QUANTITATIVE TECHNIQUES

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MATHEMATICAL
TECHNIQUES

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MATHEMATICAL TECHNIQUES

▶ OBJECTIVES

At the end of this chapter, one should be able to:

- Interpret data in a Venn diagram
- Apply matrix algebra to input-output analysis and Markovian process
- Apply calculus to economic models

▶ INTRODUCTION

Linear algebra has extensive applications innatural sciences and social sciences, since nonlinear models are often approximated by linear ones.

One of the applications of linear algebra is in finding solutions of simultaneous linear equations. The simplest case is when the number of unknowns is the same as the number of equations. One could begin with the problem of solving n simultaneous linear equations for n unknowns.

Fast Forward: Linear algebra is the branch of mathematics concerned with the study of vectors, vector spaces, linear maps and systems of linear equations.

▶ DEFINITION OF KEY TERMS

Set theory - Set theory is the branch of mathematics that studies sets, which are collections of objects.

Function - The mathematical concept of function expresses a relationship between two variables i.e the dependence between two quantities, one of which is known (theindipendent variable, argument of the function, or its 'input') and the other produced (the dependent variable, value of the function, or 'output').

Matrix - In mathematics, a matrix (plural matrices) is a rectangular table of elements (or entries), which may be numbers or, any abstract quantities that can beadded and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformations and to record data that depend on multiple parameters. Matrices can be added, multiplied, and decomposed in various ways.

Determinant – This is a characteristic of a matrix and is obtained from the elements of a matrix by specified calculation. The determinant is only specified for a square matrix.

Equilibrium state – This is a situation in a Markov process when there is no further gain in market share. It is also known as the steady state or the long-run state.

▶ **INDUSTRY CONTEXT**

In practice, Business Calculus presents some of the mathematical tools that are useful in the analysis of business and economic problems. A few topics are compound interest, annuities, differential and integral calculus.

▶ **EXAM CONTEXT**

The algebra and calculus has been emphasized by examiners in various sittings as shown below:

■

■

1.1 NEED FOR QUANTITATIVE TECHNIQUES IN THE BUSINESS WORLD

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Mathematics is logical and precise as applied to measurable phenomena. Measurable quantities include:

   -Output

   -Revenue

   -Commissions

   -Costs

   -Profits etc

Rationale: Management needs to be able to influence factors which can be manipulated or controlled to achieve objectives e.g. sales level is determined by a sales function which has determinants such as level of advertising, price, income etc.

Optimization theory i.e. maximising or minimising some measure of revenue or costs calls for the use of mathematical analysis.

   Maximise: Revenue, profits, productivity, and motivation

   Minimise: Costs, risks, lateness, fatigue

Mathematical analysis also helps in marginal analysis i.e. the conversion of marginal function to a total function.

Drawback: It is not applicable to non-measurable phenomena even if such a factor is crucial to the success of an organization.

1.2 FUNCTIONS

Definitions

1. A constant – This is a quantity whose value remains unchanged throughout a particular analysis e.g fixed cost, rent, and salary.

2. A variate (variable) – This is a quantity which takes various values in a particular problem

Illustration 1.1.

Suppose an item is sold at Sh 11 per unit. Let S represent sales rate revenue in shillings and let Q represent quantity sold.

Then the function representing these two variables is given as:

S = 11Q

S and Q are variables whereas the price - Sh 11 - is a constant.

Variables

Types of variables

Independent variable – this is a variable which determines the quantity or the value of some other variable referred to as the dependent variable. In Illustration 1.1, Q is the independent variable while S is the dependent variable.

An independent variable is also called a predictor variable while the dependent variable is also known as the response variable i.e Q predicts S and S responds to Q.

3. A function – This is a relationship in which values of a dependent variable are determined by the values of one or more independent variables. In illustration 1.1 sales is a function of quantity, written as S = f(Q)

Demand = f(price, prices of substitutes and complements, income levels,….)

Savings = f(investment, interest rates, income levels,….)

Note that the dependent variable is always one while the independent variable can be more than one.

Types of functions

Some of the more commonly encountered functions in the business world include:

- Polynomials
- Multivariate
- Logarithmic
- Discrete Vs Continuous
- Step function
- Exponential

1.   Polynomial Functions

These are functions of the form $y = a + b_1x + b_2x^2 + \ldots\ldots + b_nx^n$

Where:

y   represents dependent variable

x   represents Independent variable

$a, b_1, b_2, \ldots, b_n$ are constants

$b_1, b_2, \ldots, b_n$ coefficient constants attached to a variate

n   highest power of X (the degree of polynomial)

Categories of polynomials

1st degree of polynomials (the linear function)

$y = a + bx$

e.g $y = 20 + 2x$

$y = 5x$

$y = 15 - 0.3x$

2nd degree (quadratic functions)

General form $y = a + b_1x + b_2x^2$ (where $b_2 \neq 0$)

e.g $y = 23 - 10x + 0.2x^2$

$y = x^2$

$y = 100 - x^2$

3rd degree (cubic functions) – the highest power = 3

General form: $y = a + b_1x + b_2x^2 + b_3x^3$ ($b_3 \neq 0$)

e.g $y = 12 - 0.1x + x^2 + 3x^3$

$y = 3x^3$

$y = x^3 - x^2$

2. Multivariate Functions

These are functions with more than one independent variable

Note:

1. One independent variable implies univariate function.

2. Two independent variable implies bivariate function.

3. Three independent variable implies multivariate function.

e.g

$y = 21 - 3x + 4x^2$ – bivariate function

$y = 2x - 12 + N^4$ – multivariate

3. Logarithmic Functions

These are functions which have at least one term being in logarithmic form

e.g

$y = 7 \log x$

$\log y = 0.6x$

$3\log y = 5z$ (multivariate logarithmic function)

4. Exponential Functions

These are functions where the predictor variable is at least part of an exponent or power

e.g.

$y = 122x$

$y = 0.2x + 16\ 4z + n$ (Multivariate exponential function)

5. Continuous vs discrete functions

This is a function which has the values of dependent variable defined for all values of their independent variable (s) i.e there are no gaps

e.g $y = x2$

Table 1.1

x	-3	-2	0	1	2	3
y	9	4	0	1	4	9

Diagram 1.1
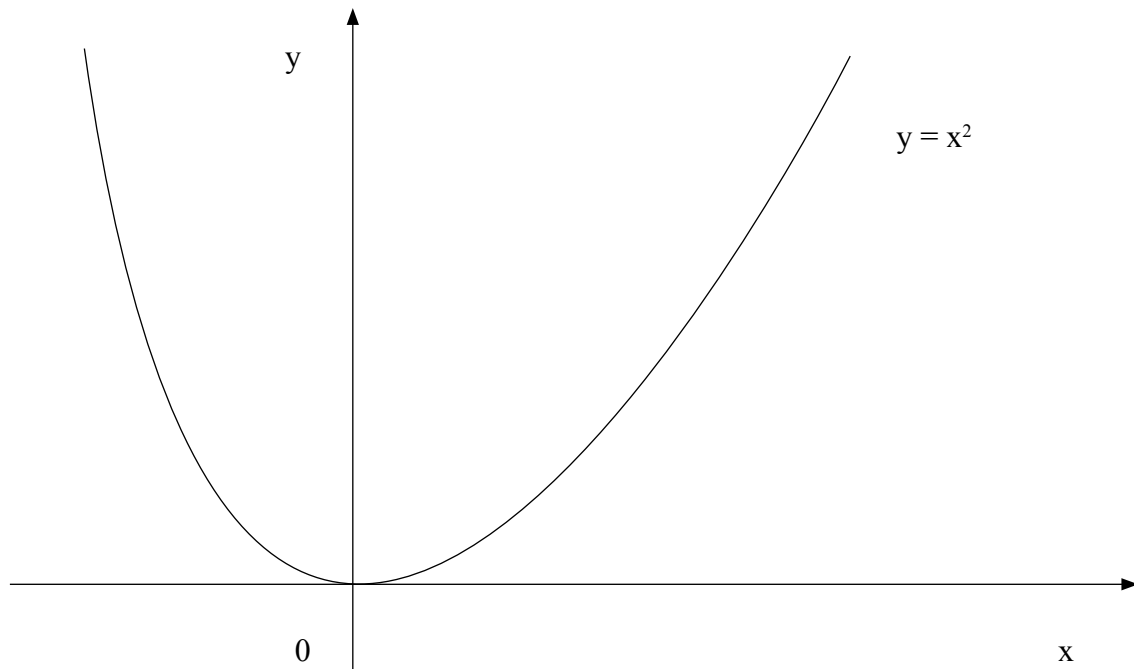
Discrete function – Is a function in which the dependent variable is constant for particular values of the independent variable. It might change to another level for different values of the independent variable and so on e.g fixed cost for various production, capacities or levels.

Fixed cost is Ksh 1000 for   0 – 100 units

             Ksh 1,500 for 101 – 150 units

             Ksh 3,000 for 101 – 150 units

             Ksh 4,000 there after

Diagram 1.2

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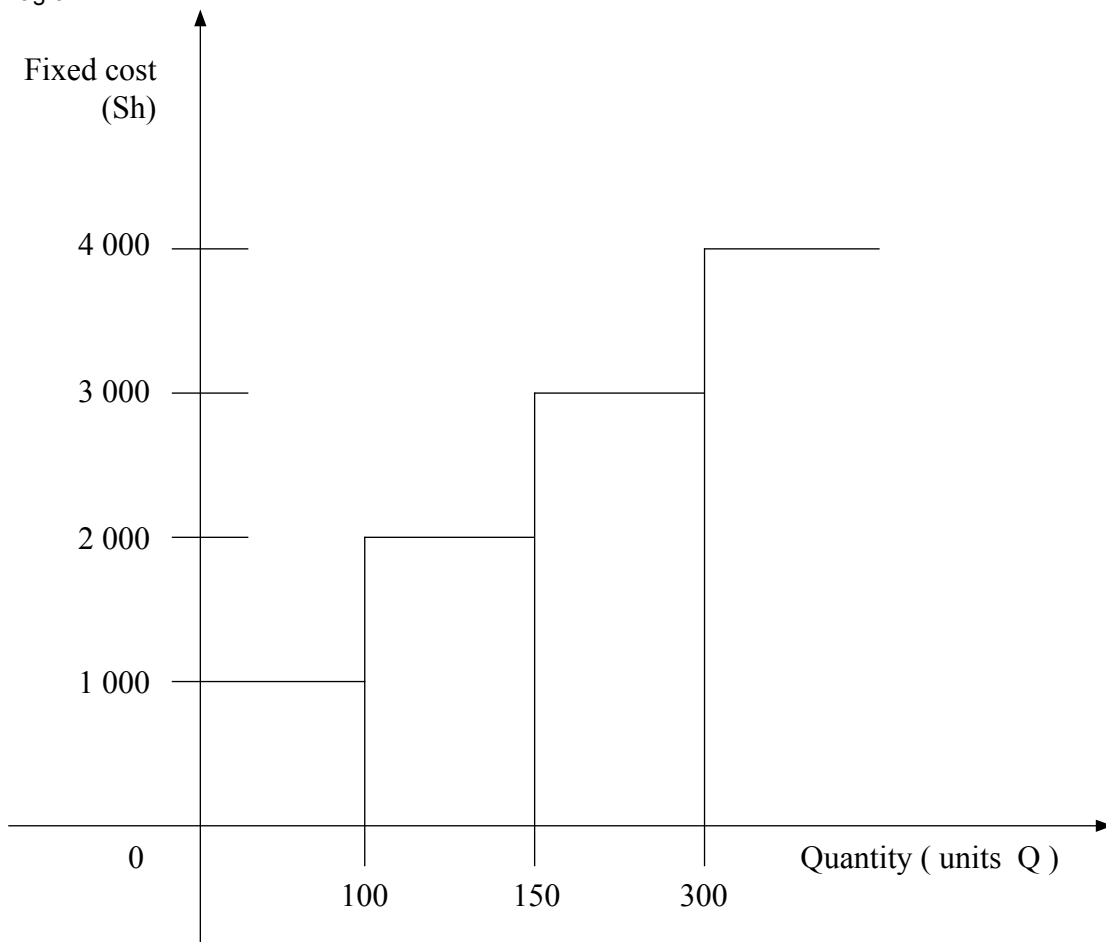
Polynomial Functions and their applications

Linear Functions

The linear function is a polynomial of the form: y = a + bx whereby a represents y – intercept i.e value of y when x is zero, b represents the slope or gradient i.e the value that y changes when x changes by one unit.

Sketches of linear function
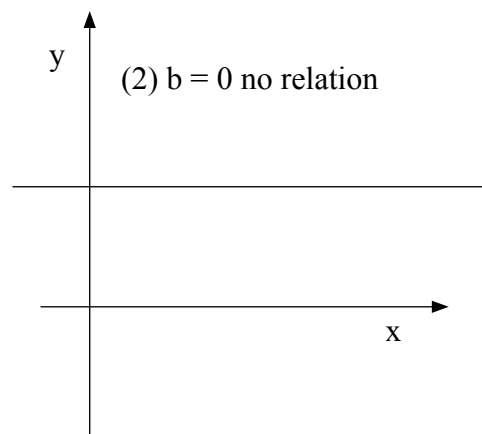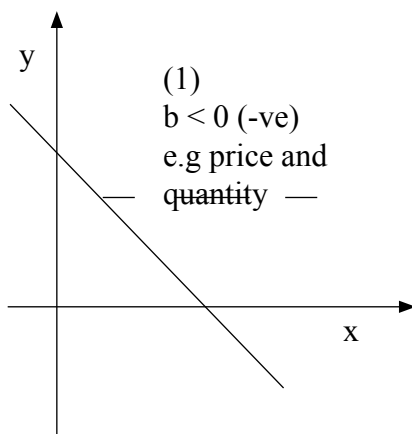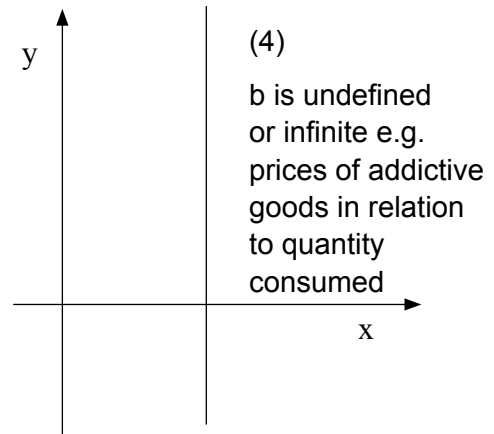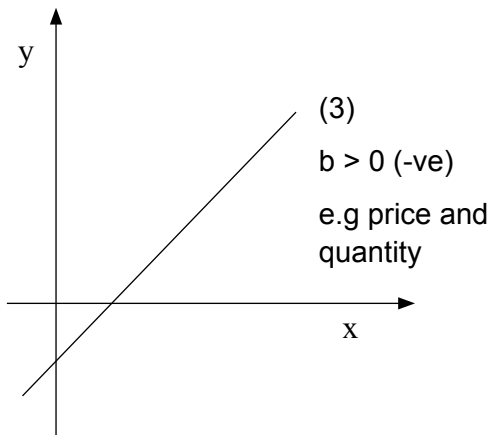
Characteristics of the straight line

1)   It has only one solution i.e can cross the x axis only once, $y = a + bx$

Therefore $0 = a + bx$

$$x = -\frac{a}{b}$$

2)   It has no turning point. The turning point is also known as critical or stationary point

3)   It is completely defined when critical

    i)   Either two points on the line are given, or

    ii)   One point and the slope are specified.

Illustration 1.2

1.   Determine the linear function that goes through the following points

X	2	5
Y	5	17

2.   What is the straight line which has slope b = -0.5 and goes through the point (x,y) = (10,18)

Solution

Slope = $\dfrac{\Delta Y}{\Delta X} = \dfrac{17 - 5}{5 - 2} = \dfrac{12}{3} = 4$

Y = a+ bx

5 = a+ 4x2

5 = a +8

a = -3

Therefore $y = -3 + 4x$

$y = a + bx$

$b = -0.5$

at $(x, y) = (10, 18)$

$18 = a + (-0.5)(10)$

$18 = a - 5$

$a = 23$

$y = 23 - 0.5x$

Applications of linear functions

1. Analysis of commission and wages

2. Demand and supply function (market equilibrium)

3. Fixed assets accounting – depreciation – straight line method

4. Cost volume profit (c-v-p) analysis (profit planning)

Profit = f (price, output, cost)

Profit is a function of costs, prices, volume etc. The problem is how does management manipulate factors which determine profit in order to maximize that profit. For linear C-V-P analysis, we make the following assumptions:

1. Linearity – all functional (mathematical) relationships are linear with respect to the activity level such as quantity produced and so on. Revenue, cost and profit functions are linear.

2. Price per unit is constant e.g. there are no quantity discounts.

3. Unit variable cost is constant e.g. - direct material costs do not change

                - Wages rates are constant

4. Fixed costs do not change

5. All costs can be categorised as either fixed or varied i.e. there are no semi variables cost.

6. All units produced are sold i.e. inter period changes are negligible

7. The only factor that influences revenues, costs and profits is the level of activity

8. There are no demand or other restrictions i.e. no constraints

9. All factors under consideration are known with certainty.

Equations

Let   R = total sales revenue in monetary terms

   P = unit price

   V = unit variable costs

   f = fixed cost

   x = sales in physical units

  Vx = total variable cost

  C = total cost = f + Vx

  Π = profit

Sales in physical units

1. Profit function

   $\Pi = R - C$

   $R = Px$

   $C = Vx + f$

   $\Pi = Px - (Vx - f)$

   $\Pi = Px - Vx - f$

   $\Pi = (P - V)x - f$

   P – V = Contribution margin (C.M)

   $\Pi = CMx - f$

2.   Unit sales x for target profit sh T

From equation 1, $\Pi = T = CMx - f$

$$X = \frac{T + f}{CM}$$

3.   Break even sales units, $X_{b/e}$

At Break Even Point (BEP), R = C so that $\Pi = 0$

   From equation 1

   $0 = CMx - f$

   $f = CMx$

   $X_{b/e} = \dfrac{f}{CM}$

Sales in monetary terms e.g shs R

1. Profit function

$$\Pi = (P - V)x - f \quad R = Px, \text{ therefore } x = \frac{R}{P}$$

$$\Pi = (P - V)\frac{R}{P} - f$$

$$\Pi = \left(\frac{P - V}{P}\right)R - f \quad \left(\frac{P - V}{P}\right) \quad = \text{contribution margin ratio (CMR)}$$

Therefore, Π = CMR×R – f

2. Level of sales revenue for target profit sh. T

From equation 1

$$T = CMR \times R - f$$

$$f + T = CMR \times R$$

$$R = \frac{f + T}{CMR}$$

3. Break even sales revenue, $R_{b/e}$

At B.E.P, Π = 0

From equation 2, T = 0

Therefore, $R_{b/e} = \frac{f}{CMR}$

Note

1). $CMR = \frac{P - V}{P}$

$CMR = \frac{P}{P} - \frac{V}{P} \quad = \textit{variable cost ratio (VCR)}$

$CMR = 1 - VCR$

$CMR + VCR = 1$

2). $VCR = \frac{V}{P} = \dfrac{\textit{Total variable cost ratio}}{\textit{Total Sales Revenue}}$

$= \frac{V \times x}{P \times x}$

Diagram 1.3

Graphical representation
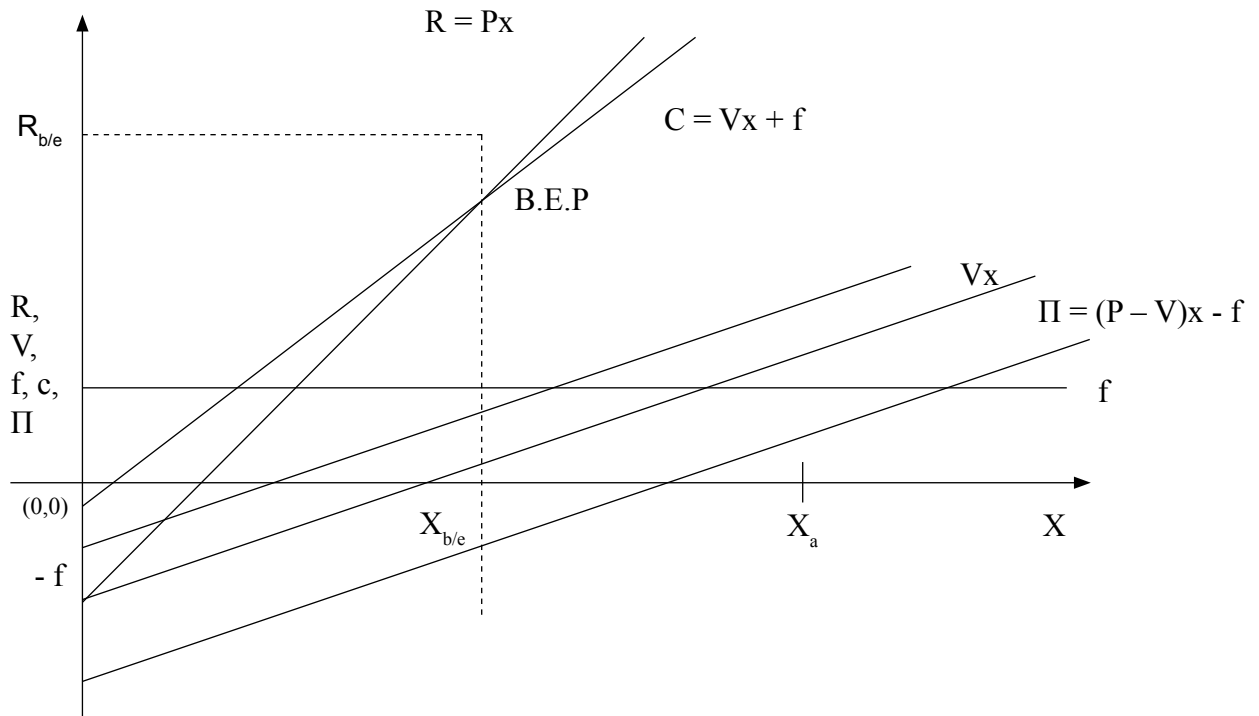
Let actual sales be Xa > xb

Xa –xb = the margin of safety (Mos) i.e the extent by which actual sales exceeds break even sales or conversely, it is the extent by which sales would have to fall before the firm begins to make losses.

Illustration 1.3

Consider a product with the following data.

P = sh 200

V = sh 140

f = sh 800,000

Required

Determine the break even sales units

The profit if sales are 10,000 units

Sales units required to make profit of sh 200,000

Solution

a) X = $\dfrac{F}{CM}$ CM = P-V = 200 - 140 = 60

   X = $\dfrac{800,000}{60}$ = 13,333 units

b) ∏   = CMX –F   = 60 × 10,000 – 800,000

     = (SH 200,000)

c) Target profit = SH 2,000,000

Sales unit

   $X = \dfrac{T + F}{CM}$

   $X = \dfrac{2,000,000 + 800,000}{60}$ = 46,667

▊ Illustration 1.4

A firm sales a product whose data in two periods follows:

Period	Sales	Variable cost	Profit
1	100,000	60,000	20,000
2	150,000	90,000	40,000

Assume the price, unit variable cost and fixed are the same in the two periods

Required :

   1) Determine the fixed cost

   2) Determine the break even sales revenue

   3) What is the profit when sales are sh 600,000

   4) What is the sales revenue required for a profit of sh 110,000

   5) Determine the profit if variable cost incurred is sh 300,000

Solution

1) Fixed cost

$\Pi = CMR \times R - f$

$CMR = \dfrac{P - V}{P}$

$\Pi = R - C$
$\Pi = Px - (Vx + f)$
$20,000 = 100,000 - (60,000 + f)$
$f = 40,000 - 20,000 = 20,000$

2) $R_{b/e} = \dfrac{f}{CMR}$

$\Pi = CMR \times R - f$

$\dfrac{40,000}{100,000} = CMR$ Therefore $CMR = \dfrac{2}{5} = 0.4$

$R_{ble} = \dfrac{2,000}{0.4} = Sh\ 50,000$

3) $\Pi = CMR \times R - f$

$= 0.4 \times 600,000 - 20,000$

$= 220,000$

4) $R = \dfrac{f + T}{CMR}$

$= \dfrac{20,000 + 110,000}{0.4} = sh\ 325,000$

5) $\Pi = CMR \times R - f$
Let sales revenue consistent with this level of variable cost be R

$\dfrac{\text{Total variable cost}}{\text{Total sales revenue}} = VMR$

$R = \dfrac{300,000}{0.6} = sh\ 500,000$

$\Pi = 0.4 \times 500,000 - 20,000$

$\Pi = sh\ 180,000$

Quadratic functions

General formula: $y = a + b_1 x + b_2 x^2$

Where x = independent variable

y = dependent variable

a, b_1, b_2 = constants

Note $b_2 \neq 0$

Properties of quadratic functions

1). Number of solutions (roots) is 2, i.e. it can cross the x-axis twice.

Recall: if $ax^2 + bx + c = 0$

2). A quadratic function has a single turning point.

3). A quadratic function is completely specified once any three points which lie on this curve are given.

Quadratic sketches
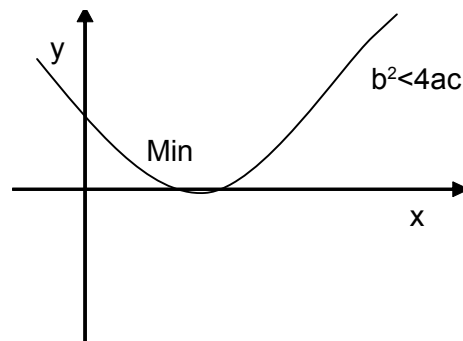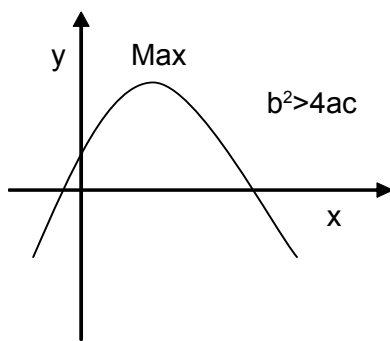
Cubic functions

These are 3rd degree polynomials.

General form: $y = a + b_1 x + b_2 x^2 + b_3 x^3$

Where $b_3 \neq 0$

Properties of a cubic function

i. It will have at least one and at most three real roots i.e. the number of times it crosses the x – axis

ii. It has either two turning points (one max , the other min) or a point of inflexion

iii. It is completely defined once we have four points which lie on the curve

Exponential functions

Exponential functions

These are functions whose, at least, one term as independent variable is part of an exponent or power.

e.g. $y = 15^{2x}$

$y = 72 - 23^x$

$y = 2x + 0.5^x$

▮ Base exponential functions

A special class of exponential functions is of the form: $y = ae^{mx}$ where a, e, m are constants e = Euler's constant, which is associated with natural growth and natural decay.

$$e = \left(1 + \frac{1}{n}\right)^n \quad \text{as } n \to \infty$$

When n = 1

$e = (1+1)^1 = 2$

n = 10

$$e = \left(1 + \frac{1}{10}\right)^{10} = 2.5937$$

Applications for Exponential functions

Growth process

1. Population growth

2. Rate of inflation

3. Growth in value of certain types of assets e.g. land

4. Growth in uses of certain resources e.g. petroleum

Decay process

1. Assets depreciation

2. Decline in purchasing power of the shilling

3. Decline in the rate of incidences of certain disease as medical research and technology

4. Decline in the efficiency of a machine through time.

5. Learning curve. Increased efficiency over time, decrease in time taken to produce units as an employee learns over time.

Illustration 1.5

Population growth

The population growth of a country was 100 million in 1990. It has since grown at 4% per annum. The growth rate is exponential of the form $p = ae^{kt}$

p = population

k = growth rate

t = time in years

a = a constant

Required:

a).   What is the function which describes population growth through time?

b).   What is the population in the year 2010?

Solution

It is of the form $p = ae^{kt}$

Let 1990 be t = 0

Constant % growth rate is the coefficient of t which is k = 4%

For t = 0 (1990) the population, P = 100

Replacing these values in our equation we get.

$100 = ae^{0.04 \times 0}$

a = 100 million   since $e^0 = 1$

Therefore, $P = 100e^{0.04t}$

For year 2010, t = 20

$P = 100e^{0.04t}$

$P = 100e^{0.04 \times 20}$

P = 222.5541 million (4 d. p)

Asset valuation

The resale value v, of a certain piece of industrial equipment has been found to be described by the function

$V = 100 \times 4^{+0.1t}$ or $100(4^{+0.1t})$ *where*   t = time

depreciation rate = 10%

Required :

a)   What was the purchase cost of the equipment?

b)   What is the expected re-sale value after: i) 5 years, ii) 10 years

Solutions

a)   Purchase cost – it is the value when t = 0 which is 100

b)   When t = 5

    V = 100×e-0.5

    V = 60.653

When t = 10

V = 100×e-0.1×10

V = 36.788

Debt collection and management

For a certain type of credit card, the collection percentage of credit issued and exponential function of the time (months) since credit was issued, specifically, the function which approximates this relationship is P = 0.95 (1-e$^{-0.7t}$)

Where t = time   and t ≥ 0 (in months)

    p ≡ percentage of debtors (in shillings)

Required:

a)   Calculate the percentage of debts recovered after

    (i) 3 months   (ii) 7 months

b)   What should be the provision for bad debts?

Solutions

a)   (i)   3 months

        P = 0.95(1-e$^{-0.7×3}$)

         = 0.95(1-e$^{-2.1}$)

        = 83%

    (ii)   7 months

        P = 0.95(1-e$^{-0.7×7}$)

        P = 0.95(1-e$^{-4.9}$)

        P = 94%

b)   As t→∞, e$^{-0.7t}$→0

    Therefore, (1-e$^{-0.7t}$) →1 hence p→0.95

    Interpretation: Eventually 95% of debts will be collected and 5% will be bad debts.

Diagram 1.7

$$y = 0.95(1-e^{0.7t})$$
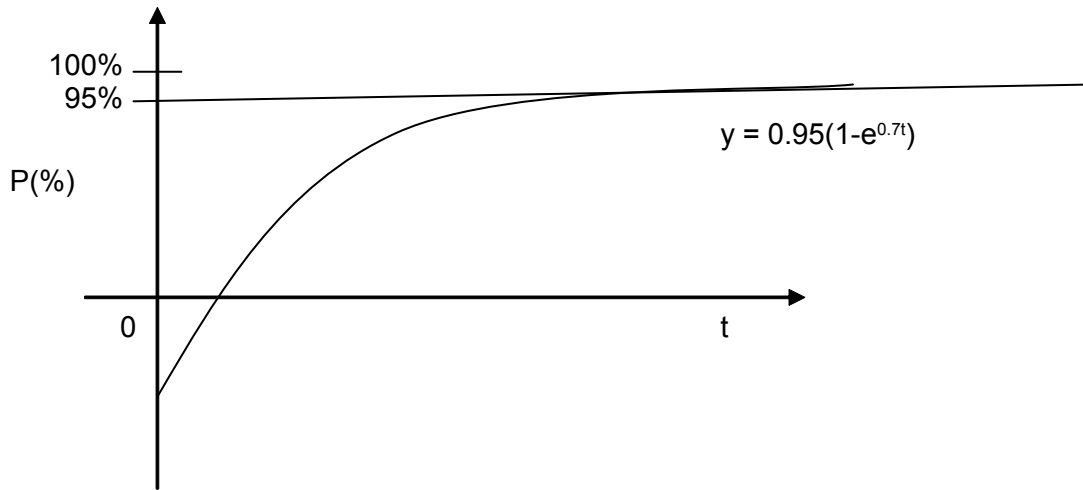
1). Compound interest

If a certain amount of money Sh p is deposited in an account which earns a compound interest at a rate of I % per annum, the amount after n years (interest + principal) is given as $S = P(1 + I)^n$, if P = sh 1,000 and I = 8%

What is the sum after 25 yrs if interest is compounded annually?

Solution

S = 1000(1+0.08)^{25}

S = sh. 6,848

Logarithmic functions

A logarithmic is a power to which a base must be raised in order to give a certain number i.e. a logarithm is an exponent. Consider the following equation.

$2^3 = 8$

The table below shows the logarithmic and exponential form

Exponential equations   Logarithmic equation

$10^4 = 10,000$ $4 = \log_{10} 10,000$

$2^4 = 16$ $4 = \log_2 16$

$5^2 = 25$ $2 = \log_5 25$

$10^2 = 100$ $2 = \log_{10} 100$

1.3 MATRICES

Fast Forward: Due to their widespread use, considerable effort has been made to develop efficient methods of matrix computing.

A matrix is a rectangular array of numbers in rows and columns enclosed in brackets. A matrix that has m rows and n columns is said to be an m × n matrix. For example

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$ is a 2 × 2 matrix.

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 2 \end{bmatrix}$$ is a 2×3 matrix.

Matrix names are usually represented by capital letters while elements of a matrix are represented by a lower case subscripted letters : a_{ij} stands for real numbers where a_{ij} is the element in the ith row and j^{th} column.

Types of matrices

Row matrix – is a matrix which has only one row.

For example (2, 2), (2, 3, -2, 5)

Column matrix – this is a matrix which has only one column.

For example, $$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$ $$\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Square matrix – this is a matrix in which the number of rows is equal to the number of columns.

$$A = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$$

Diagonal matrix – this is a square matrix that has zeros everywhere except on the main diagonal - that is the diagonal running from the upper left to the lower right.

For example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A matrix s diagonal if a_{ij} is equal to zero for all elements when i≠j and a_{ij} is not equal to zero and at least i = j

1).   Identity matrix – this is a square matrix with the leading diagonal elements all equal to one and all other elements equal to zero i.e. it is a diagonal matrix whose diagonal matrix is equal to one.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If you multiply a matrix by an identity matrix, you will get the same matrix regardless if you pre-multiply or post-multiply.

$$A_{mm}\, I_{mm} = A_{mm} = I_{mm}\, A_{mm}$$

2).   Zero or null matrix – this is a square matrix where every element is zero.

Note:

   a)   When null matrix is added or subtracted from another matrix that matrix remains unchanged.

   b)   Pre or post-multiplying a matrix with a null matrix results in another matrix.

3).   Scalar matrix – is a diagonal matrix whose diagonal elements are equal.

$$A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular matrix – A square matrix whose element a_{ij} is equal to zero. Whenever i<j, it is called a lower triangular matrix, whenever i>j, it is called an upper triangular matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 8 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 10 & 2 & 4 \\ 0 & 4 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices are used because they are able to summarise data. Through matrices, operations, formulation and solution of data are simplified which would almost be impossible or complicated in conventional or algebraic operations. Knowledge of matrices can be used in solving problems that arise in various fields of operations such as:

1. Simultaneous equations
2. Markov processes
3. Input-output analysis
4. Linear programming
5. Game theory

Operations of matrices

The following operations can be carried out in matrices:

1. Addition
2. Subtraction
3. Multiplication
4. Determinant
5. Transposition
6. Matrix Inversion

Matrix addition and subtraction

If A and B are two matrices of the same order then the addition of A and B is defined to be the matrix obtained by adding the corresponding elements of A and B.

Illustration 1.6

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 9 \\ 5 & 6 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 8 \\ 3 & 7 & 6 \\ 2 & 6 & 4 \end{bmatrix}$$

$$A + B = C = \begin{bmatrix} 3 & 2 & 10 \\ 7 & 10 & 15 \\ 7 & 12 & 12 \end{bmatrix}$$

To subtract one matrix from another, the rule is to subtract corresponding elements just like matrix addition. (Note: order of subtraction is important).

Multiplication of two matrices

The main rule is that, for it to be possible to multiply two matrices, the number of columns in the first matrix should equal to the number of rows in the second. For instance, 2×2 and 1×2 cannot be multiplied because there are two columns in first and one row in the second. On the other hand, 2×2 and 2×1 can be multiplied and it should give 2×1 matrix.

Rule :

$$
\begin{bmatrix}
\text{Matrix size (a × b)} \quad × \quad \text{Matrix size (b × c)} \\
\\
\text{gives} \\
\\
\text{Matrix (a × c)}
\end{bmatrix}
$$

Illustration 1.7

$$
A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 7 & 4 \end{bmatrix} \quad
B = \begin{bmatrix} 8 & 0 & 5 & 4 \\ 3 & 2 & 1 & 1 \end{bmatrix}
$$

A = 3×2 while B = 3 × 4 hence C = 3×4

$$
\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 7 & 4 \end{bmatrix} \times
\begin{bmatrix} 8 & 0 & 5 & 4 \\ 3 & 2 & 1 & 1 \end{bmatrix}
$$

The general method of multiplication is that the elements in row *m* of the first matrix are multiplied by the corresponding elements in columns *n* of the second matrix and the products obtained are then added giving a single number.

$$
= \begin{bmatrix}
3×8+1×3 & 3×0+1×2 & 3×5+1×11 & 3×4+1×1 \\
2×8+4×3 & 2×0+4×2 & 2×5+4×11 & 2×4+4×1 \\
7×8+4×3 & 7×0+4×2 & 7×5+4×11 & 7×4+4×1
\end{bmatrix}
= \begin{bmatrix}
27 & 2 & 26 & 13 \\
28 & 8 & 54 & 12 \\
68 & 8 & 79 & 32
\end{bmatrix}
$$

Note:

1. The size of the final matrix AB is 3×4 that is, it has the same number of rows of A and same number of columns of B.
2. A×B ≠ B×A
3. If AB is possible then it does not necessarily mean BA is possible unless the two matrices A and B are square matrices.

Transposition

The transpose of a matrix m×n is an n×m matrix denoted by A' or A^T whose rows are the columns of A and whose columns are the rows of A.

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{1n} \\
a_{21} & a_{22} & a_{2n} \\
a_{m1} & a_{m2} & a_{mm}
\end{bmatrix}
\qquad
A' \text{ or } A^T = \begin{bmatrix}
a_{11} & a_{21} & a_{m1} \\
a_{12} & a_{22} & a_{m2} \\
a_{1n} & a_{2n} & a_{mn}
\end{bmatrix}
$$

Note:

The transpose of a diagonal matrix is the same diagonal matrix.

If a square matrix and its transpose are equal, that is, if $a_{ij} = a_{ji}$ for all *i* and *j* the matrix is said to be symmetric about its main diagonal.

$$A = \begin{bmatrix} -1 & 5 & -3 \\ 5 & 0 & 4 \\ -3 & 4 & 9 \end{bmatrix}$$

$$A' = \begin{bmatrix} -1 & 5 & -3 \\ 5 & 0 & 4 \\ -3 & 4 & 9 \end{bmatrix}$$

Hence matrix A is a symmetric matrix.

A symmetric matrix that reproduces itself when multiplied by itself is said to be idempotent; that is, A is idempotent if

A' = A

AA = A

Determinant of a matrix

This is a scalar obtained from the elements of a matrix by a specified operation, which is a characteristic of a matrix. The determinant is only specified for a square matrix.

For a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_2^2 \end{bmatrix} \qquad |A| \text{ or } A^{det} = a_{11}a_{22} - a_{12}a_{21}$$

Illustration 1.8

$$A = \begin{bmatrix} 8 & 9 \\ 6 & -5 \end{bmatrix} \quad |A| = 8×-5 - 9×6$$

$$= -40-54$$
$$= -94$$

$$B = \begin{bmatrix} 1 & 2 \\ 1 & -10 \end{bmatrix} \quad |B| = (1×-10)-(2×1)$$

$$= -10-2$$
$$= -12$$

For a 3×3 matrix.

There are two methods to get a determinant:

  a)   Artistic method
  b)   Co-factor method/ Laplace method

◼Artistic method

Given A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The following steps are followed:

1). Rewrite the first two columns of the matrix to the right of the original matrix.
2). Locate the elements of the three primary diagonal P_1, P_2 and P_3 and those of the secondary diagonal S_1, S_2 and S_3.
3). Multiply the elements of each primary and each secondary diagonal.
4.) The determinant equals the sum of the products for these three primary diagonals minus the product of these three secondary diagonals.
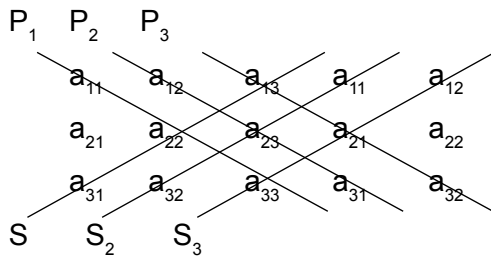
$|A| = [a_{11} \quad a_{22} \quad a_{33} + a_{12} \quad a_{23} \quad a_{31} + a_{13} \quad a_{21} \quad a_{33}] - [a_{31} \quad a_{22} \quad a_{13} + a_{32} \quad a_{23} \quad a_{11} + a_{33} \quad a_{21} \quad a_{12}]$

Illustration 1.9

A = $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 4 \\ 3 & -2 & 1 \end{bmatrix}$
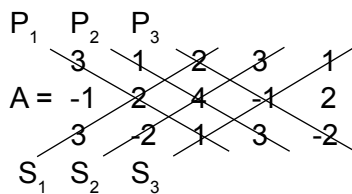
$|A| = [(3×2×1) + (1×4×3) + (2× -1× -2)] - [(3×2×2) + (-2×4×3) + (1× -1×1)]$
$= (6+12+4) - (12-24-1)$
$= 22-(-13)$
$= 35$

Note: It only works for a 3×3 matrix and cannot be extended to a 4×4 matrix or higher order matrix.

Cofactor (Laplace method)

For any square matrix there can be found a matrix of cofactors, 'A_c'. The matrix of cofactors will have dimensions of A and will consist of elements Cij which are known as cofactors of each element aij obtained in A. The corresponding cofactors Cij will be determined as follows:

Either mentally or with a pencil cross off row i and column j in the original matrix. Focus on the remaining elements. The remaining elements will form a sub-matrix of the original matrix.

Find the determinant of the remaining sub-matrix. This determinant is called the minor of element a_{ij}.

The cofactor a_{ij} is found by multiplying minor by either positive one or negative one depending on the position of element a_{ij} that is (-1)i+j(minor).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a11 \begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12 \begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13 \begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

Illustration 1.10

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix} \quad signs = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$A = \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} \quad -1 \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} \quad +2 \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}$$

A = 3|(2×1) − (-2×4)| -1|(-1×1) − (3×4)| +2|(-1×-2) − (3×2)|

A = 3|2+8| -1|-1−12| +2|2−6|

A = 30+13-8

A = 35

$$A_c = \begin{array}{ccc} + \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} & - \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} & + \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} \\ - \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 1 \\ 3 & -2 \end{vmatrix} \\ + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & - \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} & + \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \end{array}$$

Therefore

$|A| = (1×3) + (2×-3) + (-2×-14)$

$|A| = 35$

Inverse of a matrix

If for an n×n matrix, A there is another n× n matrix, B such that their product is the identity matrix of product n× n.

