is the same irrespective of how fast it is moving.

1.3.2 Linear Charge Density As Seen By Different Observers In Different Frames of Reference

You just learned in the last sub-section that the net charge on a body is Lorentz invariant. How about charge density? Of course, we do not expect it to be. Why? Well, remember that charge density depends on the dimensions of the object on which there is a charge distribution. For example, linear charge distribution is the charge per unit length. And you have seen that the lengths of objects appear shortened relative to an observer in a frame of reference in which it is in motion (length contraction). So, linear charge density varies with the variation of the length of the charge distribution.

Consider a string of equally spaced charges at rest in the S frame as shown in figure 1.1

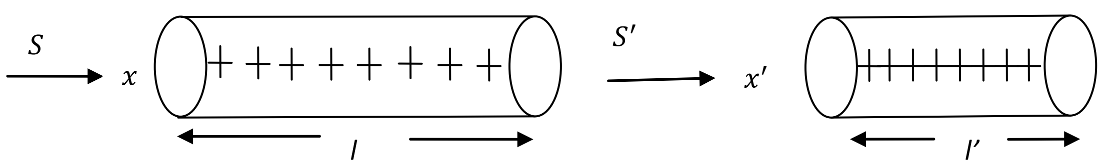

Fig.1.1a String s seen by an observer at rest in S

Fig.1.1b String as seen by an observer in motion in S'

We will assume that the charges are close enough to form a continuous string of charge of length l. If the charge of the system is $Q = ne$, where $N = 1,2,3, \ldots$, then, the linear charge density λ is given as

$$\lambda = \frac{Q}{l}$$

in coulomb/metre (C/m)

or

$Q = \lambda l$

in coulomb (C).

If $l = 1\ m$, then $\lambda = Ne$ C/m.

Now, as seen in the S' frame which is in motion to the right along the +x-axis at constant velocity v relative to S, the length l of the charge string is shortened by the factor $\beta = \sqrt{1 - v^2/c^2}$. That is $l' = l\sqrt{1 - v^2/c^2}$, where l' is the contracted length as seen in the S' frame. The linear charge density λ' in the S' frame is therefore

$$\lambda' = \frac{Q}{l'} = \frac{Q}{l\sqrt{1 - v^2/c^2}}$$

1.1

Thus, for a unit length, i.e. $l = 1\ m$, we have

$$\lambda' = \frac{Ne}{\sqrt{1 - v^2/c^2}} = \gamma Ne$$

1.2

Thus, The N charges appear squashed into a contracted length $\frac{1}{\gamma} = \beta = \sqrt{1 - v^2/c^2}$ as depicted in equation 1.2. We therefore conclude that *linear charge density is not Lorentz invariant.* Of course, by extension, surface as well as volume charge density is also not Lorentz invariant.

1.3.3 Electric and Magnetic Forces

You could remember that parallel conductors carrying current in the same direction exert an attractive magnetic force on each other. Now, in a current-carrying conductor, electrons are in motion relative to the apparently stationary positive "ionic cores" or the protons. Conventionally, the direction of the electric current in the conductor is taken as the direction of the motion of these positive cores. With these reminders at the back of your mind, let us proceed to show how magnetism arises, given electrostatics and relativity. We want to calculate the force between a current- carrying conductor and a moving charge without recourse to the laws of magnetism.