$A_{n×n} × B_{n×n} = I_{n×n}$

Then B is said to be inverse or reciprocal of A. A matrix which has an inverse is known as a non-singular matrix. A matrix which has no inverse is said to be a singular matrix.

The inverse of a 2×2 matrix is calculated by interchanging the elements in the leading diagonal and multiplying the elements in the other diagonal by -1, the resultant matrix is then multipliead by the inverse of the original matrix's determinant $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a22 & -a12 \\ -a21 & a11 \end{bmatrix}$$

Illustration 1.11

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$$

For a 3×3 matrix

Procedure:

(1)   Determine the matrix of cofactors A_c from matrix A.

The cofactor of any element a_{ij} (known as c_{ij}) is the signed minor associated with that element.

The sign is not changed if (i+j) is even and it is changed if (i+j) is odd. Thus the sign alternated whether vertically or horizontally, beginning with a plus in the upper left hand corner.

i.e. 3 x 3 signed matrix will have signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

For matrix A   = $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{bmatrix}$

Hence the cofactor of element a_{11} is $m_{11} = 3$, cofactor of a_{12} is $-m_{12} = -1$ the cofactor of element a_{13} is $+m_{13} = 3$ and so on.

Matrix of cofactors of A: A_c = $\begin{bmatrix} 3 & -1 & 3 \\ -4 & 2 & -5 \\ -2 & 1 & -2 \end{bmatrix}$

in general for a matrix M   = $\begin{pmatrix} a & b & c \\ e & d & f \\ g & h & i \end{pmatrix}$

Cofactor of a is written as A, cofactor of b is written as B and so on.

Hence matrix of cofactors of M is written as

= $\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$

The determinant of a n×n matrix can be calculated by adding the products of the element in any row (or column) multiplied by their cofactors. If we use the symbol Δ for determinant.

Then $\Delta = aA + bB + cC$

or

= dD + eE + fF e.t.c

Note: Usually for calculation purposes we take $\Delta = aA + bB + cC$

Hence in the example under discussion

$\Delta = (4 \times -3) + (2 \times 2) + (3 \times 3) = 1$

(2)   Determine the adjoint matrix which is the transpose of A_c.

The adjoint of matrix $\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$ is written as

$\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$ i.e. change rows into columns and columns into rows (transpose)

The inverse of A is found by multiplying the adjoint matrix by the reciprocal of determinant of A.

The inverse of the matrix $\begin{pmatrix} a & b & e \\ d & e & f \\ g & h & i \end{pmatrix}$

is written as $\dfrac{1}{\text{determinant}}$ $\times$ (adjoint of the matrix of cofactors

i.e. $A^{-1} = \dfrac{1}{\Delta}$ $\times$ $\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

Where $\Delta = aA + bB + cC$

Hence inverse of $\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix}$ is found as follows

$\Delta = (4 \times -3) + (2 \times 2) + (3\,(3) = 1$

A = -3   B = 2   C = 3

D = 9   E = -6   F = -8

G = -16   H = 11   I = 14

Properties of a determinant

1. If two rows or columns are interchanged a matrix will retain absolute value of the determinant but it will change its sign.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If rows are changed into columns and columns into rows then the determinant will remain unchanged.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

3. If two rows or columns are identical in a matrix then the determinant vanishes.

$$A = \begin{vmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 1 & 4 & 5 \end{vmatrix} \quad |A| = 0$$

4. If any row or column of a matrix is multiplied by a constant k the determinant obtained is k times the original determinant.

$$\begin{vmatrix} a1 & a2 & a3 \\ kb_1 & kb_2 & kb_3 \\ c1 & c2 & c3 \end{vmatrix} = k \begin{vmatrix} b1 & b2 & b3 \\ a1 & a2 & a3 \\ c1 & c2 & c3 \end{vmatrix}$$

5. If to any row or column is added k times the corresponding element of another row or column the determinant remain unchanged.

$$\begin{vmatrix} a1 + kb1 & a2 + kb2 \\ b1 & b2 \end{vmatrix} = \begin{vmatrix} a1 & a2 \\ b1 & b2 \end{vmatrix}$$

6. If any row or column is the sum of two or more elements then the determinant can be expressed as a sum of two or more determinants.

$$\begin{vmatrix} a_1+k_1 & a_2+k_2 & a_3+k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c2 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7. If determinant vanishes by putting x = a then x-a is a factor of determinant.

$$A = \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \qquad B = \begin{matrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{matrix}$$

Then $AB| = |A| \times |B|$

■ Applications of matrices

1. Solving simultaneous equations.
2. Markov processes
3. Input-output analysis

1. Solving simultaneous equations.

Given a set of simultaneous equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}a_3 + \ldots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n = b_2$$

$$a_{n1}x_{1+}a_{n2}x_2 + a_{n3}x_3 + \ldots a_{nn}x_n = b_n$$

This can be expressed in matrix form:

$$\begin{bmatrix} a_{11} & a & a_1 & a_{1n} \\ a_{21} & a_{22} & a & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}$$

Coefficient matrix   Unknowns   Constants

Simultaneous equations can be solved by one of the following methods:-
1. elimination
2. graphical
3. substitution
4. matrix:-

(a) **Cramer's rule method**

Using this method the solution of

A n× n X n× n = Cn×1

Can be obtained through the ratio:

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{1n} \\ b_2 & a_{22} & a_{2n} \\ a_n & a_{n2} & a_{nn} \end{vmatrix}}{|A|}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$$

$$X_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{n1} \\ a_{21} & b_2 & a_{n2} \\ a_{n1} & b_n & a_{nn} \end{vmatrix}}{|A|}$$

$$X_n = \frac{\begin{vmatrix} a_{11} & b_{12} & b_1 \\ a_{21} & b_{22} & b_2 \\ a_{n1} & a_{n2} & b_n \end{vmatrix}}{|A|}$$

Note: The denominator is the coefficient matrix; the numerator is the determinant of the coefficient matrix with the *j* column replaced by the column of the constant term from the right hand side of the equation.

Illustration 1.12

    2x –y=7

    x+3y=14

Solution:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$x = \dfrac{\begin{vmatrix} 7 & -1 \\ 14 & 3 \end{vmatrix}}{7} = \dfrac{35}{7} = 5$$

$$y = \dfrac{\begin{vmatrix} 2 & 7 \\ 1 & 14 \end{vmatrix}}{7} = \dfrac{21}{7} = 3$$

(b) Inverse method

Given

$$A_{n \times n} X_{n \times 1} = b_{n \times 1}$$
$$A^{-1}_{n \times n} A_{n \times n} X_{n \times 1} = A^{-1}_{n \times n} b_{n \times 1}$$
$$I_{n \times n} X_{n \times 1} = A^{-1}_{n \times n} b_{n \times 1}$$
$$X_{n \times 1} = A^{-1}_{n \times n} b_{n \times 1}$$

Illustration 1.13

$$2x - y = 7$$
$$x + 3y = 14$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{matrix} x \\ y \end{matrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$\frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 35 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

2. Markov processes

Markov analysis is a technique that deals with probabilities of future occurrences by analysing presently known probabilities. The technique has numerous applications in business including market share analysis, bad debt prediction, university enrolment prediction, determining whether a machine will break down in future, and so on.

Markov analysis makes the assumption that the system starts at initial state of condition for instance two competing manufacturers might have 40% and 60% of the market sales respectively as initial state. Perhaps in two months the market shares of the two companies will change to 45% and 55% of the market share respectively. Predicting these future states involves knowing the systems likelihood or probability of changing from one state to another. For a particular problem these probabilities can be collected and placed in a matrix table. This matrix of transition probability shows the likelihood of the system would change from one period to the next. This is the Markov process and enables us to predict future states and conditions.

Assumptions

1. There are a limited or finite number of states.
2. The states in the system are collectively exhaustive. This means that an object can only belong to or subscribe to a state within that system but not outside.
3. The states in the system are mutually exclusive that is an object in the system can be found in one and only one state.
4. The probability that an object will shift from one state to another during any specified period of time remains constant from period to period or transition to transition.
5. We can predict any future state from the previous state (initial state) and the matrix of transition probability.

Transition matrix, T

TO

$$S_1 \quad S_2 \quad S_M$$

$$\text{FROM} \quad \begin{bmatrix} S_1 & P_{11} & P_{12} & P_{1m} \\ S_2 & P_{21} & P_{22} & P_{2m} \\ S_M & P_{1m} & P_{2m} & P_{mm} \end{bmatrix}$$

Where: Si/Sj represent state of a given phenomena

Pij represent the probability that the object shift from state i to state j (transition matrix)

The transition probabilities for Markov processes are normally organised in a matrix called transition matrix.

Properties of a transition matrix

(1) The sum of probabilities across a row is one.

(2) The row indicates one source of object to all its destinations.

(3) Transition matrix is a square matrix.

(4) Probabilities Pij are obtained empirically that is by observation and data collection.

Illustration 1.14: For 2×2 matrix

In a certain country there are two daily newspapers: The Citizen and The Mirror. A researcher interested in the reading habit of this country found the following:

Of the readers who read Citizen on a given day 50% do so following day while the rest change to the Mirror. Of those who read Mirror on a given day 40% change to the Citizen the following day.

Yesterday the readership levels were 30% Citizen and 70% Mirror. Assume all conditions hold.

Required:

(a) Determine the readership levels of both dailies:-

    (i). Today

    (ii). Tomorrow

(b) If this process persists long enough, what will be the eventual readership?

Solution

Initial state

    C   M

    (0.3 0.7)

Transition matrix

	To		
		C	M
From	C	0.5	0.5
	M	0.4	0.6

(a) (i) Today

$$(0.3\ 0.7)\begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} = (0.43\quad 0.57)$$

    C = 0.43

    M = 0.57

  (ii) Tomorrow

$$(0.43\quad 0.57)\begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} = (0.443\ 0.557)$$

Equilibrium state

In a Markov process a situation reaches when there is no further gain in market share. This situation is known as equilibrium state or steady state or the long-run state.

By definition equilibrium condition exist if state probabilities do not change after a large number of periods. At equilibrium, state probability from next period equals state probability of this period.

p = long-run readership level of Citizen

q = long-run readership level of Mirror.

$(p \quad q)T = (pq)$

$$(p \quad 1-p)\begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} = (p \ 1-p)$$

0.5p + 0.4 − 0.4p = p

0.1 + 0.4 = p

0.9p = 0.4

p = 4/9 = 0.444

Citizen = 44.45

Mirror = 55.6%

Applications

(1) Marketing to future market shares.
(2) Finance to predict share prices in stock exchange market.
(3) Human resource management to analyse shifts of personnel among various organisation units like departments, divisions and so on.
(4) Financial accounting to estimate provision for bad debts.

3. Input-output analysis

The sectors in the economy are interdependent for continuity. In an economy the sectors depend on each other for the continued production of output.

In input-output analysis the main task is to determine the output required from each sector in the economy in order to satisfy both inter-sectorial and external demand. The technique was developed by Wassily Leontief in 1961.

Types of input-output models

Input-output models are of two types:-

1. The closed in which the entire production is consumed by those participating in production.
2. Open in which some of the products are consumed by an external body.

In the closed model, one seeks the income of each participant in the system. In the open model, one seeks the amount of production needed to achieve a future demand when the amount of production needed to achieve current demand is known.

The most useful application of input-output analysis in the economy or the common broker is the ability to tell how the change in demand for one industry affects the entire economy.

Illustration 1.15

Input Output table

Table 1.2

| | To | | | Final | Total |
FROM	Agric	Industry	Service	Demand	Demand (output)
Agric	300	360	320	1080	2060
Industry	450	470	410	800	2130
Service	610	500	520	270	1900
Primary inputs	700	800	650	-	-
Total inputs	2060	2130	1900	-	-

NB: In the above table, one should be able to interpret the table e.g. of the total demand of 2060 metric tonnes from the agricultural sector, 300 is produced for the agricultural sector, 360 for industrial sector, 320 for the service sector and 1080 metric tones makes up the <u>final demand.</u>

The final demand is the additional demand besides the sectoral demand which is normally made by other users e.g. government, foreign countries, other manufacturers not included in the other sectors.

For production of items besides the inputs from other sectors namely labour capital e.t.c

Technical coefficients : ('to' sectors)

Agriculture $\quad \dfrac{300}{2060} \quad = \quad 300 \quad = \quad 0.14$

$450 = \dfrac{450}{2060} = 0.22$

$610 = \dfrac{610}{2060} = 0.30$

Industry $\quad 360 = \dfrac{360}{2130} = 0.7$

$470 = \dfrac{470}{2130} = 0.22$

$500 = \dfrac{500}{2130} = 0.23$

Service $\quad 320 = \dfrac{320}{1900} = 0.17$

$410 = \dfrac{410}{1900} = 0.22$

$520 = \dfrac{520}{1900} = 0.27$

The matrix of technical coefficients is:

From	To			Final Demand	Total Demand (output)
	Agric	Industry	Service		
Agric	0.14	0.7	0.17	1080 (y$_1$)	2060
Industry	0.22	0.22	0.22	800 (y$_2$)	2130
Service	0.30	0.23	0.27	270(y$_3$)	1900
Primary inputs	X	x	x	-	-
	2060(x$_1$)	2130(x$_2$)	1900(x$_3$)	-	-

From the above table, we may develop the following equations

$$0.14x_1 + 0.7x_2 + 0.17x_3 + y_1 = x_1$$

$$0.22x_1 + 0.22x_2 + 0.22x_3 + y_2 = x_2$$

$$0.30x_1 + 0.23x_2 + 0.27x_3 + y_3 = x_3$$

$$\underbrace{\begin{pmatrix} 0.14 & 0.17 & 0.17 \\ 0.22 & 0.22 & 0.22 \\ 0.30 & 0.23 & 0.27 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x3 \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X}$$

Let the coefficient matrix be represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

∴ Equation (*) may be written as

AX + Y = X

Y = X – AX

Y = X (I-A)

⇒ $(I – A)^{-1} Y = X$

The matrix I – A is known as Leontief Matrix

Technical Coefficients

These show the units required from each sector to make up one complete product in a given sector e.g. in the above matrix of coefficients it may be said that one complete product from the agricultural sector requires 0.14 units from the agricultural sector itself, 0.22 from the industrial sector and 0.30 from the service sector

NB: The primary inputs are sometimes known as "value added"

Illustration 1.16

Three clients of Disrup Ltd P, Q and R are direct competitors in the retail business. In the first week of the year P had 300 customers Q had 250 customers and R had 200 customers. During the second week, 60 of the original customers of P transferred to Q and 30 of the original customers of P transferred to R. similarly in the second week 50 of the original customers of Q transferred to P with no transfers to R and 20 of the original customers of R transferred to P with no transfers to Q.

Required

a) Display in a matrix the pattern of retention and transfers of customers from the first to the second week
(4 marks)

b) Re-express the matrix that you have obtained in part (a) showing the elements as decimal fractions of the original numbers of customers of P, Q and R
(2 marks) *Refer to this re-expressed matrix as B*

c) Multiply matrix B by itself to determine the proportions of the original customers that have been retained or transferred to P, Q and R from the second to the third week.
(4 marks)

d) Solve the matrix equation $(xyz)B = (xyz)$ given that $x + y + z = 1$ (8 marks)

e) Interpret the result that you obtain in part (d) in relation to the movement of customers between P, Q and R
(2marks)
(Total 20 marks)

Solution

a) Think of each row element as being the point from which the customer originated and each column element as being the destination e.g. 210 customers move from P to P, 60 move from P to Q and 30 move from P to R. The sum of the elements of the first row totalling 300, that is the number of customers originally with P.

Hence required matrix is

$$
\begin{array}{cc}
 & \begin{array}{ccc} P & Q & R \end{array} \\
\begin{array}{c} P \\ \text{From} \quad Q \\ R \end{array}
\begin{pmatrix} 210 & 60 & 30 \\ 50 & 200 & 0 \\ 20 & 0 & 180 \end{pmatrix}
\begin{array}{c} \text{To} \\ \text{row total } 300 \\ \text{row total } 250 \\ \text{row total } 200 \end{array}
\end{array}
$$

b). The requirement of this part is to express each element as a decimal fraction of its corresponding row total. The second row, first element is therefore 50/250, that is 0.2 and the second element is therefore 200/250 that is 0.8.

$$
\text{Hence B} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix}
$$

c).
$$
\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix}
\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix}
=
\begin{pmatrix} 0.54 & 0.30 & 0.16 \\ 0.30 & 0.68 & 0.20 \\ 0.16 & 0.02 & 0.82 \end{pmatrix}
$$

STUDY TEXT

The result can be checked by the normal rules of matrix multiplication.

d) $(x \quad y \quad x) \quad X \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} = \quad (x \quad y \quad z)$

This produces from the first row

$0.7x + 0.2y + 0.1z = x$

Which reduces to $0.2y + 0.1z = 0.3x$

Or $2y + z = 3x$ ………………………...(i)

Or

The second row produces, $0.2x + 0.8y = y$

Reducing to $0.2x = 0.2 y$

$X = y$ ………………………..(ii)

Or

The third row produces $0.1x + 0.9z = z$

Reducing to $0.1x = 0.1z$

$X = z$ ………. (iii)

At this point you will notice that condition h (ii) and condition (iii) produce $2x + x = 3x$ when substituted into condition (i), we therefore need extra condition $x + y + z = 1$ to solve the problem.

Thus $x + x + x = 1$

Or $3x = 1$

That is $x = \frac{1}{3}$

Leading to $x = \frac{1}{3}, \quad y = \frac{1}{3}, \quad z = \frac{1}{3}$

e).   In proportion terms this solution means that P, Q, and R will in the long term each have one third of the total customers

To apply the input-output model the sectors in the economy should be (assumptions):-

1. Finite.
2. Mutually exclusive
3. Output from all sectors in the economy can be expressed in the same monetary unit.
4. Input requirement per unit for each sector remains constant irrespective of the number of units produced from the sector and are known and are estimated in reliable degree off accuracy that is A = technology matrix.
5. External demand from each sector is known.

Technology matrix

The technology matrix (technical coefficient matrix) is usually denoted by A and will describe the relation a sector has with others. The technology matrix A will be a matrix such that each column vector represents a different industry and each corresponding row vector represents what that industry inputs as a commodity in the column industry for example the technology matrix A below represents the relationship between industries of construction and farming and clothing.

<div align="center">

Inputs

Farming   construction   clothing

</div>

$$\text{Outputs} \quad \begin{matrix} \text{Farming} \\ \text{Construction} \\ \text{Clothing} \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The relationships between these industries in the example are as follows:

The entry a_{11} holds the number of units the farmer uses from his own products for producing 1 unit.

The entry a_{21} holds the number of units the farmer needs for construction to produce one more unit of farming.

The entry a_{31} holds the number of units the farmer needs for clothing to produce 1 more unit of farming.

Input-output model derivation

Let X_1 = total output required from sector 1

     X_2 = total output required from sector 2

     a_{ij} = number of units required in sector i as input to sector j.

Table 1.3

	To		Total output = total input to each sector	External demand
	S_1	S_2		
From $\quad S_1$	$a_{11}x_1$	$a_{12}x_2$	X_1	d_1
S_2	$a_{21}x_1$	$a_{22}x_2$	X_2	d_2

Let the external demand from sector 1 be D_1 and from sector 2 be D_2.

Sector 1 $x_1 = a_{11}x_1 + a_{12}x_2 + d_1$

Sector 2 $x_2 = a_{21}x_1 + a_{22}x_2 + d_2$

$$\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$X = AX + D$

$\qquad$ X-AX = D

$\qquad$ (I-A)X = D

$\qquad$ $X = (I-A)^{-1}D$

Where A is the technological matrix

$\qquad$ D is the external demand

$\qquad$ (I-A) is called Leontief matrix,

$\qquad$ $(I-A)^{-1}$ is called Leontief inverse matrix

Note: The primary purpose of input-output analysis is to calculate the level required at a particular level of intermediate and final demand.

Input-output analysis can also be used to study sectors of an economy either using the closed model or the open model.

In a closed economy the sum of the elements in a column will always total 1.

In general each entry in the technological matrix is represented as a_{ij} which is equal to xij/xj where xj represent the physical output of sector j.

1.4 CALCULUS

Calculus is concerned with the mathematical analysis of change or movement. There are two basic operations in calculus:

i)   Differentiation
ii)   Integration

These two basic operations are reverse of one another in the same way as addition and subtraction or multiplication and division.

Utilities of calculus in Business

1. Often we may be interested in optimisation e.g. maximisation of revenue, profit, productivity.
    Minimisation of cost, waste

2. Calculus is used in marginal analysis e.g. change of marginal cost (MC) to total cost (TC).
   Marginal profit (MP) to total profit (TP)
   Marginal revenue (MR) to total revenue (TR) etc

Origin:German: Wilhem Godfried Von Loibnitz (17th C), used this technique to solve problems in astronomy

Briton: Sir Isaac Newton, used to solve problems in physics, mathematics etc

Differentiation

It is concerned with rates of change e.g. profit with respect to output

   i)   Revenue with respect to output
   ii) Change of sales with respect to level of advertisement
Savings with respect to income, interest rates

Rates of changes and slope (gradients)

Linear functions

Sketch the following functions on the same graph and comment on their relationships

(1) y = -1 +3x   (2) y = 4 + 3x

x	-1	0	1	2	3
y	-4	-1	2	5	8
y	1	4	7	10	13

Diagram 1.8
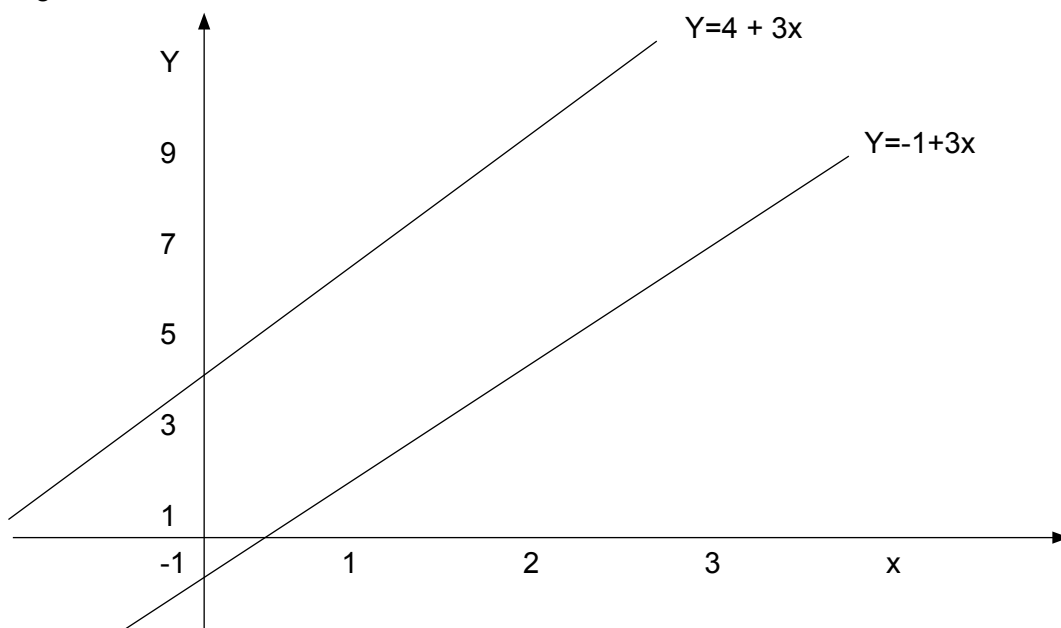
Comment: the two lines are parallel i.e. the rate of change of y with respect to x for the two is the same. As x changes by one unit y changes by 3 units in both cases.

Mathematical sign for change is the Greek letter , delta, Δ

Therefore $\dfrac{\Delta y}{\Delta x}$ = 3 Δ

$$= \frac{y_2 - y_1}{x_2 - x_1} \text{ e.g. for the 1}^{st} \text{ line } \frac{\Delta y}{\Delta x} = \frac{7-1}{1-(-1)} = 3$$

$\dfrac{\Delta y}{\Delta x}$ is also known as the slope or gradient

For any linear function the slope is constant and is equal to b in the form: y = a + bx

Forms of slope of linear functions
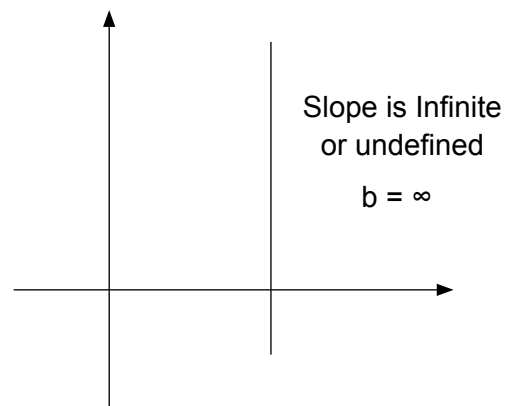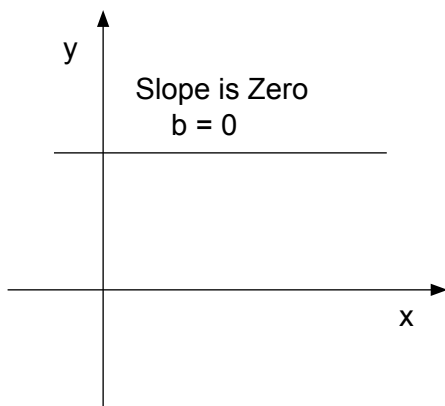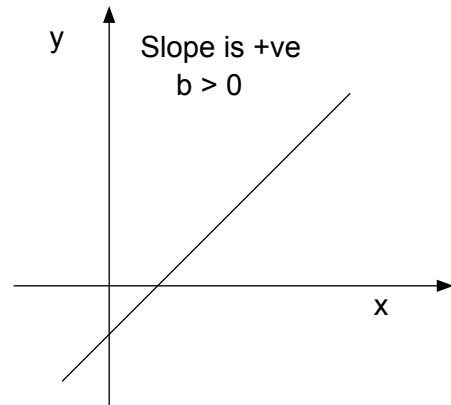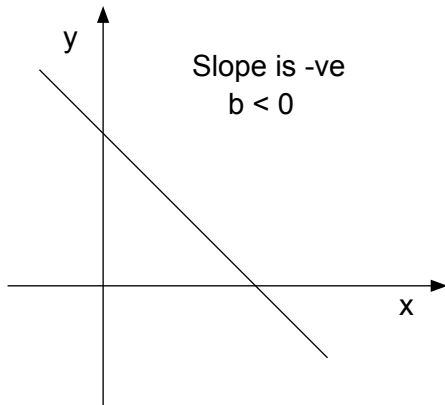
Exercise

What is the slope of line joining the points (x,y) = (2,5) and (x,y) = (5,18)?

$$\frac{\Delta y}{\Delta x} = \frac{18 - 5}{5 - 2} = \frac{13}{3} = 4 \frac{1}{3}$$

Non-Linear Functions

A slope of straight line is constant however slopes of other (non-linear) functions are different at different sections/ points of the function

Consider the general function, y = f(x)

Diagram 1.8
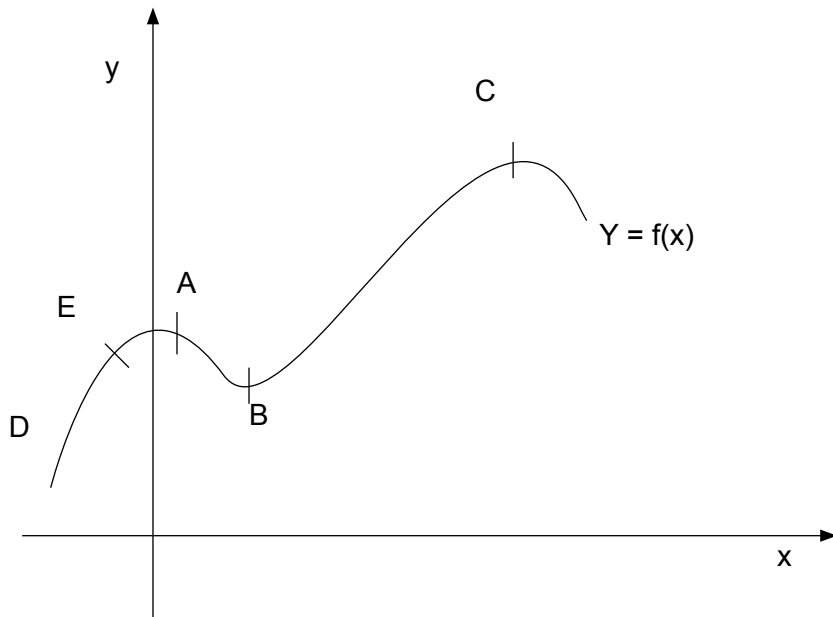
Slope of Y f(x) varies at different points along the curve e.g. to the left of point A, slope is positive

Between points A and B slope is negative

Between points B and C slope is positive

Slope at point D is greater than at point E

Problem: to calculate the slope of non –linear functions at any point of interest given the function

Integration

It is the reversal of differentiation.

An integral can either be indefinite (when it has no numerical value) or definite (have specific numerical values).

It is represented by the sign ∫f(x)dx.

Rules of integration

i. The integral of a constant

∫adx = ax +c   where a = constant

Example

Find the following

    a) $\int 23dx$

    a) $\int \gamma^2 dx$. (where γ is a variable independent of x, thus it is treated as a constant).

Solution

    i. $\int 23dx = 23x + c$

           $\int \gamma^2 dx. = \gamma^2 x + c$

    ii.   The integral of x raised to the power n

$$\int x^n dx = \frac{1}{n+1} x^{n+1} = + c$$

Example

Find the following integrals

    a) $\int x^2 dx$

    b) $\int x^{-5/2} dx$

Solution

  –   i) $\int x^2 dx = \frac{1}{3} x^3 + c$

    ii) $\int x^{-5/2} dx = -\frac{2}{3} x^{-3/2} + c$

    iii).   Integral of a constant times a function
           $\int af(x)dx = a\int f(x)dx$

Example

Determine the following integrals

    i. $\int ax^3 dx$

    ii. $\int 20x^5 dx$

Solution

    a) $\int ax^3 dx = a\int x^3 dx$

              $= \frac{a}{4} x^4 + c$

    b) $\int 20x^5 dx = 20\int x^5 dx$

              $= \frac{10}{3} x^6 + c$

iv). Integral of sum of two or more functions

$\int \{f(x) + g(x)\} \, dx = \int f(x)dx + \int g(x) \, dx$

$\int \{f(x) + g(x) + h(x)\}dx = \int f(x)dx + \int g(x)dx + \int h(x)dx$

Example

Find the following

$\int(4x^2 + \frac{1}{2} x^{-3})\, dx$

$\int(x^{3/4} + 3/7\, x^{-\frac{1}{2}} + x^5)$

Solution

  i) $\int(4x^2 + \dfrac{1}{2}x^{-3})dx = \int 4x^2 dx + \int \dfrac{1}{2}x^{-3}dx$

        $= 4x^3 - \dfrac{1}{4}\, x^2 + c$

  ii) $\int(X^{3/4} + \dfrac{3}{7}X^{-1/2} + X^5)\, dx = \int x^{3/4}dx + \int \dfrac{3}{7}x^{-1/2}dx + \int x^5 dx$

        $= \dfrac{4}{7}x^{7/4} + \dfrac{6}{7}\, x^{1/2} + \dfrac{1}{6}x^6 + c$

  v)   Integral of a difference

      $\int\{f(x) - g(x)\}\, dx = \int f(x)dx - \int g(x)\, dx$

◼ Definite integration

Definite integrals involve integration between specified limits, say a and b

The integral $\int_a^b f(x)dx$ is a definite integral in which the limits of integration are a and b

The integral is evaluated as follows

1. Compute the indefinite integral $\int f(x)dx$. Supposing it is F(x) + c
2. Attach the limits of integration
3. Substitute b (the upper limit) and then substitute a (the lower limit) for x.
4. Take the difference and the result is the numerical value for the definite integral.

Applying these steps to the definite integral

    $\int f(x)dx = \Big[F(x) + c\Big]_a^b$

    $= \Big\{\big[F(b) + c - F(a) + c\big]\Big\}$

    $= F(b) - F(a)$

Example 1

Evaluate

  i. $\int^3(3x^2 + 3)dx$

  ii. $\int_0^5(x + 15)dx$

Solution

$$\int_1^3 (3x^2 + 3)dx = [(x^3 + 3x + c)]$$

$$= \quad (27 + 9 + c) - (1 + 3 + c)$$

$$= \quad 32$$

b. $\int_0^5 (x + 15)dx = [(\tfrac{1}{2}x^2 + 15x + c)]_0^5$

$$= (12\tfrac{1}{2} + 75 + c) - (0 + 0 + c)$$

$$= 87\tfrac{1}{2}$$

The numerical value of the definite integral $\int_a^b f(x)dx$ can be interpreted as the area bounded by the function f(x), the horizontal axis, and x=a and x=b see figure below

Diagram 1.9
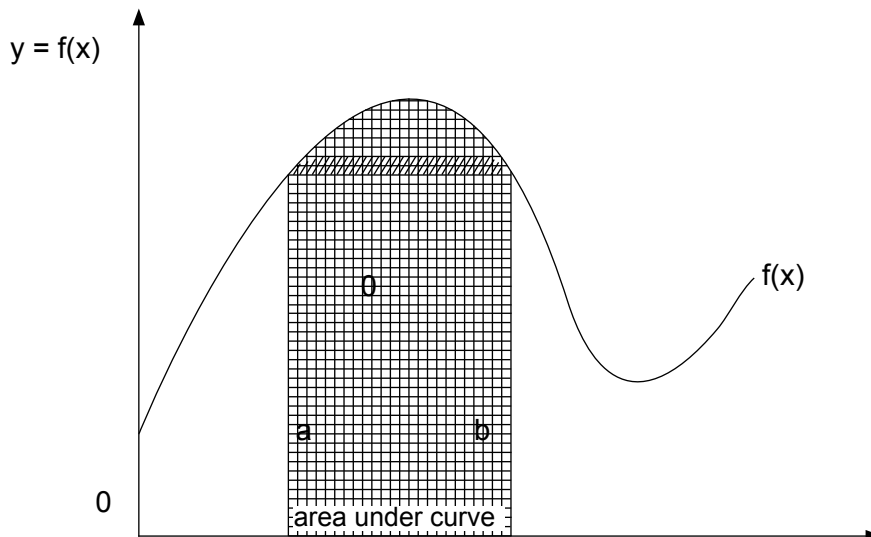
Therefore $\int_2^b f(x)dx = A$ or area under the curve

Example 2

1. You are given the following marginal revenue function
    $MR = a + a_1q$

Find the corresponding total revenue function

Solution

Total revenue $= \int MR.dq = \int (a + a_1q)dq$

$$= aq - \frac{1}{2}a_1q^2 + c$$

Example 3

A firm has the following marginal cost function

$$MC = a - a_1q + a_2q^2$$

Find its total cost function.

Solution

The total cost C is given by

$$C = \int MC.dq$$

$$= \int (a + a_1q + a_2q^2).dq$$

Example 4.

Your company manufacturers large scale units. It has been shown that the marginal (or variable) cost, which is the gradient of the total cost curve, is $(92 - 2x)$ Shs. thousands, where x is the number of units of output per annum. The fixed costs are Shs. 800,000 per annum. It has also been shown that the marginal revenue which is the gradient of the total revenue is $(112 - 2x)$ Shs. thousands.

Required

i. Establish by integration the equation of the total cost curve
ii. Establish by integration the equation of the total revenue curve
iii. Establish the break even situation for your company
iv. Determine the number of units of output that would
    a) Maximize the total revenue and
    b) Maximize the total costs, together with the maximum total revenue and total costs

Solution

i) First find the indefinite integral limit points of the marginal cost as the first step to obtaining the total cost curve
Thus $\int (92 - 2x) dx = 92x - x^2 + c$
Where c is constant

Since the total costs are the sum of variable costs and fixed costs, the constant term in the integral represents the fixed costs, thus if Tc are the total costs then,

$Tc = 92x - x^2 + 800$
or $Tc = 800 + 92x - x^2$

ii) As in the above case, the first step in determining the total revenue is to form the indefinite integral of the marginal revenue
Thus $\int (112 - 2x) dx = 112x - x^2 + c$
Where c is a constant

The total revenue is zero if no items are sold, thus the constant is zero and if Tr represents the total revenue, then

$Tr = 112x - x^2$

iii) At break even the total revenue is equal to the total costs
Thus $112x - x^2 = 800 + 92x - x^2$
$20x = 800$
$x = 40$ units per annum

iv.

a)   Tr = 112x – x²

$$\frac{d(Tr)}{dx} = 112x - 2x$$

$$\frac{d^2(Tr)}{dx^2} = -2$$

at the maximum point

$$\frac{d^2(Tr)}{dx^2} = 0 \qquad \text{that is } 112 - 2x = 0$$

x = 56 units per annum

Since this confirms the maximum

The maximum total revenue is Shs. (112 x 56 – 56 x 56) x 1000   = Shs. 3,136,000

b)   Tc = 800 + 92 x – x2

$$\frac{d(Tc)}{dx} = 92 - 2x$$

$$\frac{d^2(Tc)}{dx^2} = -2x$$

At this maximum point
$$\frac{d(Tc)}{dx} = 0$$

92 − 2x = 0
92 = 2x
x = 46 units per annum
since

$$\frac{d^2(Tc)}{dx^2} = -2x \quad \text{this confirms the maximum}$$

the maximum costs are Shs. (800 + 92 x 46 - 46 x 46) x 1000

= Shs. 2,916,000

■ Rules /techniques of differentiation

1. The derivative of a constant is zero e.g. if y = 20 then

2. The derivative of the function $y = ax^n$ is

3. The derivative of a sum of difference. Aggregate the derivatives of the various terms which make the function i.e. if y = U + V where U, V ≡ functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

4. The derivative of a product (product rule)

   If y = UV (U, V≡ functions of x) then $\dfrac{dy}{dx} = U\dfrac{dv}{dx} + V\dfrac{du}{dx}$ product rule

5. Derivative of a quotient

Suppose $y = \dfrac{U}{V}$ where U, V≡ functions of x

$$\text{then} \frac{dy}{dx} = \frac{U\dfrac{dv}{dx} - V\dfrac{du}{dx}}{v^2} \text{ - Quotient rule}$$

6. Derivative of a function of a function
   If y = f(u) and u = f(u)

   $$\text{Then } \frac{dy}{dx} = \frac{du}{dx} \times \frac{dv}{dx} \qquad \text{- Chain rule}$$

7. Derivative of an exponential function
   If $y = ae^{fx}$ (a and e are constants)

   $$\text{Then} \frac{dy}{dx} = f\,'(x)ae^{fx}$$

8. Derivative of logarithmic function
   If y = ℓnf(x)

   $$\text{Then} \frac{dx}{dx} = \frac{f\,'(x)}{f(x)}$$

1st order and 2nd order conditions

In most situations one is faced with the process of differentiation for example what is the maximum level of profit, at what points you have max. Revenue applied in NSE statistics

Diagram 1.10
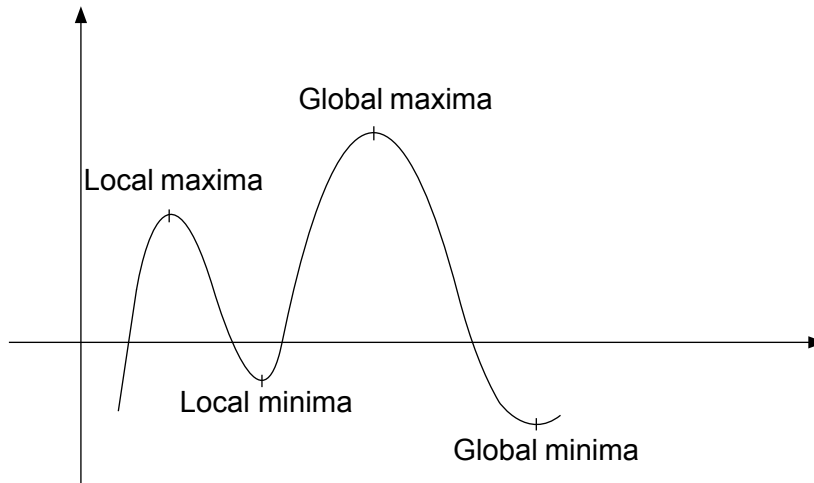
Maxima and minima are found at the turning point of the curve, when the curve is parallel to x-axis, at this point the gradient = 0 i.e the derivative = 0

2nd Order derivatives

2nd order derivative is a result of differentiating a function twice. It is usually donated as y″ or

$\dfrac{d^2 y}{dx^2}$ or f ″ (x)

$y = 3x + x^2$

$y' = 3 + 2x$

$y'' = 2$

Example.

Determine the maxima or minima of the graph $y = 3x^2 - 2x$

Solution

We use 1st and 2nd order derivatives to determine if a point is min. or max.

1st derivative $\dfrac{dy}{dx} = 6x + 2$, $\dfrac{dy}{dx} = 0$ but at min or max point:

therefore 6x + 2 = 0 ; at max or min

Which gives x = - 1/3

2nd derivative $\dfrac{d^2y}{dx^2} = 6$

Note: When 2nd derivative is +ve → minimum point

If -ve → maximum point

Conclusion: The graph $y = 3x^2 - 2x$ has a minimum point when x = -1/3.
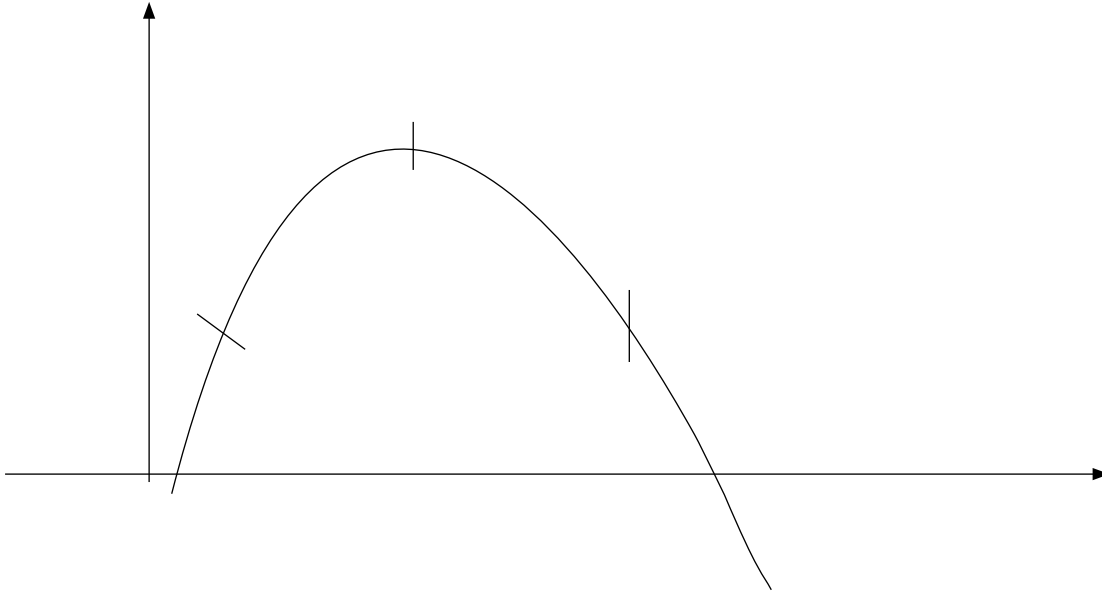
Determine the critical values and state whether these are maximum or minimum functions

(i). $y = \dfrac{1}{3}x3 + x2 - 35x + 10$

(ii). $y = x^2 - 9x$

Solution

(i) $\dfrac{dy}{dx} = x^3 + x^2 - 35$

$x^2 + 2x - 35 = 0$

$x^2 - 5x + 7x - 35 = 0$

$x(x-5) + 7(x-5) = 0$

$(x + 7)(x - 5) = 0$

Either x = -7 or x = 5

2nd derivative

$$\frac{d^2y}{dx^2} = 2x + 2$$

When x = -7

$$\frac{d^2y}{dx} = -14 + 2 = -12 \text{ max.}$$

When x = 5

$$\frac{d^2y}{d^2x} = 10 + 2 = 12 \text{ min}$$

(ii). $\frac{dy}{dx} = 2x - 9n$

2x – 9 =0
2x = 9
X = 4.5

$$\frac{d^2y}{dx} = 2 \text{ therefore minimum.}$$

Application of derivatives

(1) Profit maximization

Π = TR – TC

$$\frac{d\Pi}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ}$$

When MR = MC or MR – MC = 0 means that profit is at maximum.

(2) Revenue maximization

This is mainly concerned with maximizing returns

Differentiate revenue function to get optimal value

R = PQ

$$\frac{dTR}{dP} \text{ or } \frac{dTR}{dQ} = 0$$

or at critical point.

2nd derivative $\frac{d^2TR}{dP^2}$ or $\frac{d^2TR}{dQ^2}$ the value should be negative.

3) Cost minimisation

1st derivative

$$\frac{dC}{dQ} = 0$$

2nd derivative

$$\frac{d^2C}{dQ^2} = +ve$$

Illustration 1

Revenue and average cost function for a given firm are given as

AR = 4 - ¼ Q

AC = 4/Q + 2 − 0.3Q + 0.05Q²

Required: Find the level of Q and P that would max the profit and compute the max profit.

Hint: Use 2nd condition to confirm whether it's maximum

Solution

TR = (AR)Q

TC= (AC)Q
TC = 4 + 2Q − 0.3Q² + 0.05Q³
Π = TR − TC

$$TR = 4Q - \frac{1}{4}Q^2$$

$$\Pi = (4Q - \frac{1}{4}Q^2) - (4 + 2Q - 0.3Q^2 + 0.05Q^3)$$

Π = 2Q +0.05Q² − 0.05Q³ − 4

$$\frac{d\pi}{dQ} = 2 + 0.1Q - 0.15Q^2$$

2 + 0.1Q − 0.15Q² = 0

Q is either -3.3 or 4

$$\frac{d^2\pi}{dQ^2} = 0.1 - 0.3Q$$

Therefore, when Q =4 ,Π = maximum.

Partial Differentiation

Exponential functions

$$y = 3e^{0.5\,t}$$

Differentiate:

$$\ell ny = \ell n3 + \ell ne^{0.5\,t}$$

$$\ell ny = \ell n3 + 0.5t\ell ne \text{ but } \ell ne = 1$$

Partial differentiation is used in multivariate function i.e. functions that have more than one independence variables

E.g. Y = 3x + 2 x Z

Since we have numerous independent variables we can only derive or compute the instantaneous rate of change of the function with respect to one independent variable as such the remaining independence variables are assumed to be constant.

Such a derivative of a multivariate function is donated as partial differentiation denoted as

$$f(x_1, x_2, x_3) = \frac{dy}{dx_1}$$

$$= \frac{dy}{dx_2} \text{ and so on.}$$

$$y = 3x_1 + 2x_2^2$$

$$\frac{dy}{dx_1} = 3 \qquad \frac{dy}{dx_2} = 4x_2$$

Rules

1. Sum rule

The partial derivative of the sum of two functions f (x, y) and g(x, y) with respect to x is the sum of partial derivatives of the two functions w.r.t. x, y

$$y = f(x_1, x_2) + g(x_1, x_2)$$

$$\frac{dy}{dx} = f\,'\,x_2 + g'\,x_2$$

Example

$$Z = 2x^2 + 3y$$

$$\frac{dz}{dx} = 4x \qquad \frac{dz}{dy} = 3$$

2. Difference – partial derivative of the difference of functions f(x, y) and g(x, y) w.r.t x is the difference of the partial derivative of the functions w.r.t x

$$Z = f(x, y)\text{-}g(x, y)$$

$$\frac{dz}{dx} = f\,'\,x - g'\,x \qquad \frac{dz}{dy} = f\,'\,y - g'\,y$$

Example

$$Y = 3x_1{}^2 - x_2{}^3$$

$$\frac{dy}{dx_1} = 6x_1 \qquad \frac{dy}{dx_2} = -3x_2^2$$

3. Product rule.

Y = UV

Example

$$Y = x_1 x_2$$

$$\frac{dy}{dx_1} = x_2 \qquad \frac{dy}{dx_2} = x_1$$

4. Quotient rule.

$$y = \frac{U}{V} \qquad y' = \frac{U'V - V'U}{V^2} \qquad = \frac{-U\frac{dv}{dx} + V\frac{du}{dx}}{V^2}$$

Example

$$Y = \frac{U(x_1 x_2)}{V(x_1 x_2)}$$

$$\frac{d^2 y}{dx} = \frac{U'(x_1) V - V'(x_2)U}{V^2}$$

Partial derivative of a quotient $y = \frac{f(x_1 x_2)}{g(x_1 x_2)}$ w.r.t. x_1 equals partial derivative of the numerator

w.r.t.x_1 times the denominator minus partial derivative of denominator w.r.t x_1 times the numerator all divided by the square of denominator.

Example

$$y = \frac{x_1{}^2 + x_2{}^2}{x^1 x^2}$$

First order condition

$$\frac{dy}{dx_1} = \frac{2x_1(x_1 x_2) - x_2(x_1{}^2 + x_2{}^2)}{(x_1 x_2)^2}$$

Second order condition (SOC)

$$\frac{d^2y}{dx_1^2} = \frac{dy}{dx_1}\left(\frac{dy}{dx_1}\right)$$

$$\frac{dy}{dx_2^2} = \frac{dy}{dx_2}\left(\frac{dy}{dx_2}\right)$$

Example

$Y = 3x_1^2 + x_2^3 + 3x_1x_2 + 6$

Solution

F.O.C $\frac{dy}{dx_1} = 6x_1 + 3x_2$ $\frac{dy}{dx_2} = 3x_2^2 + 3x_1$

S.O.C $\frac{d^2y}{dx_1^2} = 6$ $\frac{d^2y}{dx_2^2} = 6x_2$

Applications of partial differentiation

1. In marginal costs – if costs in a function are of two or more elements.
2. Related commodities be it substitutes or complimentary where demand is not only influenced by product price but the price of other commodities.
3. Partial elasticity of demand.
4. Profit maximization, revenue optimization and cost optimization.

SET THEORY

Introduction

Study of sets is popular in economic and business world since the basic understanding of concept in sets algebra provides a form of logical language through which the business specialists can communicate important concepts and ideas.

It is also used in solving counting problems of a logical nature

A further study of set algebra provides a solid background to understanding probability and statistics which are important business decision-making tools

Any well defined collection of group of objects is a set e.g. set of all courses offered in Strathmore University, all first year students at Egerton University, student studying medicine in Kenya.

Requirement of a set

1. A set must be well defined i.e. it must not leave any room for ambiguities e.g sets of all students- which? Where? When?
   A set must be defined in terms of space and time

2. The objective (elements or members) from a given set must be distinct i.e each object must appear once and only once, Must appear but not more than once
3. The order of the presentation of elements of a given set is immaterial
4. e.g 1,2,3 = 1,3,2 = 3,2,1

Specifying (Naming) sets

By convention sets are specified or named using a capital letter. Further the elements of a given set are designed by either listing all the elements or by using descriptive characteristics or patterns e.g.

A = (0, 1, 2, 3, 4, 5, 6) : Listing all
A = (whole No's from zero to six)   : descriptive characteristics
A = (0, 1, 2….6)   : Pattern

1. Set membership- set membership is expressed by using the Greek letter epsilon, $\in$ e.g. $3 \in A$; $0, 5 \in A$
2. A finite set consist of a limited or countable number of elements e.g. A has 7 members therefore A is a finite set. An infinite set consists of unlimited or uncountable number of members e.g. set of all odd numbers.
3. A subset S of some other set A is such that all elements in S are members of A e.g. if S = [1, 5] then S is a subset of A denoted: $S \subset A$ equally A is a superset to S denoted $A \supset S$.
4. Equality of sets – if all elements in set D_1 are also in D_2 and all elements in D_2 are also in D_1 then D_1 and D_2 are equal e.g. let $D_1 = (a, c, f)$ $D_2 = (c, a, f)$ Then $D_1 = D_2$.Further $D_1 \subset D_2$ and $D_2 \subset D_1$ i.e. each set is subset to itself.
5. Members of universal set are denoted by U or ξ. The universal set is that which contains all elements under consideration by the analyst or the researcher.
   Let U = set of all students at universities in Kenya in the year 2005
   S1 = Student of Strathmore University
   S2 = Engineering Students
   S3 = Students above 25years at Kenyatta University
   S4 = All female students below 20years of age.
6. The null or the empty set- this is the set with no elements denoted:
   E = ( ) or E = Ø
7. Compliment of a set. If U = Universal set and A is a subset of the universal set, then the compliment of A denoted A' or A^c represents all elements in the universal set which are not members of A e.g
   A = (whole No's from 0-6)
   U = (whole No's from 0-10)
   Then A' = (whole No's from 7-10)
8. Pictorial or Diagrammatic representation of sets- This is done using Venn diagrams (named after the 18th C. English Logician, John Venn)

   ◯ = a single set/ordinary set (Not a universal set)

   ▭ = universal set

Diagram 1.11

$$U = \{ 0, 1, 2, \ldots 10\}$$
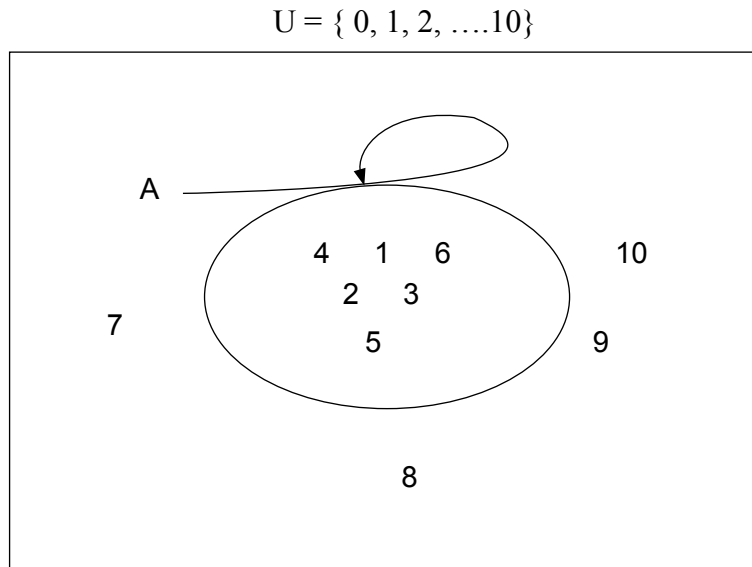
Set Algebra

This consists of ways or operations whereby sets are combined in order to obtain other sets of interest.

Basic Set Operations

1.   Let P = (1, 2, 3) and Q = (1, 3, 5,6)

Union of sets, denoted U .Therefore PUQ represents all elements in P or Q.

Note: "Or" could be used in place of U

PUQ = (1,3,2,5,6)
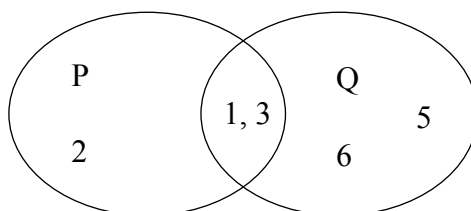
2.   Intersection of sets denoted ∩ This consists of elements in both P and Q. It is the common area.

P∩Q = (1,3)

Note: "and" could be used in place of ∩

3.   Set difference or set disjunction denoted by (-) e.g P-Q consist of elements in P but not Q i.e (2). Whereas Q – P = (5, 6)

4. Symmetric Difference, Denoted by Δ Greek letter "delta". These are elements in P but not Q or in Q but not in P i.e

P Δ Q = (P − Q)U ((Q − P) = (2,5,6)

Laws of Set Algebra.

1. Commutative Laws   (i) PUQ=QUP

(ii) P ∩Q=Q∩P

The order in which sets are combined with union or interaction is irrelevant

2. Associative laws   (i) PU (QUR) = (PUQ)U R

(ii) P∩ (Q∩R) = (P∩Q)∩R

The selection of three sets for grouping in a union or intersection is immaterial

3. Distributive laws   (i) PU (Q∩R) = (PUQ) ∩ (PUR)

(ii) P U (QUR) = (PUQ) U(P∩R)

4. Idempotent laws   (i) QUQ = Q

(ii)   Q∩Q=Q

5.   PUØ = P

6. P∩Ø = Ø

7. PU U = U   (U = universal)

8. P∩U = P

9. PUP'= U compliments each other

10. P∩P' = Ø   no common area

11. De Morgan's Laws

(i)   (QUR)' = Q'∩R'

(ii)   (Q∩R)' = (Q'UR')

Example –

1. Rewrite the following

(i)   (A'UB')'

= A∩B

(ii)   (A'∩B)'

= AUB'

2. Simplify the following

(i)   PU (P'∩Q)

= PUQ

(ii)   (AUB) U(A∩B)

= (AUB)

3. The sets L, M and N in a universal set consisting of the first 10 lower case letter of the alphabetical are: L = (a,b,c) M = (b,c,a,e) N= (a,d,e,f)

Required

Determine the members of the following sets:

i) MUN,   ii) LUN,   iii) L'   iv) L∩M∩N'   v) (LUMUN)' vi) (M∩N)

vii) M∩N∩L'   viii) (M∩N∩L)'

Diagram 1.12

N(ξ) = (a, b,…,j)

i)   MUN = {b,c,a,e,d,f}
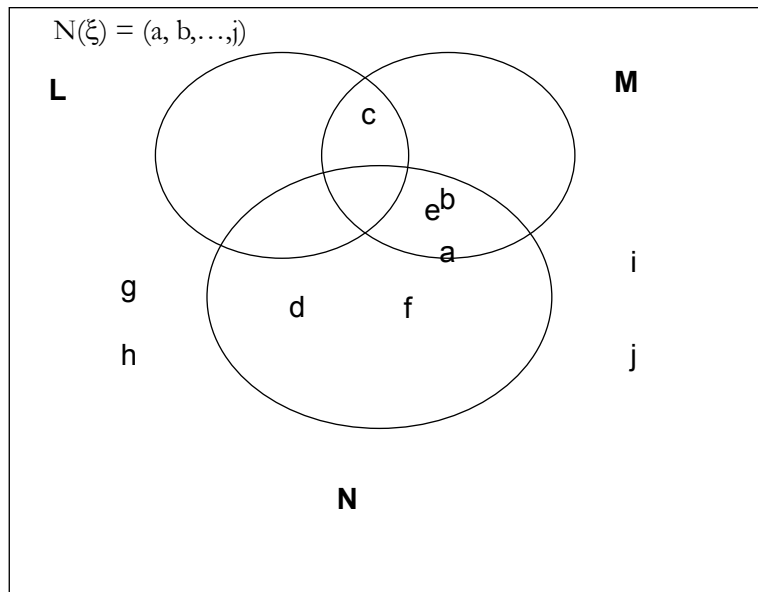
ii)   LUN = { a,b,c,d,e,f}

iii)   L' = {d,e,f,g,h,i,j}

iv)   L∩M∩N' = { b,c}

v)   (LUMUN)' = {g,h,i,j)

vi)   M∩N = {a,e}

vii)   M∩N∩L'= {e}

viii) (M∩N∩L)' = {b,c,d,e,f,g,h,i,j}

Logical Counting Problems

Number of elements in a set.

Two sets

In any set S if S contains K elements, we show this on n(s)= K e.g if s= {10,0,17,2,12} then

n(s) =5

Generally given any two sets S_1 and S_2, $n(S_1 US_2) = n(S_1) + n(S_2) - n(S_1 \cap S_2)$

We have to subtract the No of the elements with the intersection to avoid counting them twice. However if S1 and S2 are disjoint sets then n $(S_1 \cup S_2)$ = n (S1) +n (S_2)
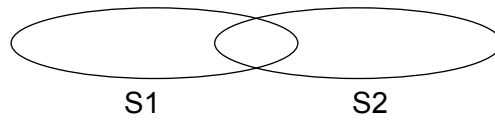
S1   S2

S1 and S2 have an intersection i.e they share some elements
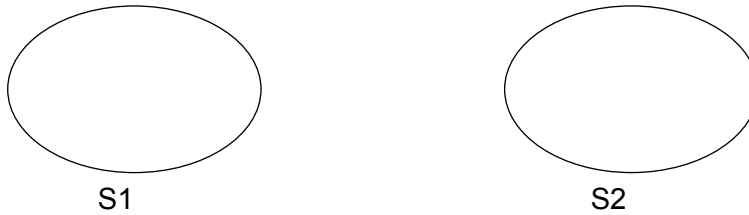
S1   S2

S1 and S2 are disjoint i.e they lack an intersection.

For symmetry difference number of elements $n(S_1 \Delta S_2) = n(S_1 \cup S_2) - n(S_1 \cap S_2)$

S1   S2
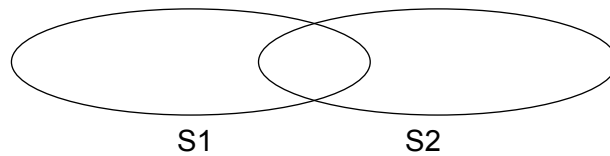
Illustration (2-sets)

In a recent survey of 400 students in a college 100 were listed as studying typing (T) and (150) were listed as doing accountancy (A), 75 registered as doing both courses.

Required: -

Find the number of students in the college who are not registered in either course.

How many students were registered for typing only.

Solution

N(T) = 100   N(A) = 150   N(T∩A) = 75

Diagram 1.13

Venn diagram

n (U) = 400

b=25

c

75

d=75
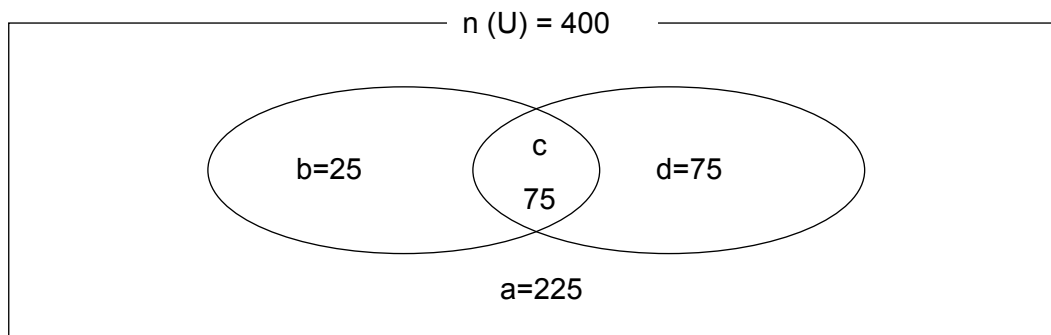
a=225

a) n(neither T nor A) = 225

b) n(T) = 25

Illustration

A survey was conducted on the newspapers readership of three dailies: Nation Daily (D), the Standard (S) and the Kenya Times (K) and the following data was obtained. The No that read: D and K – 19

S and D – 15

S and K- 14

D, K and S =4 only

D = 55

S = 45

K= 39

Required: -

a) Determine the number of people who read
   i) Daily Nation only
   ii) Standard or Kenya Times but not Daily Nation
b) Determine the total no of people interviewed if 5 people read none of the papers
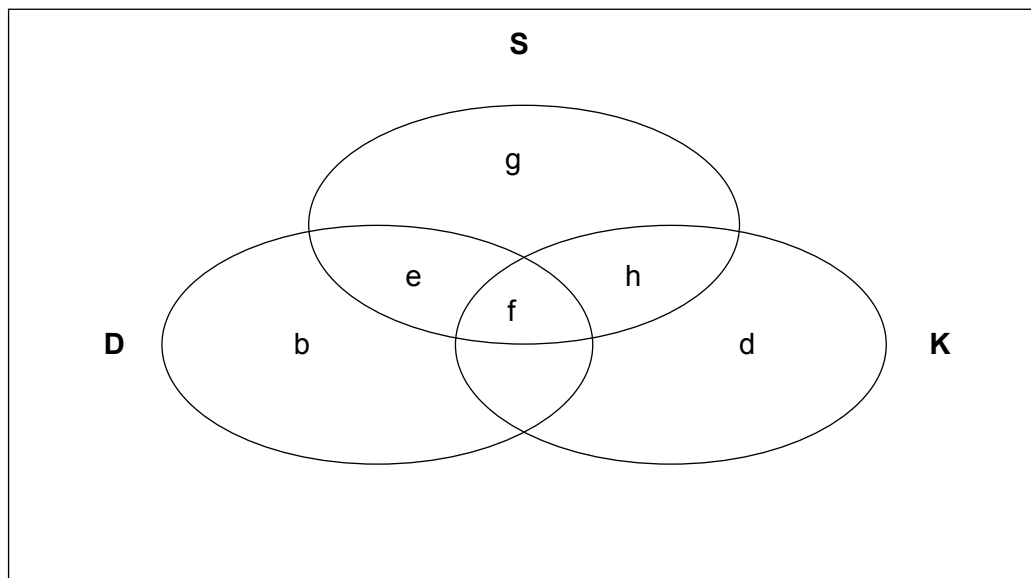
Solution

b = 55-11-4-15 = 25

g = 45-4-11 =20

d = 39-15-4-10 = 10

(i) Daily nation only b = 25

S and K only = 10+20+10=40

Total (U) = a+b+c+d+e+f+g+h = 100

1.5 CHAPTER SUMMARY

Functions of a single independent variable may either be linear or non linear.

Linear functions can be represented by:

$y = a + bx$

Whereas non – linear functions can be represented by functions such as:

$y = \alpha_0 + \alpha_1{}^3x + \alpha_2x^3$

$y^2 = 3x + 18$

$y = 2x^2 + 5x + 7$

Where α, a, b, c, d, k = constants

$ax^2 + bx + cy + d = 0$

$xy = k$

$y = a^x$

The solution of a system of linear simultaneous equations is a set of values of the variables which simultaneously satisfy all the equations of the system.

The determinant of a square matrix det (A) or A is a number associated to that matrix. If the determinant of a matrix is equal to zero, the matrix is called singular matrix otherwise it is called non-singular matrix.

Differentiation deals with the determination of the rates of change of business activities or simply the process of finding the derivative of a function.

Integration deals with the summation or totality of items produced over a given period of time or simply the reverse of differentiation

An Implicit function is one of the $y = x^2 y + 3x^2 + 50$. It is a function in which the dependent variable (y) appears also on the right hand side.

To differentiate the above equation we use the differentiation method for a product, quotient or function of a function.

A set can be classified as a finite or infinite set, depending upon the number of elements it has. A finite set has a finite number of elements whereas an infinite set has an infinite number of elements.

CHAPTER QUIZ

1. A ………….associates a single output to each input element drawn from a fixed set, such as the real numbers, although different inputs may have the same output.

2. Which one of the following is not a function:
   - (i)   Polynomials
   - (ii)   Multivariate
   - (iii)   Logarithmic
   - (iv)Exponential
   - (v)   Calculus

3. Which of the following operations cannot be applied to a matrix equation:
   (i)   Multiplication
   (ii)   Addition
   (iii)   Division
   (iv)   Subtraction

4. List the basic operations in calculus.

ANSWERS TO CHAPTER QUIZ

1. Function
2. (v) Calculus
3. (iii) Division
4. Differentiation and integration

QUESTIONS FROM PREVIOUS EXAMS

JUNE 2000 QUESTION 1

a)   Define the following terms:
   (i)   Stochastic process
   (ii)   Transition matrix
   (iii) Recurrent state
   (iv) Steady state

b)   UC Limited specialises in selling small electrical appliances. A considerable portion of the company's sales is on installment basis. Although most of the company's customers make their installment payments on time, a certain percentage of their accounts are always overdue and some are never paid at all. The Company's experience with overdue accounts has been that if a customer falls two or more installments behind schedule, then this account is generally not going to be paid; hence it is the company's policy of discontinuing credit to such customers and to write these accounts off as bad debts. At the beginning of each month, the company reviews each account and classifies them as either "paid-up", "current" (being paid on time), "overdue (one payment past due) or "bad debt". To investigate this problem, the analysts at UC Limited have constructed matrix representing the various states that each shilling in the accounts receivable can take on at the beginning of two consecutive months.

Transition Matrix of UC Limited

State of each Sh. at next month

	Paid	Current	Overdue	Bad debt
	(P_1)	(P_2)	(P_3)	(P_4)
State of each Sh.Paid (P_1)	1.00	0.00	0.00	0.00
First Month Current (P_2)	0.30	0.50	0.20	0.00
Overdue (P_3)	0.50	0.30	0.10	0.10
Bad debt (P_4)	0.00	0.00	0.00	1.00

Required:

(i)   Interpret P_{22} and P_{23} (2 marks)

(ii)   "Paid" and "bad debts" states have values of 1. interpret   (3 marks)

(iii)   If the original amount of money outstanding was Sh. 100,000, determine how much UC Limited expected to be paid back if the records indicate that for every Sh. 100 in payments due Sh. 70 are classified as "current" and Sh. 30 are classified as "overdue".

(7 marks)

(Total: 20 marks)

Note:

$$P^n = \begin{matrix} 1.000 & 0 & 0 \\ 0.950 & 0 & 0.05 \\ 0.870 & 0 & 0.13 \\ 0 & 0 & 0 & 1.00 \end{matrix}$$

JUNE 2002: QUESTION 2

a)   Define the following terms as used in Markovian analysis

    i)   Transition matrix   (2 marks)

    ii)   Initial probability vector   (1 marks)

    iii)   Equilibrium   (1 marks)

    iv)   Absorbing state   (2 marks)

b)   A company employs four classes of machine operators (A, B, C and D), all new employees are hired as class D, through a system of promotion, may work up to a higher class. Currently, there are 200 class D, 150 class C, 90 class B and 60 class A employees. The company has signed an agreement with the union specifying that 20 percent of all employees in each class be promoted, one class in each year. Statistics show that each year 25 percent of the class D employees are separated from the company by reasons such as retirement, resignation and death. Similarly 15 percent of class C, 10 per cent of class B and 5 per cent of class A employees are also separated. For each employee lost, the company hires a new class D employee.

Required

    i)   The transition matrix   (7 marks)

    ii)   The number of employees in each class two years after the agreement with the union.

(7 marks)

(Total: 20 marks)

JUNE 2003: QUESTION 2

The break-even point is that level of output at which revenue equals cost.

Required:

Graphically show the break-even point using the cost, revenue and profit functions.

(6 months)

a) ABC Manufacturers Ltd. Produces spare part for motor vehicles. The demand for this spare part is given by;

D = 15 – 0.5p; where d = demand and p = price

Required:

    i) Explain clearly why the coefficient to the price is negative.
    ii) Determine the total revenue function
    iii) Determine the price elasticity at a price of Sh. 12. Clearly interpret your answer.

                                                     (4 marks)

b) The revenue of Better Option Pink Mobile Phone Company is related to advertising (a) and phone sold (q). specifically, the relationship can be expressed as

$R = q^2 + 3qa + a^2$

However, the budget constraint on advertising and production is given by

$q + a = 100$

Required:

    i) Determine the maximum revenue with the advertising and production constraints.

                                                     (4 marks)
    ii) What does a represent economically?   (2 marks)
                                                    Total: 20 marks)

DECEMBER 2003: QUESTION 2

a) Explain the purpose of Venn diagrams   (3 marks)
b) A market study taken at a local sporting goods store, Maua Wahome Store, showed that of the 200 people interviewed, 60 owned tents, 100 owned sleeping bags, 80 owned stoves, 40 owned both tents and camping stoves and 40 owned both sleeping bags and camping stoves.

Required:

c) If 20 people interviewed owned a tent, a sleeping bag and a camping stove, determine how many people owned only a camping stove. In this case, is it possible for 30 people to own both a tent and a sleeping bag, but not a camping stove?   (6 marks)
d) "Under One Thousand Shillings" Corner Stone is planning to open a new store on the corner of Main and Crescent Streets. It has asked the Tomorrow's Marketing Company to do a market study of randomly selected families within a five kilometers radius of the store. Among the questions it wishes Tomorrow's Marketing Company to ask each homeowner are:
    i) Family income.
    ii) Family size
    iii) Distance from home to the store site
    iv) Whether or not the family owns a car or uses public transport.

Required:

"One Thousand Shilling" Corner Store. Denote which of these are discrete and which are continuous random variables.   (11 marks)

                                                (Total: 20 marks)

DECEMBER 2004: QUESTION 2

Ujumi Industries Ltd. Has two subsidiary companies; apex and Maxima. Apex is solely involved in manufacturing of Ujumi's products. Apex operates as a cost centre and supplies all its products to Maxima. The total cost of production for Apex is given by the equation $C = 3Q^3 - 30Q^2 + 50Q + 300$ where Q is the number of units (in hundreds of thousands) produced. Currently, apex does not charge Maxima for the transfer of the goods.

Maxima is involved in the distribution and marketing of the products. The total cost associated with maxima's activities is given by the equation

$C = 2Q^3 - 10Q^2 + 25Q + 100$.

The revenue generated by Maxima on selling Q units is, $R = 400Q - 250Q^2$

Ujumi's management has accused apex managers for not controlling the production cost which has caused the company's profit to fall. In response apex managers argued that much of the overhead cost is incurred in the transfer of the products to maxima. Apex Management have argued that Ujumi's management should allow them to charge maxima a transfer fee of Sh. 100 per unit, so as to cut down on the overheads.

Required

    a)   The optimal number of units that Maxima should receive from apex if no fee is charged.   (6 marks)

    b)   The optimal number of units that apex can transfer without charging any fee, to minimize total cost.   (4 marks)

    c)   The optimal policy that Ujumi Industies Ltd. as a whole should adopt to maximise profits when no transfer fee is charged.   (6 marks)

    d)   Explain clearly whether allowing Apex to charge the transfer fee, would help Ujumi Industries Ltd. as a whole to improve their optimal policy.

        (4 marks)

        (Total: 20 marks)

CHAPTER TWO

Descriptive Statistics

CHAPTER TWO
Descriptive Statistics

▶ OBJECTIVES

- At the end of this chapter, you should be able to::
- Define statistics;
- Mention uses of statistics in business and other areas;
- Explain what descriptive statistics and inferential statistics are;
- Construct unweighted and weighted indices;
- Construct a price, quantity, value and special purpose index;
- Explain how the consumer price index is constructed and used;
- Compute various measures of dispersion for data organised in a frequency distribution;
- Compute and explain the uses of the coefficient of variation and the coefficient of skewness.

Fast Forward: Descriptive Statistics are used to describe the basic features of data gathered from an experimental study in various ways.

▶ INTRODUCTION

Statistics is the art and science of getting information from data or numbers to help in decision making.

As a science, statistics follows **a systematic procedure** to reach objective decisions or solutions to problems.

As an art statistics utilises **personal judgment and intuition** to reach a solution. It depends on experience of the individual involved. It is more subjective.

Statistics provides us with tools that aid decision making. For example, using statistics we can estimate that expected returns and associated risks of a given investment opportunity.

▶ DEFINITIONS OF KEY TERMS

Statistical inference is deduction about a population based on information obtained from a sample drawn from it. It includes:

- point estimation
- interval estimation
- hypothesis testing (or statistical significance testing)
- prediction

Measures of central tendency are single numbers that are used to summarise a larger set of data in a distribution of scores. The three measures of central tendency are mean, median, and mode.

Measures of dispersion – These are important for describing the spread of the data, or its variation around a central value. Such measures of dispersion include: standard deviation, interquartile range, range, mean difference, median absolute deviation, avarage absolute deviation (or simply average deviation)

Variance is the sum of squared deviations divided by the number of observations. It is the average of the squares of the deviation of the individual values from their means.

Skewness describes the degree of symmetry in a distribution. When data are uni-modal and symmetrical, the mean, mode and median will be almost the same value.

Kurtosis describes the degree of peakedness or steepness in a distribution.

▶ EXAM CONTEXT

Statistics as an art and science of getting information provides us with tolls that aid decision-making, for example, with the aid of statistics we can estimate the expected returns and associated risks of a given investment opportunity. It has been a wealthy area for examiners as outlined below:

12/06, 12/06, 12/05, 12/04, 6/04, 6/01, 12/00, 6/00

Statistics has two broad meanings

1. Statistics refers to the mass of components or measures such as the mean, mode, standard deviation etc. that help in describing the characteristics of a given set of data or distribution. In this respect, statistics is simply, numerical facts about a given situation. This is the original meaning of statistics. Governments were the first organisations to collect vital statistics such as death rates, birth rates, economic growth etc.
2. Statistics refers to a method of study. This is commonly known as the scientific method. Statistics is the process that gives thorough systematic steps to aid problem solving or decision making.

The steps are:

a)   Problem identification and definition

Once the problem is identified, it should be formulated as clearly and as unambiguously as possible. The questions to be answered will clearly be defined at this point. This is necessary because it enables us to make appropriate conclusion from the study.

b)   Research methodology design

After formulating the problem, we need to come up with a defined plan on how the study will be conducted to solve the problem.

Research design entails the following:

i.   Determine the **population of study.** Population of study refers to the entire collection of objects or subjects for which we want to make a decision e.g. the number of customers of a supermarket; the number of students in public universities, the number of matatus in Nairobi, the number of employees in campus, the number of hospitals in Nyeri.
ii. Decide whether to use a **census or a sample study**. A census entails studying each and every element in a population. A sample study entails studying a portion of the population.

iii. Decide the **data collection techniques** to use e.g. observation, questionnaires, interviews, Internet pop-ups etc.
iv. Identify the **data analysis technique** to be used.
  • Actual data collection/fieldwork
  • Data analysis

After data is collected, it is analysed so as to enable us make decisions. Data analysis involves the following:

• **Organisation of data** - This enables us to check for completeness, accuracy and reference. Organising data can also involve coding the data for ease of analysis.
  • **Data presentation** – Diagrams, graphs, tables etc. are used to present data so as to highlight the visual impression of the data. The kinds of graphs and charts used will depend on the audience. Type of graphs and charts may also be determined by type of information being presented.

2.1 STATISTICAL INFERENCE

The four steps above are referred to as **descriptive statistics**. Descriptive statistics helps us to organise, summarise and present data in a convenient and informative way.

Inferential statistics, on the other hand, enables us to draw conclusions about the characteristics of a population. Because most studies depend on samples, we use the sample results to draw conclusions about the population from which the sample was drawn. The process of making conclusions about the population based on sample statistics is known as sample inference.

Use sample statistics   conclude on population parameters

Based on the statistical inference, then we answer the question or make the decision or solve the problem that we set out to do.

Case Study

Statistics is applied in various fields. In the pension industry, actuarial methods are used to measure the costs of alternative strategies with regard to the design, maintenance or redesign of pension plans.

The strategies are greatly influenced bycollective bargaining; the employer's old, new and foreign competitors; the changing demographics of the workforce; changes in the internal revenue code; changes in the attitude of the internal revenue service regarding the calculation of surpluses; and equally importantly, both the short and long term financial and economic trends.

It is common with mergers and acquisitions that several pension plans have to be combined or at least administered on an equitable basis. When benefit changes occur, old and new benefit plans have to be blended, satisfying new social demands and various government discrimination test calculations, and providing employees and retirees with understandable choices and transition paths. Benefit plans liabilities have to be properly valued, reflecting both earned benefits for past service, and the benefits for future service. Finally, funding schemes have to be developed that are manageable and satisfy the Financial Accounting Standards Board (FABS)

Summary measures

Fast Forward: In descriptive statistics, summary statistics are used to summarise a set of observations, in order to communicate the largest amount as simply as possible.

The purpose of computing summary measures is to have single values that describe characteristics of samples or population of interest. For example, we can compute measure of central tendency or averages that represent the typical value of the distribution.

Measures computed from population are known as parameters while those computed from samples are known on statistics e.g. population mean (μ), population std deviation (σ) proportion (π) the sample statistics are used to estimate the population parameters because most studies use samples. There are various types of summary measures including averages and measures of dispersion.

Measures of central tendency

a) Mean

It is the most popular measure of central tendency. It is the sum of all the values divided by the number of values. When the mean is computed from a sample, it is represented by the symbol ($\bar{x}$) pronounced x-bar. When computed from a population it is represented by the symbol μ (mu)

i) Arithmetic mean

Arithmetic mean represents the sum of all observations divided by the number of observations.

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \sum_{i-1}^{n} \frac{x_i}{n}$$

Illustration

Mr. Ochieng would like to know the average amount of money students carry when they go to class. From a random sample of his closest friends, he has collected the following values Shs250, 2000, 650, 4000,2000, 1050, 90, 8000, 1500,264

What is the mean amount of money carried by these students?

$$\bar{x} = \frac{250 + 2,000 + 650 + 4,000 + 2,000 + 1,050 + 90 + 8,000 + 1,500 + 264}{10}$$

$\bar{x}$ = Shs 1,980.40

Weighted mean

$$x = \frac{f_1 x_1 + f_2 x_2 \ldots + f_n x_n}{f_1 + f_2 \ldots + f_n}$$

Where **f** is / are the respective frequencies or weights of the observed values.

Example 2

The following table shows the number of drinks sold and their respective prices by Kabana Musyoka Musau. Compute the average price of the drinks sold.

Unit price (kshs)	60	100	110	120	400	1600	3500
No.of drinks	205	300	250	150	10	2	1

$$\bar{X}= \frac{60(205) + 100(300) +110(250)+120(150)+400(110)+1600(2)+3500(1)}{205+300+250+150+10+2+1}$$

$$\bar{X} = \frac{98500}{918}$$

$$= 107.298$$

On average the drinks were worth $\cong 107.30$ shillings

Mean for grouped data

When computing the mean for grouped data, we only get an estimate of the mean. This is because the individual values lose their identity in the process of constructing a grouped frequency distribution. We use the same formula as the one for weighted mean, the only difference being that the X's are now the respective class mid points instead of the individual values.

$$\bar{x} = \frac{\sum_{i-1}^{n} f_i x_1}{\sum_{i-1}^{n} f_i}$$

Example 3. Compute the mean mark scored by the students in the B-statistics class

Table 1.1

Class interval	Frequency (f)	Classical point (x)	fx	d	fd
30-40	3	35	105	-25	-60
40-50	10	45	450	-10	-100
50-60	19	55	1045	0	0
60-70	31	65	2015	10	310
70-80	11	75	825	20	220
	$\sum f = 74$		$\sum fx = 4440$	$\sum fd =$	370

$$\bar{x} = \frac{4440}{74}$$

$$\bar{x} = 60$$

We may also get the mean of grouped data using the assumed mean method. The method simplifies calculation by reducing the magnitude of numbers used in the calculation. We start by making one of the class mid-points, the assumed mean. The actual mean is then found by using the formula.

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}$$

Where d is the diff.6th the assumed mean and the respective class midpoints.

i.e. d+xi-4

A is the assumed mean.

Example 4

Let's take A in the previous example to be 55

Therefore $\bar{x}$ = 55+ $\frac{370}{74}$ =55 + 5

$\qquad \bar{x}$ = 60

(b) Median

The median is a positional measure. A median is the middle value in an ordered set of data values. There are as many values above the median as there are below it. Given a set of data, the first step is to arrange the data in order. When we have an odd number. of observations the median is the value of the middle observation. Median is denoted as $X_{0.5}$

.To locate the middle value of a distribution, we use the formula

$X_{0.5} = \frac{n+1}{2}$ Value in the distribution

Example1

The salaries of a sample of CEOs of companies in Westlands area are:

Kshs, 90,000, 150 000, 1.2m, 500 000, 600 000, 1.5m

What is the median salary for this sample of CEOs?

90 000, 150 000, 500 000, 600 000, 800 000, 1.2m, 1.5m

$X_{0.5} = \frac{7+1}{2}$

$\qquad$ = 4th observation which is Ksh 600,000

For an even number of observations, the median is the arithmetic mean of the two middle values.

Example 2

What is the median in the following values?

1200, 18, 30, 17, 60, 90, 48, 65, 120

17, 18, 30, 48, 49, 60, 65, 90, 120, 1200

$X 0.5 = \frac{49+60}{2}$ = 54.5

Median for grouped data

In grouped data, just like in ungrouped data, the median is the value or point occupying the middle position. It divides the data into 2 equal parts.

$$X_{0.5} = L_{0.5} + \left\{ \frac{n/2 - y}{f} \right\}$$

i where $L_{0.5}$ = The lower limit or lower boundary of the median class

cf is the cumulative frequency of the distribution upto the class immediately preceding the median class.

F= The frequency of the median class

i =The interval or class width of the median class

n= The total no. of observations

Steps

Identify the median class as it contains the median. The median class is the one that contains the observation slightly above n/2 observation.

Example 3

Compute the median score in the business statistics class

Class	f	cf	
30-40	3	3	0-2
40-50	10	13	3-12
50-60	19	32	13-31
60-70	31	63	31-62 (37th)
70-80	11	74	62-73

The median class contains the observation slightly above $\frac{74}{2}$ observation i.e 37th

Therefore is the 60-70 class

$$X_{0.5} = 60 + 10 \left[\frac{\frac{74}{2} - 32}{31} \right]$$

$$= 60 + 1.61$$

$$X_{0.5} = 61.61$$

c) Mode

The word originated from the French word 'La mode' meaning mostly fashionable. It represents the value that occurs most often in a distribution. "The value with the highest frequency."