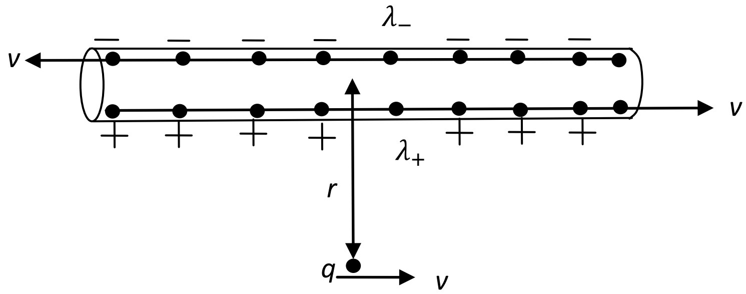

Fig.1.2a Charge distribution as seen in S frame

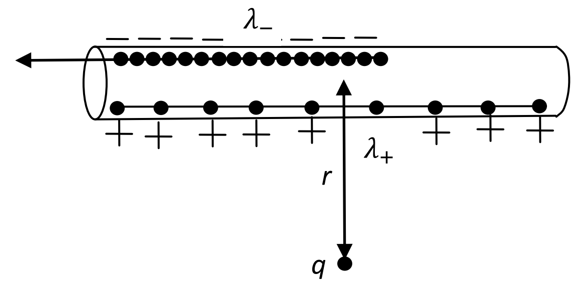

Fig.1.2b Charge distribution as seen in S' frame

Consider, once again, a string of equally spaced positive charges, this time moving steadily to the right with a constant drift speed v. We, again assume that the charges are close enough to form a continuous string of positive charges of linear charge density λ_+. Of course, you also have a string of negative charges with linear charge density λ_-, moving in the opposite direction with the same speed v. Since the string has no net charge, it is electrically neutral. The situation is illustrated in figure 1.2a. The net current to the right is $I = 2\lambda v$

Now, consider a charge q at a distance r perpendicular to the conductor and moving to the right with a velocity, which we will assume to be v_0. You know that electron drift speeds are very small compared with the relativistic speed of q. But, never mind, the procedure we will adopt here will give similar results. There is no electrical force on q. Why? Well, this is because the string, we have already agreed, is neutral as seen in the S frame.

In the S' frame, q is at rest but S' is moving at a speed v_0 relative to S. The velocity of the positive charges as seen in S' frame is

$$v_+ = \frac{v - v_0}{1 - vv_0/c^2}$$

1.3

And the velocity of the negative charges as seen in the S' frame is

$$v_- = \frac{v + v_0}{1 + vv_0/c^2}$$

1.4

As you can see, v_- is greater than v_+, the squashing up of negative charges is more severe than the positive charges in the S' frame. The wire will then carry a ne t negative charge. Thus, we have

$\lambda_+ = +\gamma_+\lambda_0$

And

$\lambda_- = -\gamma_-\lambda_0$

Here,

$$\gamma_+ = \frac{1}{\sqrt{1 - v_+^2/c^2}}$$

1.5

and

$$\gamma_- = \frac{1}{\sqrt{1 - v_-^2/c^2}}$$

1.6

Now, λ_0 is the charge density of the positive line in its rest frame, so

$$\lambda = \gamma\lambda_0$$

1.7

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

1.8

Therefore we write

$$\gamma_+ = \frac{1}{\sqrt{1 - \frac{1}{c^2}(v - v_0)^2\left(1 - \frac{vv_0}{c^2}\right)^{-2}}}$$

$$= \frac{c^2 - vv_0}{\sqrt{(c^2 - vv_0)^2 - c^2(v - v_0)^2}}$$

$$= \frac{c^2 - vv_0}{\sqrt{(c^2 - v^2)(c^2 - v_0^2)}} = \gamma\frac{(1 - vv_0/c^2)}{\sqrt{1 - v_0^2/c^2}}$$

Likewise

$$\gamma_- = \gamma\frac{(1 + vv_0/c^2)}{\sqrt{1 - v_0^2/c^2}}$$

Therefore, the net line charge as seen in the S' is

$$\lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda v_0}{c^2\sqrt{1 - v_0^2/c^2}}$$

1.9

Now, what is the conclusion of this analysis? Well, as can you see, the consequence of the unequal Lorentz contraction of the positive and the negative lines, an otherwise neutral current-carrying wire in one frame of reference becomes charged in another reference frame.

To calculate the electric field due to this resultant charge, as seen in the S' frame, we have

$$E = \frac{\lambda_{tot}}{2\pi\epsilon_0 r}$$

The electrical force due to this resultant charge is

$$F = qE = \frac{\lambda v}{\pi\epsilon_0 c^2 r}\frac{qv_0}{\sqrt{1 - v_0^2/c^2}} \tag{1.10}$$

Now, from Newton's third law of motion, forces exist in pairs. Thus, if there is a force of electrical origin in the S' frame, there must also a force in the S frame. But, what type of force is it? To answer this question, we calculate this force by applying the transformation rules, namely,

$$F = \gamma F' = \sqrt{1 - \frac{v_0^2}{c^2}}F' = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qv_0}{r} \tag{1.11}$$

Noting that $\lambda v = I$, the current and $c^2 = (\epsilon_0\mu_0)^{-1}$in equation 1.11, we write

$$F = -\left(\frac{\mu_0 I}{2\pi r}\right)qu = Bqv \tag{1.12}$$

Where $B = \frac{\mu_0 I}{2\pi r}$ is the magnetic field as you can recognize.