In grouped data, the mode will fall into the modal class. The modal class is the class with the highest frequency. The mode can be computed for any level of data.

It is most for normal data because for this level data, we cannot compute the other level of central tendency.

The mode can be used when considering the popularity of given attributes e.g. the most popular car in a given town.

A distribution can have more than one mode. When there are two modes, it is said to be bi-modal denoted Xm

Example

What is the mode in following data set 2, 11, 25, 11, 2, 5, 17, 38, 25, 17, 25, 13

$$= 25 \text{ (since it appears 3 times)}$$

Mode for grouped data

The mode for grouped data falls in the modal class. After the modal class is determined, the mode is found as:

$$Xm = L_m + \left(\frac{d_1}{d1 + d2}\right) i$$

Where L_m = The lower limit of the modal class

       d_1 = The difference between the frequency of the modal class and that of the

          class before it

       d_2 = The difference between the frequencies of the modal class and that of the class

          just after it.

       i   = The class mdth of the modal class.

Example

Compute the modal mark in the business statistics class last semester

Mark	Frequency
30 – 40	3
40 – 50	10
50 – 60	19
60 – 70	31
70 – 80	11

Modal class – 60 -70 with a frequency of 31

$$X_m = 60 + \frac{(31 - 19)}{(31 - 19) + (13 - 11)} \times 10$$

$$X_m = 63.75$$

ii) Measures of dispersion

The measures of central tendencies give us values that may be considered to be typical values samples of population from which they are computed.

Measures of dispersion enable us to know how far or how near observed values are spread from the averages. They show the extent to which such values differ from the average value (usually the mean). When observed values are close to the mean, we say there is low dispersion.

Dispersion is also known as spread, scatter or variation. Some of the most commonly used measures of dispersion include: - range, variance and standard deviation.

Range

It is the difference between the highest and the lowest values in a data set

Therefore, R= $X_{max} - X_{min}$

The range is the simplest measure of dispersion because it only uses two values. It is most useful in cases where there are erratic changes.

Example

What is the range in the following exchange rates of the shilling to the US dollar? 75, 74,77, 68, 69, 70, 73, 74, 68.5, 75.5, 69, 78.5, 70

Range = max – min

= 78.5 – 68

= 10.5

Example

The following data shows salaries earned by the top management of Kabete International Ltd.

105,000; 2,000,000; 300,000; 250,000; 120,000; 350,000, 130,000

Range   = Max – Min

= 2,000,000 – 105 000

= 1,895,000

Weakness

Range depends on only two values. This means that it can be influenced by extreme values that may be considered to be outliers.

It doesn't not give an indication as to how the values are spread in a distribution

To overcome this weakness, we use the inter-quartile range (IR).

The inter-quartile range is the difference between the top quartile and the lower quartile

$IR = Q_3 - Q_1$

Q_3– The value in the observation below which ¾ or 75% of the observation lie and above in the remaining ¼ or 25% of the observations

Q_1- Will have 25% of observation are less than and 75% above it.

Quartiles, Deciles, percentiles

Another way to describe variation in data is to determine the location of what divides a set of observation into equal parts. These values include the median, quartile, deciles and percentiles.

Quartiles

They divide an ordered set into four equal parts. The first quartile Q_1 is the value withih which 25% of the observation lie and Q_3 is the value below which 75% of the observations lie. When computing quartiles, the first step is to locate the quartile class. The location of the quartile is found as:

$Q_j = (n+1) j/4=$ Where Q_j quartile and is 1, 2, 3, 4.

The quartile value is then found as;

$$Q_j = L_j + \frac{(jt/4-cf)}{F} i$$

Where L_j - the lower limit of the quartile class

cf   - Cumulative frequency up to the class before the quartile class

f   - the frequency of the quartile class

i   - width of the quartile class

Example

Compute the values of Q_1 and Q_3 for the scores in the Business Statistics course last semester

Marks	f	cumulative	cf
30-40	3	3	0-2
40-50	0	13	3-12
50-60	19	32	13-31 (18)
60-70	31	63	31-62
70-80	11	74	62-73

Solution

Find the location of Q_1

Q_1 falls in the class width (74+1) ¼ the observation is 18.75[th] observation. This is the class 50-60

Q_1 will therefore be

$$50 + \frac{(¼ (74) - 13)}{19} \times 10$$

= 52.89

This means that 25% of the students scored less than 52.89% in the course.

Q_3 = Location = (n+1)3/4

= (74+1)3/4

75 x 3/4 = 56.25

Value $= 60 + \dfrac{(¾ (74) - 32)}{31} \times 10$

= 67.58

This means that 75% of the students scored less than 67.58% marks

Percentiles

They divide an ordered data set into 100 equal parts

Given a set distribution $X_1, X_2, X_3, \ldots\ldots\ldots X_n$ then p^{th} percentile is the value of X such that P% of the observation are less than P and (100-p) % of the observation are greater than P.

Example 1

P40 means 40% of the observations are less than P and 60% of the observation are greater than P

P40 is the 40th percentile

The location of the percentile has to be determined before we get the percentile. The percentile class is the one that contains the $(n+1) k/100^{th}$ observation where k-1, 2,3,……100

The percentile will then be found as;

$$P_k = Lk + \frac{(kn/100 - cf)}{f}$$

Where   Lk - The lower limit of the percentile class

           Cf - Cumulative frequency upto class just before the percentile

           f   - Frequency of P class

           j   - Width of P class

Example 2

Compute the following percentiles for the scores in the Business Statistics course last semester.

P10, p25, p50, p75, p60, p50

P10, location (74+1) 10/100 = 75x0.1 = 7.5

We look for the class with 7.5^{th} observation

$$40+ \frac{(10 \times 74 - 3)}{\frac{100}{10}}10 = 44.4$$

Variance and standard deviation

The two common measures of dispersion are variance and standard deviation.

A data set that is more variable will have a larger variance than one that is relatively homogeneous. The variance is the sum of the square deviations divided by the number of observations. It is the average of the squares of the deviation of the individual values from their means. For any set of values, the sum of square deviations from the mean is smaller than the sum of square from any other point.

Population variance is denoted as $\delta^2 \rightarrow$ parameter

Sample variance is denoted as $S^2 \rightarrow$ statistics

$$\delta^2 = \frac{\sum\limits_{i-1}^{n} (X_i - \mu)^2}{N}$$

Where x_i = individual observed values

μ = population mean

N = No. of observations

	Population		Sample	
Mean	μ	$= \dfrac{\sum x}{N}$		$= \dfrac{\sum x}{n}$
Sample size	n		N	
Standard deviation	σ	$= \sqrt{\dfrac{\sum (x - \mu)^2}{N}}$	s	$= \sqrt{\dfrac{\sum (x -)^2}{n - 1}}$

Standard deviation = $\sqrt{\sigma 2} - \sqrt{\text{variance}}$

Example

The following values were observed from a population:

30, 32, 40, 48, 50

Compute the variance for this data

Solution

$$\sigma^2 = \sum \frac{(x_i - u)^2}{N}$$

$$\mu = \frac{200}{5}$$

$$\mu = 40$$

	Value, x_i	Deviation $(x_i - \mu)$	$(x_i - \mu)^2$
	30	-10	100
	32	-8	64
	40	0	0
	48	8	64
	50	10	100
Total $\sum$	200		328

$$S^2 = \frac{328}{5}$$

$$= 65.6$$

Sample variance, $S^2 = \dfrac{\sum\limits_{i=1}^{n} (X_i - X)^2}{n - 1}$

$$= \frac{328}{4}$$

$$= 82$$

For group data, we only get an approximation (estimate) of the variance.

$$S^2 = \frac{\sum\limits_{i=1}^{n} f_i(X_i - X)^2}{\sum f_i - 1}$$

The standard deviation is the square root of the variance. It is expressed in the same units as the original data.

Population variance, $\sigma^2 = \dfrac{\sum\limits_{i=i}^{N} (x_i - m)^2}{N}$

Sample standard deviation, $S = \sqrt{\dfrac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}}$

For grouped data, $S = \sqrt{\dfrac{\sum\limits_{i=1}^{n} (x_i - x)^2}{\sum fi - 1}}$

Coefficient of variation

Coefficient of variation is useful when comparing the levels of variability in sets of data. It is a relative measure of variability. It is especially useful when comparing sets that are not measured in the same units e.g. in weights of people vs. income, or when comparing data with means that are of different magnitudes, or risk of projects. The coefficient of variation is dimensionless (free of units). It is generally expressed in percentage or in decimal form.

$$CV - \frac{s}{x} \text{ or } = \frac{s}{x} \times 100\%$$

The higher the coefficient of variation, the higher the variability.

Example

Which of these 2 sets of data has greater variability?

A	B
χ = 150kgs	χ = 0.85cm
S = 30.5kgs	S = 0.015cm
CV = $\frac{30.5}{150}$	CV = $\frac{0.015}{0.85}$
= 0.203	= 0.018

Set A has greater variability than set B

Measures of normality/shape

A normal distribution is data that forms a symmetrical bell curve. Measures of normality tells us more about the way data is distributed e.g. figures A, B and C below appear to be symmetrical. Distributions may have same averages and measures of dispersion but have different shapes. Measures of normality give us an idea of how the data is distributed. Measures of normality include coefficient of skewness and coefficient of kurtosis.

Skewness

Skewness describes the degree of symmetry in a distribution. When data are uni-modal and symmetrical, the mean, mode and median will be almost the same value. In a skewed distribution, we have higher frequencies occurring to one end of the distribution e.g.

A

B

C
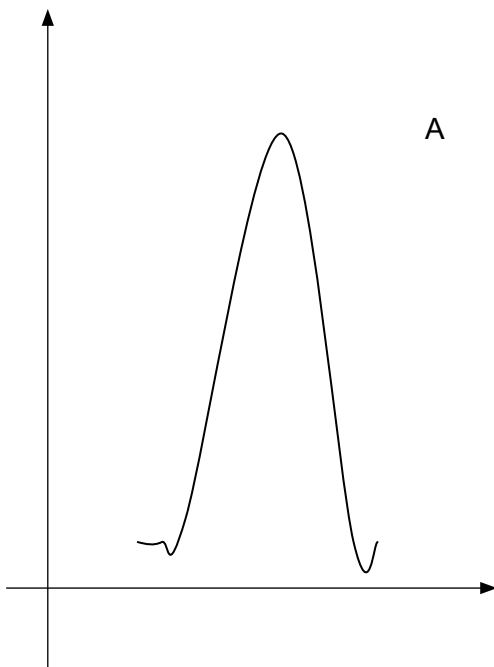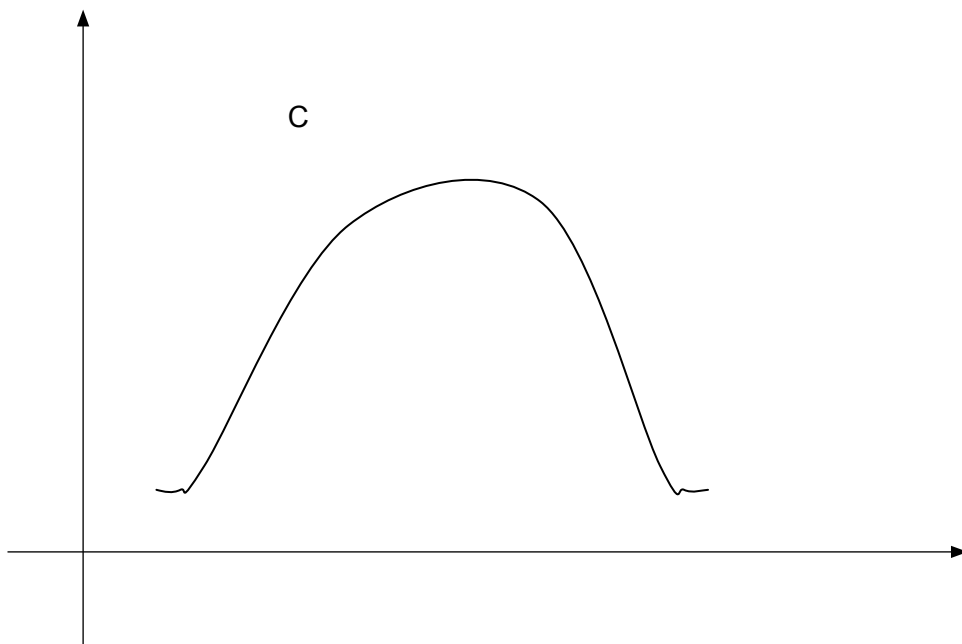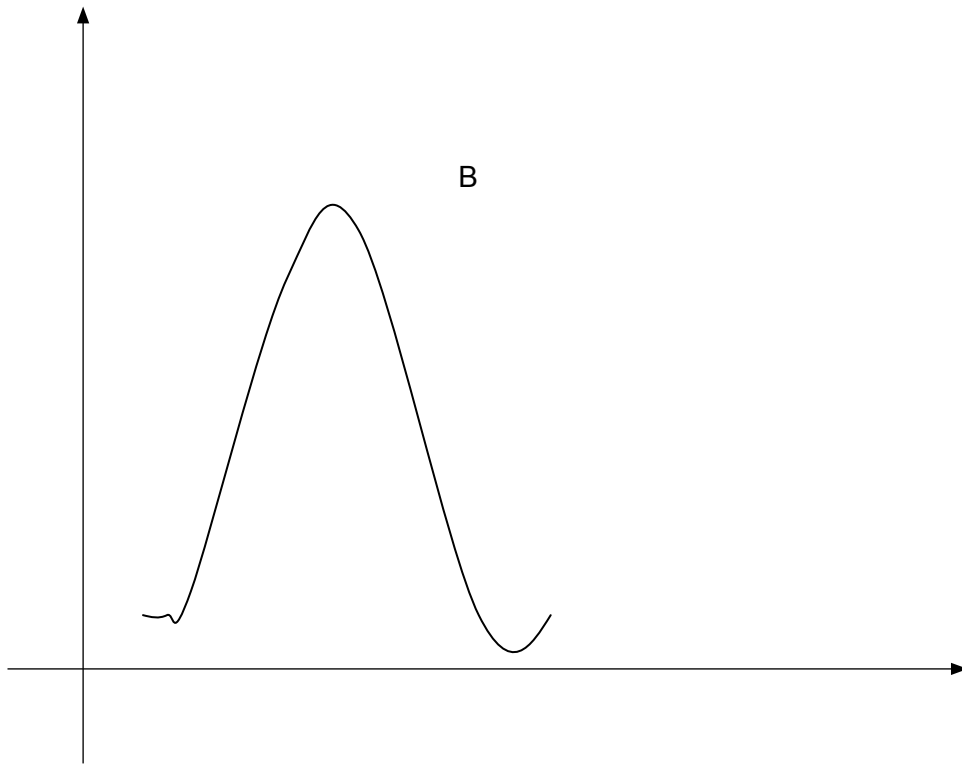
Graphs A and C represent skewed distributions;

Skewed to the right (negatively skewed)

Skewed to the left (positively skewed)

In a positively skewed distribution the mean > median >mode

In a negatively skewed distribution the mean < median < mode

When data are skewed, the mean will be pulled towards the skew. The degree of skewness is composed by using either the 1st Pearsonian coefficient of skewness or the 2nd

First coefficient of skewness, SK1 = $\dfrac{X - X_m}{S}$ where

Second coefficient of skewness, SK2 = $3\dfrac{\left(X - X_{0.5}\right)}{S}$

x_m = mode

S = standard deviation

$X_{0.5}$ = median

If SK1 or SK2 = 0, the distribution is normally distributed or is symmetrical.

If SK>0, the distribution is positively skewed.

If SK<0, the distribution is negatively skewed.

Kurtosis

Kurtosis describes the degree of peakedness or steepness in a distribution.

Example

Leptokurtic
(Highly peaked)

Mesokurtic
(Normal distribution)
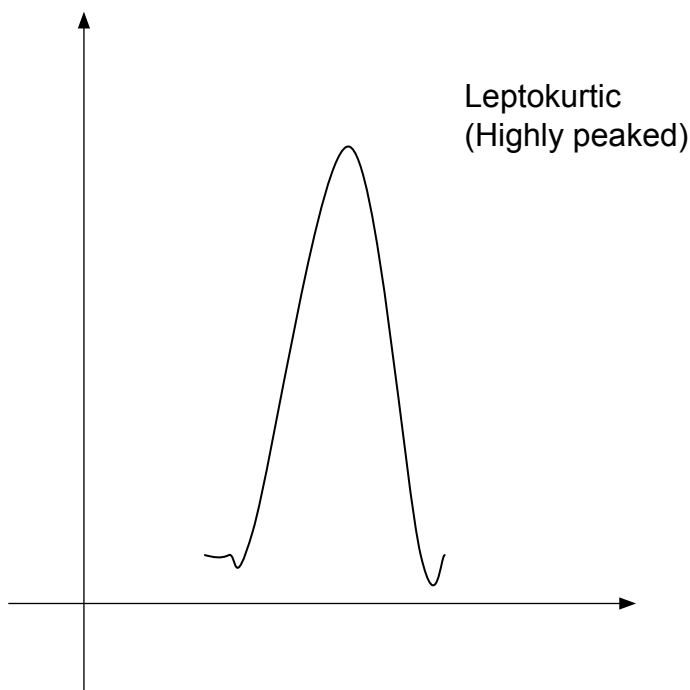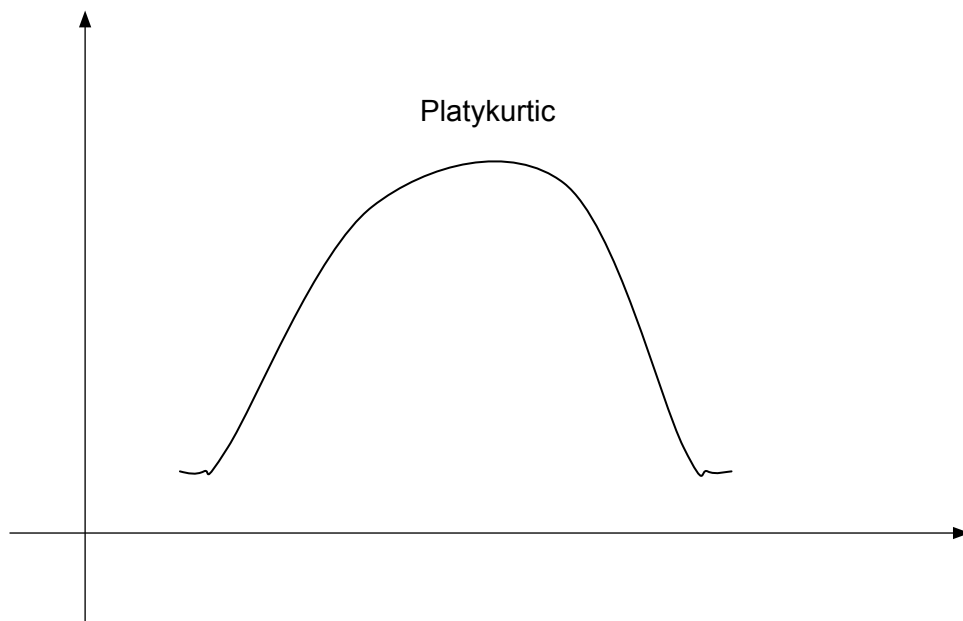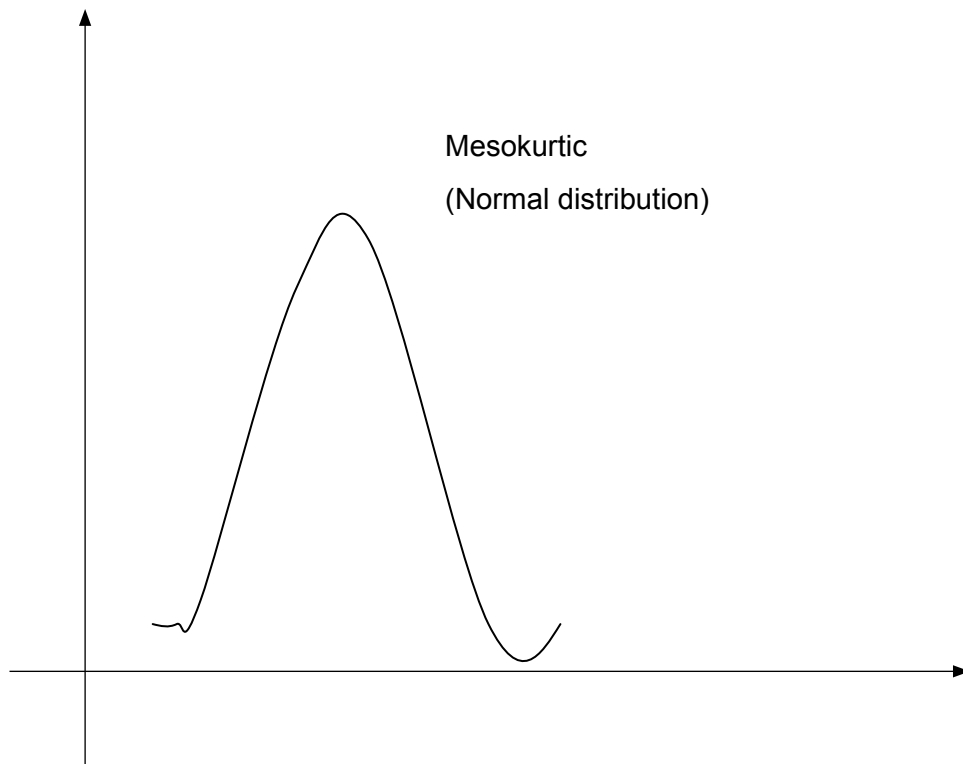
Platykurtic

For normal distribution, k ≈ 0.25 i.e. mesokurtic.

If K< 0.25, the distribution is platykurtic.

If K> 0.25, the distribution is leptokurtic.

$$K = \frac{\frac{1}{2}(Q_3 - Q_1)}{P_{90} - P_{10}}$$

Empirical Rule

i.   The empirical rule says that if a sample or population of measurement has a normal distribution
ii.   Approximately 68% of the observations lie within one standard deviation of the mean
iii. Approximately 95% of the observations lie within two standard deviations of the mean
iv.   Approximately 99.7% of the observations lie within three standard deviations of the mean.

Diagram 1.1
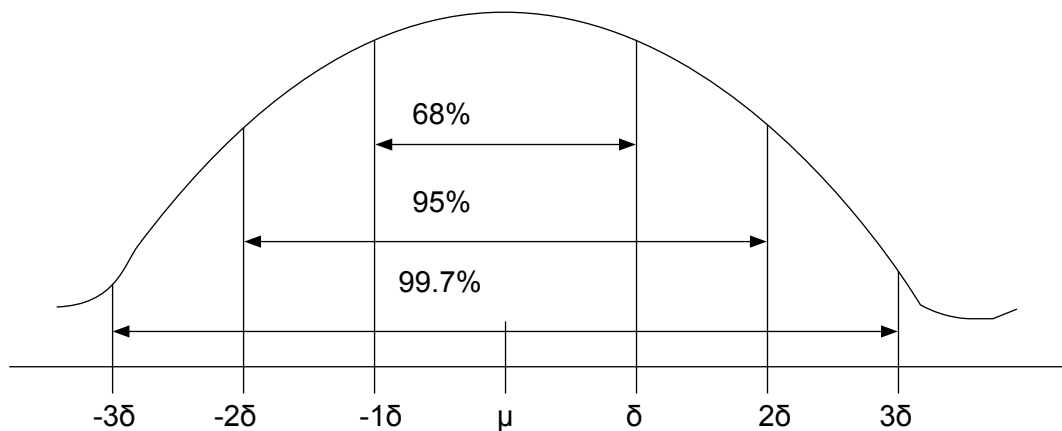
CHAPTER SUMMARY

Statistics is the art and science of getting information from data or numbers to help in decision making.

The following are some characteristics of index numbers

1. They are specialised averages to obtain a typical measure of central tendency like an average. The items must both be comparable and the unit of measurement must be the same

2. Measure the change in the level of a phenomenon3. Measure the effect of changes over a period of time

Counting techniques may be classified into:

i.   Probability trees
ii.   Permutations
iii. Combinations

CHAPTER QUIZ

1. Define Mean
2. Which of the following is the odd one out?
   i. Mean
   ii. Mode
   iii. Median
   iv. Range
3. What is the importance of Kurtosis?
4. ………… describes the degree of symmetry in a distribution when data are uni-modal and symmetrical, the mean, mode and median will be almost the same value.
5. List three counting techniques.

ANSWERS TO CHAPTER QUIZ

1. It is the sum of all the values divided by the number of values.
2. Range – it is not a central tendancy measure
3. Kurtosis describes the degree of peakedness or steepness in a distribution.
4. Skewness
5. i) Probability trees
    ii) Permutations
    iii)Combinations

QUESTIONS FROM PREVIOUS EXAMS

1. The weights of 15 parcels recorded at the GPO were as follows:
    16.2, 17, 20, 25(Q1) 29, 32.2, 35.8, 36.8(Q2) 40, 41, 42, 44(Q3) 49, 52, 55 (in kgs)

Required

Determine the semi interquartile range for the above data

2. The following table shows the levels of retirement benefits given to a group of workers in a given establishment.

Retirement benefits £ '000	No of retirees (f)	UCB	cf
20 – 29	50	29.5	50
30 – 39	69	39.5	119
40 – 49	70	49.5	189
50 – 59	90	59.5	279
60 – 69	52	69.5	331
70 – 79	40	79.5	371
80 – 89	11	89.5	382

Required

a) Determine the semi interquartile range for the above data
b) Determine the minimum value for the top ten per cent.(10%)
c) Determine the maximum value for the lower 40% of the retirees

3. The following information was obtained from an NGO which was giving small loans to some small scale business enterprises in 1996. the loans are in the form of thousands of Kshs.

Loans	Units (f)	Midpoints(x)	x-a=d	d/c= u	fu	Fu²	UCB	cf
46 – 50	32	48	-15	-3	-96	288	50.5	32
51 – 55	62	53	-10	-2	-124	248	55.5	94
56 – 60	97	58	-5	-1	-97	97	60.5	191
61 –65	120	63 (A)	0	0	0	0	0	0
66 –70	92	68	5	+1	92	92	70.5	403
71 –75	83	73	10	+2	166	332	75.5	486
76 – 80	52	78	15	+3	156	468	80.5	538
81 – 85	40	83	20	+4	160	640	85.5	57.8
86 – 90	21	88	25	+5	105	525	90.5	599
91 – 95	11	93	30	+6	66	396	95.5	610
Total	**610**				**428**	**3086**		

Required

Using the Pearsonian measure of skewness, calculate the coefficients of skewness and comment briefly on the nature of the distribution of the loans.

4. a) Distinguish between discrete and continuous data.

    b) What is dispersion and what is the formula for the standard deviation?

    c) What is the measure of relative dispersion?

CHAPTER THREE

Probability

CHAPTER THREE
Probability

▶ **OBJECTIVES**

At the end of this chapter, you should be able to:

i.   Define probability.
ii.   Describe the classical, the relative frequency, and the subjective approaches to probability.
iii. Calculate probabilities, applying the rules of addition and multiplication.
iv.   Determine the number of possible permutations and combinations.
v.   Calculate a probability using Baye's Theorem.
vi.   Define a probability distribution.
vii. Distinguish between a discrete probability distribution and a continuous probability distribution.
viii Calculate the mean, variance and standard deviation of a probability distribution.
ix.   Construct binomial and poisson distribution.
x.   Determine which probability distribution to use in a given situation.

Fast Forward: Probability, or chane, is a way of expressing knowledge or belief of the possibility or the strength of the possibility that an event will occur.

▶ **INTRODUCTION**

Probability is a measure of likelihood, the possibility or chance that an event will happen in future. It can be considered as a quantification of uncertainty.

Uncertainty may also be expressed as likelihood, chance or risk theory. It is a branch of mathematics concerned with the concept and measurement of uncertainty.

Much in life is characterised by uncertainty in actual decision making.

Probability can only assume a value between 0 and 1 inclusive. The closer a probability is to zero the more improbable that something will happen. The closer the probability is to one the more likely it will happen.

▶ **DEFINITIONS OF KEY TERMS**

Random experiment results in one of a number of possible outcome e.g. tossing a coin

Outcome is the result of an experiment e.g. head up, gain, loss, etc. Specific outcomes are known as events.

Trial- Each repetition of an experiment can be thought of as a trial which has an observable outcome e.g. in tossing a coin, a single toss is a trial which has an outcome as either head or tail

Sample space is the set of all possible outcomes in an experiment e.g. a single toss of a coin, S=(H,T). The sample space can be finite or infinite. A finite sample space has a finite number of possible outcomes e.g. in tossing a coin only 2 outcomes are possible.

An infinite sample space has an infinite number of possible outcomes e.g. time between arrival of telephone calls and telephone exchange.

An Event of an experiment is a subset of a sample space e.g in tossing a coin twice S= (HH, HT, TH, TT) HH is a subset of a sample space.

Mutually exclusive event - A set of events is said to be mutually exclusive if the occurrence of any one of the events precludes the occurrence of other events i.e the occurrence of any one event means none of the others can occur at the same time e.g. the events head and tail are mutually exclusive

Collectively exclusive event - A set of events is said to be collectively exclusive if their union accounts for all possible outcomes i.e. one of their events must occur when an experiment is conducted.

Favourable events refers to the number of possible occurrences of a given event in an experiment e.g. if we pick a card from a deck of 52 cards the number favorable to a red card is 26, in tossing a coin the number favourable to a head is one.

Independent events – Events are independent if the happening or non-happening of one has no effect on the future happening of another event. E.g. in tossing two times of a coin, the outcome of 1st toss does not affect 2nd toss.

Equally likely events – Events are equally likely if the happening of one is not favoured over the happening of others. In tossing a coin the tail and head are equally likely.

▶ OTHER CONCEPTS

Unconditional and conditional probabilities – with unconditional probability we express probability of an event as a ratio of favourable outcomes to the number of all possible outcomes.

A conditional probability is the probability that a second event occurs if the first event has already occurred.

Joint probability – joint probability gives the probability of the joint or simultaneous occurrence of two or more characteristics.

Marginal probability – is the sum of two or more joint probabilities taken over all values of one or more variables. It is the probability that the results when we ignore one or more criteria of classification when computing probability.

▶ **INDUSTRY CONTEXT**

Probability is used throughout business to evaluate financial risks and decision-making. Every decision made by management carries some chance for failure, so probabiity analysis is conducted formally.

In many natural processes, random variation conforms to a particular probability distribution known as the **normal distribution**, which is the most commonly observed probability distribution.

▶ **EXAM CONTEXT**

The probability topic has been examined previously:

6/06, 12/05, 6/05, 12/04, 6/04, 6/03, 12/02, 6/02, 12/01, 12/00, 6/00

Laws of Probability

1.   Rules of Addition

a)   Special rule of addition
If two events A and B are mutually exclusive the probability of one or other occurring is equal to the sum of their probability

P (A or B) = P (A) + P (B)

P (A or B or C) = P (A) + P (B) + P(C)

Illustration

An automatic plastic bag - a mixture of beans, broccoli and other vegetables, most of the bags contain the correct weight but because of slight variations in the size of beans and other vegetables. A package may be slightly under or overweight. A check of 4,000 packages of past reveals the following:

Weight	Event	No of packages
Underweight	A	100
Satisfactory	B	3600
Overweight	C	300

What is the probability that a particular package will be either underweight or overweight?

The two events are mutually exclusive

P (A or C) = P (A) + P (C)

P(A) =100/4000

P(C) =300/4000

P(A or C) =400/4000 = 1/10 =0.1

Note: The probability that a bag of mixed vegetable is selected underweight P(A) plus the probability that it is not underweight P(NA) must logically be equal to one. This is referred to as Complement rule.

b)　General rule of Addition

The outcome of an experiment may not be mutually exclusive. This rule is used to combine events that are not mutually exclusive.

P(A or B) = P(A) + P(B)- P(A and B)

Example

a)　What is the probability that a card chosen at random from a pack of well shuffled deck will either be a king or a heart

P(king or heart) = P(king) + P(heart) - P(king and heart)

$$\frac{4}{52} + \frac{13}{52} + \frac{1}{52} = \frac{16}{52}$$

b)　Routine physical examination is conducted annually as part of the programme of a particular organisation. It was discovered that 8% needed correcting shoes, 15% needed major dented work and 3% needed for both correcting shoes and major dental work.

What's the probability that an employee selected at random will either need correcting shoes or major dental work?

2. Rule of Multiplication

a) Special Rule of multiplication

For two events A and B the probability that A and B will both occur is found by multiplying the probability

P(A and B)=P(A) × P(B)

P(A and B and C) = P(A) × P(B) × P(C)

This rule is applicable for independent events i.e. the occurrence of one does not depend on the occurrence of the other.

Example

i)　Two coins are tossed. What is the probability that both will land tails up

The two events are independent

P (T and T) = P (T) × P (T)

=1/2 × 1/2 = ¼ = 0.25

ii) From long experience firestone tyres have a 0.8 probability that their xB.70 will last 40,000 miles before it becomes bald and adjustments made. If you purchase four xB.70

What's the probability that all tyres will last 40,000 miles?

What's the probability that at least two will last 40,000 miles?

Solution

$$P(L \text{ and } L \text{ and } L \text{ and } L) = P(L) \times P(L) \times P(L) \times P(L)$$

$$= 0.8 \times 0.8 \times 0.8 \times 0.8$$

$$= 0.4096$$

Probability of lasting, P(L) = 0.8; Probability of not lasting, P(N)= 0.2

Probability of at least two tyres lasting 40,000 miles is given by the following combinations:

b)　General rules of Multiplication

It states that of two events A and B the joint probability that both events will happen is found by multiplying the probability that A will happen by the conditional probability of event B happening

$$P(A \text{ and } B) = P(A) \times P(B|)$$

Where P(B/A) stand for probability that B will occur given that A has already occurred.

Example

1. Assume that there are 10 rolls of film in a box 3 of which are defective. Two rolls are to be selected one after another. What's the probability of selecting a defective roll followed by another defective roll?

   $$P (D \text{ and } D) = P (D) \times P (D \mid D)$$

   $$3/10 \times 2/9 = 6/90 = 0.07$$

2. Three effective toothbrushes were accidentally shipped to a chemist by Clean Brand Product along with 17 non-effective ones

What's the probability that of the first 2 toothbrushes sold one will be effective and the other one will not.

Solution

$$P (\text{One defective} = P (D \text{ and } N) \text{ or } P (N \text{ and } D)$$

$$= 2/20 \text{ x } 17/19 + 17/20 \text{ x } 3/19$$

▉ Contingency Table

A two dimensional contingency table is formed by classifying two factors. One factor determines the row categories and the other determines the column categories. Each element in the table belongs to the two classifications known as a cell e.g. classifying subjects by gender (M and F) and smoking status (Current/Former/Never). Such categories are said to be mutually exclusive and collective. Exhaustive mutually exclusive means the categories include all possibilities and so there is a category for everyone.

Table 3.1

Gender		Smoking Status		
		Current	Former	Never
	Male	a	b	c
	Female	d	e	f

The two factors here are smoking status and gender and the cells are a,b,c,d,e and f.

Example

A survey of executives dealt with their loyalty to the company. Out of the questions asked "If you were given an offer by another Company equal or slightly better than the present will you remain in the Company or take another position. The responses were classified with their level of service within the company.

Length of Service

Loyalty	Less than 1 year	1-5 years	6-10 years	More than 10 years	Total
Remain	10	30	5	75	120
Not Remain	25	15	10	30	80
					200

What's the probability of randomly selecting an executive who is loyal to the company (who will remain) and who has been in the service in more than 10 years?

P(R and >10) = P(R) × P (>10|R)

$$= \frac{120}{200} \times \frac{75}{120}$$

$$= \frac{75}{200}$$

What's the probability of selecting at random an executive who will remain loyal to the company and has less than 6 years of service

$$= \frac{10 + 30}{200}$$

$$= \frac{40}{200}$$

$$= 0.2$$

■ Tree Diagram (Probability Trees)

Is very useful for portraying conditional and joint probabilities. It is particularly useful for analysing business decisions where there are several stages of problems.

Bayes Theorem

In the 18th Century Rev Thomas Bayes, an English Presbyterian Minister, ordered this question, "Does God really exists" Being interested in mathematics he attempted to develop a formula to arrive at the probability that God really exists based on the evidence available on earth.

Later, **Laplace** refined Bayes work and gave it the name Bayes Theorem in a workable form, the Bayes Theorem is

$$P(A/B) = \frac{P(A)P(B)}{P(B)}$$

$$= \frac{P(A_1) \times P(B/A_1)}{P(A) \times P(B/A) + P(A_2) \times P(B/A_2)}$$

Note: $P(PA \text{ and } B) = P(A) \times P(B/A)$

Example 1

Suppose 5% of the population of a third world country has a disease peculiar to that country. Let A_1 refer to the event that a person has the disease and A_2 refer to the event that a person does not have the disease. Therefore, if a person is chosen at random, the probability that the individual chosen will have a disease will be $P(A_1) = 0.05$ thus $P(A_2) = 0.95$.

These probabilities are called prior probabilities (they are assigned before any empirical data is obtained). It's the probability based on the present level of information. Assume further that there is a diagnostic technique to detect the disease but it is not accurate. Let B denote the event test shows disease is present. Assume that historical evidence shows that if a person actually has the disease the probability that the test will indicate presence of a disease is 0.9.

$P(B|A_1) = 0.9$

Assume that there is probability of 0.15 that a person does not actually have the disease but the test indicates that the disease is present = $(P(B/A_2) = 0.15$.

Let us randomly select a person from the country and perform a test. The test indicates that the disease is present. What's the probability that the person actually has the disease?

$$P(A_1 \mid B) = \frac{P(A_1) \times P(B/A_1)}{P(A_1) \times P(B/A_1) + P(A_2) \times P(B/A_2)}$$

Note: $P(B) = P(A_1) \times P(B/A_1) + P(A_2) \times P(B/A_2)$

Example 2

An electronic firm purchases its supplies from 4 different suppliers. A Ltd supplies 20%, B Ltd 30% C Ltd 25% and D Ltd 25%.

A Ltd tends to have the best quality. Only 3% of their supplies are defective. B Ltd supplies are 4% defective, C Ltd 7.0% and D Ltd 6.5% defective

Required:

a) What is the probability of selecting a defective item?

b) A defective supply was discovered in two shipments. What's the probability that it came from A Ltd

What's the probability that the defective supply came from A ltd, C ltd and D ltd

Solution

a)   What is the probability of selecting a defective item

P (D) = P(A and D) + P(B and D) + P(C and D)+P(D and D)

0.006+0.012+0.0175+0.01625

=0.05175

b) $P(A/D) = \dfrac{P(A \text{ and } D)}{P(D)}$

$= \dfrac{0.006}{0.05157}$

c) $P(A/D) = P(A) \times P(D/A)$

$= \dfrac{P(A) \times P(D/A) + P(B) \times P(D/B)) + P(C) \times P(D/C)) + P(D)) \times P(D/D))}{P(D)}$

� Principle of Counting

If the number of possible outcomes in an experiment is relatively easy to count and list all the possible events e.g. there are six possible events resulting from a roll of a dice. If however, there a large number of possible outcomes such as number of boys and girls for families with 10 children it would be tedious to list and count all the possibilities: we could have two boys and eight girls, 1 boy and 9 girls etc. This situation is further complicated when we consider the order of arrangements.

To facilitate counting, counting rules can be employed. These include multiplication rule, permutation rule, and combination rule.

Multiplication Rule

If there are M ways of doing one thing and N ways of doing another thing, there are

M x N ways of doing both; that is, the total number of arrangements equals M×N. This can be extended to more than two events e.g. for three events M, N and O the total number of arrangements equal M × N × O.

Example I

An automobile dealer wants to advertise that for Kshs 500,000 you can buy a convertible, a two-door or a four-door model with your choice as either a saloon or a station wagon. How many different arrangements of models and car types can the dealer offer?

Solution

Convertible Saloon   Convertible S.W

Two-door saloon   2-door S.W

Four-door Saloon   4-door S.W

We can employ multiplication rule as a check when M- No of models

N- No of car types

Solution = 6

Example 2

The marketing manager of ABC Ltd would like to send a representative to the branches of a company in Central Kenya. The company has three regions in Central Kenya, each with 5-branches. The sales representative will be based in one of the sub-branches. How many ways can this be done?

M – No. of regions

N – No. of branches

M x N = 3x5 = 15

Solution =15 ways

Example 3

Pioneer manufactures 3 models of stereo receivers, 2 cassettes decks, 4 speakers and 3CD-carousels. When the 4 types of compatible are sold together they form a system. How many different systems can electronic firm offer?

2 x 4 x3=24syatems

Permutation Rule

The multiplication formula is applied in finding the number of possible arrangements for two or more groups. The permutation formula is applied to find the number of possible arrangements where there is only one group of objects.

Permutation is thus any arrangement of r objects from n possible objects. However this can be calculated using the permutation formula:

$$^nP_r = \frac{n!}{(n-r)!}$$

where; ^nP_r = number of arrangements

n = number of items to be arranged

r = number of elements to be contained in the arrangement.

! = *factorial e.g.;* 4! = 4 × 3 × 2 × 1 = 24

Example 1

In how many ways can you arrange A, B and C?

Solution

In this case, note that the order in which the elements appear matters i.e ABC is different from ACB.

Therefore

ABC   BAC   CAB

ACB   BCA   CBA

Answer: Six ways.

Using the permutation formula

$$^nP_r = \frac{n!}{(n-r)!}$$

where; ^nP_r = number of arrangements

n   = number of items to be arranged

r   = number of elements to be contained in the arrangement.

!   = *factorial e.g.;* 4! = 4 × 3 × 2 × 1 = 24

In this example

$$^3P_3 = \frac{3!}{(3-3)!}$$

= 3 × 2 × 1 = 6 ways

Note that 0! = 1

ii)   3 electronic points are to be assembled in a plug-in unit for a TV set. The parts can be assembled in any order. In how many different ways can the 3 parts be assembled?

All the objects (electronic parts) are to be assembled and therefore r=3. .

Solution

n =3, r =3

$$^nP_r = \frac{n!}{(n-r)!} = \frac{3!}{(3-3)!}$$

$$= \frac{6}{1}$$

$$= 6$$

Example 2

There are 10 numbers; 0 through to 9, which are to be used in code group of four to identify an item of clothing e.g. code 1083 is to identify blue blouse, code 1030 is identify a pair of socks and so on. Repetition of numbers is not permitted i.e. same number can't be used twice or more in a total sequence e.g. 2256 or 2872

How many different code groups can be designed?

n = 10   r = 4

^nP_r = 5.040

Permutation allowing repetition

If repetitions are permitted the permutation formula $^nP_r = n^r$ e.g assume a 2-letter code where 'a' and 'b' are to be taken each at a time with repetitions such as aa allowed.

In this case n = 2, r = 2

The no. of permutations $^2P_2 = 2^2 = 4$

Combination Rule

In determining the number of permutations of n-different things taken at a time, the order of the items is important.

Combination determines the number of ways to choose r objects from group of n objects without regard to order.

Combination Formula=

$$^nC_r = \frac{n!}{(n-r)!}$$

Example

The sales department has been asked to design 42 colour codes for 42 different parts. A colour code should consist of three colours and if 3 colours are used for one part, the same cannot be used to identify a different part. Would 7 different colours be adequate to generate the 42 parts colour codes?

Solution

n=7 r=3

$^7C_3 = 35$

35 colour codes would be inadequate. Thus 7 colours would not be adequate.

As an alternative to the use of three colour combination it has been suggested that only two colours (r) be used for one colour code. Would 10 colours be adequate to colour code the 42 different parts?

n = 10   r = 2

$^{10}C_2 = 45$

This number would be adequate.

Probability Distributions

Statistical inference has the objective of making inferences or statement about a population based on a sample selected from a population.

Probability distribution – Is a listing of all the outcomes of an experiment and the probability associated with each outcome.

Example:

Suppose we are interested in determining the number of heads showing upon tossing a coin three times.

Table 3.2: Enumerating the samples

Options	1st coin toss	Second	Third	No. of heads	Prob
1	T	T	T	0	1/8
2	T	T	H	1	1/8
3	T	H	T	1	1/8
4	T	H	H	2	1/8
5	H	H	H	3	1/8
6	H	T	H	2	1/8
7	H	H	T	2	1/8
8	H	T	T	1	1/8

x = outcome   P(x) = Probability of outcome

x	P(x)
0	1/8
1	2/8
2	3/8
3	1/8

Note:

Two important characteristics of probability distribution

The probability of a particular outcome must always be between zero and one.

The sum of probabilities of all mutually exclusive probabilities in a distribution is one

Exercise

Assuming you roll a die and observe the possible outcome of an experiment involving a six side die;

Develop a probability distribution for this outcome

Portray the probability distribution graphically

Random variable

In any experiment of chance the outcomes occur randomly e.g. rolling a single dice is an experiment in which any of the six possible outcomes can occur. Some experiments result in outcomes that are quantitative such as shs, weight or number of children while others result in outcomes that are qualitative e.g. colour, or religious preference.

A random variable is a quantity resulting from a random experiment that by chance can assume different values, for example:

1. If we count the number of students absent (the random variable) from class on a Monday morning we can have 0, 1, 20, 30 etc.
2. If we toss two coins and count the number of heads we could have 0,1 or 3 heads. Since the exact number of heads resulting from this experiment is due to chance, it will be called a random variable.

Other random variables might be the number of defective light bulbs produced during a week, the weight of members of DMS 201 regular class.

A random variable can either be discrete or continuous.

Discrete random variable

A discrete random variable can assume only a certain number of separate values of an interval; for example, if there are 100 employees, the count of number absent on Friday can be 0, 2, and 100

In most cases a discrete random variable is usually a result of counting something.

Note: A discrete variable can assume fractional or decimal values in some cases. These values must however be separated or countable.

Continuous random variable

A continuous random variable can assume any value in an interval. It is usually the result of measuring something e.g. height of a person. It can assume one of an infinitely large number of values within certain limitations.

E.g. the weight of a cloth can be 67, 67.2, 67.24kg, 67.241kg depending on the accuracy of measuring instrument

Logically, if we organise a set of discrete random in a probability distribution the distribution is called a discrete probability distribution.

How to determine mean, variance, standard deviation of probability distribution

The mean represents the central location of the data

The variance describes the spread in the data

In a similar way, a probability distribution is summarised by its mean and its variance

Mean (μ)

The mean is a typical value to represent a probability distribution. It is also the long-run average value of the random variable. It is also referred to as expected value E(x).

E (x) can be computed as

E (x) = μ = Σx.P(x)

For a continuous variable

$$E (x) = μ = \int_{-\infty}^{\infty} x.P(x)dx$$

Where P (x) is the probability of various outcomes of x

Variance and standard deviation

The mean does not describe the amount of spread (variation) of a distribution. The variance does this.

$$s^2 = \sum (x - μ)^2 P(x)$$

The standard deviation = square root of Variance

Example

Bill sells new cars for GM. Bill usually sells the largest number of cars on Saturday. He has established the following probability distribution for the cars he expects to sell on a particular Saturday.

No. of cars = x	Probability P(x)
0	0.1
1	0.2
2	0.3
3	0.3
4	0.1

1.   On a typical Saturday how many cars should Bill expect to sell?

2.   What's the variance and standard deviation of the distribution?

Solution

No. of cars = x	Probability P(x)	x.P(x)	x- μ	(x- μ)²	(x- μ)² P(x)
0	0.1	0	-2.1	4.41	0.441
1	0.2	0.2	-1.1	1.21	0.242
2	0.3	0.6	-0.1	0.01	0.003
3	0.3	0.6	0.9	0.81	0.243
4	0.1	0.4	1.9	3.61	0.361
Total		μ = 2.1			σ² = 1.290

1) $E(x) = \Sigma x. P(x) = 2.1$

2) $\Sigma (x- \mu)^2 P(x) = 1.290$

Standard deviation = $\sqrt{1.29}$ = 1.358 cars

How to interpret standard deviation

Suppose another salesperson Wendi also sold a mean of 2.1 cars on Saturday and standard deviation of her sales was 1.9 cars. We would conclude that there is more variability of Saturday sales of Wendi than Bill

Therefore Bill is doing better i.e he is more consistent and the number of cars he sells is not so different from the mean as compared to Wendi.

Discrete Probability Distribution

There are 3 common Probability Distribution

Binomial Probability Distribution

Poisson Probability Distribution

Continuous Probability

Binomial probability distribution

This is an example of a discrete probability distribution. Here, there are only two possible outcomes e.g. tossing a coin, the answer in a true /false question is either true or false. These two outcomes are mutually exclusive.

More examples

A product is either classified as acceptable by the quality control department or not acceptable.

A sales call results in a customer either purchasing or not purchasing a product. Frequently we classify the possible two outcomes as "success and failure" however, this classification does not imply one outcome is good and the other is bad.

The information obtained in a binomial distribution is as a result of count i.e. we count the number of success in the total number Of trials e.g. if we flip a coin five times, we could count the number of times a head appears.

The probability of a success remains the same from one trail to another e.g. the probability that you will give the first question of a true /false test correctly is one half (1/2). This is the 1st trial. The probability that you will guess in the 2nd question (2nd trial) is also ½ and so on.

One trial in a binomial distribution is independent of another trial. In fact this is the same as saying that there is no rhythmic pattern with respect to the outcomes. As an example, the answers to the true/false question are not arranged in any order e.g. FFFF *TTTTTTTT* FFFF and so on

The number of trials is fixed at a certain value

Formula in binomial distribution

To construct a binomial Probability Distribution we must know

1.   The number of trials

2.   The probability of success on each trial

$$P(r) = \left[\frac{n!}{r!(n-r)}\right] p^r q^{n-r}$$

$$= {}^nC_r p^r q^{n-r}$$

Where   n - Number of trials

r - Number of observed outcomes

p - Probability of success of each trial

q - Probability of failure = 1-p

Example

The answer to a true/false question is either true or false. Assume that;:

1.   An examination consists of four true/false questions

2.   A student has no knowledge of subject matter. The chance (probability) that the student will guess the correct answer to the first question is ½. Likewise the probability of guessing right the remaining number of question is 0.5 for each question. What's the probability that:

a)   The student will get none out of the four questions correct;

b)   The students will get exactly one out of the four correct?

n= 4

r = ?

p = 0.5

q = 0.5

r = 0

$$P(0) = \left[\frac{4! \times 0.5 \times 0.5^4}{0!(4-0)}\right]$$

= 0.0625

Poisson Probability Distribution

Fast Forward

The binomial probability distribution for probabilities of success P less than 0.05 can be computed but the calculations would be quite time consuming especially for a large n say 100 or more. The distribution of probabilities would become more and more skewed as the probability of success becomes smaller. Where the probability of success is very small and n is larger, we refer to it as as Poisson probability distribution named after Simeon Poisson. The probability of a particular event happening is quite small.

The poisson distribution is also a discrete probability distribution because it is formed by counting something. An important characteristic of such a distribution is that we can count only the occurrences, we cannot count the non-occurrences. For example, we can only count the number of telephone calls coming to a telephone exchange (switch board) each minute. We cannot count the number of calls that we did not make. We can count accidents within a given time but we cannot count the number of accidents that did not occur within a given period.

A poisson situation can be recognised by the following characteristics:

a)   The existence of events that
- occur randomly
- are rare

b)   An interval of time, space or distance is defined within which events can occur

Assumptions

The probability of occurrence is the same everywhere in the interval

The probability of multiple occurrences at precisely same point is negligible

The occurrence rate is a random variance

Formula

$$p(x) = \frac{e^{-x} | x}{x!}$$

Where x equals the arithmetic mean number of occurrences (success) in a particular interval. It's the specific value of the random variable.

e - is an exponential.

x - is the number of occurrences.

Continuous Probability Distribution

In continuous probability distribution, random variable can take any value within a given range of values by chance. The distribution of continuous random variable is characterised by a probability density function which represents all the possible outcomes in an experiment. The probability density function can be a graph, an equation or a curve, which represents these possible values. A probability of an experiment = 1.

Given a probability density function of a random variable x we can determine the probability that x has a value between two given points A and B. This probability is equal to the total area enclosed by the curve of the function, two perpendicular lines erected at points A and B and the x-axis as a proportion of the total area enclosed by the function curve and x-axis.

Daigram 3.3
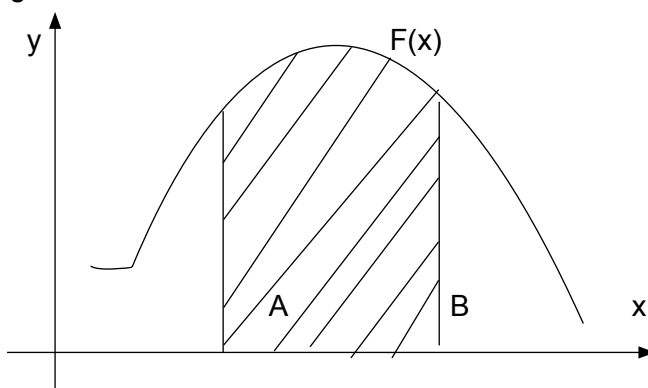
There are several types of continuous probability distribution:

Normal distribution

The normal distribution curve is symmetrical and bell-shaped. Most observations in the distribution are close to the mean with fewer observations further away. A normal distribution can be determined by the values of the mean and the standard deviation.

The normal curve is asymptotic to the x-axis i.e. its tails continuously approach the x-axis but never quite touch it to infinity.

In a normal distribution the following observations are true:

68.26% of all observations lie within 1 standard deviation of the mean

95.44% of all observations lie within 2 standard deviations of the mean

99.73% of all observations lie within 3 standard deviations of the mean

Diagram 3.4
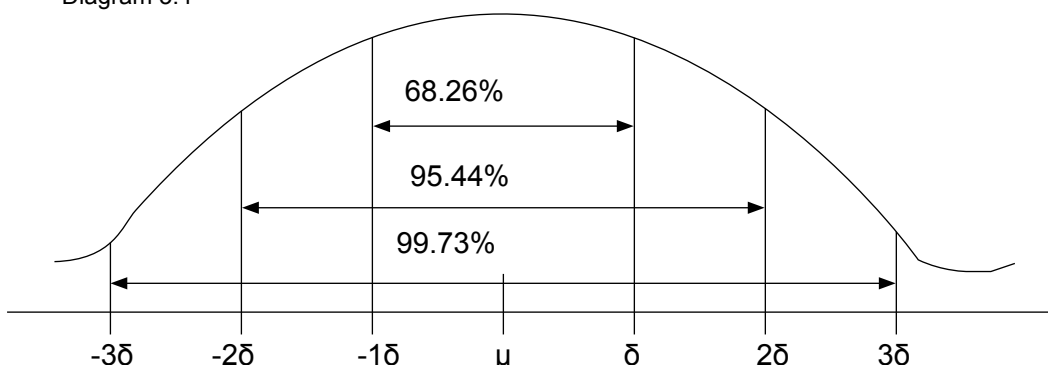
The normal distribution is a family of distribution e.g. standard normal distribution. Standard normal distribution is a special case of the normal distribution. It has a mean of zero and a standard deviation of 1 given any normal distribution. We may convert it to the standard normal distribution by simply converting its µ to o and ó equal to 1. We convert each observation to the standard normal variation as follows:

$$Z = \frac{c - m}{\sigma}$$

Where   Z is the standardised random variable. Also known as standard normal deviate

orZ-score

x is the individual observed value of the random variable

µ is the mean

σ is standard deviation

By standardising any normally distributed random variable we can use the normal distribution to estimate the area enclosed by the normal curve and true z-value. In the normal table, the values of z only go to about z=3

If we have a list of observations of a normaly distributed random variable, what proportion of the observation will be expected to be?

Example

If x is a continuous random variable with a mean of 50 and standard deviation of 2, what is the probability that an observation picked at random is

1. <u>Less than 52</u>
2. <u>Less than 46</u>
3. <u>Between 44 and 48</u>

Solution

1. µ = 50

σ= 2

when x = 52

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{52 - 50}{2}$$

Calculate z = 1

= 0.3413

= 0.5 + 0.3413

= 0.8413

Less than 46

$x = 46$

$x = \dfrac{46-50}{2} = -4/2 = -2$

Calculate 2 = -2

From table 2 = 0.4772

Less than 46

= 0.5 -0.4772

= 0.0228

2. Between 44 and 48

$z = \dfrac{44-50}{2}$

=6/2 = -3

z = 0.49865

$z = \dfrac{48-50}{2}$

= 2/2 = 1

z = 0.3413

3. Between 44 and 48

0.49865 – 0.3413 = 0.15735

CHAPTER SUMMARY

A set of events is said to be mutually exclusive if the occurance of any one of the events precludes the occurrence of any of the other events e.g. when tossing a coin, the events are a head or a tail these are said to be mutually exclusive since the occurrence of heads for instance implies that tails cannot and has not occurred.

Binomial probability distribution is a set of probabilities for discrete events. Discrete events are those whose results or outcomes can be counted. Binomial probabilities are commonly encountered in business situations e.g. in quality control activities when determining the probability of having a certain number of defective items in a given consignment.

General rules of Multiplication – It states that of two events, A and B, the joint probability that both events will happen is found by multiplying the probability that A will happen by the conditional probability of event B happening

P(A and B) = P(A) x P(B/A)

Where P(B/A) stand for probability that B will occur given that A has already occurred.

CHAPTER QUIZ

1. The mean does not describe the amount of spread (variation) of a distribution. The ………… does this.
2. The probability of multiple occurrences at precisely same point is negligible

True ▢

False ▢

3. The ………………. curve is symmetrical and bell-shaped.
4. Which distribution is represented by the formula

$$\left[\frac{n!}{r!(n-r)}\right] p^r q^{n-r}$$

■

■
ANSWERS TO CHAPTER QUIZ
■

1. Variance
2. True
3. Normal distribution
4. Binomial distribution

■

■
QUESTIONS FROM PREVIOUS EXAMS
■

JUNE 2000: QUESTION 7

i)   Define probability as used in Quantitative Techniques.   (3 marks)

ii)   What is Bayes Theorem? Explain how Bayes Theorem can be utilised practically

(5 marks)

iii) KK accounting firm has noticed that of the companies it audits, 85% show no inventory shortages, 10% show small inventory shortages and 5% show large inventory shortages. KK firm has devised a new accounting test for which it believes the following probabilities hold:

P (company will pass test/no shortage)   =   0.90

P (company will pass test/small shortage) =   0.50

P (company will pass test/large shortage) =   0.20

Required:

(i)   Determine the probability if a company being audited fails this test has large or small inventory shortage.   (7 marks)

(ii)   If a company being audited passes this test, what is the probability of no inventory shortage?   (5 marks)

(Total: 20 marks)

December 2001 Question *4*

a)   The past records of Salama Industries indicate that about 4 out of 10 of the company's orders are for export. Further, their records indicate that 48 per cent of all orders are for export in one particular financial quarter. They expect to satisfy about 80 orders in the next financial quarter.

Required:

(i)   Determine the probability that they will break their previous export record. (7 marks)

(ii)   Explain why you have used the approach you have chosen to solve part (i) above

(2 marks)

b) Gear Tyre Company has just developed a new steel-belted radial tyre that will be sold through a national chain of discount stores. Because the tyre is a new product, the company's management believes that the mileage guarantee offered with the tyre will be an important factor in the consumer acceptance of the product. Before finalising the tyre mileage guarantee policy, the actual road test with the tyres shows that the mean tyre mileage is 36,500 kilometres and the standard deviation is 5,000 kilometres. In addition, the data collected indicate that a normal distribution is a reasonable assumption.

Required:

(i)   Gear Tyre Company will distribute the tyres if 20 per cent of the tyres manufactured can be expected to last more than 40,000 kilometres. Should the company distribute the tyres?   (4 marks)

(ii)   The company will provide a discount on a new set of tyres if the mileage on the original tyres does not exceed the mileage stated on the guarantee.
What should the guarantee mileage be if the company wants no more that 10% of the tyres to be eligible for the discount.   (4 marks)

c)   Explain briefly some of the advantages of the standard normal distribution.   (3 marks)
(Total: 20 marks)

JUNE 2002: QUESTION *3*

a)   State clearly what is meant by two events being statistically independent.   (2 marks)

b)   In a certain factory which employs 500 men, 20% of all employees have a minor accident in a given year. Of these, 30% had safety instructions whereas 80% of all employees had no safety instructions.

Required:

Find the probability of an employee being accident-free given that he had:

(i)   No safety instructions   (5 marks)

(ii)   Safety instructions   (5 marks)

c)   An electric utility company has found that the weekly number of occurrences of lightning striking the transformers is a Poisson distribution with mean 0.4.

Required:

(i)   The probability that no transformer will be struck in a week.   *(*3 marks)

(ii)   The probability that at most two transformers will be struck in a week.   (5 marks)
(Total: 20 marks)

JUNE 2003: QUESTION 7

a)   The J.R Muchemi Computer Company is considering a plant expansion that will enable the company to begin production of a new computer product. The company's executive director must determine whether to make the expansion a medium or large scale project. Uncertainly exists in the demand for the new product, which for planning purposes may be low, medium or high demand. The probability estimates for demand are 0.20, 0.50 and 0.30 respectively. The firm's accountants have developed the following annual profit (in thousands of shillings) forecast for the medium and large scale expansion projects:

		Medium scale profit	Expansion profitability	Large scale profit	Expansion profitability
	Low	50	0.20	0	0.20
Demand	Medium	150	0.50	100	0.50
	High	200	0.30	300	0.30

Required:

(i)   Which decision is preferred for the objective of maximising the expected profit

(4 marks)

(ii)   Which decision is preferred for the objective of minimising the risk or uncertainty?

(4 marks)

From your answers in (i) and (ii) above, should the company go for medium scale expansion or the large scale expansion? Explain.   (2 marks)

b)   A manufacturing firm based in Nairobi Kenya receives shipment of parts from two different suppliers from UK and Japan. Currently, 65% of the parts purchased by the company are from suppliers 1 (UK) and the remaining 35% are from supplier 2 (Japan). The quality of the purchased parts varies with the source of supply. Historical data suggest that the quality rating of the two suppliers are as shown below:

	Percent of good parts	Percent of bad parts
Supplier 1	98	2
Supplier 2	95	5

The parts from the two suppliers are used in the firm's manufacturing process and during the processing, a machine breaks down because it attempts to process a bad part.

Required:

(i)   Given the information that a part is bad, determine the probability that it came from Supplier 1 and it came from Supplier 2.   (5 marks)

(ii)   Show the above information in a probability tree.   (5 marks)

(Total: 20 marks)

DECEMBER 2004: QUESTION 7

a)   Indicate which of the following statements you agree with and which you disagree with and defend your opinion.

(i)   When performing a Bayesian decision analysis, the prior probabilities must inevitably be subjective probabilities   (3 marks)

(ii)   The Bayes action will be the same regardless of whether it is selected using expected monetary pay-offs or expected utilities.   (3 marks)

(iii)   For a given decision situation, the maximin criterion, on one hand, and the maximax criterion on, on the other, will always point towards different actions.

(3 marks)

c) In a flow-chart diagram, show the steps involved in a standard posterior analysis.

(6 marks)

With reference to probability theory, briefly but clearly, explain the statement "there is only one thing certain and that is that nothing is certain" (5 marks)

(Total: 20 marks)

CHAPTER FOUR

Sampling and
Estimation

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Sampling and Estimation

▶ OBJECTIVES

At the end of this chapter, you should be able to:

- Explain why in many situations a sample is the only feasible way to learn something about a population.
- Explain the various methods of selecting a sample.
- Distinguish between probability sampling and non-probability sampling.
- Define and construct a sampling distribution of sample means.
- Explain the Central Limit theorem and its importance in statistical inference.
- Calculate confidence intervals for means and proportions.
- Determine how large a sample should be for both means and proportions.

Fast Forward: Sampling is that part of *statistical* practice concerned with the selection of individual observations intended to yield some knowledge about a *population* of concern, especially for the purposes of *statistical inference.*

▶ INTRODUCTION

A population consists of all the items with which a particular study is concerned. A sample is a much smaller number chosen from this population. The sample must be chosen randomly. The data collected in the sample is used to draw inferences about the corresponding population parameter.

The three types of distributions:

1. Population distribution
   Population distribution is the distribution of the individual values of population. Its mean is denoted by "μ".
2. Sample distribution
   It is the distribution of the individual values of a single sample. Its mean is generally written as 'm.' It is extremely unlikely that it will be the same as "μ".
3. Distribution of sample means
   A sample of size n is taken from the parent population and mean of the sample is calculated. This is repeated for a number of samples so that a population of sample means is obtained. This population approaches a normal distribution as n increases and is the distribution of sample means.

▶ DEFINITION OF KEY TERMS

1. **Estimate** – an approximate calculation of quantity or degree or worth; an estimate of what it would cost; a rough idea how long it would take.

2.   **Sample** - a small part of something intended as representative of the whole. In statistics, a sample is a subset of a population.
3.   **Probability** – Probability is the likelihood or chance that something is the case or will happen. Probability theory is used extensively in areas such as statistics, mathematics, science and philosophy to draw conclusions about the likelihood of potential events and the underlying mechanics of complex systems.
4.   **Proportion** – The quotient obtained when the magnitude of a part is divided by the magnitude of the whole; a quantity of something that is part of the whole amount or number.
5.   **Null hypothesis** – describes some aspect of the statistical behaviour of a set of data. This description is treated as valid unless the actual behaviour of the data contradicts this assumption.
6.   **An alternative hypothesis** is one that specifies that the null hypothesis is not true. The alternative hypothesis is false when the null hypothesis is true, and true when the null hypothesis is false. The symbol H_1 is used for the alternative hypothesis.

▶ INDUSTRY CONTEXT

With the realisation of the fact that in business time is money, dynamic technologies for forecasting

have been a necessary toolin a wide range of managerial decisions.. In making strategic decisions

under uncertainty, we all make forecasts. We may not think that we are forecasting, but our

choices will be directed by our anticipation of results of our actions or inactions.

Indecision and delays are the parents of failure. For instance, budgets are intended to help

managers and administrators do a better job of anticipating, and hence a better job of managing

uncertainty, by using effective forecasting and other predictive techniques.

▶ EXAM CONTEXT

Sampling and estimation has been a popular field for examiners. The student must understand the formulae for the previous tests to avoid confusion during an exam. Previous exam papers where the topic has featured are:

12/02, 6/06, 12/05, 6/05, 12/04, 6/04, 6/03, 12/02, 12/00, 6/00

Fast forward: Sampling is that part of *statistical* practice concerned with the selection of individual observations intended to yield knowledge about a *population* of concerns, especially for the purpose of *statistical inference*.

4.1 METHODS OF PROBABILITY SAMPLING

There is no one 'best' method of selecting a probability sample from a population of interest. A method used to select a sample of invoices in a file drawer might not be the most appropriate method to use when choosing a national sample of voters. However, all probability sampling methods have a similar goal, namely, to allow chance to determine the items or persons to be included in the sample.

These sampling methods include:

1. Simple random sampling

A sample is formulated in such a manner that each item or person in the population has the same chance of being included. For instance suppose a population consists of 576 employees of Yana Tires. A sample of 63 employees is to be selected from that population. One way is to first write all their names, put the names in a box, mix them thoroughly then make the first selection. Repeat this process until the 63 employees are selected.

The other method which is convenient is to use the identification number of each employee and a table of random numbers. As the name implies, these numbers have been generated by a random process (in this case by a computer). Bias is therefore completely eliminated from the selection process.

2. Systematic random sampling

The items or individuals of the population are arranged in some way- alphabetically, in a file drawer by date received, or some other method. A random starting point is selected and then every kth number of the population is selected for the sample. A systematic sample should not be used, however, if there is a predetermined pattern to the population.

3. Stratified random sampling

A population is first divided into subgroups called strata and a sample is selected from each stratum. After the population has been divided into strata, either a proportional or non-proportional sample can be selected. A proportional sampling procedure requires that the number of items in each stratum be in the same proportion as found in the population.

In a non-proportional stratified sample, the number of items studied in each stratum is disproportionate to their number in the population. We then weight the sample results according to the stratum's proportion of the total population.

4. Cluster sampling

It is often employed to reduce the cost of sampling a population scattered over a large geographic area. Suppose you want to conduct a survey to determine the views of industrialists in a state. Selecting a random sample of industrialists in the state and personally contacting each one would be time-consuming and very expensive. Cluster sampling would be useful by subdividing the state into small units of regions often called primary units.

Standard error of the mean.

The standard deviation of the sample mean from the overall mean μ is called the standard error (S_e)

For large samples, $S_e = \dfrac{s}{\sqrt{n}}$

Where s is the standard deviation of the population and n is the size of the population.

Note: In general, the standard deviation of the population is not known. In such cases, the standard deviation of the sample (provided it is large) is a good estimate of the population standard deviation (s).

The standard error of the mean then becomes

Standard $\dfrac{DeviationOf\ Sample}{\sqrt{n}}$

Proportions

There are times when information cannot be given as a mean or as a measure but only as a fraction or percentage.

Examples:

Percentage of female in a certain population.

Proportion of defective production in total production.

In these cases, we are faced with estimating the population proportion from a single sample.

The sampling theory states that if repeated large random samples are taken from a population, the sample proportion 'p' will be normally distributed with mean equal to the population proportion and standard error equal to

$$\sqrt{\dfrac{p(1-p)}{n}}$$

where n is sample size.

The procedure for estimating a proportion is similar to that for estimating a mean.

The Sampling distribution of sample proportions

Population proportions are used in business particularly in market research, where we might investigate the proportions of populations displaying a particular characteristic.

Provided p, (1-p) are both greater than 5, we may use

$$\sqrt{\dfrac{p(1-p)}{n}}$$

to represent the standard error of the sampling distribution of sample proportions

Confident intervals

Usually we use a single sample, to produce an estimate of population parameter. It is important to know how reliable this estimate is based on sample results.

The standard error of the sampling distribution (or proportions) will give us an indication of reliability – the smaller the standard error, the less variable the sample statistic.

However it is simpler to appreciate the reliability of an estimate if we set up a range of values within which we can be reasonably sure that the population parameters lie. This range of values is called confidence interval.

Confidence limits

Confidence limits are the outer limits to a confidence interval. This is a zone of values within which we may be confident that the true population mean (or the parameter being considered) does lie.

Example: the 95% confidence interval for the population proportion is:

$$p \pm 1.96 \sqrt{\frac{p(1 - p)}{n}}$$

Note: It is usual to write 1-p = q

Test of significance

In practice, sizes of population parameters such as mean and proportions, are generally unknown. However a claim may be made that the size of a given parameter is equal to 'x' (say). Such a claim may be true or false. There is need for such a claim to be put to test.

To test this claim we take a sample. It will be a miracle if the sample taken from the population will have the same mean or standard deviation as the corresponding population parameters. The difference could be:

a)   because the original belief was wrong, or

b)   because the difference was purely due to ordinary chance.

If the difference cannot be explained solely due to ordinary chance, then the difference is said to be statistically significant.

4.2 THE NULL HYPOTHESIS

Fast Forward

A common convention is to use the symbol H_0 to denote the null hypothesis.

It is an assumption that nothing has changed i.e. what we have assumed to be true (H_0), is actually true. In other words, there is no contradiction between the believed mean and the sampled mean and the difference is solely due to chance.

Testing the null hypothesis

We know that 95% of the means of all samples, will fall within 1.96 standard errors of the population mean (or believed mean). If the sample mean lies more than 1.96 standard error from the believed mean, one can reject the null hypothesis at the 95% level of confidence and say that there is a contradiction between the believed population mean and the sample. If the null hypothesis is not rejected, then we say there is not enough evidence to prove that the true mean is not as we believe.

Sampling theory for small samples (t-Distribution)

In applying sampling theories to small samples (n ≤30), we assume:

    a)   the parent population is normal or near normal
    b)   In this case the sample means do not follow a normal distribution.

This distribution is similar to the standard normal distributions in that, it is symmetrical and continuous. The major difference between the two is that in the case of normal distribution there is only one defined distribution; there are however many t-distributions depending upon the sample size.

The value of t is obtained from the t- tables depending upon the degrees of freedom and the level of confidence.

Type I and Type II Errors

If we reject a hypothesis when it should be accepted, we say that a type I error has been made. If on the other hand, we accept a hypothesis when it should be rejected, we say that a type II error has been made.

In either case a wrong decision or error in judgment has occurred. For a given sample size, an attempt to decrease one type of error is accompanied in general by an increase in another type of error.

While testing hypothesis (H_0) and deciding to either accept or reject a null hypothesis, there are four possible occurrences.

    a)   Acceptance of a true hypothesis (correct decision) – accepting the null hypothesis and it happens to be the correct decision. Note that statistics does not give absolute information, thus its conclusion could be wrong only that the probability of it being right are high.
    b)   Rejection of a false hypothesis (correct decision).
    c)   Rejection of a true hypothesis – (incorrect decision) – this is called type I error, with probability = α.
    d)   Acceptance of a false hypothesis – (incorrect decision) – this is called type II error, with probability = β.

In practice, type I error is considered more serious than the type II errors. The maximum probability with which we would be willing to risk a type I error is called the level of significance of the test.

One-tailed and two –tailed test

When we are interested in extreme values of the statistics S or its corresponding Z score on both sides of the statistic, we perform a test called a two-tailed test (or two-sided test). Similarly we may be interested only in extreme values to one side of the mean, such tests are called one-tailed tests (or one-sided tests)

Levels of significance

A level of significance is a probability value which is used when conducting tests of hypothesis. A level of significance is basically the probability of one making an incorrect decision after the statistical testing has been done. Usually such probability used are very small e.g. 1% or 5%

0.5000

0.4900

1% provision for errors

Critical Value

0

0.45

5% = 0.05

Critical Region

Critical Value = −1.65

0
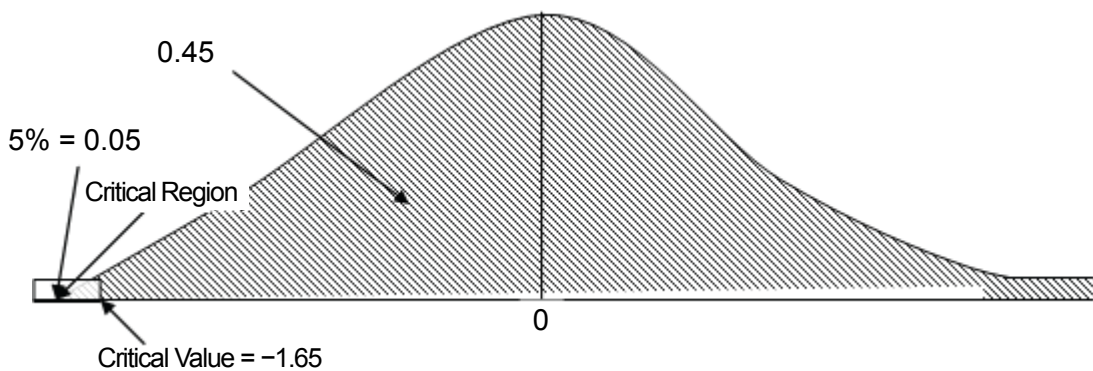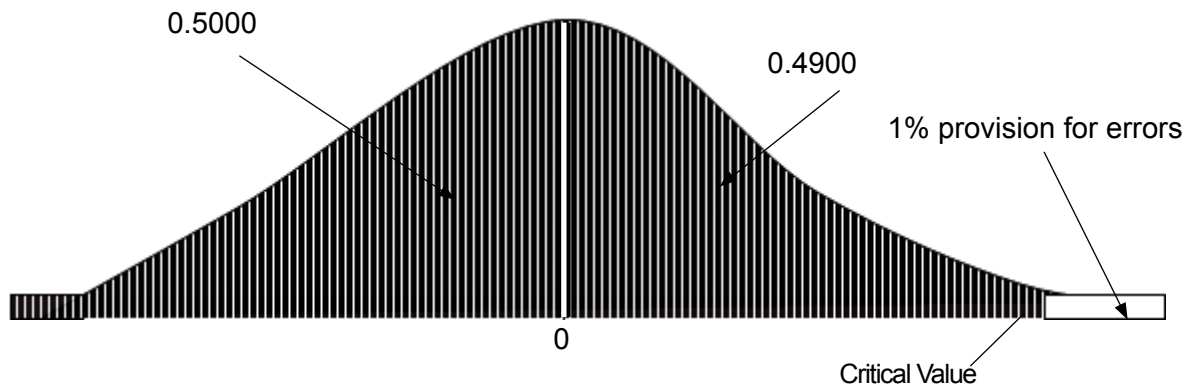
NB: If the standardised value of the mean is less than –1.65 we reject the null hypothesis (H_0) and accept the alternative hypothesis (H_1) but if the standardised value of the mean is more than –1.65 we accept the null hypothesis and reject the alternative hypothesis

The above sketch graph and level of significance are applicable when the sample mean is less than the population mean)

The following is used when sample mean > population mean
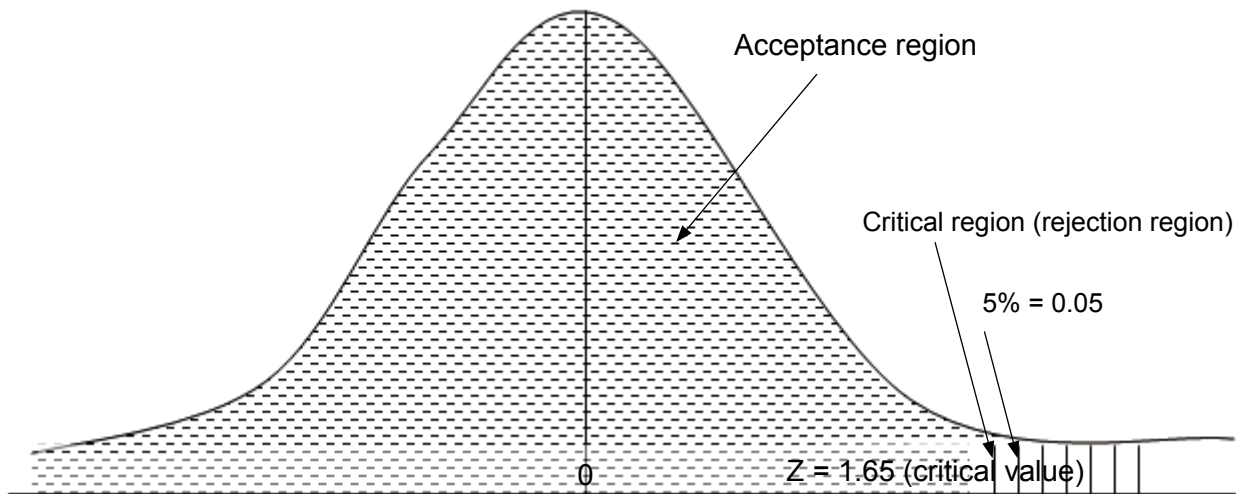
NB: If the sample mean standardized value < 1.65, we accept the null hypothesis but reject the alternative. If the sample mean value > 1.65 we reject the null hypothesis and accept the alternative hypothesis

The above sketch is normally used when the sample mean given is greater than the population mean
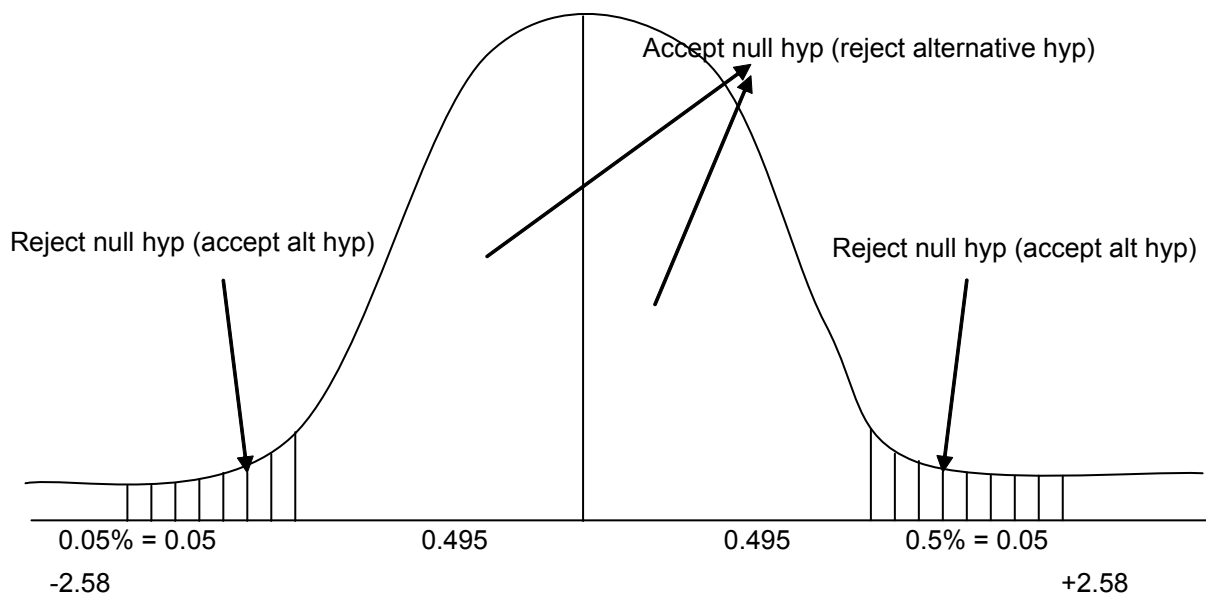
NB: If the standardized value of the sample mean is between –2.58 and +2.58 accept the null hypothesis but otherwise reject it and therefore accept the alternative hypothesis

Two-tailed tests

A two-tailed test is normally used in statistical work (tests of significance) e.g. if a complaint lodged by the client is about a product not meeting certain specifications i.e. the item will generate a complaint if its measurements are below the lower tolerance limit or above the upper tolerance limit
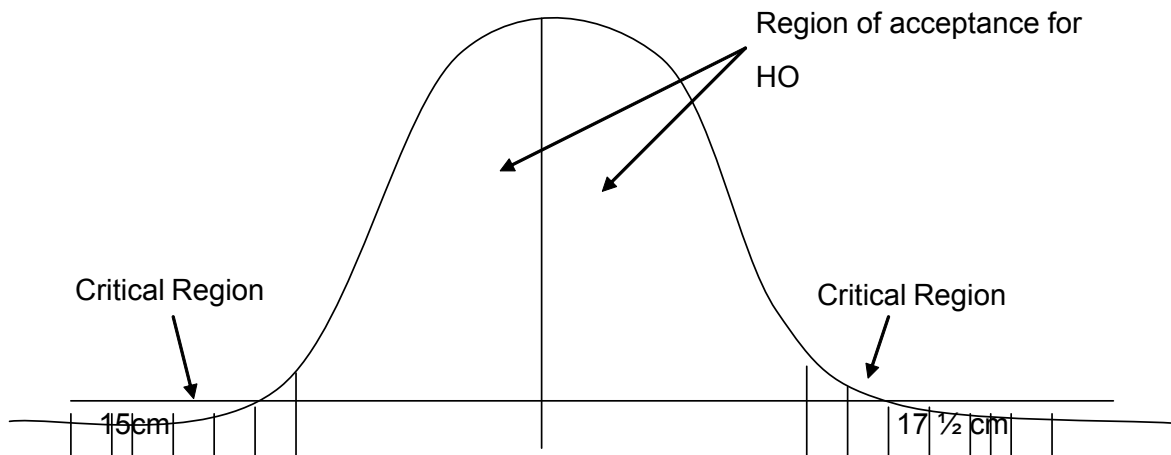
NB: Alternative hypothesis is usually rejected if the standardized value of the sample mean lies beyond the tolerance limits (15cm and 17 ½ cm).

One-tailed test

This is a test where the alternative hypothesis (H_1:) is only concerned with one of the tails of the distribution e.g. to test a business complaint if the complaint is above the measurements of an item being shorter than is required.