So, the force as seen in the S frame is a magnetic force.

Thus, we can conclude that given electrostatics and relativity, we obtain magnetism.

Summary

- Different observers in different frames of reference will count the same number of electronic charges in any macroscopic object. Therefore, the net charge of any macroscopic body is invariant under Lorentz transformation.

- Linear charge density varies with the variation of the length of the charge distribution. Therefore conclude that *linear charge density is not Lorentz invariant.* Of course, by extension, surface as well as volume charge density is also not Lorentz invariant.
- Given electrostatics and relativity, we obtain magnetism.

Conclusion

Magnetism results from a combination electrostatics and relativity.

Tutor Marked Assignments

Explain, with suitable diagrams how magnetism arises given electrostatics and special relativity.

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1. Introduction to Special Relativity by Wolfgang Rindler, Oxford University Press, 1990.

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The Four-Vector Formulation of Electrodynamics

CONTENTS

Introduction

We have pointed out earlier on in this course that all of electrodynamics is contained in Maxwell's equations together with the Lorentz force equation. Now, we want to investigate how these equations behave under relativistic transformation from one inertial frame to another. In order to do this, we need to know the transformation properties of the differential operators with which these field equations are written. In other words, we must express the field equations in the invariant four-vector form. So, let us get around to doing just that.

1.3.0 Transformation Properties of the Differential Operator

To establish the invariant forms of the differential operators, what we are going to do is to apply the rules of partial differentiation to differentiate the four-dimensional space-time coordinates. First of all, the rule of partial differentiation is as follows:

$$\frac{\partial}{\partial x_1'} = \frac{\partial x_1}{\partial x_1'}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x_1'}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial x_1'}\frac{\partial}{\partial x_3} + \frac{\partial x_4}{\partial x_1'}\frac{\partial}{\partial x_4}$$

Recall your Lorentz coordinate transformation.

$$x_1 = \frac{x_1' - i\beta x_4'}{\sqrt{1-\beta^2}}$$

$$x_2 = x_2'$$

$$x_3 = x_3'$$

$$x_4 = \frac{x_4' + i\beta x_1'}{\sqrt{1-\beta^2}}$$

Now,

$$\frac{\partial x_1}{\partial x_1'} = \frac{\partial}{\partial x_1'}\left(\frac{x_1' - i\beta x_4'}{\sqrt{1-\beta^2}}\right)\frac{\partial}{\partial x_1} \qquad = \frac{1}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x_1}$$

$$\frac{\partial x_2}{\partial x_1'} = 0$$

$$\frac{\partial x_3}{\partial x_1'} = 0$$

$$\frac{\partial x_4}{\partial x_1'} = \frac{1}{\sqrt{1-\beta^2}}\left(\frac{\partial}{\partial x_1} + i\beta\frac{\partial}{\partial x_4}\right)$$

And now, if you compare our result with Lorentz transformation coordinate transformation equations, you will easily discover that they the same in form, except that x_μ is replaced by $\frac{\partial}{\partial x_\mu}$ for $\mu = 1,2,3,4$.

We leave it as an exercise for you to write the corresponding equations for the other components, namely $\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}$. You will also find out that they too transform just in the same way as the Lorentz coordinate transformation. Thus, we conclude that $\frac{\partial}{\partial x_\mu}$ is a four-vector. Just like we did for x_μ, we can obtain the invariant scalar product of $\frac{\partial}{\partial x_\mu}$. This gives

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_4^2}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2$$

You have seen this in connection with the wave equation propagating at the speed of light c through the free space. This four-dimensional operator is known as d'Alembertian and is denoted by $\square^2$.

1.3.1 The Four-vector Form of the Continuity Equation

Having studied the invariant form of the differential operator, we now proceed to show that charge density and current density are four-vectors and then write the equation of continuity in the invariant form.

Recall that charge is invariant and does not depend on the relative motion of the reference frame. Also, you know that charge is conserved. The statement of conservation of charge is expressed quantitatively as

$$\nabla.\vec{J} = -\frac{\partial \rho}{\partial t}$$

or

$$\nabla.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

where $\vec{J}$ is the current density and ρ the volume charge density.

This equation takes on the invariant form if the current density is expressed in its four-vector, J_μ, which is made of the current density as the space-like part and the charge density as the time-like part.

Thus, if ρ is the charge density in the $S-$ frame in which the charges are at rest, then the current density four-vector J_μ may be defined as the product of ρ and the four-velocity v_μ. That is

$$J_\mu = \begin{pmatrix} \gamma\rho v_x \\ \gamma\rho v_y \\ \gamma\rho v_z \\ \gamma\rho ic \end{pmatrix} = \begin{pmatrix} \rho' v_x \\ \rho' v_y \\ \rho' v_z \\ \rho' ic \end{pmatrix} = \begin{pmatrix} J_x \\ J_y \\ J_z \\ \rho ic \end{pmatrix}$$

Here, $\rho' = \gamma\rho$, which shows that the charge density has increased as a result of the change in volume element (Lorentz contraction).

As you can see, J_μ can be written as $J_\mu = (\vec{J}, J_t)$

where $\vec{J} = (J_x, J_y, J_z)$ represents the spatial components or space-like part and $J_t = ic\rho$ and represents the time-like part of the four-vector. But still, we could use the notation introduced for four-vectors, namely,

$$J_\mu = (J_1, J_2, J_3, J_4),$$

where J_i for $i = 1,2,3$ represent J_x, J_y, J_z and $J_4 = ic\rho$.

We can now write the continuity equation in terms of J_μ as follows:

$$\nabla.\vec{J} + \frac{\partial \rho}{\partial t} = \frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0$$

Or simply,

$$\frac{\partial J_\mu}{\partial x_\mu} = 0$$

It is quite clear that in this form, the equation of continuity is, without doubt, invariant. Also, you can see that in the S' frame translating at constant speed v along the common x-axis relative to the S frame, our ρ' should be expressed as

$$\rho' = \gamma\left(\rho - \frac{vJ_x}{c^2}\right)$$

Thus, in the S frame in which we assume the charges to be at rest, we have $J_x = 0$ and therefore $\rho' = \gamma\rho$ as before.

1.3.2 The Four-vector Form of Maxwell's Equations

We have seen that for two observers in relative motion, what one sees as an electric field might be seen by the other as a magnetic field. We will now take a look at a general rule for transforming between the two fields. We may ask, "Given a field in S frame, what is the field in S' frame in relative motion at constant velocity along the common x-axis?" Perhaps you want to guess that $\vec{E}$ is one spatial part of a four-vector while $\vec{B}$ is the time-like part. If so, we afraid, your guess is not correct. *Electric and magnetic fields are not the space-like and time-like parts of a four-vector*. Rather, they form some of the components of a quantity called the *four-tensor.* We are not going to bother you with tensors as the fields and therefore Maxwell's equations can be put in the Lorentz invariant form by expressing them in terms of potentials and formulate the equations in the four-vector form.

First of all, let us write down the Maxwell's the equations in rationalized mks system of units.

$$\nabla.\vec{E} = \frac{\rho}{\epsilon_0} \quad (1)$$

$$\nabla.\vec{B} = 0 \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

4

You are familiar with these equations from your course in electromagnetism. And, now, the electric and magnetic fields can be expressed in terms of the scalar and vector potentials as follows:

$$\vec{E} = -\nabla \emptyset = \frac{\partial \vec{A}}{\partial t}$$

5

$$\vec{B} = \nabla \times \vec{A}$$

6

Let us first deal with equations 2 and 3. If we use the vector identities

$$\nabla.\left(\nabla \times \vec{A}\right) = 0$$

and

$$\nabla \times (\nabla \emptyset) = 0,$$

equation 2 becomes

$$\nabla.\vec{B} = \nabla.\left(\nabla \times \vec{A}\right) = 0$$

and equation 3 becomes

$$\nabla \times \vec{E} = \nabla \times \left(-\nabla \emptyset - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \nabla \times (-\nabla \emptyset) + \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$= 0 + \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\frac{\partial\left(\nabla \times \vec{A}\right)}{\partial t}$$

$$= -\frac{\partial \vec{B}}{\partial t}$$

Thus, equations 2 and 3 are automatically satisfied.