E.g. a manufacturer of a given brand of bread may state that the average weight of the bread is 500gms but if a consumer takes a sample and weighs each of the pieces of bread and happens to have a mean of 450 gms he will definitely complain about the bread which is underweight. The statistical analysis to be done will concentrate on the left tail of the normal distribution in which one will have to establish whether 450gms being less than 500g is statistically significant. Such a test is referred to as one-tailed test.
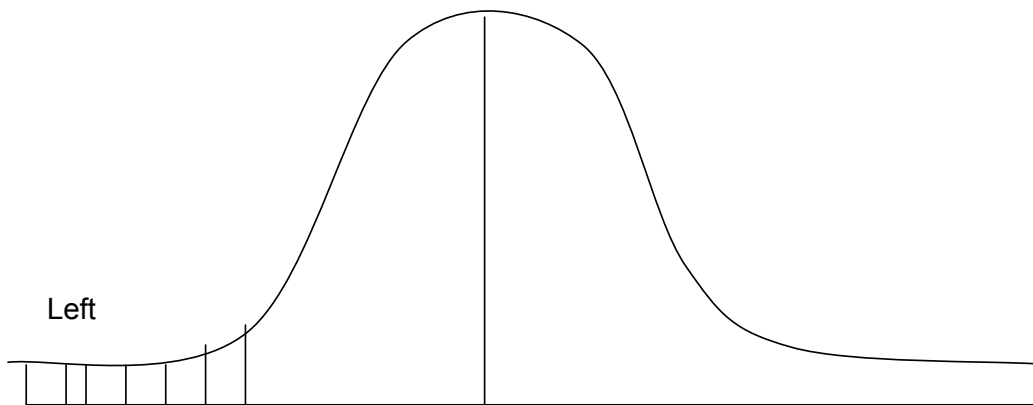
On the other hand, the test may tend towards the right hand tail of the normal distribution. When this happens, the major complaint is likely to do with oversize items bought. The test is known as one-tailed as the focus is on one end of the normal distribution.

	Number of standard errors	
	Two-tailed test	One-tailed test
5% level of significance	1.96	1.65
1% level of significance	2.58	2.33

Hypothesis Testing Procedure

Whenever a business complaint comes up there is a recommended procedure for conducting a statistical test. The purpose of such a test is to establish whether the null hypothesis or alternative hypothesis is to be accepted.

The following are steps normally adopted:

1. Statement of the null and alternative hypothesis
2. Statement of the level of significance to be used.
3. Statement about the test statistic i.e. what is to be tested e.g. the sample mean, sample proportion, difference between sample means or sample proportions
4. Type of test whether two tailed or one tailed.
5. Statement on critical values using the appropriate level of significance
6. Standardising the test statistic
7. Conclusion showing whether to accept or reject the null hypothesis

The F-Distribution (The Variance Ratio Test)

The F-test is based on the ratio of the variances of two samples.

The F-distribution has the following shape

Diagram 4.1

Critcal Piont   F - Value
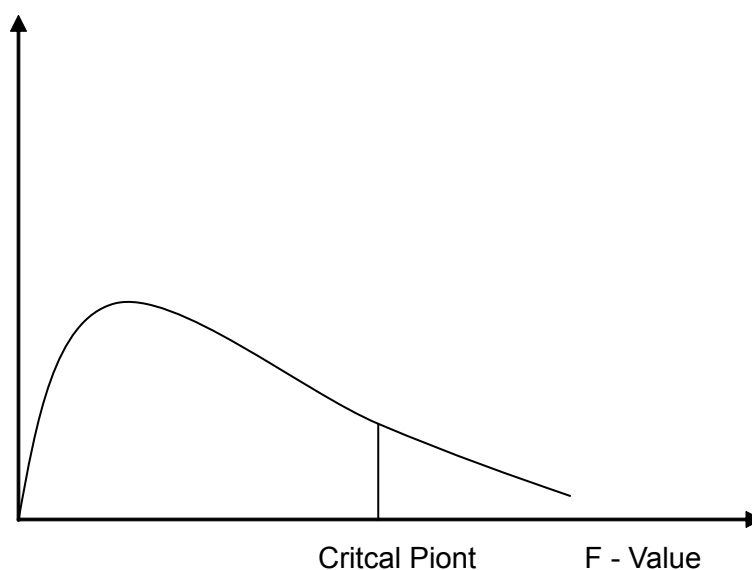
The null-hypothesis is that there is no significant difference between the variances of the two samples.

How to use the F-test

Step I: Calculate the variance for each sample and use these to estimate the population variance.

Step II: Obtain the F-value which is

$$\frac{\text{Larger variance}}{\text{Smaller variance}}$$

Step III: Find F – value from F-tables using degrees of freedom and level of confidence.

4.3 THE CHI –SQUARE DISTRIBUTION X²

This distribution can be used to test whether an observed series of values differs significantly from what is expected.

Formula:

$$x^2 = \sum \frac{(observed\ value - expected\ value)^2}{expected\ value} = \sum \frac{(o - E)^2}{E}$$

Testing the difference between two sample means (large samples)

A large sample is defined as one which contains 30 or more items ($n \geq 30$ where n is the sample size)

In a business those involved are constantly observant about the standards or specifications of the item which they sell e.g. a trader may receive a batch of items at one time and another batch at a later time. At the end, he may have concluded that the two samples are different in certain specifications e.g. mean weight, mean lifespan, mean length e.t.c. Further it may become necessary to establish whether the observed differences are statistically significant or not. If the differences are statistically significant then it means that such differences must be explained i.e. there are known causes but if they are not statistically significant then it means that the differences observed have no known causes and are mainly due to chance.

If the differences are established to be statistically significant then it implies that the complaints, which necessitated that kind of test, are justified.

Let X_1 and X_2 be any two samples whose sizes are n_1 and n_2 and mean $\overline{X}_1$ and $\overline{X}_2$. Standard deviation S_1 and S_2 respectively. In order to test the difference between the two sample means, we apply the following formulas

$$Z = \frac{X_1 - X_2}{S(X_1 - X_2)} \qquad where = S(X_1 - X_2) = \sqrt{\frac{S_1^2 + S_2^2}{n_1\ n_2}}$$

Example 1

An agronomist was interested in a particular fertilizer yield output. He planted maize on 50 equal pieces of land and the mean harvest obtained later was 60 bags per plot with a standard deviation of 1.5 bags. The crops grew under natural circumstances and conditions without the soil being treated with any fertilizer. The same agronomist carried out an alternative experiment where he picked 60 plots in the same area and planted the same plant of maize but a fertilizer was applied on these plots. After the harvest it was established that the mean harvest was 63 bags per plot with a standard deviation of 1.3 bags

Required

Conduct a statistical test in order to establish whether there was a significant difference between the mean harvest under the two types of field conditions. Use 5% level of significance.

Solution

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

Critical values of the two-tailed test at 5% level of significance are 1.96

The standardised value of the difference between sample means is given by Z where

$$Z = \left| \frac{\bar{X}_1 - \bar{X}_2}{S(X_1 - X_2)} \right| \qquad \text{where} = S(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{1.5^2 + 1.3^2}{50 \quad 60}}$$

$$Z = \left| \frac{(60 - 63)}{\sqrt{50 \quad 60}} \right|$$

$$= 11.11$$

$$-1.96 \qquad 0 \qquad +1.96$$
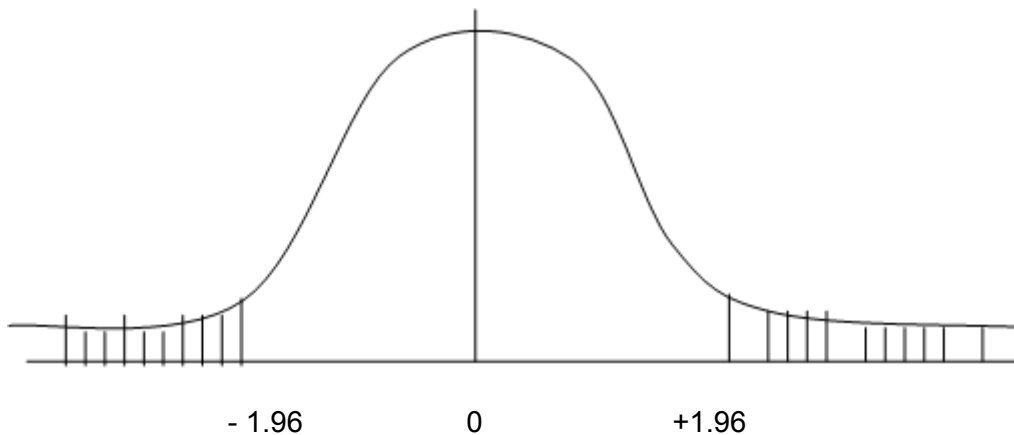
Since 11.11 < -1.96, we reject the null hypothesis but accept the alternative hypothesis at 5% level of significance i.e. the difference between the sample mean harvest is statistically significant. This implies that the fertilizer had a positive effect on the harvest of maize

Note: You don't have to illustrate your solution with a diagram.

Example 2

An observation was made about reading abilities of males and females. The observation led to a conclusion that females are faster readers than males. The observation was based on the times taken by both females and males when reading out a list of names during graduation ceremonies.

In order to investigate the observation and the consequent conclusion a sample of 200 men were given lists to read. On average, each man took 63 seconds with a standard deviation of 4 seconds. A sample of 250 women was also taken and asked to read the same list of names. It was found that they took 62 seconds on average with a standard deviation of 1 second.

Required

By conducting a statistical hypothesis testing at 1% level of significance establish whether the sample data obtained supports the earlier observation.

Solution

$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

Critical values of the two tailed test is at 1% level of significance is 2.58.

$$Z = \left| \frac{X_1 - X_2}{S(X_1 - X_2)} \right|$$

$$Z = \left| \frac{63 - 62}{\sqrt{\frac{4^2}{200} + \frac{1^2}{250}}} \right|$$

STUDY TEXT
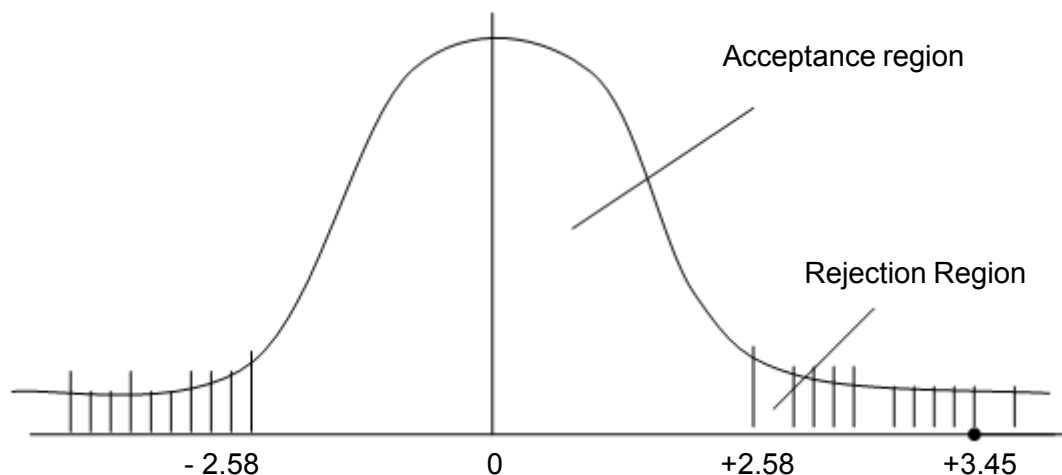
Since 3.45 > 2.33 reject the null hypothesis but accept the alternative hypothesis at 1% level of significance i.e. there is a significant difference between the reading speed of Males and females, thus females are actually faster readers.

Test of hypothesis on proportions

This follows a similar method to the one for means except that the standard error used in this case is:

$$Sp = \sqrt{\frac{Pq}{n}}$$

Z score is calculated as, $Z = \dfrac{P - \pi}{Sp}$ Where P = Proportion found in the sample.

Π – the hypothetical proportion.

Example

A member of parliament (MP) claims that in his constituency only 50% of the total youth population lacks university education. A local media company wanted to ascertain that claim thus they conducted a survey taking a sample of 400 youths, of these 54% lacked university education.

Required:

At 5% level of significance, confirm if the MP's claim is wrong.

Solution

Note: This is a two-tailed test since we wish to test that the hypothesis is different (≠) and not against a specific alternative hypothesis e.g. less than or more than.

H_0 : π = 50% of all youth in the constituency lack university education.

H_1 : π ≠ 50% of all youth in the constituency lack university education.

$$Sp = \sqrt{\frac{pq}{n}} \quad = \sqrt{\frac{0.5 \times 0.5}{400}} = 0.025$$

$$Z = \left| \frac{0.54 - 0.50}{0.025} \right| = 1.6$$

at 5% level of significance for a two-tailed test the critical value is 1.96 since calculated Z value < tabulated value (1.96).

i.e. 1.6 < 1.96 we accept the null hypothesis.

Thus the MP's claim is accurate.

Hypothesis testing of the difference between proportions

Example

Ken industrial manufacturers have produced a perfume known as "fianchetto." In order to test its popularity in the market, the manufacturer carried a random survey in Back rank city where 10,000 consumers were interviewed after which 7,200 showed preference. The manufacturer also moved to area Rook town where he interviewed 12,000 consumers out of which 10,000 showed preference for the product.

Required

Design a statistical test and use it to advise the manufacturer regarding the differences in the proportion, at 5% level of significance.

Solution

$H_0 : \pi_1 = \pi_2$

$H_1 : \pi_1 \neq \pi_2$

The critical value for this two-tailed test at 5% level of significance = 1.96.

Now $Z = \left| \dfrac{(P_1 - P_2) - (\Pi_1 - \Pi_2)}{S(P_1 - P_2)} \right|$

But since the null hypothesis is $\pi_1 = \pi_2$, the second part of the numerator disappears i.e.

$\pi_1 - \pi_2 = 0$ which will always be the case at this level.

Then $Z = \left| \dfrac{(P_1 - P_2)}{S(P_1 - P_2)} \right|$

Where:

	Sample 1	Sample 2
Sample size	$n_1 =$ 10,000	$n_2 =$ 12,000
Sample proportion of success	$P_1 = 0.72$	$P_2 = 0.83$
Population proportion of success.	Π_1	Π_2

Now $S(p_1 - p_2) = \sqrt{\dfrac{pq}{n_1} + \dfrac{pq}{n_2}}$

Where $P = \dfrac{p_1 n_1 + p_2 n_2}{n_1 + n_2}$

And $q = 1 - p$

$\therefore$ in our case

$P = \dfrac{10,000(0.72) + 12,000(0.83)}{10,000 + 12,000}$

$= \dfrac{84,000}{22,000}$

$= 0.78$

$\therefore q = 0.22$

$$S(P_1 - P_2) = \sqrt{\frac{0.78(0.22)}{10,000} + \frac{0.78(0.22)}{12,000}}$$

$$= 0.00894$$

$$Z = \left| \frac{0.72 - 0.83}{0.00894} \right| = 12.3$$

Since 12.3 > 1.96, we reject the null hypothesis but accept the alternative. The differences between the proportions are statistically significant. This implies that the perfume is much more popular in Rook town than in Back rank city.

Hypothesis testing about the difference between two proportions

Is used to test the difference between the proportions of a given attribute found in two random samples.

The null hypothesis is that there is no difference between the population proportions. It means two samples are from the same population.

Hence

$H_0 : \pi_1 = \pi_2$

The best estimate of the standard error of the difference of P1 and P2 is given by pooling the samples and finding the pooled sample proportions (P) thus

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2}$$

Standard error of difference between proportions

$$S(p_1 - p_2) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

And $Z = \left| \frac{P_1 - P_2}{S(p_1 - p_2)} \right|$

Example

In a random sample of 100 persons taken from village A, 60 are found to be consuming tea. In another sample of 200 persons taken from a village B, 100 persons are found to be consuming tea. Do the data reveal significant difference between the two villages so far as the habit of taking tea is concerned?

Solution

Let us take the hypothesis that there is no significant difference between the two villages as far as the habit of taking tea is concerned i.e. $\pi_1 = \pi_2$

We are given

$P_1 = 0.6;\ n_1 = 100$

$P_2 = 0.5;\ n_2 = 200$

Appropriate statistic to be used here is given by

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2}$$

$$= \frac{(0.6)(100)+(0.5)(200)}{100+200} = \frac{60+100}{300}$$

$$= 0.53$$

$$q = 1 - 0.53$$

$$= 0.47$$

$$S(P_1 - P_2) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

$$= \sqrt{\frac{(0.53)(0.47)}{100} + \frac{(0.53)(0.47)}{200}}$$

$$= 0.0608$$

$$Z = \left| \frac{0.6 - 0.5}{0.0608} \right|$$

$$= 1.64$$

Since the computed value of Z is less than the critical value of Z = 1.96 at 5% level of significance; therefore we accept the hypothesis and conclude that there is no significant difference in the habit of taking tea in the two villages A and B

t-distribution (student's t distribution) tests of hypothesis (test for small samples n < 30)

For small samples n < 30, the method used in hypothesis testing is similar to the one for large samples except that t values are used from t distribution at a given degree of freedom v, instead of z score, the standard error Se statistic used is also different.

Note that v = n – 1 for a single sample and $n_1 + n_2 - 2$ where two samples are involved.

a)   Test of hypothesis about the population mean

When the population standard deviation (S) is known then the t statistic is defined as

$$t = \left| \frac{\overline{X} - m}{S_{\overline{X}}} \right| \text{ where } S_{\overline{X}} = \frac{S}{\sqrt{n}}$$

Follows the students t distribution with (n-1) d.f. where:

$\overline{X}$ = Sample mean

μ = Hypothesis population mean

n = sample size

and S is the standard deviation of the sample calculated by the formula:

$$S = \sqrt{\frac{\sum (X - \overline{X})^2}{n-1}} \qquad \text{for } n < 30$$

If the calculated value of t exceeds the table value of t at a specified level of significance, the null hypothesis is rejected.

Example

Ten oil tins are taken at random from an automatic filling machine. The mean weight of the tins is 15.8 kg and the standard deviation is 0.5kg. Does the sample mean differ significantly from the intended weight of 16kgs. Use 5% level of significance.

Solution

Given that n = 10; $\overline{x}$ = 15.8; S = 0.50; μ = 16; v = 9

$H_0 : μ = 16$
$H_1 : μ ≠ 16$

$$= S_{\overline{X}} = \frac{0.5}{\sqrt{10}}$$

$$t = \left| \frac{15.8 - 16}{\frac{0.5}{\sqrt{10}}} \right|$$

$$= \left| \frac{0.2}{0.16} \right|$$

$$= -1.25$$

The table value for t for 9 d.f. at 5% level of significance is 2.26. The computed value of t is smaller than the table value of t. therefore, difference is insignificant and the null hypothesis is accepted.

b)   Test of hypothesis about the difference between two means

The t test can be used under two assumptions when testing hypothesis concerning the difference between the two means; that the two are normally distributed (or near normally distributed) populations and that the standard deviation of the two is the same or at any rate not significantly different.

Appropriate test statistic to be used is

$$t = \frac{\overline{X}_1 - \overline{X}_2}{S_{(\overline{X}_1 - \overline{X}_2)}} \qquad \text{at } (n_1 + n_2 - 2) \text{ d.f.}$$

The standard deviation is obtained by pooling the two sample standard deviation as shown below.

$$S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

Where S_1 and S_2 are standard deviation for sample 1 and 2 respectively.

Now $S_{\overline{X}_1} = \dfrac{Sp}{\sqrt{n_1}}$ and $S_{\overline{X}_2} = \dfrac{Sp}{\sqrt{n_2}}$

$S_{(\overline{X}_1 - \overline{X}_2)} = \sqrt{S_{\overline{X}_1}^2 + S_{\overline{X}_2}^2}$

Alternatively $S_{(\overline{X}_1 - \overline{X}_2)} = Sp \sqrt{\dfrac{n_1 + n_2}{n_1 n_2}}$

Example

Two different types of drugs – A and B – were tried on certain patients for increasing weights, 5 persons were given drug A and 7 persons were given drug B. The increase in weight (in pounds) is given below

Drug A	8	12	16	9	3		
Drug B	10	8	12	15	6	8	11

Do the two drugs differ significantly with regard to their effect in increasing weight? (Given that v= 10; $t_{0.05}$ = 2.23)

Solution

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

$t = \left| \dfrac{\overline{X}_1 - \overline{X}_2}{S_{(\overline{X}_1 - \overline{X}_2)}} \right|$

Calculate $\overline{X}_1$, $\overline{X}_2$ and S

X_1	$X_1 - \overline{X}_1$	$(X_1 - \overline{X}_1)^2$	X_2	$(X_2 - \overline{X}_2)$	$(X_2 - \overline{X})^2$
8	-1	1	10	0	0
12	+3	9	8	-2	4
13	+4	16	12	+2	4
9	0	0	15	+5	25
3	-6	36	6	-4	16
			8	-2	4
			11	+1	1
ΣX_1 = 45	$\Sigma(X_1 - \overline{X}_1)$ = 0	$\Sigma (X_1 - \overline{X}_1)^2$= 62	ΣX_2= 70	$\Sigma (X_2 - \overline{X}_2)$ = 0	$\Sigma (X_2 - \overline{X}_2)^2$= 54

$$X_1 = \frac{\sum X_1}{n_1} = \frac{45}{5} = 9 \qquad X_2 = \frac{\sum X_2}{n_2} = \frac{70}{7} = 10$$

$$S_1 = \sqrt{\frac{62}{4}} = 3.94 \qquad S_2 = \sqrt{\frac{54}{6}} = 3$$

$$S_p = \sqrt{\frac{(4)15.4 + (6)9}{10}}$$

$$= 3.406$$

$$S_{(\overline{X}_1 - \overline{X}_2)} = \sqrt{\frac{11.6}{5} + \frac{11.6}{7}} \text{ or } 3.406 \sqrt{\frac{7+5}{5(7)}}$$

$$= 1.99$$

$$t = \left| \frac{\overline{X}_1 - \overline{X}_2}{S_{(\overline{X}_1 - \overline{X}_2)}} \right| = \left| \frac{9-10}{1.99} \right|$$

$$= 0.50$$

Now $t_{0.05}$ (at v = 10) = 2.23 > 0.5

Thus we accept the null hypothesis.

Hence there is no significant difference in the efficacy of the two drugs in the matter of increasing weight.

Example

Two salesmen A and B are working in a certain district. From a survey conducted by the head office, the following results were obtained. State whether there is any significant difference in the average sales between the two salesmen at 5% level of significance.

	A	B
No. of sales	20	18
Average sales in shs	170	205
Standard deviation in shs	20	25

Solution

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

Where

$$S_p = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$$S_{(\overline{X}_1 - \overline{X}_2)} = S_p \sqrt{\frac{n_1 + n_2}{n_1 n_2}}$$

Where: $\overline{X}_1 = 170$, $\overline{X}_2 = 205$, $n_1 = 20$, $n_2 = 18$, $S_1 = 20$, $S_2 = 25$, $V = 36$

$$S_p = \sqrt{\frac{(19)(20^2)+(17)(25^2)}{20+18-2}}$$

$$= 22.5$$

$$S_{(\overline{X}_1-\overline{X}_2)} = 22.5\sqrt{\frac{38}{360}}$$
$$= 7.31$$

$$t = \left|\frac{170-205}{7.31}\right|$$
$$= 4.79$$

$t_{0.05}(36) = 1.9$ (Since d.f > 30 we use the normal tables)

The table value of t at 5% level of significance for 36 d.f. when d.f. >30, that t distribution is the same as normal distribution, 1.9. Since the computed value of t is more than the table value, we reject the null hypothesis. Thus, we conclude that there is significant difference in the average sales between the two salesmen.

Testing the hypothesis equality of two variances

The test for equality of two population variances is based on the variances in two independently selected random samples drawn from two normal populations

Under the null hypothesis $\acute{o}_1^2 = \acute{o}_2^2$

$$F = \frac{\dfrac{s_1^2}{\acute{o}_1^2}}{\dfrac{s_2^2}{\acute{o}_2^2}} \quad \text{Now under the } H_0 : \acute{o}_1^2 = \acute{o}_2^2 \text{ it follows that}$$

$$F = \frac{S_1^2}{S_2^2} \quad \text{which is the test statistic.}$$

Which follows F – distribution with V_1 and V_2 degrees of freedom. The larger sample variance is placed in the numerator and the smaller one in the denominator.

If the computed value of F exceeds the table value of F, we reject the null hypothesis i.e. the alternate hypothesis is accepted

Example

In one sample of observations the sum of the squares of the deviations of the sample values from sample mean was 120 and in the other sample of 12 observations it was 314; test whether the difference is significant at 5% level of significance

Solution

Given that $n_1 = 10$, $n_2 = 12$, $\Sigma(x_1 - \overline{X_1})^2 = 120$

$\Sigma(x_2 - \overline{X_2})^2 = 314$

Let us take the null hypothesis that the two samples are drawn from the same normal population of equal variance

$H_0 : \acute{o}_1^2 = \acute{o}_2^2$

$H_1 : \acute{o}_1^2 \neq \acute{o}_2^2$

Applying F test i.e.

$$F = \frac{S_1^2}{S_2^2}$$

$$= \frac{\frac{\sum(X_1 - \overline{X}_1)^2}{n_1 - 1}}{\frac{\sum(X_2 - \overline{X}_2)^2}{(n_2 - 1)}}$$

$$= \frac{\frac{120}{9}}{\frac{314}{11}}$$

$$= \frac{13.33}{28.55}$$

since the numerator should be greater than denominator

$$F = \frac{28.55}{13.33} = 2.1$$

The table value of F at 5% level of significance for $V_1 = 9$ and $V_2 = 11$. Since the calculated value of F is less than the table value, we accept the hypothesis. The samples may have been drawn from the two populations having the same variances.

Chi square hypothesis tests (Non-parametric test)(X^2)

They include among others

Test for goodness of fit

Test for independence of attributes

Test of homogeneity

Test for population variance

The Chi square test (χ^2) is used when comparing an actual (observed) distribution with a hypothesized, or explained distribution.

It is given by: $\chi^2 = \sum \frac{(O - E)^2}{E}$ Where O = Observed frequency

E = Expected frequency

The computed value of χ^2 is compared with that of tabulated χ^2 for a given significance level and degrees of freedom.

i)   Test for goodness of fit

These tests are used when we want to determine whether an actual sample distribution matches a known theoretical distribution

The null hypothesis usually states that the sample is drawn from the theoretical population distribution and the alternate hypothesis usually states that it is not.

Example

Mr Nguku carried out a survey of 320 families in Ateka district, each family had 5 children and they revealed the following distribution

No. of boys	5	4	3	2	1	0
No. of girls	0	1	2	3	4	5
No. of families	14	56	110	88	40	12

Is the result consistent with the hypothesis that male and female births are equally probable at 5% level of significance?

Solution

If the distribution of gender is equally probable then the distribution conforms to a binomial distribution with probability $P(X) = \frac{1}{2}$.

Therefore

H_0 = the observed number of boys conforms to a binomial distribution with $P = \frac{1}{2}$

H_1 = the observations do not conform to a binomial distribution.

On the assumption that male and female births are equally probable, the probability of a male birth is $P = \frac{1}{2}$. The expected number of families can be calculated by the use of binomial distribution. The probability of male births in a family of 5 is given by

$P(x)$ $= {}^5c_X\, P^x\, q^{5-x}$ (for x = 0, 1, 2, 3, 4, 5,)

$= {}^5c_X\, (\frac{1}{2})^5$ (Since $P = q = \frac{1}{2}$)

To get the expected frequencies, multiply P(x) by the total number N = 320. The calculations are

shown below in the tables

x	P(x)	Expected frequency = NP(x)
0	$^5C_0\,(\,\frac{1}{2}\,)^5 \quad = \frac{1}{32}$	$320 \times \frac{1}{32} = 10$
1	$^5C_1\,(\,\frac{1}{2}\,)^5 \quad = \frac{5}{32}$	$320 \times \frac{5}{32} = 50$
2	$^5C_2\,(\,\frac{1}{2}\,)^5 \quad = \frac{10}{32}$	$320 \times \frac{10}{32} = 100$
3	$^5C_3\,(\,\frac{1}{2}\,)^5 \quad = \frac{10}{32}$	$320 \times \frac{10}{32} = 100$
4	$^5C_4\,(\,\frac{1}{2}\,)^5 \quad = \frac{5}{32}$	$320 \times \frac{5}{32} = 50$
5	$^5C_5\,(\,\frac{1}{2}\,)^5 \quad = \frac{1}{32}$	$320 \times \frac{1}{32} = 10$

Arranging observed and expected frequencies in the following table and calculating x^2

O	E	$(O-E)^2$	$(O-E)^2/E$
14	10	16	1.60
56	50	16	0.72
110	100	100	1.00
88	100	144	1.44
40	50	100	2.00
12	10	4	0.40
			$\Sigma(O-E)^2/E = 7.16$

$$\chi^2 \quad = \quad \Sigma\frac{(O-E)^2}{E}$$

= 7.16

The table of χ^2 for V = 6 – 1 = 5 at 5% level of significance is 11.07. The computed value of $\chi^2 = 7.16$ is less than the table value. Therefore the hypothesis is accepted. Thus it can be concluded that male and female births are equally probable.

ii)   Test of independence of attributes

This test discloses whether there is any association or relationship between two or more attributes. The following steps are required to perform the test of hypothesis.

1.   The null and alternative hypothesis are set as follows

H_0: No association exists between the attributes

H_1: An association exists between the attributes

2.   Under H_0 an expected frequency E corresponding to each cell in the contingency table is found by using the formula

$$E = \frac{R \times C}{n}$$

Where R = a row total, C = a column total and n = sample size

3.   Based upon the observed values and corresponding expected frequencies the χ^2 statistic is obtained using the formular

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

4.   The characteristics of this distribution are defined by the number of degrees of freedom (d.f.) which is given by

d.f. = (r-1) (c-1),

Where r is the number of rows and c is number of columns corresponding to a chosen level of significance, the critical value is found from the chi squared table

5.   The calculated value of χ^2 is compared with the tabulated value χ^2 for (r-1) (c-1) degrees of freedom at a certain level of significance. If the computed value of χ^2 is greater than the tabulated value, the null hypothesis of independence is rejected. Otherwise we accept it.

Example

In a sample of 200 people where a particular device was selected, 100 were given a drug and the others were not given any drug. The results are as follows

	Drug	No drug	Total
Cured	65	55	120
Not cured	35	45	80
Total	100	100	200

Test whether the drug will be effective or not, at 5% level of significance.

Solution

Let us take the null hypothesis that the drug is not effective in curing the disease.

Applying the χ^2 test

The expected cell frequencies are computed as follows

$$E_{11} = \frac{R_1 C_1}{n} = \frac{120 \times 100}{200} = 60$$

$$E_{12} = \frac{R_1C_2}{n} = \frac{120 \times 100}{200} \quad = \quad 60$$

$$E_{21} = \frac{\dfrac{R_2C_1}{n}}{} = \frac{\dfrac{80 \times 100}{200}}{} = \quad 40$$

$$E_{22} = \frac{\dfrac{R_2C_2}{n}}{} = \frac{\dfrac{80 \times 100}{200}}{} = \quad 40$$

The table of expected frequencies is as follows

60	60	120
40	40	80
100	100	200

O	E	$(O-E)^2$	$(O-E)^2/E$
65	60	25	0.417
55	60	25	0.625
35	40	25	0.417
45	40	25	0.625
			$\Sigma(O-E)^2/E = 2.084$

Arranging the observed frequencies with their corresponding frequencies in the following table we get

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

= 2.084

$V = (r-1)(c-1) = (2-1)(2-1) = 1$; $c^2_{tabulated(0.5)} = 3.841$

The calculated value of χ^2 is less than the table value. The hypothesis is accepted. Hence the drug is not effective in curing the disease.

iii) Test of homogeneity

This is concerned with the proposition that several populations are homogenous with respect to some characteristic of interest e.g. one may be interested in knowing if raw materials available from several retailers are homogenous. A random sample is drawn from each of the population and the number in each of sample falling into each category is determined. The sample data is displayed in a contingency table.

The analytical procedure is the same as that given for the test of independence.

Example

A random sample of 400 persons was selected from each of three age groups and each person was asked to specify which types of TV programmes they preferred. The results are shown in the following table

Type of programme

Age group	A	B	C	Total
Under 30	120	30	50	200
30 – 44	10	75	15	100
45 and above	10	30	60	100
Total	140	135	125	400

Test the hypothesis that the populations are homogenous with respect to the types of television programmes they prefer, at 5% level of significance.

Solution

Let us take the hypothesis that the populations are homogenous with respect to different types of television programmes they prefer

Applying χ^2 test

O	E	$(O - E)^2$	$(O - E)^2 / E$
120	70.00	2500.00	35.7143
10	35.00	625.00	17.8571
10	35.00	625.00	17.8571
30	67.50	1406.25	20.8333
75	33.75	1701.56	50.4166
30	33.75	14.06	0.4166
50	62.50	156.25	2.500
15	31.25	264.06	8.4499
60	31.25	826.56	26.449
			$\Sigma(O - E)^2 / E = 180.4948$

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

The table value of χ^2 for 4 d.f. at 5% level of significance is 9.488.

The calculated value of χ^2 is greater than the table value. We reject the hypothesis and conclude that the populations are not homogenous with respect to the type of TV programmes preferred, thus the different age groups vary in choice of TV programmes.

STUDY TEXT

CHAPTER SUMMARY

Methods of sampling

a.   Random or probability sampling methods

These include

Simple random sampling

Stratified sampling

Systematic sampling

Multi stage sampling

b.   Non random probability sampling methods

These consist of

Judgment sampling

Quota sampling

Cluster sampling

- A hypothesis is a claim or an opinion about an item or issue. Therefore it has to be tested statistically in order to establish whether it is correct

- When testing a hypothesis, one must fully understand the 2 basic hypothesis to be tested namely

The null hypothesis (H_0)

The alternative hypothesis(H_1)

Standard hypothesis tests

Normal test

Test a sample mean ($\overline{X}$) against a population mean (μ) (where samples size n > 30 and population variance σ^2 is known) and sample proportion, P (where sample size np >5 and nq >5 since in this case the normal distribution can be used to approximate the binomial distribution).

t test

Tests a sample mean ($\overline{X}$) against a population mean, especially where the population variance is unknown and n < 30.

Variance ratio test or f test

It is used to compare population variances   with samples of any size drawn from normal populations.

Chi squared test

It can be used to test the association between attributes or the goodness of fit of an observed frequency distribution to a standard distribution

CHAPTER QUIZ

1. Which is the odd one out?
    (a) Simple random sampling
    (b) Stratified sampling
    (c) Systematic sampling
    (d) Continuous sampling
    (e) Multi stage sampling
2. If the difference cannot be explained solely due to ordinary chance, then the difference is said to be _____
3   _____ is an assumption that nothing has changed.
4. What is the formula for T- Distribution?
5. Which one is not in this category?
    a) Judgment sampling
    b) Quota sampling
    c) Systematic sampling
    d) Cluster sampling

ANSWERS TO CHAPTER QUIZ

1. (d) Continuous sampling
2. Statistically significant.
3. Null Hypothesis

4. $t = \dfrac{x - \mu}{\dfrac{\sigma}{\sqrt{n}}}$

5. (c) Systematic Sampling

QUESTIONS FROM PREVIOUS EXAMS

▶ DECEMBER 2000 QUESTION 4

a) Briefly explain each of the following distributions indicating whether it is a discrete or a continuous distribution.
   (i) Binomial distribution.   (3 marks)
   (ii) Poisson distribution   (3 marks)
   (iii) Normal distribution   (3 marks)
   (iv) Chi-square distribution   (3 marks)
   (v) Fisher (F) distribution   (3 marks)
b) Give one example in the accounting profession where each of the above distributions can be applied.   (5 marks)
                                                          (Total: 20 marks)

▶ DECEMBER 2002 QUESTION 2

a) Transparency and Certified Public Accountants (CPAs) have been appointed to audit accounts of Health National Hospital. Due to the large number of accounts, the consultants have decided to audit a random sample of the accounts.

Required

   (i) State the advantages of auditing a sample of the accounts.   (3 marks)
   (ii) Describe briefly the sampling technique you would recommend.   (5 marks)
   (iii) What are the advantages and disadvantages of the sampling technique recommended in (ii) above?   (4 marks)
b) State and briefly explain the qualities of a good point estimator.   (8 marks)
                                                          (Total: 20 marks)

▶ **JUNE 2003 QUESTION 4**

a) Nation Standard Newspaper poll for the year 2002 presidential campaign in Kenya sampled 491 potential voters in October 2002. A primary purpose of the poll was to obtain an estimate of the promotion of potential voters who favour each candidate. Close to December 2002 elections, better precision and smaller margins of error were desired. Assume a planning value for the population proportion of p = 0.50 and that a 95% confidence level is desired.

Required:

Determine the recommended sample size for each of the following surveys:

Survey	Margin of error
Early December	0.02
Pre-election day	0.01

(8 marks)

b) Future Computer Company has developed a new computer accounting software package to help accountancy analysts reduce the time required to design, develop and implement an accounting system. To evaluate the benefits of the software package, a random sample of 24 accountancy analysts is selected. Each analyst is given specifications for a hypothetical accounting system. Then 12 of the analysts are instructed to produce the accounting system by using the current technology. The other 12 analysts are trained in the use of the software package and then instructed to use it to produce the accounting system. The 24 analysts complete the study and the results are shown below:

Completion Time and Summary Statistics for the Software Testing Study

Current technology	New technology
300	276
280	222
344	310
385	338
372	200
360	302
288	317
321	260
376	320
290	312
301	334
283	265

Sample size $n_1 = 12$ $n_2 = 12$

Sample mean $X_1 = $ 325 $X_1 = 288$

Sample standard deviation $S_1 = 40$ $S_2 = $ 44

Required:

Determine whether the new software package should be adopted at 95% confidence level.

(7 marks)

Note: $S^2 = \dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$

and

$t = \dfrac{(X_1 - X_2) - (u_1 - u_2)}{\left\{ S^2 \left[\dfrac{1}{n_1} + \dfrac{1}{n_2} \right] \right\}^{\frac{1}{2}}}$

(Total: 20 marks)

CHAPTER FIVE

Regression,
Time Series and
Forecasting

CHAPTER FIVE
Regression, Time Series and Forecasting

▶ **OBJECTIVES**

At the end of this chapter, you should be able to:

- Establish relationship between two or more variables.
- Understand a particular situation, explain it and then analyse it.
- Discuss assumptions underlying analysis of the linear regression model.
- To build a model qualitatively about which factors are likely to influence the dependent variable.

Fast Forward: Time series forecasting is the use of a model to forecast future events based on known past events to forecast future data points before they are measured.

▶ **INTRODUCTION**

All businesses have to plan their future activities, both short and long term. Managers will have to make forecasts of the future values of important variables such as sales, interest rates, costs etc. In this chapter, we will look at ways of using past information to make these forecasts.

For example, we may wish to explain the variability in sales by looking at the ways in which they have changed with time, ignoring other factors. If we can explain the past pattern, we can use it to forecast future values. A data set, in which the independent variable is time, is referred to as time series.

Care is required since the historical pattern is not always relevant for particular forecasts. A company may deliberately plan to change its pattern if, for example it has been making a loss. There may be large external factors which completely modify the pattern. There may be a major change in raw material prices, world inflation may suddenly increase or a natural disaster may affect the business unpredictably.

In this section we begin by looking at time series which contain components such as trend, seasonal variation and cyclical variation. The components can be combined in a number of ways. We will look at two specific models: the additive components model and the multiplicative component model. As the names imply, the components are added or multiplied, respectively. For each of these models, there are different ways of calculating the trend component. We will use a combination of moving averages and linear regression.

You should note that the techniques described in this chapter are not the only ones, nor necessarily the best, forecasting methods for any particular forecasting situation. There are many more sophisticated statistical techniques; there are qualitative methods which must be used when there is little or no past data. The Delphi technique and the scenario writing method are examples of these.

▶ DEFINITIONS OF KEY TERMS

1. **Time series -** A time series is a sequence of data points, measured typically at successive times, spaced at (often uniform) time intervals.
2. **Trend –** This is a pattern manifested by collected information over a period of time oftenly used to predict future events. It could also be used to estimate uncertain events in the past.
3. **Forecast** - Forecasting is the process of estimation in unknown situations. Prediction is a similar, but more general term. Both can refer to estimation of time series, cross-sectional or longitudinal data. Usage can differ between areas of application: for example in hydrology, the terms "forecast" and "forecasting" are sometimes reserved for estimates of values at certain specific future times.

▶ INDUSTRY CONTEXT

All businesses have to plan their future activities. When making both short and long term plans, managers will have to make forecasts of the future values of important variables such as sales, interest rates, costs, etc. Care should be taken since historical pattern is not always relevant for particular forecasts. A company may deliberately plan to change its pattern if it has been making a loss.

▶ EXAM CONTEXT

Time is money. Students should clearly have an indepth understanding of time series. The topic has been previously examined as follows:

12/06, 6/06, 12/05, 6/05, 12/04, 6/04, 12/03, 6/03, 6/02, 12/01, 6/01, 12/00, 6/00

5.1 LINEAR REGRESSION

Introduction

Consider a company which regularly places advertisements for one of its products in a local newspaper. The company keeps records, on a monthly basis, of the amount of money spent on advertising and the corresponding sales of this product. If advertising is effective at all, then we can see intuitively that there is likely to be a relationship between the amount of money spent on advertising and the corresponding monthly sales. We would expect that the larger the sum spent on advertising, the greater the sales, at least within certain limits. There are a number of factors which will work together to determine the exact value of sales each month such as the price of a competitor's product, perhaps the time of the year, or the weather conditions, Nevertheless, if the expenditure on advertising is thought to be a major factor in determining sales, knowledge of the relationship between the two variables would be of greater use in the estimation of sales and related budgeting and planning activities.

The term association is used to refer to the relationship between variables. For the purposes of the statistical analysis two aspects of the problem are defined. The term regression is used to describe the nature of the relationship, while the term correlation is used to describe the strength of the relationship.

We need to know, for example, whether the monthly advertising expenditure is strongly related to the monthly sales and therefore will provide a reliable estimate of sales, or whether the relationship is weak and will provide a general indicator only.

The general procedure in the analysis of the relationship between variables is to use a sample of corresponding values of the variables to establish the nature of the relationship. We can then develop a mathematical equation, or model, to describe this relationship.

From the mathematical point of view, linear equations are the simplest to set up and analyse. Consequently, unless a linear relationship is clearly out of the question, we would normally try to describe the relationship between the variables by means of a linear model. This procedure is called **Linear Regression.**

A measure of the fit of the linear model to the data is an indicator of the strength of the linear relationship between the variables and, hence, the reliability of any estimates made using this model. For Example:

Diagram 5.1

A linear relationship
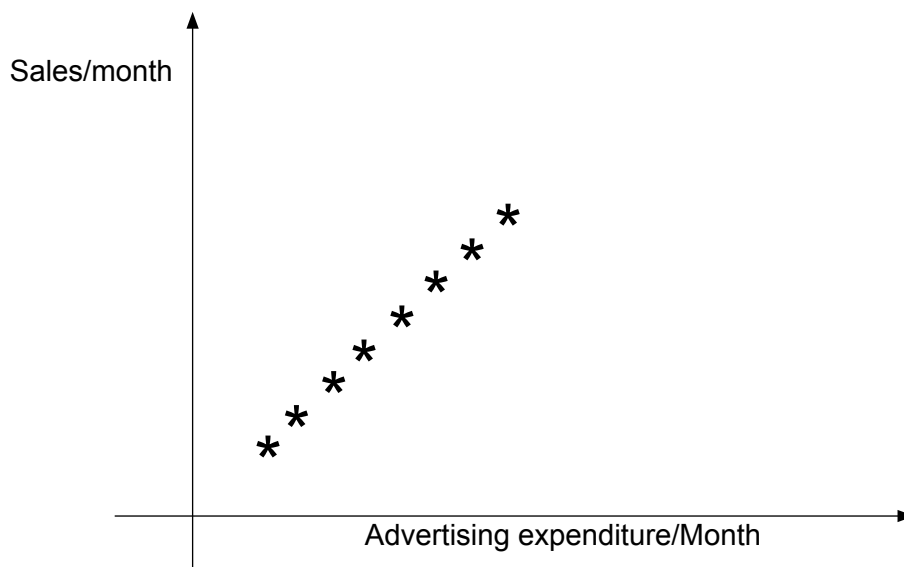
This graph indicates that a linear regression model will be a suitable way of assesing the relationship between sales and advertising expenditure.