Now, let us turn our attention to the remaining equations, namely, equations 1 and 4.

Equation 1 becomes,

$$\nabla.\vec{E} = \nabla.\left(-\nabla\emptyset - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\frac{\partial\left(\nabla.\vec{A}\right)}{\partial t} - \nabla^2\emptyset$$

$$= \frac{\rho}{\epsilon_0}$$

We, then use the Lorentz condition

$$\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial\emptyset}{\partial t} = 0$$

7

or

$$\nabla.\vec{A} = -\frac{1}{c^2}\frac{\partial\emptyset}{\partial t}$$

to write the above equation as

$$-\nabla^2\emptyset + \frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

8

On the other hand, equation 4 can be written as

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \nabla \times \left(\nabla \times \vec{A}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\emptyset - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\nabla^2\vec{A} + \nabla\left(\nabla.\vec{A}\right) + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\emptyset) + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$= \mu_0 \vec{J}$$

Applying Lorentz condition the above becomes

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = -\nabla^2 \vec{A} + \nabla\left(-\frac{1}{c^2}\frac{\partial \emptyset}{\partial t}\right) + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \emptyset) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$= -\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$

$$= \mu_0 \vec{J}$$

Thus,

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$$

9

Let us write equations 8 and 9 together

$$-\nabla^2 \emptyset + \frac{1}{c^2}\frac{\partial^2 \emptyset}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$$

Take a close look at these equations. You can easily see that, apart from the constant factors, the right hand side of both equations are components of a four-vector

$J_\mu = (\vec{J}, ic\rho)$.

Besides, the operator on the left hand side is a four-vector of the form

$$-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x_\mu^2},$$

which is Lorentz invariant.

Both sides of the equation transform the same way, provided we note that A_μ is a four-vector, i.e.,

$$A_\mu = \left(\vec{A}, \frac{i\emptyset}{c}\right).$$

Now, let us make the substitution $A_\mu = \frac{i\emptyset}{c}$ and $J_\mu = i\rho c$. We obtain,

$$\frac{1}{c}\left\{-\nabla^2\emptyset + \frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2}\right\} = \mu_0\rho c$$

$$-\nabla^2\emptyset + \frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2} = \mu_0\rho c^2$$

$$-\nabla^2\emptyset + \frac{1}{c^2}\frac{\partial^2\emptyset}{\partial t^2} = \frac{\rho}{\epsilon_0},$$

10

where use has been made of the fact that

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

Notice that the resultant equation 10 which we have obtained is Gauss' law.

So, we can write Ampere's and Gauss' law together using a single four-vector equation, namely,

$$-\frac{\partial^2 A_\mu}{\partial x_\mu^2} = \mu_0 J_\mu$$

11

Notice also that the Lorentz condition is invariant and can be written in terms of a four-vector as

$$\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial\emptyset}{\partial t} = 0$$

or

$$\frac{\partial A_\mu}{\partial x_\mu} = 0$$

12

We have shown that Maxwell's equations are Lorentz invariant as required by the theory of special relativity. Besides, they are also in agreement with the second postulate of special relativity which requires that all observers measure the same speed of light c. We nave also found that A_μ and J_μ are four-vectors, so that if have their values one frame of reference S, we can work out the corresponding values in another reference frame S' in uniform motion translation relative to S using he Lorentz transformation. For instance,

$$\emptyset' = \gamma\left(\emptyset - \frac{vA_x}{c^2}\right)$$

$$A'_y = A_y$$

$$A'_z = A_z$$

$$A'_x = \gamma\left(A_x - \frac{v\emptyset}{c^2}\right)$$

13

The inverse of the above transformation can be obtained in the usual way.

To make our four-vector formulation of electrodynamics complete, we have to show that the Lorentz force is Lorentz invariant. To do this, we have to first of all learn how to transform the $\vec{E}$ and $\vec{B}$ fields.

1.3.3 Transformation of the Fields

Having learnt to transform the four potential A_μ and the differential operator $\frac{\partial}{\partial x_\mu}$, we now in a position to learn how to transform the fields.

Recall that

$$\vec{E} = -\nabla\emptyset = \frac{\partial\vec{A}}{\partial t}$$

So that,

$$E'_x = \frac{\partial\emptyset'}{\partial x'} - \frac{\partial A'_x}{\partial t'}$$

Now,

$$A_\mu = \left(\vec{A}, \frac{i\emptyset}{c}\right)$$

and

$$\frac{\partial}{\partial x_\mu} = \left(\nabla, \frac{\partial}{\partial t}\right).$$

Thus,

$$E'_x = ic\left\{\frac{\partial A'_4}{\partial x'_1} - \frac{\partial A'_1}{\partial x_4}\right\}$$

Since, A_μ and $\frac{\partial}{\partial x_\mu}$ Lorentz invariant, we can write

$$A'_4 = \gamma(A_4 - i\beta A_1)$$

$$A'_1 = \gamma(A_1 + i\beta A_4)$$

$$\frac{\partial}{\partial x'_1} = \gamma\left(\frac{\partial}{\partial x_1} + i\beta\frac{\partial}{\partial x_4}\right)$$

Therefore,

$$E'_x = ic\left[\gamma\left(\frac{\partial}{\partial x_1} + i\beta\frac{\partial}{\partial x_4}\right)\{\gamma(A_1 + i\beta A_4)\}\right]$$

$$= ic\gamma^2\left[\frac{\partial A'_4}{\partial x_1} + i\beta\frac{i\partial A_4}{\partial x_4} - i\beta\frac{\partial A_1}{\partial x_1} + \beta^2\frac{\partial A_1}{\partial x_4} - \frac{\partial A_1}{\partial x_4} + i\beta\frac{\partial A_1}{\partial x_1} - \beta^2\frac{\partial A_4}{\partial x_4}\right]$$

$$= ic\gamma^2(1-\beta^2)\left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}\right)$$

$$= ic\left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}\right)$$

$$= E_x$$

Similarly, we can derive the other components of E' and B'. They here given below

$$E'_x = E_x$$

$$E'_y = \gamma(E_y - vB_z)$$

$$E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_x\right)$$

$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Of course, the inverse transformation can similarly be written but with v replaced by – v.

SAQ 1

Show that $E_z' = \gamma\left(E_z + vB_y\right)$

1.3.4 Electric Field of a Point Charge in Uniform Motion

Consider a point charge q at the origin of a reference frame S' in uniform translation along the common x-axis relative to the S frame. The electric field due to this point charge as measured in S' is

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}'}{r'^3}$$

In terms of its components, we have

$$E_x = E_x' = \frac{q\gamma y}{4\pi\epsilon_0(x'^2 + y'^2 + z'^2)^{3/2}}$$

$$E_y = \gamma E_y' = \frac{q(x - vt)}{4\pi\epsilon_0(\gamma^2(x - vt)^2 + y'^2 + z'^2)^{3/2}}$$

$$E_z = \gamma E_z' = \frac{q\gamma x}{4\pi\epsilon_0(\gamma^2(x - vt)^2 + y'^2 + z'^2)^{3/2}}$$