Diagram 5.2

A non-Linear relationship
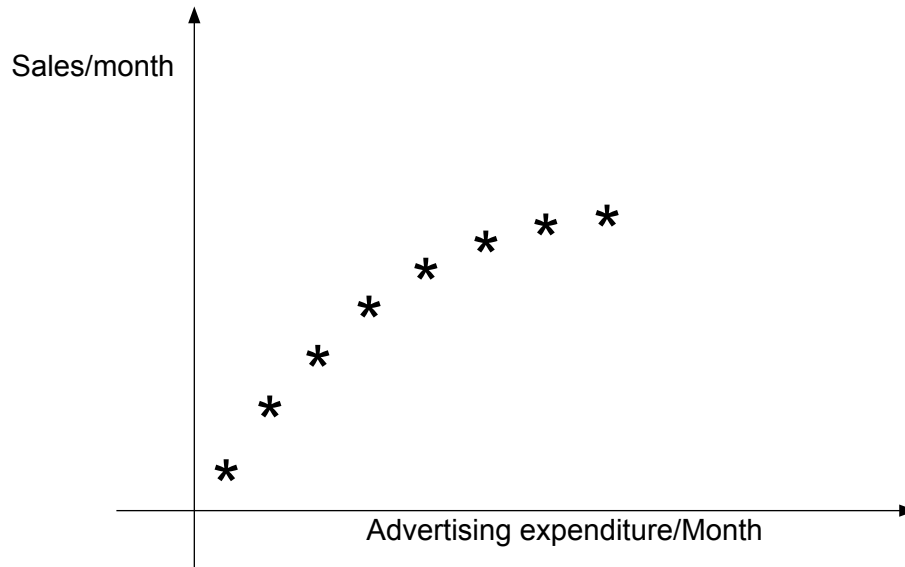
This graph indicates that a linear model would not be suitable in describing the relationship between sales and advertising expenditure.

Linear regression is our first example of the use of mathematical models. The purpose of a model is to help us to understand a particular situation, possibly to explain it and then analyse it. We may use the model to make forecasts or expeditions. A model is usually a simplification of the real situation. We have to make assumptions so that we can construct a manageable model. Models range from the simple twovariable type to the complicated multivariable models. These models are widely used because there are many easy-to-use computer programmes available to carry out the required calculations.

This chapter will cover the steps in the analysis of a simple linear regression model. We will take one sample of data and use it to illustrate each of the steps.

This chapter concludes with sections on multiple regression models, examples of non-linear relationships and finally a non-parametric measure of correlation, Spearman's rank correlation coefficient.

Simple linear Regression Model

We are interested in whether there is any linear relationship between the two variables. For example, we may be interested in the heights and weights of a number of people, the price and quantity of a product sold, employees age and salary, chicken's age and weight, weekly departmental costs and hours worked or distance traveled and the time taken.

As an example, let us say that a poultry farmer wishes to predict the weights of the chicken he is rearing. Weight is the variable which we wish to predict, therefore weight is the dependent variable. We will plot the dependent variable on the y axis. It is suggested to the farmer that weight depends on the chicken's age. Age is said to be an independent variable. The independent variable will be plotted on the x-axis. If we can establish the nature of the relationship between the age and the weight of the chickens, then we can predict the weight of a chicken by looking at the age.

Setting up of a simple linear regression model: Illustration

We are running a special delivery service for short distances in a city. We wish to cost the service and to do this we must estimate the time for deliveries of any given distance.

There are factors, other than the distance travelled, which will affect the times taken –traffic congestion, time of day, road works, the weather,, the road system, the driver, his transport. However the initial investigation will be as simple as possible. We consider distance only, measured as the shortest practical route in miles and the time taken in minutes.

The relevant population is all of the possible journeys, with their times, which could be made in the city. It is an infinite population and we require a random sample from this population. For simplicity, let us use a systematic sample design for this preliminary sample. We will measure the time and distance for every tenth journey starting from randomly selected day and a randomly selected hour of next week. The firm works a 6-day week, excluding Sundays. The random number, chosen by throwing a dice is 2, so next Tuesday is the chosen day. The service runs from 8am to 6pm. A random number, between 0-9 is chosen from random number tables to select the starting time. The number chosen is 6, so the first journey chosen is the first one after 1pm (i.e. this is the sixth hour, beginning at 8am), then we take every tenth delivery after that.

The sample data for the first 10 deliveries will be used for the analysis.

Table 5.1 Sample data for delivery distances and times

Distance, mile	Time, Minutes
2.5	16
3.4	13
1.9	19
t.2	18
3.0	12
α.3	11
3.0	8
3.0	14
1.5	9
4.1	16

We wish to explain variations in the time taken, the dependent variable (y) by introducing distance as the independent variable(x). Generally we would expect the time taken to increase as distance increases. If this data was plotted it would not appear on a straight line. It also looks through the plotted points cluster around a straight line. This means that we could use a linear model to describe the relationship between these two variables. The points are not exactly on a line. It would be surprising if they were, in view of all the other factors which we know can affect journey time. A linear model will be an approximation only to the true relationship between journey time and distance, but the evidence of the plot is that is the best available.

In the population from which we have taken our sample, for each distance, there are numerous different journeys and numerous different times for each of the journeys. In fact, for any distance there is a distribution of possible delivery times. Our sample of 10 journeys is, in effect, a number of different samples, each taken from these different distributions. We have taken a sample of size 1 from 1.0 mile deliveries and the 1.3 mile deliveries, a sample of size 2 from the 3.0 mile deliveries and we have taken no samples from the distributions for distance not on our list.

We now require a method of finding the most suitable line to fit through our sample of points. The line is referred to as the line of best fit.

The line of best fit is called the least squares regression line.

The equations for the slope and the intercept of the least squares regression line are:

Slope, $b = \dfrac{n\sum xy}{n\sum x^3} - \dfrac{\sum x \sum y}{(\sum x)^2}$

Where n is the sample size.

Intercept, $\alpha = \dfrac{\sum y^2}{n} - \dfrac{b\sum x}{n}$

The calculations for our sample of size n=10 are given below. The linear model is:

$y = x + ab$

Table 5.2 calculations for the regression line

Distance Time

X miles	Y miles	xy	x^2	y^2
3.5	16	56.0	12.25	256
2.4	13	31.2	5.76	169
4.9	19	93.1	24.01	361
4.2	18	75.6	17.64	324
3.0	12	36.0	9.0	144
1.3	11	14.3	1.69	121
1.0	8	8.0	1.0	64
3.0	14	42.0	9.0	196
1.5	9	13.5	2.25	81
4.1	16	65.6	16.81	256
Totals 28.9	**136**	**435.3**	**99.41**	**1972**

The final column is used in later calculations

The slope, $b = \dfrac{10 \times 435.3 - 28.9 \times 136}{10 \times 99.41 - 28.9^2}$

$= \dfrac{422.6}{158.9}$

$= 2.66$

We now insert these vales in the linear model, giving

$y = 5.91 + 2.66x$

Or

Delivery time (Min) = 5.91+2.66×delivery distance (miles)

The slope of the regression line (2.66 minutes per mile) is the estimated number of minutes per mile needed for a delivery. The intercept (5.91 minutes) is the estimated time to prepare for the journey and to deliver the goods, that is, the time needed for each journey other than the actual travelling time. The intercept gives the average effect on journey time of all of the influential factors except distance which is explicitly included in the model. It is important to remember that these values are based on a small sample of data. We must determine the reliability of the estimates, that is, we must calculate the confidence intervals for the population parameters.

Strength of the linear relationship- the correlation coefficient, r

Assessment of how well the linear model fits the data. Let us consider two variables, x and y. We consider that a relationship is likely between these variables.

The ratio of the explained variation to the total variation is used as a measure of the strength of the linear relationship. The stronger the linear relationship, the closer this ratio will be to one. The ratio is called the coefficient determination and is given the symbol r^2 where

$$r^2 = \frac{\sum (y - \bar{y})^2}{\sum (y - \bar{y})}$$

The coefficient of determination is frequently expressed as a percentage and tells us the amount of the variation in y which is explained by the introduction of x into the model. A perfect linear relationship at all means $r^2 = 0$ or 0%. The coefficient of determination does not indicate whether y increases or decreases as x increases. This information may be obtained from the **Pearson product moment correlation coefficient, r.** This coefficient is the square root of the coefficient of determination:

$$r = \sqrt{\frac{\left(\sum y - \bar{y}\right)^2}{\left(\sum y - \bar{y}\right)^2}}$$

For the purpose of calculations it is useful to re-arrange this expression algebraically to give:

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left(n\sum x^2 - \left(\sum x\right)^2\right)\left(n\sum y^2 - \left(\sum y\right)^2\right)}}$$

This is the sample correlation coefficient.

The value of r always lies between -1 and +1. The sign of r is the same as the sign of the slope, b. If b is positive, showing a positive relationship between the variables, then the correlation coefficient, r, will also be positive. If the regression coefficient, b, is negative then the correlation coefficient, r, is also negative.

As the strength of the linear relationship between the variables increases, the plotted points will lie more closely along a straight line and the magnitude of r will be closer to 1. As the strength of the linear relationship diminishes, the value of r is closer to zero. When r is zero, there is no linear relationship between the variables. This does not necessarily means that there is no relationship of any kind. Figure 8 and 9 below will both give values of the correlation coefficient which are close to zero.

Diagram 5.8

No relationship of any kind between the variable r ≈0
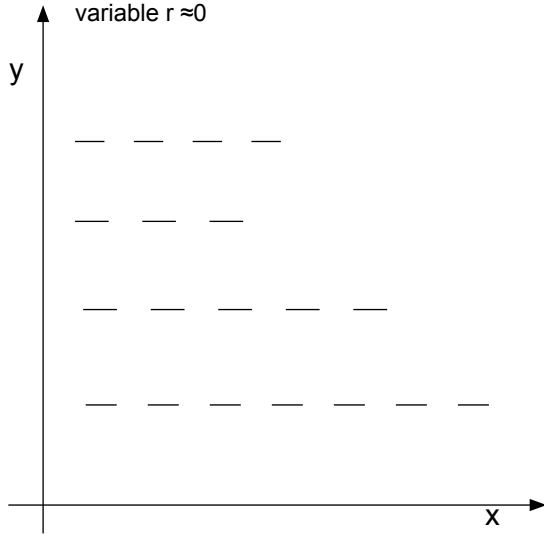
Diagram 5.9

A very strong linear relationship relationship between the variables

r≈1
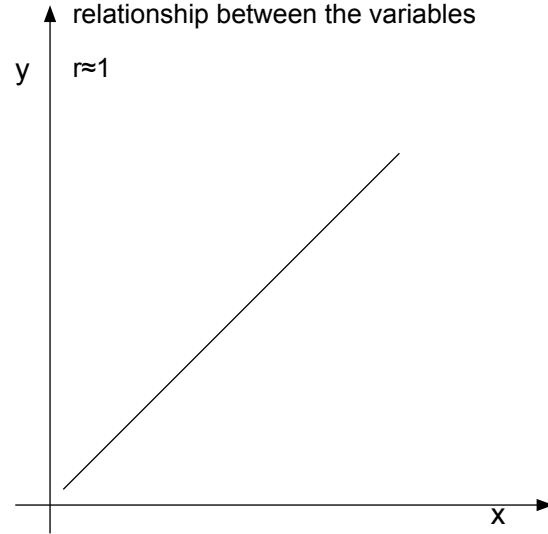
Return to the example above, in which a model was set up to predict delivery times for journeys of a given distance within a city. The correlation coefficient, r is calculated as follows:

$$ r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left(n\sum x^2 - \left(\sum x\right)^2\right)\left(n\sum y^2 - \left(\sum y\right)^2\right)}} $$

Notice that we have already evaluated the top line and part of the bottom line in the calculation for the slope of the regression line, b

$$ r = \frac{10 \times 435.2 - 28.9 \times 136}{\sqrt{(10 \times 99.41 - 28.9^2)(10 \times 1972 - 136^2}} = \frac{422.6}{158.9 - 1224} $$

r = 0.958

This value of the correlation coefficient is very close to +1 which indicates a very strong linear relationship between the delivery distance and the time taken. This conclusion confirms the subjective assessment made from the scatter diagram.

The coefficient of determination (r^2 x 100%) gives the percentage of the total variability in delivery time which we have explained in terms of the linear relationship with the delivery distance. In this example, the coefficient of determination is high:

r^2 = 0.958 x 100 =91.8%

The sample model: *time (minutes) = 5.91 + 2.66 × distance in miles* has explained 91.8% of the variability in the observed times. It has not explained 8.2% of the variability in the journey time but have not included in the model.

Prediction and estimation using the linear regression model

Predictions within the range of the sample data

We can use the model to predict the mean journey time for any given distance. If the distance is 4.0 miles, then our estimated mean journey time is:

$$\hat{Y} = 5.91 + 2.66 \times 4.0 = 16.6 minutes$$

A word of caution: It is good practice to use the model to make predictions for values of the independent variables which are outside the range of the data used.

The relationship between time and distance may change as distance increases. For example, a longer journey might include the use of a high speed motorway, but the sample data was drawn only from slower city journeys. In the same way, longer journeys may have to include meal or rest stops which will considerably distort the time taken.

The information from which we have been working is a sample from the population of journey times within the distance range 1 to 4.9 miles. If we wish to extrapolate to distance outside this range, we should collect more data. If we are unable to do this, we must be very careful using the model to predict the journey times. These predictions are likely to be unreliable.

Estimation, error and residuals

How accurate are our predictions likely to be? In the next section, we will consider this question in terms of the familiar idea of confidence intervals. However, it is also to assess the reliability of the predictions by mean of the differences between the observed value of the dependent variable, y, and the predicted value $\hat{y}$, for each value of the independent variable x. These errors or residuals, e, are the unexplained part of each observation and are important for two reasons. Firstly, they allow us to check that the model and its underlying assumptions are sound. Secondly, we can use them to give a crude estimate of the likely errors in the predictions made using the line.

The table below gives the residuals for the example above.

Table 5.3 Calculation of residuals (y-$\hat{y}$)

Distance in Miles x	Observed Time, Mins, y	Estimated times =5.91+2.66x Mins, $\hat{y}$	Residual e=(y-$\hat{y}$)
3.5	16	15.22	0.78
2.4	13	12.29	+0.71
4.9	19	18.94	+0.06
4.2	18	17.08	+0.92
3.0	12	13.89	-1.89
1.3	11	9.37	+1.63
1.0	8	8.57	-0.57
3.0	14	13.89	+0.11
1.5	9	9.90	-0.90
4.1	16	16.82	-0.82

We can examine the suitability of the model by plotting the residuals on the y-axis, against either the calculated values of y or, in bivariate problems, the x-values. This procedure is particularly important in multiple regressions, when the original data cannot be plotted initially on a scatter diagram so that the linearity of the proposed relationship may be assessed. If the linear model is a good fit, the residuals will be randomly and closely scattered about zero. There should be no pattern apparent in the plot.

If the underlying relationship had in fact been a curve, then the residual pattern would have shown this very clearly. An example of the effect of fitting a linear model when the relationship between the variables is actually curvilinear. The residential also allow us to assess the spread of the errors. One of the basic assumptions behind the last method is that the spread of data about the line is the same for all of the values of x, that is, the amount of variability in the data is the same across the range of x.

The residual pattern for the journey time example shows two large residuals. This may indicate that the data used do not conform to the assumptions of uniform spread. The consequences of this will be that the confidence limits, described in the next section, will be unusable.

The only way to continue with the statistical analysis of confidence intervals and hypothesis testing in this case, is to transform the data (often by taking logs of the x values) until the residual plot gives a random scatter of points about e = 0, with no larger values.

Computer generated solution

The setting up and evaluation of a linear regression model can be a lengthy task, especially if the data set is moderately large. Fortunately there are many computer packages available which will take drudgery out of the work by performing the entire arithmetic task to produce the required statistics. Unfortunately, it is all too easy to collect data and, without any further thought, to enter it into a computer. The programme will produce a linear model, no matter how unsuitable that may be.

Illustration

Leisure Publishers Ltd. recently published 20 romantic novels by 20 different authors. Sales ranged from just over 5,000 copies for one novel to about 24,000 copies for another novel. Before publishing, each novel had been assessed by a reader who had given it a rating between 1 and 10. The managing director suspects that the main influence on sales is the cover of the book. The illustrations on the front covers were drawn either by artist A or artist B. The short description on the back cover of the novel was written by either editor C or editor D.

A multiple regression analysis was done using the following variables:

 Y   sales (millions of shillings).

 X_1 1 if front cover is by artist A.

         2 if front cover is by artist B.

 X_2 readers' rating

 X_3 1 if the short description of the novel is by editor C.

         2 if the short description of the novel is by editor D.

The computer analysis produced the following results:

Correlation coefficient r = 0.921265

Standard error of estimate = 2.04485

Analysis of variance

	Degrees of freedom	Sum of squares	Mean square	F ratio
Regression	3	375.37	125.12	29.923
Residue	16	66.903	1.1814	

Individual analysis of variables:

Variable	Coefficient	Standard error	F Value
Constant	15.7588	2.54389	38.375
1	-6.25485	0.961897	42.284
2	0.0851136	0.298272	0.081428
3	5.86599	0.922233	40.457741

Correlation Coefficients

1	- 0.307729	0	- 0.674104
	1	0.123094	0.310838
		1	0.627329
			1

Required:

(a) The regression equation.
(b) Does the regression analysis provide useful information? Explain.
(c) Explain whether the covers were more important for sales than known quality of the novels.
(d) State with 95% confidence the difference in sales of a novel if its cover illustrations were done by artist B instead of artist A.
(e) State with 95% confidence the difference in sales of a novel if its short description was by editor D and not editor C.

Solution

(a) Regression equation, $\hat{y}$ = 15.7588 – 6.25485X_1 + 0.0851136X_2 + 5.86599X_3
(b) The regression analysis provide useful information
$r = 0.921265 \Rightarrow R^2 = 0.85$
The regression equation explains about 85% of the variation in sales.
There is a high positive linear relationship between the sales and the independent variables.
(c) Generally the bigger the value of F, the better the predictor readers rating have a small value of F compared with the other decision variables. Curves have a bigger F value. This implies that curves are important for sales prediction.
(d) $\propto$ = 5%

$$\left.\begin{array}{l}\propto/2 = 0.025 \\ \text{dif} = 20 - 4 = 16\end{array}\right\} \text{t} - \text{Critical} = 2.12$$

95% confidence interval   = - 6.25485 ± 2.12 × 0.961897

$$= - 6.25485 ± 2.03922$$

$$= - 8.29407 < B_1 < - 4.21563$$

We are 95% confident that a cover by artist B instead of artist A reduces sales by between 4216 and 8294 copies.

(e) 95% confidence interval = 5.86599 ± 2112 × 0.922233

$$= 5.86599 ± 1.95513$$

$$= 3.911 < B_3 < 7.821$$

We are 95% confident that using short descriptions by Editor D instead of Editor C will increase sales by 3911 and 7821 copies.

Statistical influence in linear regression analysis

The underlying assumptions

In this section, we will discuss some of the necessary assumptions underlying the further analysis of the linear regression model. The data from which a linear regression model is constructed is a sample of the population of pairs of x and y values. Essentially we are using the sample to build a model which we hope will represent the relationship in the population as a whole. The relationship between the dependent variable, y and the independent variable, x, is described by:

$y = a + bx + e$

where ε is the deviation of the actual value of y from the line:

$y = a + bx$

for given value of x.

$y = a + bx$

is the linear model we would set up if we had all the population data.

For any given x, the population of y value is assumed to be normally distributed about the population line, with the same variance for all x's

$\hat{y}$ is the mean of all the y values for a given x. As before, Greek letters refer to the population parameters such as μ and alfa. E is the error, or residual, the difference between the actual y value and the mean value from the line. If the least squares method is used to determine the line of best fit, then we are minimising summation of $\sum e^2$ the linear model which we calculate from the sample is:

Where y is an estimate of the population mean y for a given value of x, and a and b are the sample statistic used to estimate the population parameters α and .β

As in sampling situation, if we take a second sample, different values of a and b will arise. There is an exact analogy between the use of $\bar{x}$ to estimate μ and the use of a to estimate beta. By making assumptions about the sampling distribution of x, we can find confidence intervals for the value of the population mean μ. Exactly the same procedure may be used for alpha and beta by making inference from the sample values of a and b. Our basic model is;

$y = a + bx + e$

Assumptions

1. The underlying relationship is linear
2. The independence
3. The errors or residuals, ε are normally distributed
4. For any given x, the expected value of e is zero, i.e $E(\varepsilon) = 0$
5. The variance of ε is constant for all values of x, ie the variance $\varepsilon = s^2$
6. The errors re independent.

These assumptions hold if the population of y values, for a given x, is normal, with mean;

$$\mu_{y/x} = \alpha + \beta x$$

Where μ denotes the mean of y for a given x, and variance = δ^2

The line, set up from the sample data, is the estimate of this population line, with a as the best estimate of α and b as the best estimate of β. Since there are many possible samples which could be drawn from a given population, it is not possible to be sure that a particular set of sample data is actually drawn from the given population. Hypothesis tests show how confident we may be on the sample used to asses the compatibility of the sample with population. The tests show how confident we may be about the linearity of the parent population. If there is no linear relationship in the population r, the population correlation coefficient, will be zero, and β, the slope of the regression line, will also be zero. Once we have tested the overall linearity, we may wish to calculate confidence intervals for the slope β, for the intercept α, for the mean value of y given a value of x,or for individual values of y for a given x. We will use a random sample to calculate sample statistics and to estimate the corresponding population parameters.

Hypothesis tests to assess the overall linearity of the relationship

We are using sample data which has been drawn at random from a population in order to estimate a suitable linear relationship for the population. We do not actually know that the underlying relationship in the population is linear. The random sampling process could, quite legitimately result in a sample which exhibits linear properties but which was actually drawn from a population in which the underlying relationship is not linear.

We require some means of assessing the likelihood that a linear relationship in the sample implies a linear relationship in the population. Hypothesis tests help with this assessment. As in any situation in which hypothesis tests are used, we can never prove beyond all doubt that the population relationship is compatible with the relationship derived from the sample. We simply determine the consistency, or otherwise, of the sample evidence with the given null hypothesis. Linear regression generates several statistical figures and it is possible to perform separate hypothesis tests on these. We therefore build up an accumulative picture of the evidence for or against the basic hypothesis of linear relationship in the population.

We will now look at this hypothesis test in turn. The null hypothesis is essentially the same for all of the tests: That there is no linear relationship between the dependent and independent variables in the population.

Testing the population correlation coefficient

The evaluation of the Pearson product moment correlation coefficient, r, depends on the size of the sample. The interpretation of the value of r is independent of size from the point of view of the sample, but the implications for the population relationship are different sample sizes. A different inference will be drawn when considering a correlation coefficient of, 0.90, which arises from a sample of 6 items, compared to the same value which arises from a sample of 20 items. We can feel more confident that the underlying relationship is linear in the second case, since the chance of obtaining a sample which exhibits linearity, from a population which does not decrease as the size of the sample increases. The correlation coefficient is assessed using a t test:

H_o; there is no linear relationship between the y and x variables. The independent variable does not help in predicting the values of y, i.e. $r = 0$

H_1: $r \neq 0$ there is some linear relationship between the x and y variables. X does help to predict the y values.

Using this alternative hypothesis, we have a two-sided test. If we had decided that only positive value for r would be sensible, then H_1: $r > 0$ and we would now use a one-sided test. The test statistic is:

$$t = \sqrt{\frac{r^2(n-2)}{(1-r^2)}}$$

The number of degrees of freedom is (n-2), because we have calculated x and y to find r, using up two degrees of freedom, n is the number of pairs of values in the sample. If we wish to test at the 5% level using a two-tail test statistic would be compared with $t_{0.025,(n-2)}$ found from the tables.

To illustrate the procedure, we will return to an example which was concerned with the estimation of journey times from the journey distance. Previously we have found that r=0.958. Therefore the test statistic is:

$$t = \sqrt{\frac{0.958^2 \times 8}{(1-0.958^2)}} = \sqrt{\frac{7.342}{0.082}}$$

= 9.45

The number of degrees of freedom is: (10-2) =8

From the tables

$t_{0.025,\ 8}$=2.306

The test statistic (9.45) is greater than 2.306; therefore, we reject H_0 at the 5% level of significance and choose to accept H_1. The evidence is not consistent with the null hypothesis at this level. We assume that the correlation coefficient in the population is not zero and that there is a linear relationship between journey time and distance. This is the result we would expect with such a high value of the sample correlation coefficient, r.

Hypothesis test on the slope of the simple regression line

In simple linear regression, the hypothesis test on the slope of the line does exactly the same job as the test on the correlation coefficient. We do either one test or the other, but not both. In multiple regressions, however, where we have a regression coefficient for each of the independent variables, the two tests fulfill different functions and both are needed.

H_0: There is no linear relationship between the variables, x does not help predicting y, ie β=0

H_1: β≠0 i.e. there is a linear relationship and x does help to predict the y values.

In this case a two sided test is used. However, as in the test on r, we can alter this to a one sided test if we think that β>0 or B<0 is a more appropriate alternative hypothesis. When the population variance is unknown, the test statistic for a sample mean is:

$$t = \frac{\text{(sample statistics parameter assumed in } H_0}{\text{best estimate of standard error of statistic}}$$

$$= \frac{(\bar{x} - \mu)}{s/\sqrt{n-1}}$$

The test statistic for the regression coefficient, b, is:

$$t = \frac{(b - o)}{\text{Estimated standard error of b}}$$

The estimated standard error of b is:

$$se_b = \frac{\hat{\sigma}_e}{\sqrt{\sum(x - \bar{x})^2}}$$

Where is the variance of the distribution of the residuals about the population regression line. Remember, we are assuming that this variance is the same for all values of x. σ_e^2 is the best estimate of this population variance σ_e^2

$$\sigma_e^2 = \frac{\sum e^2}{(n-2)} = \frac{\sum(y - \hat{y})2}{(n-2)}$$

For computational purposes, this expression may be re-arranged algebraically as:

$$\hat{\sigma}_e^2 = \frac{\left(\sum y^2 - a\sum y - b\sum xy\right)}{(n-2)}$$

Again to illustrate the procedure, refer to the example about the journey times and distances. Using the first expression for $\hat{\sigma}_e^2$

$$\hat{\sigma}_e^2 = \frac{0.78^2 + 0.71^2...(-0.82)^2}{8} = \frac{10.01}{8} = 1.25$$

Therefore: $\hat{\sigma}_e^2$ = 1.12minutes

And:

$$\sqrt{\sum (x - \bar{x})^2} = \sqrt{\left(\sum x^2 - \frac{\left(\sum x \right)^2}{n} \right)} = \sqrt{15.889} = 3.99$$

Therefore:

$$se_b = \frac{1.12}{3.99} = 0.281$$

The test statistic for β is:

$$t = \frac{2.66}{0.281} = 9.47$$

If allowance is made for rounding errors, this value of t is the same as the t statistic obtained in the test on the correlation coefficient, 9.47 compared with 9.45.

To test at the 5% level using a two-sided test, we compare the test statistic with the boundary value found from the tables.

$$t_{0.025,\ 8} = 2.306$$

Since 9.47>2.306, we reject H_0 and choose to accept H_1. At the 5% decision level, the evidence is not consistent with the null hypothesis. This is the same conclusion as before. We choose to assume that there is a linear relationship between the journey time and distance. i.e. β≠0, x does help to explain the variability in y.

Confidence intervals in linear regression analysis

Confidence interval for the slope of the population regression line β

The (1-p) 100% confidence interval for the slope, β, defines a range of values about b, the sample estimate of β, within which we may be (1-p) 100% confident that the actual value of β lies. Put another way, for (1-p) 100% of the sample, the true value of β will lie within the confidence interval.

$$b \pm t_{(p/2),(n-2)} se_b$$

From the above we know that:

$$se_b = \frac{\hat{\sigma}_e}{\sqrt{\sum (x - \bar{x})^2}}$$

Let us calculate the 95% confidence interval for the slope of the regression model derived in the example about journey times and distance

$$b \pm t_{0.025,\ 8} se_b = 2.66 \pm 2.31 \times 0.281 = 2.66 \pm 0.65$$

We are 95% confident that the population slope, β lies between 2.01 and 3.31 minutes per mile. There is a 5% chance that β lies outside the range.

Confidence for the mean value of y for a given value of x

We now return to the basic assumption in regression analysis that for a given value of x, which we will call x0, the possible values of y are normally distributed. The mean value of these normal distributions is the value of y on the population regression line. We will call this mean value μy/x. The (1-p) 100% confidence interval for $\mu_{y/x}$ is :

$$\hat{y} \pm t_{(p/2),(n-2)} \; \hat{\sigma} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x - \bar{x})^2}}$$

Where $\hat{y}$ is the estimated value, calculated from the sample regression, $\hat{y} = a + b$

Notice that this confidence interval depends on the value of x_0. The width of the interval about the regression line, therefore, varies as x varies. The interval is at its narrowest when $x_0 = x$, the sample mean of the x's. The interval is then the more familiar:

$$\hat{y} \pm t_{(p/2),(n-2)} \; \hat{\sigma}_e / \sqrt{n}$$

As x_0 moves further away from $\bar{x}$, in either direction, the width increases.

In the example which we have been following, the 95% confidence interval for $\mu_{y/x}$ is

$$\hat{y} \pm 2.31 \times 1.12 \sqrt{\frac{1}{10} + \frac{(x_0 - 2.89)^2}{15.89}}$$

$$\hat{y} = 5.91 + 2.66 x_0$$

Values for this interval will be calculated in the next section.

Confidence interval for individual values of y for given value of x

A further assumption of the model is that y values are distributed about the regression line with variance, S^2_e, which is the same for all values of x, since we are using a sample, there are two elements of variability for individual y values. One arises from the estimated position of the mean, $\mu_{y/x}$ and the other arises from the availability of the individual values about this mean.

The two elements are different in that the first is due to the fluctuations inherent in sampling and the effect can be reduced if the sample is increased. The second is due to the nature of the variables and is unavoidable. It can be argued, therefore, that the confidence interval for individual values of y is not like other confidence intervals which arise entirely due to sampling fluctuation. Some books refer to this as 'prediction' interval rather than confidence interval. Whatever name is used, it is important to understand the distinction between the (1-p) 100% interval for $\mu_{y/x}$ and that for the individual y's, when x is given. The confidence interval expressions look similar; the only difference is that the variance for individual y's given x is increased by S^2_e. Similarly, the (1-p) 100% confidence interval for individual y values, given x = x_0 is

$$\hat{y} \pm t_{(p/2),(n-2)} \hat{\sigma}_e \sqrt{\left[1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum (x - \bar{x})^2} \right]}$$

Where:

$$\hat{y} = a + b x_0$$

The 95% confidence interval in example above is:

$$\hat{y} \pm 2.31 \times 1.12 \sqrt{\left[1 + \frac{1}{10} + \frac{(x_0 - 2.89)^2}{15.89}\right]}$$

The table below illustrates the behaviour of the two confidence intervals, as x_0 changes

Table 5.4 Calculation of confidence intervals for .$\mu_{y/x}$ and y given x_0, for example above

	Estimated time	$\hat{y}$ =5.91+2.66x095%confidence intervals for y given $x_{0\pm}$ mins	
Distance x_0 Miles	Minutes	$\mu_{y/x}$ ± mins	
1.0	8.57	1.47	2.98
2.0	11.23	1.00	2.77
2.89($\overline{x}$)	13.60	0.82	2.71
3.0	13.89	0.82	2.71
4.0	16.55	1.09	2.81
4.9	18.94	1.54	3.01

These are long and complicated calculations for a sample of only 20 values. Much of the work can be taken away by the computer package but it is important to understand what the package is doing and how to interpret its output. Unfortunately different packages use slightly different terms and symbols. If you have a regression analysis package available to you, it will help if you work through a simple example, like the one in this book, by hand/manually, and then use the package. Compare the computer output with the hand/manual calculation until you thoroughly understand it. A clear understanding of the two variable linear models will also help considerably when we tackle multiple regression, the calculation for which are always done by computer.

Multiple linear regression models

In the previous section it was mentioned that the chosen independent variable is unlikely to be the only factor which affects the dependent variable. There will be many situations in which we can identify more than one factor we feel must influence the dependent variable. For example, we wish to predict cost per week for a production department. It is reasonable to suppose that departmental costs will be affected by production hours worked, raw material used, and number of items produced and maintenance hours worked.

It seems sensible to use all of the factors we have identified to predict the departmental costs for a sample week, we can collect the data on costs, production hours, raw material usage, etc, but we will no longer be able to investigate the nature of the relationship between costs and other variables by means of a scatter diagram. In the absence of any evidence to the contrary, we begin by assuming a linear relationship and only if this proves to be unsuitable, will we try to establish a non-linear model. The linear model for multiple linear regressions is:

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_n x_n + \varepsilon$$

The variation in y is explained in terms of the variation in a number of independent variables which, ideally, should be independent of each other. For example, if we decide to use 5 independent variables, the model is:

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \varepsilon$$

As with simple linear regression, the sample data is used to estimate α, β_1, β_2 etc. The line of best fit for the sample data is:

$$y = a + b_1 x_1 + b_2 x_2 + ... b_n x_n$$

These are the same as those for the simple regression case. However, here they lead to very complex calculations. Fortunately, a computer package performs these calculations leaving us free to concentrate on the interpretation and evaluation of the multiple linear regression models. In the following section, we will indicate the steps which should be taken when setting up and using a multiple regression model, but at all stages, we assume that the actual calculations will be done by computer.

The time series components

The value of a variable, such as sales will change over time due to a number of factors. For example, a company may be expanding hence there will be an upward movement in the sales of the product as time progresses. The general change in the value of a variable over time is referred to as the **trend T**. In time series analysis the trend is the line of best fit. The model is then used to forecast future trend. In practice, there may be no trend at all, with demand fluctuating about a fixed value, or more likely, there may be a non-linear trend. The graphs below represent the trend demand at different stages of a product's life cycle.

There is underlying upward trend for the newly launched product and a dying curve of the old product reaching the end of its economic life. It is difficult to fit the equation to these trend curves.

The moving average technique, described in the following sections, can be used to separate the trend from the seasonal pattern. The technique uncovers the historical trend by smoothing away the seasonal fluctuations. However, the moving average trend is not used to forecast future trend values because there is too much uncertainty involved in extending such a series.

Diagram 5.10 Sales of a successful new product
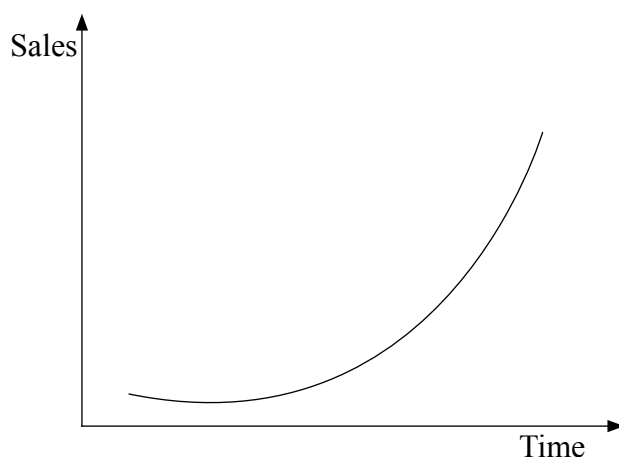
Diagram 5.12 Sales of product nearing the end of the life
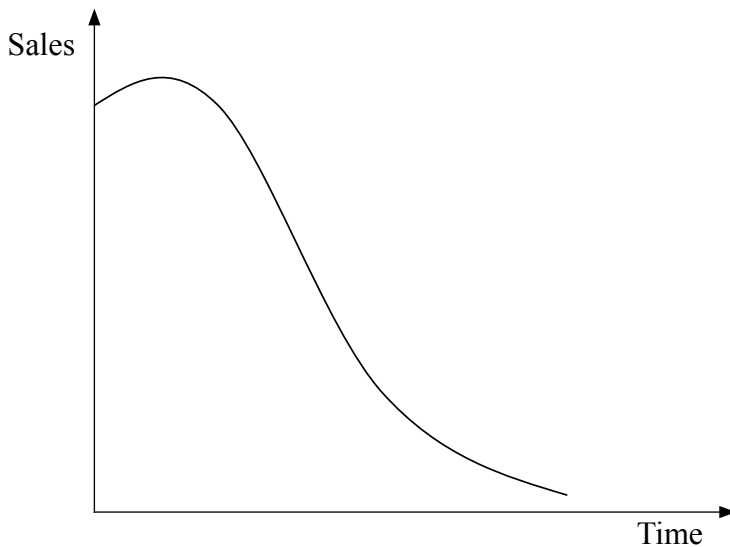
Generally, it is found that values do not indicate a trend only. If these regular fluctuations occur in a short term, they are referred to as seasonal variation,'S.' Longer term fluctuation is called **cyclical variation.**

The seasonal pattern in examples used in this chapter refers to the traditional seasons, but in forecasting generally, the term 'season" is applied to any systematic pattern. It may be the pattern of retail sales during the week, in which case the 'season' is a day. We may be interested in a seasonal pattern of traffic flow during the day and during the week. This will give us an hourly 'seasonal' pattern, superimposed on a daily 'seasonal' pattern, which both fluctuate about a daily trend. If we use annual data, we cannot identify a seasonal pattern. Any fluctuation about the annual trend data would be described by the cyclical component. This cyclical factor will not be included in our examples. It is seen only in long-term data, covering 10, 15 or 20 years, where large scale economic factors cause additional fluctuations about the trend.

These cyclical factors were apparent economic data from about 1960 to 1975. This was the time when many of these forecasting ideas were being developed, but since then, the overall economic pattern has changed. We will concentrate on short-term models which exclude the cyclical component.

The final term in our model also arises in the linear regression model. It is **error** or **residual,** the part of the actual observation that we cannot explain using the model. We can use the errors to give us measure of how well our model fits the data. Two measures are usually used. These are the **mean absolute deviation**.

$$MAD = \frac{\sum |Actual - Forecast|}{n} = \frac{\sum |E_t|}{n}$$

This is the sum of all the errors, ignoring their sign, divided by number of forecasts, and the mean square error

$$MSE = \frac{\sum (E_t)^2}{n}$$

This is the sum of the squares of the errors, divided by the number of forecasts. This second measure emphasises the large errors.

In the analysis of a time series, we attempt to identify the factors which are present and to build a model which combines them in an appropriate way.

Example: To illustrate the choice of an appropriate time series model.

The data below are quantities of a product sold by Lewplan Plc during the last 13-three monthly periods.

Table 5.11 Quantities of product sold over last 13-three monthly periods.

Date		Quantity sold 000
Jan – May	19 x 6	239
Apr – Jun	201	
Jul – Sep	182	
Oct – Dec	297	
Jan – Mar	19 x 7	324
Apr – Jun	278	
Jul – Sep	257	
Oct – Dec	384	
Jan – Mar	19 x 8	401
Apr – Jun	360	
Jul – Sep	335	
Oct – Dec	462	
Jan – Mar	19 x 9	481

We wish to analyzse this data set to see if we can find historical pattern. If there is a consistent pattern, we will use it to forecast the quantity sold in subsequent three monthly periods.

Solution: The figures are plotted below. In time series diagrams it is customary to join the points with straight lines so that any pattern can be seen more clearly.

Diagram 5.10 Lewplan plc, sales per 3 months.
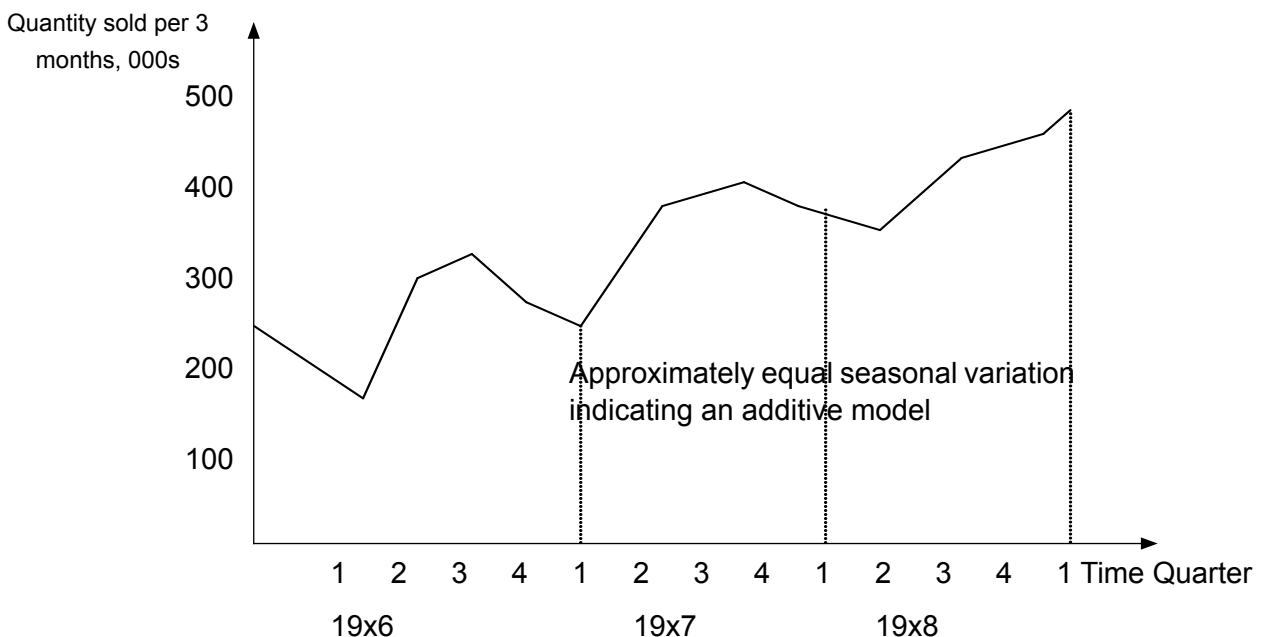
The diagram suggests that there may be an increasing trend, overlaid by seasonal fluctuations. The sales in the winter seasons, 1 and 4 are consistently higher than those in the summer seasons 2 and 3. The seasonal component appears to be fairly constant over the 3 years. The trend is for the sales to increase overall from around 230 in 19x6 to 390, but the seasonal fluctuations have not increased. This indicates that the additive component model should be more suitable. See section 9.3.