$$B_x = 0$$

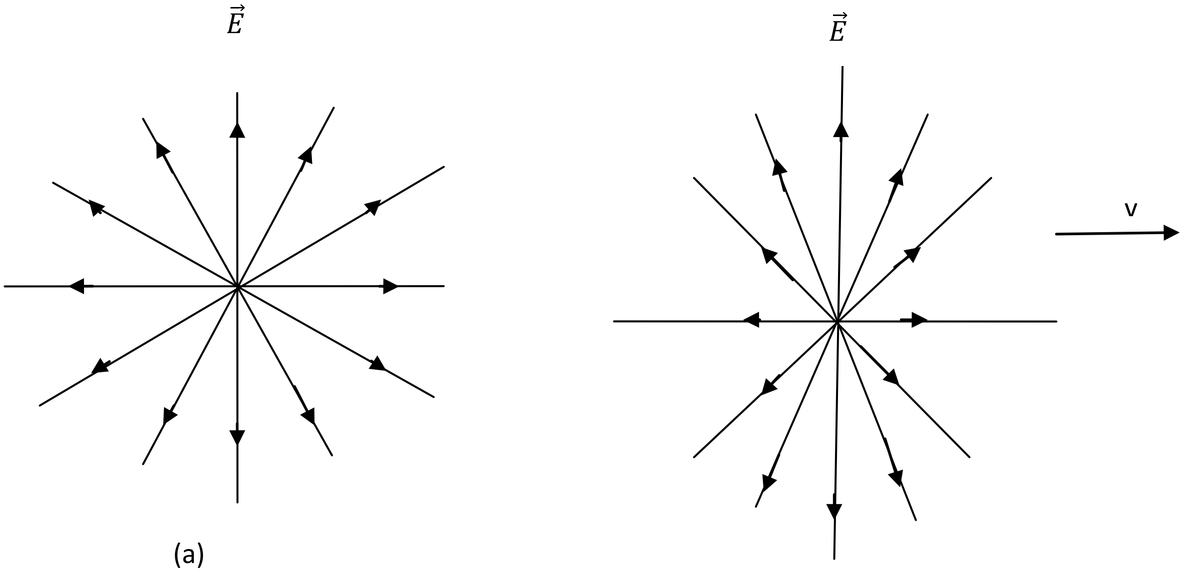
$$B_y = -\frac{\gamma v E_z'}{c^2} = -\frac{vE_z}{c^2}$$

$$B_z = -\frac{\gamma v E_y'}{c^2} = -\frac{vE_y}{c^2}$$

or equivalently,

$$\vec{B} = \frac{1}{c^2}\vec{v} \times \vec{E}$$

The field lines are still straight and radiate from the charge but in the direction of motion, the field pattern is squashed up. This is shown in figure 1.1.



(a)

(b)

Figure 1.1: Electric field of a point charge at rest fig.(a) compared to a point charge in motion at relativistic speed v ,fig. (b).

1.3.5 Transformation of the Forces

To complete our four-vector formulation of electrodynamics we now investigate the Lorentz invariance force,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

You have to remember that $\vec{F} = q\vec{E}$ and that the transformation of the $\vec{E}$ field is given by equation 14, namely,

$$E'_x = E_x , \qquad thus, \qquad F'_x = qE'_x = qE_x$$

$$E'_y = \gamma(E_y - vB_z) \qquad thus, \qquad F'_y = qE'_y = q\gamma(E_y - vB_z)$$

$$E'_z = \gamma(E_z + vB_y) \qquad thus, \quad F'_z = qE'_z = q\gamma(E_z + vB_y) \qquad 15$$

On the other hand, the components of $\vec{F}$ in the S frame are

$$F_x = qE_x = F'_x$$

$$F_y = qE_y = q(E_y - vB_z) = \frac{F'_y}{\gamma}$$

$$F_z = qE_z = q(E_z + vB_y) = \frac{F'_z}{\gamma}$$

This completes our task of the four-vector or, if you like, the covariant formulation of electrodynamics. The term covariant implies the existence of the relativistic principle for a physical phenomenon.

Summary

- All of electrodynamics is contained in Maxwell's equations together with the Lorentz force equations.
- The differential operators, with which the field equations are written, are expressible in four-vector form and are Lorentz invariant.
- The equation of continuity is expressible in four-vector notation and is also Lorentz invariant.
- The electric and the magnetic fields can be written in terms of vector and scalar potentials which are respectively the space-like and the time-like parts of a four-vector.
- Maxwell's equations are Lorentz invariant as required by the theory of special relativity. Besides, they are also in agreement with the second postulate of special relativity which requires that all observers measure the same speed of light c.
- When transformed from one inertial frame to another, what is observed as a pure electric field in one frame might appear as a magnetic field or a combination of both in another frame. Also, what is observed as a pure magnetic field in one frame might appear as an electric field or a combination of both in another frame.
- The field lines of a point charge are straight and radiate from the charge but in the direction of motion, the field pattern is squashed up.

Conclusion

Under the transformation from one inertial frame to another, what is one observer's electric field might be another's magnetic field.

Tutor Marked Assignments

1. Show that $(\vec{E}.\vec{E})$ is relativistically invariant.

2. Show that $(E^2 - c^2B^2)$ is relativistically invariant.

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