The analysis of an additive component model A = T + S + E

The addictive component model is one in which the variation of the value of the variable overtime can be described by adding the relevant components. Assuming that cyclical variation is not included the actual value of the variables A may be modeled by:

Actual value = trend + seasonal variation + error

That is

$$A = T + S + E$$

In both the addictive and multiplicative components modeled, the general analysis procedure is the same:

Step 1: Calculate the seasonal components

Step 2: Remove the seasonal component from the actual values. This is called deseasonalising the data.

Calculate the trend from these deseasonalised figures.

Step 3: Deduct the trend figures from deaseasonalised figures to leave the errors

Step 4: Calculate the mean average deviation (MAD) or the mean square error (MSE) to judge whether the model is reasonable, or to select the best from different models.

Calculate the seasonal components for the addictive model

Example: Setting up the addictive component model for a time series

Refer to the example in the previous section which relates to the quarterly sales of Lewplan plc. We have already decided that an addictive model is appropriate for these data; therefore the actual sales may be modeled by:

$$A = T + S + E$$

To eliminate the seasonal components we will use the method of moving averages. If we add together the first 4 data points, we obtain the total sales for 19x6; dividing this by 4 gives the quarterly average for 19 x 6, i.e.

(239 + 201 + 183 + 297)/4 = 229.75

This figure contains no seasonal component because we have averaged them out over the year, that is, for mid-point of quarters 2 and 3. If we move on for 3 months, we can calculate the average quarterly figure for April 19x6 – March 19x 7 (251), for July 19x6 –June 19 x 7 (270.5) and so on. This process generates the 4 point moving averages for this set of data. The set of moving averages represent the best estimate of the trend in the demand.

We now use these trend figures to produce estimates of the seasonal components. We calculate:

$$A – T = S + E$$

Unfortunately, the estimated values of the trend given by the 4 point moving averages are for points in time which are different from those for the actual data. The first value, 229.75, represents a point in the middle of 19 x x, exactly between the April June and the July- September quarters. The second value, 251, falls between the July-September and the October-December actual figures. We require a deseasonalised average which corresponds with the figure for an actual quarter. The position of the deseasonalised averages is moved by reaveraging pairs of values. The first and second values are averaged, centring them on July-September 19 x 6, ie:

(229.75 + 251)/2 = 240.4

This is the deseasonalised average for July-September 19 x 6. This deseasonalised value, called the centred moving average can be compared directly with the actual value for July-September 19 x 6 of 182. Notice that this means that we have no estimated trend figures for the first 2 or last 2 quarters of the time series. The results of these calculations are shown in the table below:

Table 5.12 Calculations of the centered 4 point moving average trend values for the model A = T + S= E

Date	Quantity 000's A	4 Quarter Total	4quarter moving average	Centered moving average	Estimated seasonal component A – T = S + E
Jan-Mar 19x6	239				
Apr – Jun	201				
		919	229.75		
Jul – Sep	182			240.4	- 58.4
		1004	251		
Oct –Dec	297			260.6	+ 36.4
		1081	270.25		
Jan -Mar 19x7	324			279.6	+44.4
	1156	289			
Apr-Jun	278			299.9	-21.9
		1243	310.75		
Jul-Sep	257			320.4	-63.4
		1320	330		

Oct – Dec	384			340.3	+43.8
		1402	350.5		
Jan-Mar 19x8	401			360.2	+40.8
		1480	370		
Apr-Jun	360			379.8	- 19.8
		1558	389.5		
Jul-Sep	335			399.5	-64.5
		1638	409.5		
Oct –Dec	462				
	481				

For each quarter of the year, we have estimates of the seasonal components, which include some error or residual. There are two further stages in the calculations before we have usable seasonal components. We average the seasonal estimates for each season of the year. This should remove some of the errors. Finally we adjust the averages, moving them all up or down by the same amount, until their total zero. This is done because we require the seasonal components to average out over the year. The correction factor required is: (the sum of the estimated seasonal values)/4. The estimates from the last column in Table.2 are shown under their corresponding quarter numbers. The procedure is shown in the table below.

Table 5.13 Calculations of the average seasonal components

		1	2	3	4
	19X6	-	-	- 58.4	+ 36.4
	19X7	+44.4	-21.9	-63.4	+ 43.8
	19X8	+40.8	-19.8	-64.5	
Total		+ 85.2	-41.7	-186.3	+ 80.2
Average		85.2÷ 2	-41.7 ÷2	-186.3÷3	80.2÷2
Estimated seasonal component		+ 42.6	-20.8	-62.1	Sum = -0.2
Adjusted seasonal component		+ 42.6	-20.7	-62.0	Sum = 0

In this example, two of the seasonal components have been rounded up, and two have been rounded down, so that the sum equals zero.

The seasonal components confirm our comments on the diagram at the end of the section above. Both winter quarters are above the trend by about 40 thousand units and the two summer quarters are below the trend by 21 and 62 thousand units, respectively.

A similar procedure is used for seasonal variation over any period. For example, if the seasons are days of the week, take a 7-point moving average to remove the daily 'seasonal' effect rather than a 4-point moving average. This average will represent the trend at the middle of the week, that is, on day 4, therefore it is not necessary to centre these moving averages.

Deseasonalise the data to find the trend.

Fast Forward

The term "trend analysis" refers to the concept of collecting information and attempting to spot a pattern.

Step 2 is to deseasonalise the basic data. This is shown below by deducting the appropriate seasonal component from each quarter's actual sales, i.e. A – S = T + E

Table 5.14 Calculation of the deseasonalised data

Date	Quarter no	Quantity sold 000 A	Seasonal component S	Deseasonalised quantity, 000 A – S = T +E
Jan – Mar 19x6	1	239	(+ 42.6)	196.4
Apr – Jun	2	201	(- 20.7)	221.7
July – Sep	3	182	(- 62.0)	244.0
Oct-Sep	4	297	(+ 40.1)	256.9
Oct – Dec	5	324	(+ 42.6)	281.4
Jan- Mar 19x8	6	278	(- 20.7)	298.7
Jul- Sep	7	257	(- 62.0) actuals	319.0
Oct – Dec	8	384	(+ 40.1)	343.9
Jan-Mar 19x8	9	401	(+ 42.6)	358.4
Apr – Jun	10	360	(- 20.7)	380.7
July-Sep	11	335	(- 62.0)	397.1
Oct-Dec	12	462	(+ 40.1)	421.9
Jan – Mar 19x9	13	481	(+ 42.6)	438.4

These re-estimated trend values, with errors, can be used to set up a model for the underlying trend. The values are plotted on the original diagram, which now shows a clear linear trend:

Diagram 5.11 Lewplan Plc, actual and deseasonalise sales per 3 months
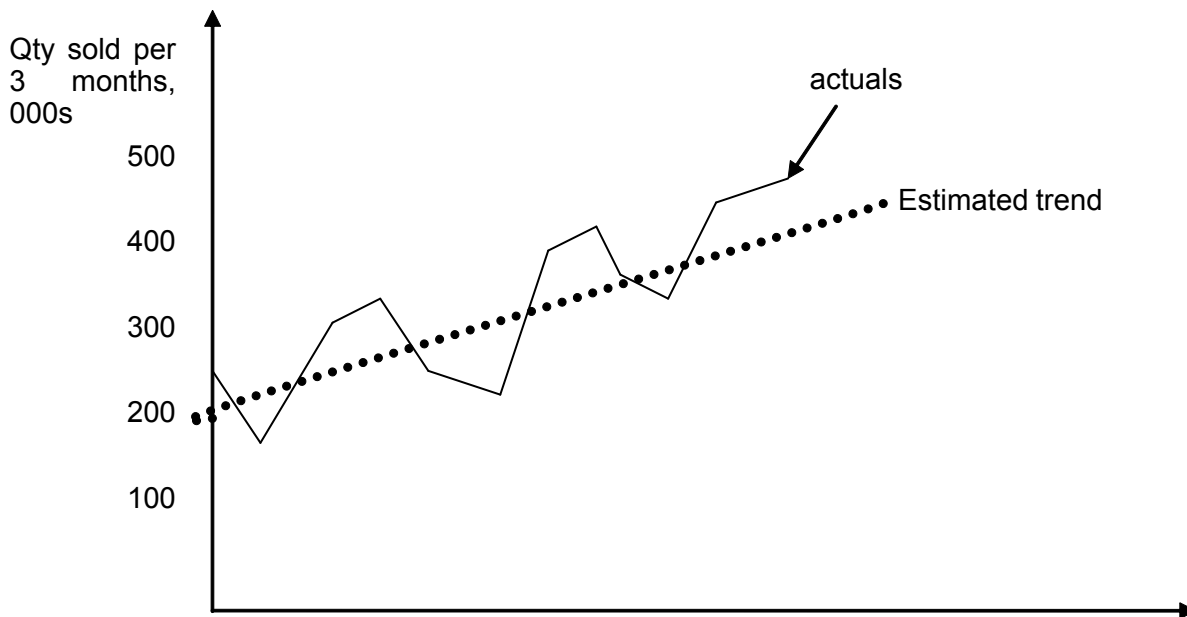
The equation of the trend line is:

T = a + bx quarter number

Where a and b represent the intercept and slope of the line. The least squares method can be used to determine the line of best fit, therefore the equation for a and b, from the previous chapter on linear regression are:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} \quad \text{and} \quad a = \frac{\sum y}{n} - \frac{b\sum x}{n}$$

Where X is the quarter number and y is (T + E) in the above table. Using a calculator, we find:

$\sum x = 91$ $\sum x^2 = 819$ $\sum y = 4158.7$ $\sum xy = 32747.1$ n = 13

It follows by substitution that:

b = 19.978 and a = 180.046

Hence the question of the trend model may be written.

Trend quantity (000s) = 180.0 + 20.0 x quarter number.

Calculate the errors

Step 3 in the procedure, before using a model to forecast, is to calculate the errors or residuals. The model is:

A = T + S + E

We have calculated S and T. We can now deduct each of these from A the actual quantity, to find the errors in the model.

Table 5.15 Calculation of errors for the addictive component model

Date	Quarter no	Quantity sold 000 A	Seasonal component S	Trend component, `000 T	Error, 000 A–S=T+E
Jan-Mar 19x6	1	239	(+ 42.6)	200	-3.6
Apr – Jun	2	201	(- 20.7)	220	+1.7
July – Sep	3	182	(- 62.0)	240	+4.0
Oct-Sep	4	297	(+ 40.1)	260	-3.1
Oct – Dec	5	324	(+ 42.6)	280	+.4
Jan- Mar 19x8	6	278	(- 20.7)	300	-1.3
Jul- Sep	7	257	(- 62.0)	320	-1.0
Oct – Dec	8	384	(+ 40.1)	340	+3.9
Jan-Mar 19x8	9	401	(+ 42.6)	360	-1.6
Apr – Jun	10	360	(- 20.7)	380	+0.7
July-Sep	11	335	(- 62.0)	400	-3.0
Oct-Dec	12	462	(+ 40.1)	420	+1.9
Jan-Mar 19x9	13	481	(+ 42.6)	440	-1.6

The final column in the table can be used for step 4, the calculation of the mean absolute deviation (MAD) or the mean square error (MSE) of the errors.

$$MAD = \frac{\sum |E_t|}{n} = \frac{28.7}{13} = 2.2$$

$$MSE = \frac{\sum (E_t)^2}{n} = \frac{78.85}{13} = 6.1$$

The errors are small at about 1% or 2%. The historical pattern is highly consistent and should give a good short-term forecast.

Forecasting using the addictive model

The forecast for this addictive component model is:

F = T +S (000 units per quarter)

Where the trend component T = 180 + 20 x quarter number, and the seasonal components, S are + 42.6 for January, March – 20.7 for April- June – 62.0 for July-September and + 40.1 October- December

The quarter number for the next three monthly periods, April-June 19 x9, is 14, therefore the forecast trend is:

T_{14} = 180 + 20 x 14 = 460   (000 units per quarter)

The appropriate seasonal component is – 20.7 (000 units). Therefore the forecast for this quarter is:

F (April – Jun 19 x9) = 460 – 20.7 = 439.3 (00 units)

It is important to remember that the further ahead the forecast, the more unreliable it becomes. We are assuming that the historical pattern continues uninterrupted. This assumption may hold for short periods but is less and less likely to be true the further we go into the future.

◼ The analysis of multiplicative component model: A = T ×S × E

In some time series, the seasonal component is not a fixed amount each year. Instead it is a percentage of the trend values. As the trend increases, so does the seasonal variation.

Example: Setting up a multiplicative component model for a time series.

CD Plc sells a range of products. The quarterly sales of one product for the last 13 quarters are given below

Table 5.16 CD Plc quarterly sales.

Date	Quarter number	Quantity sold A
Jan-Mar 19x6	1	70
Apr – Jun	2	66
July – Sep	3	65
Oct-Sep	4	71
Oct – Dec	5	79
Jan- Mar 19x8	6	66
Jul- Sep	7	67
Oct – Dec	8	82
Jan-Mar 19x8	9	84
Apr – Jun	10	69
July-Sep	11	72
Oct-Dec	12	87
Jan-Mar 19x9	13	94

The scatter diagram for these data is:

This product has a similar seasonal pattern to the previous example, with high winter values and low summer values, but the size of variations about the trend line are increasing. A multiplicative component model should be suitable for these data.

Actual values = trend x seasonal variation x error

That is: A = T X S X E

In this example, the trend looks linear but this will become clearer when we have smoothed the series.

Increasing seasonal variations as the trend increases indicates a multiplication
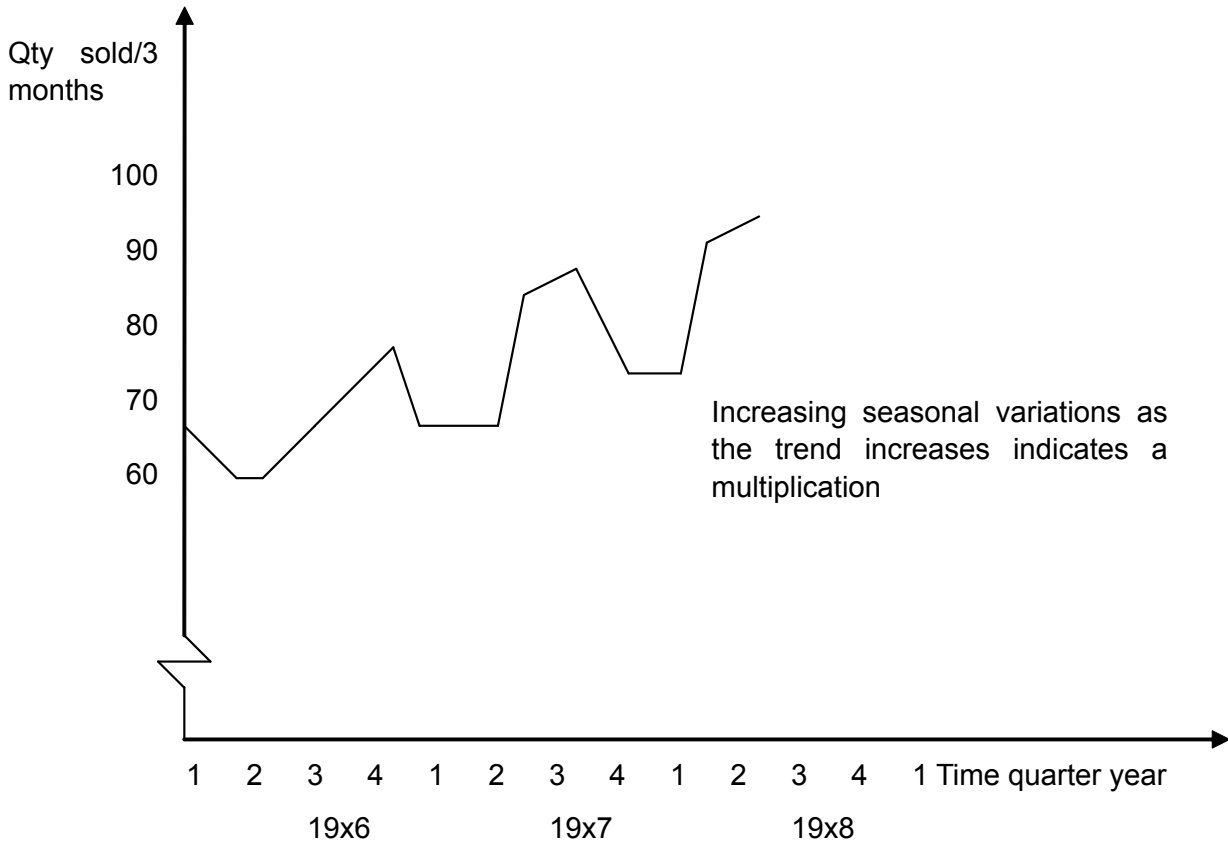
Calculations of the seasonal components

Initially the same procedure is followed as for the addictive model. The centred moving average trend values are calculated but the estimated seasonal components are ratios derived from A/T = S X E. The calculations are shown in the table below:

The seasonal components ratios are derived from the quarterly estimates in a similar way to those for the addictive model. Since the seasonal values are ratios and there are 4 seasons, we require the seasonal components to total to 4 rather than zero. (If the data comprised 7 daily seasons in each week, then the seasonal components would be required to total to 7). If the total is not 4, the values are adjusted as before. The estimates from the last column above are shown under their corresponding quarter numbers below.

Table 5.17 Calculations of the seasonal components, CD Plc

Date	Quarter no	Quantity sold 000 A	Centred 4 point moving average	Centred 4 point moving average	Seasonal component A/T
Jan-Mar 19x6	1	70			
Apr – Jun	2	66			
			68.00		
July – Sep	3	65		69.13	0.940
			70.25		
Oct – Dec	4	71		70.25	1.011
			70.25		
Jan-Mar 19x7	5	79		70.50	
			70.75		
Apr-Jun	6	66		72.13	0.915
			73.50		
Jul- Sep	7	67		74.13	0.904
			74.75		
Oct – Dec	8	82		75.13	1.092
			75.50		
Jan-Mar 19x8	9	84		76.13	1.103
			76.75		
Apr – Jun	10	69		77.38	0.892
			78.00		
July-Sep	11	72		79.25	0.909
			80.50		
Oct-Dec	12	87			
Jan-Mar 19x9	13	94			

Table 5.18 Calculations of seasonal components, CD Plc

	Year	1	2	3	4
	19 X 6			0.940	1.011
	19 X 7	1,121	0.915	0.904	1.092
	19 X 8	1.103	0.892	0.909	
Total		2.224	1.807	2.753	2.103
Average		2.224 ÷2	1.807 ÷2	2.753 ÷3	2.103 ÷2
Estimated seasonal		1.112	0.903	0.918	1.051 sum=3.984
Adjusted seasonal factor		1.116	0.907	0.922	1.055 sum = 4

The adjusted value is obtained by multiplying each estimated component ratio by (4/3.984)

The seasonal effect on the sales of the January- March quarter is estimated to increase sales by 11.6% of the trend value (1.116). Similarly effect of the October-December quarters is to raise sales by 5.5% of the trend. For the other 2 quarters, the seasonal effect is to depress the sales below the trend values to 90.7% and the 92.2% trend respectively.

Deseasonalise the data and fit the trend line

We have now found estimates of the seasonal component and can deseasonalise the data by calculating $A/S = T \times E$. these estimated trend values are calculated below.

Table 5.19 Calculations of the trend for CD plc

	Quarter number	Quantity sold A	Seasonal component ratio S	Deaseasonalised quantity 000 A/S = TXE
Jan-Mar 19x6		70	1.116	62.7
Apr – Jun		66	0.907	72.8
July – Sep		65	0.922	70.6
Oct-Dec		71	1.055	67.3
Jan- Mar 19x7		79	1.116	70.8
Apr-Jun		66	0.907	72.8
Jul- Sep		67	0.922	72.7
Oct – Dec		82	1.055	77.7
Jan-Mar 19x8		84	1.116	75.2
Apr – Jun		69	0.907	76.1
July-Sep		72	0.922	78.2
Oct-Dec		87	1.055	82.4
Jan-Mar 19x9		94	1.116	84.2

The trend values are superimposed on the original scatter diagram:

Diagram 5.12 CD Plc actual and deseasonalised sales per 3 months
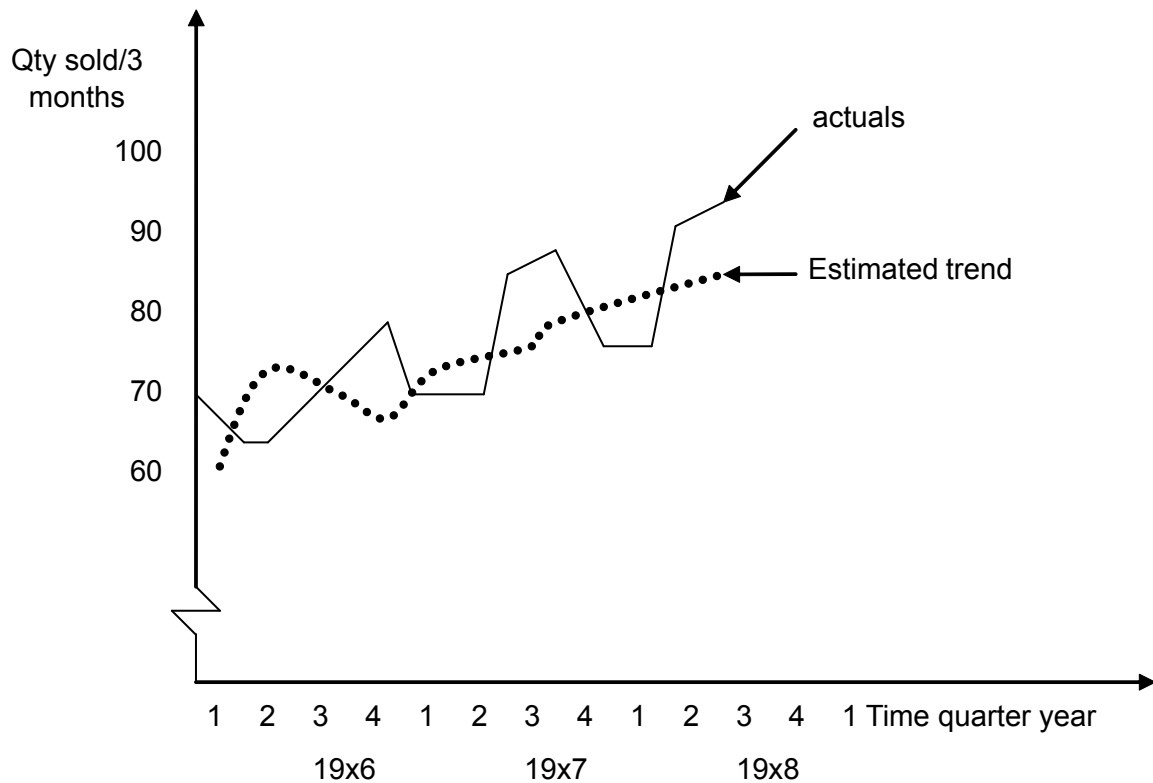
The trend which emerges is erratic. The sales values in this time series are not consistent like the ones in the first example for Lewplan Plc. CD plc is probably a more realistic example.

We now have to decide how to model the trend. It is not a curve but looks roughly linear even though the values are erratic, particularly in 19 x 6. For simplicity, we will assume that the trend is linear and use the least squares method to fit the best-line to the data. The trend line, using the same procedure is:

T = 64.6 + 1.36 X quarter number   (000 units per 3 months)

We use this equation to estimate the value of the trend sales for each of the periods.

Calculation of the errors: A /(T X S) = E OR A – (T XS) = E

We have now calculated the trend and seasonal components. We can use these to find the errors between the observed sales, A, and the sales which are forecast by the model, T x S. The table below gives the errors, both as a proportion, E = A/(T x S), and as absolute values, A – (T x S)

Table 5.19 Errors for CD Plc

Date	Qtr no	Qty sold 000 A	Seasonal component, 000 S	Trend component, 000 T	T×S	Errors A/(T×S)	A-(T×S)
Jan-Mar 19x6		70	1.116	66.0	73.7	0.95	-3.7
Apr–Jun		66	0.907	67.3	61.0	1.08	+5.0
July–Sep		65	0.922	68.7	63.3	1.03	+ 1.7
Oct-Dec		71	1.055	70.0	73.9	0.96	- 2.9
Jan-Mar 19x7		79	1.116	71.4	79.7	0.99	- 0.7
Apr-Jun		66	0.907	72.8	06.0	1.00	0
Jul- Sep		67	0.922	74.1	68.3	0.98	1.3
Oct-Dec		82	1.055	75.5	79.7	1.03	+ 2.3
Jan-Mar 19x8		84	1.116	76.8	85.7	0.98	- 1.7
Apr- Jun		69	0.907	78.2	70.9	0.97	- 1.9
July-Sep		72	0.922	79.6	73.3	0.98	- 1.3
Oct-Dec		87	1.055	80.9	85.4	1.02	+ 1.6
Jan-Mar 19x9		94	1.116	82.3	91.9	1.02	+ 2.1

The errors are high in the first year, as we could have guessed from the plot of the deseasonalised figures. However, from the January-March 19 x 7 quarter, the errors are all 2 or 3% of the actual values and the model looks reasonably satisfactory.

Forecasting with the multiplicative component model

Forecasting with either model assumes that we can fit an equation to the trend values. In both of the examples used we have been lucky. The trend has been clearly linear. If the trend had been a curve, we would have had to guess the relationship, and use some of the techniques mentioned in the previous chapter for dealing with non- linear relationships. Once we have established the trend equation, the calculations of the forecast is straightforward. The forecast is:

F = T x S

Where:

T= 64.6 + 1.36 X quarter number (000 units per 3 months)

And the seasonal component ratios are 1.116 for quarter 2, 0.922 for quarter 3, and 1.055 for quarter 4. The next quarter is April- June19 x 9, which is quarter 14 in the series and quarter 2 in the year. The forecast sales are:

F = T X S

= (64.6 + 1.36 X 14) X 0.907

= 83.64 X 0.907 = 75.9 (000 units per quarter)

Given the errors for the model, we would hope that this estimate will be within 2% or 3% actual value. Similarly, the forecast for October-December 19 x 9 is found using quarter number is 16 and the seasonal component for quarter 4:

F= T X S

= (64.6 + 1.36 X16) X 1.055

= 86.64 X 1.055 = 91.1 (000 units per quarter)

We expect the error on this forecast to be larger than previous one because it is further into the future.

CHAPTER SUMMARY

A time series is any set of data which is recorded over time. It may, for example, be annual, quarterly, monthly or weekly data. Models use the historical pattern of the time series to forecast how the variable will behave in the future. The short term forecast will be more accurate than the longer term ones. The further ahead we forecast, the less likely it is that the historical pattern will remain unchanged, and the larger will be the errors.

There are two basic models. In both cases, it is assumed that the value of the variable is made up of a number of components. The series may contain trend –general movement in the value of the variable, seasonal variation – short term periodic fluctuations in the variable values: cyclical variation – long term periodic fluctuations in the variable values, and error components – residual term. Data sets enough to include the cyclical component were not considered in this text.

The component models are:

Addictive A = T + S + E

Multiplicative A = T × S × E

In both cases, moving averages are used to deseasonalise the time series. These deseasonalised data are used to set up a model to describe the trend. The model is used to forecast the best model. Two measures give guidance on how well a model fits the past data. These are:

Mean absolute deviation (MAD) = $\dfrac{\sum |E_t|}{n}$

Mean square error (MSE) = $\dfrac{\sum (E_t)^2}{n}$

■

1. A data set, in which the independent variable is time, is referred to as…………..
2. …………. is the process of estimation in unknown situations.
3. Variance is that part of the actual observation that we cannot explain using the model.
   (a) True
   (b) False
4. Is the following component model right?   A + T= S + E

5. The formula for mean absolute deviation is $\dfrac{\sum (E_t)^2}{n}$

   (a) True
   (b) False

■

■
ANSWERS
■

1. Time series
2. Forecasting
3. (b) False. This is called Error or residual
4. Wrong. It is A – T = S + E

5. False. MAD = $\dfrac{\sum |E_t|}{n}$

■

■
QUESTIONS FROM PREVIOUS EXAMS
■

DECEMBER 2000 QUESTION 5

An increasing number of organisations operate sophisticated corporate planning models. Quite often these models feature integrated models for the production, marketing and financing sub-systems. While linkage between these models is important, forecasts of the external environment are also needed and econometric models are frequently used to provide these forecasts.

Required

a) Under production, marketing and finance sub-systems, what information about the environment might be provided by the econometric models? (9 marks)
b) The monthly electricity bill at the Chez Paul Restaurant over the past 12 months has been as follows:

Month	Amount (Sh.)
December	30,660
January	27,190
February	30,570
March	30,640
April	29,730
May	31,530
June	29,720
July	33,070
August	30,010
September	27,550
October	30,130
November	27,940

Paul is considering using exponential   α = 0.5 to forecast future electricity sales (index α smoothing with.)

Required:

Determine next January's forecast.   (11 marks)

                                                                        (Total: 20 marks)

JUNE 2002 QUESTION 5

The following regression equation was calculated for a class of 24 CPA II students.

$$\hat{Y} = 3.1 + 0.021X_1 + 0.075X_2 + 0.043X_3$$

Standard error   (0.019)   (0.034)   (0.018)

Whereby: Y =   Standard score on a theory examination

            X_1 =   Students rank (from the bottom) in high school

            X_2 =   Students verbal aptitude score

            X_3 =   a measure of the students character

Required:

a) Calculate the t ratio and the 95% confidence interval for each regression coefficient.
                                                                        (8 marks)
b) What assumptions did you make in (a) above? How reasonable are they?
                                                                        (4 marks)
c) Which regressor gives the strongest evidence of being statistically discernible?
                                                                        (2 marks)
d) In writing up a final report, should one keep the first regressor in the equation, or drop it? Why?   (6 marks)
                                                                        (Total 20 marks)

JUNE 2003 QUESTION FIVE

a) For an additional time series model, what does the term "residual variation" mean? Describe briefly its main constituents.   (4 marks)
b) Explain the moving average centering and why it is employed.   (4 marks)
c) Given a time series with trend figures already calculated, describe in words only, the method for calculating seasonal variation values using the additive model.   (4 marks)
d) When projecting a moving average trend, what basis would make the choice of the following appropriate?
    (i)   Projecting 'by eye'.   (2 marks)
    (ii) Using the method of semi-averages   (1 mark)
    (iii) Using the average change in trend per period from the range   (1 mark)
d) Explain how management of an organisation might use seasonal variation figures and seasonally adjusted data.   (4 marks)
                                                                        (Total: 20 marks)

DECEMBER 2003 QUESTION 5

a) Explain the purpose of selecting participants from different functional fields to participate in a Delphi Study. Could the strategy backfire?   (5 marks)

b) Business at M and K Sunshine Boutique can be viewed as falling into three distinct seasons:
   (1) Christmas (November – December)
   (2) Rainy Season (April – June)
   (3) All other times

The average weekly sales in thousands of shillings during each of these seasons in the past four years has been as follows:

Season	Year			
	1999	2000	2001	2002
1	1,856	1,995	2,241	2,280
2	2,012	2,168	2,306	2,408
3	985	1,072	1,105	1,120

An approximate software was used to analyse the above data and the following output is provided:

Period	Sales	Moving Average (Sh.)	Seasonal-Irregular Component (Sh.)	Deseasonalised Data (Sh.)
1	1,856	-	1.156	1,605.667
2	2,012	1,618	1.225	1,642.648
3	985	1,664	0.586	1,679.630
4	1,995	1,716	1.162	1,716.612
5	2,168	1,745	1.236	1,753.594
6	1,072	1,827	0.599	1,790.576
7	2,241	1,873	1.226	1,827.558
8	2,306	1,884	1.237	1,864.539
9	1,105	1,897	0.581	1,901.521
10	2,280	1,931	1.176	1,938.503
11	2,408	1,936	1.219	1,975.485
12	1,120	-	0.557	2,012.467

Number of moving periods: 3

Number of periods: 12

Mean Absolute Deviation (MAD) = 493.3333

Trend line of the deseasonalised data: $T_1 = 1,580.11 + 33.96t$

Required:

(i)   Explain the above output   (3 marks)
(ii) Determine the seasonal indexes and explain what they mean   (8 marks)
(iii) Determine the short range forecast for the 13th, 14th, and the 15th periods.   (4 marks)
                                                                    (Total: 20 marks)

JUNE 2004 QUESTION 5

State the principal components of a time series.   (2 marks)

a)   (i)   Explain the difference between multiplicative and additive models as used in time
           series.   (2 marks)
     (ii) State the conditions under which each model is used.   (2 marks)
b)   The table below shows the sales of new cars by quarters during a period of three
     years.

Year	Quarter 1 Sh. "million"	Quarter 2 Sh. "million"	Quarter 3 Sh. "million"	Quarter 4 Sh. "million"
2001	55.0	76.5	61.2	77.8
2002	54.4	65.9	52.7	81.4
2003	59.3	83.2	78.5	93.0

Required:

(i)   Explain the purpose of the seasonal index   (2 marks)
(ii) The seasonal index for each quarter assuming an additive model.   (12 marks)
                                                                    (Total: 20 marks)

CHAPTER SIX

DECISION THEORY

CHAPTER SIX
Decision Theory

▶ OBJECTIVES

- • At the end of this chapter, you should be able to:
- • Define such terms as state of nature, event, act and payoff.
- • Organise data into a payoff table.
- • Determine the expected payoff of an act.
- • Compute opportunity loss and expected opportunity loss.
- • Assess the value of perfect information.

Fast Forward: Decision theory in mathematics and statistics is concerned with identifying the values, uncertainities and other issues relevant in a given decision and the resulting optimal decision.

▶ INTRODUCTION

Decision making can be regarded as an outcome of mental processes (cognitive processes) leading to the selection of a course of action among several alternatives. Every decision making process produces a finalchoice. The output can be an action or an opinion of choice.

▶ DEFINITIONS OF KEY TERMS

Decision – is a commitment to irrevocably allocate valuable resources. It is a commitment to act and action is the irrevocable allocation of valuable resources.

Outcome/consequences – A consequence is a result of a course of action or decision taken by a decision maker.

Decision theory – is a body of knowledge related analytical techniques of different degrees of formalities designed to help a decision maker choose a set of alternatives in light of possible consequences. It is a theory that applies in conditions of uncertainty, risks and certainty.

Choice of judgement – it is a tough decision e.g. I will take Maria as my wife. It can be defined simply as firmness of conviction. These are results arrived at by judges.

Decision makers – A group of persons who make final choice among alternatives.

Alternative – option – this is one of the mutually exclusive courses of action in attaining the objective.

Course of action – strategy or means available to a decision maker by which the objectives may be attained.

Objective – something that a decision maker seeks to accomplish or to obtain by means of his decision. It is short term.

Goal – a general objective. It is long term. Goal is used to denote a very general objective

Cyclic variation - Any change in economic activity that is due to some regular and/or recurring cause, such as the business cycle or season.

▶ INDUSTRY CONTEXT

We apply decision theory models to political science with the intent of creating a positive theory of politics. We use models of self-interested individuals to consider various voting mechanisms used to generate group decisions.

Several important applications of decision theory with uncertainty, applying decision theory to investments, insurance, and search.

▶ EXAM CONTEXT

For successful organizations and institutions, good ideas have to be made. In this case the examiners have previously examined the students as outlined below:

12/06, 6/06, 6/06, 12/05, 12/05, 6/05, 12/03, 12/02, 6/02, 12/01, 12/00

Decision Theory

Almost everything that a human being does involves decisions. We make choices all the time. Some decisions are easy others are not.

Consider the following and the problem they give:

1. Shall I bring an umbrella today? - This decision depends on something I don't know, namely whether it will rain.
2. I am looking for a house, shall I buy this house? This house looks fine, but perhaps I'll find a better house if I go searching
3. Shall I smoke the next cigarette? One single cigarette is no problem, but if I make the same decision sufficiently many times it may kill me.

A decision maker may have more than one objective. The term goal is sometimes used to denote a very general objective, usually long term.

Decision making process.

Is the process of choosing among alternative courses of actions which are feasible.

Process:

- Identify the objectives
- Search for alternative courses of action
- Gather data about the alternatives
- Select alternative courses of action
- Compare actual and planned outcome
- Respond to divergencies for the plan.

Decision theory generally involve 4 steps

Example: Consider a manufacturing company that is thinking of several methods to increase its production.

Steps:

1. List all the viable alternatives for the company considered above. There may be the following options.
   a) Expand the present plant
   b) Construct a new plant
   c) Subcontract the plan for extra demand.
2. Identify the expected future events. Often it's possible to identify most of the events that can occur i.e. states of nature.

The difficulty is to identify which particular event will occur. For the manufacturing company, the greatest uncertainty will be about product demand. The future events related to a demand will be
   i)   high demand
   ii)   Moderate demand
   iii) Low demand
   iv)   No demand
3. Construct a payoff table – The decision maker makes a payoff table representation profits or benefits for each combination course of action of a data and states of nature. The possible payoffs for the manufacturing company expansion.

Table 6.1

States of nature

Alternatives	High	Moderate	Low	Nil
Expand	50,000	25,000	(25,000)	(45,000)
Construct	70,000	30,000	(40,000)	(80,000)
Subcontract	30,000	15,00	(1,000)	(10,000)

4. Select optimum decisions criteria - Decision makers will choose criteria which result in target profit. The criteria may be economic, qualitative or quantitative.

Decision making environment

1. **Certainty** – In this environment, there exists only one state of nature i.e there is complete certainty about the future. Complete information is also available as to which state of nature is going to occur. It is thus easy to analyse the situation and make good decisions. The decision making process is just picking the best alternative.

   Unfortunately certainty environment is a very simplistic environment which is rarely applicable in real life situation.

   Examples: Linear programming, transportation and assignment techniques, I/O analysis and EOQ

   Few complex managerial problems ever enjoy the luxury of complete information about the future and thus decision making under certainty is of little consequential interest.

2. **Uncertainty** – More than one state of nature exists but the decision maker lacks sufficient knowledge to allow him assign probabilities with the various states of nature. Uncertain events are those events that cannot be predicted with statistical confidence i.e the decision maker does not know the relevant variables nor their probability therefore decision making in this environment depends on the risk attitude of the decision maker i.e. risk averse or a risk seeker or risk neutral.

   A risk averse avoids risk at all cost; the individual is conservative and assumes the worst situation will occur.

   A risk seeker takes high risks in expectation of high returns. He assumes that the best outcome will occur.

   A risk neutral person is not affected by risks. Such people will make decisions based on something else but not risk.

3. **Risks** – Here more than one state of nature exists and the decision maker has sufficient information to allow him probabilities to each of these states of nature. Therefore it involves situations or events which may/may not occur but its probability of occurrence can be calculated statistically. There are frequency of occurrence predicted from first records.

4. **Competition –** In this environment, the decisions of the firm are affected by the decision of other firms with opposing interests.

Decision making under uncertainty

When the probability of occurrence of each state of nature can be assessed the expected monetary value (EMV) of the expected opportunity loss (EOL) decision criteria are usually appropriate. When a manager can't assess the outcome probability with confidence or virtually no probability data are available other decision criteria may be applied. These include:-

i. Maxmax

ii. Maxmin

iii. Laplace (equally likely/ Criteria of rationality)

iv. Hurcwiz (criteria of realism)

v. Minimax regret criteria (savage criterion)

1.   Maxmax

Finds alternatives that maximise the max outcome or consequence of every alternative. The decision maker first locates the maximum number since this decision criterion locates the alternatives with the highest possible gain it has been called optimistic decision criterion. The criterion appeals to risk takers for optimism who are ready to make huge profits if they occur.

Table 6.2

Alternative	high	medium	low	No Action	Max. of row
Expand	50,000	25,000	-25,000	-45,000	50,000
Contract	70,000	30,000	-40,000	-80,000	70,000
Subcontract	30,000	15,000	-10,000	-10,000	30,000

Maxmax payoff is sh70,000 corresponding to contract alternative.

2. Maxmin

Finds alternatives that maximise the min or consequences of each alternative. Under this rule, the decision maker picks the worst possible outcome under each alternative and then the best of these worst outcomes. Since this decision maker locates the alternatives that has the least possible loss it has been called pessimist decision criteria. The criterion appeals to risk averse decision makers because it's a criterion of extreme caution. It assumes that the worst outcome will occur.

Table 6.3

Alternative	high	medium	low	No Action	min. of row
Expand	50,000	25,000	-25,000	-45,000	-45,000
Contract	70,000	30,000	-40,000	-80,000	-80,000
Subcontract	30,000	15,000	-10,000	-10,000	-10,000

The maxmin payoff is sh (10,000) to the company. Corresponding to the decision subcontract.

3. Laplace (criterion of rationality)

It holds that if a decision maker does not know the probability of the various states of nature and has reason to think otherwise then the states of nature will be considered equally likely. This criterion is based upon the principle of insufficient reason. The criterion assigns equal probabilities to all the events of each alternative decision and selects the alternatives with the max expected payoff.

Symbolically if n denotes the no of events and P1 donates the payoffs the expected values of strategy say $S_1 = 1/n (p_1 + p_2 \ldots + p_n)$

Table 6.4

Alternative	high	medium	low	No action	expected payoff
Expand	50	25	-25	-45	¼ (50 + 25 -45)=1250
Construct	70	30	-40	-80	(500)
Sub construct	30	15	-10	-10	8,500

NB: Units in '000'

The alternative subcontract will be the best to take since it results in max average payoff of sh 8,500.

4. Hurcwicz criterion (criterion of realism / weighted average criterion)

Is a compromise between maxmax and maxmin decision criteria. The criterion is based on Hurwicz concept of optimism or pessimism. This concept allows the decision maker to take into account both the maximum and minimum of each alternative and assigns the weight according to the degree of pessimism or optimism

The alternative which maximises the sum of these payoffs is then selected.

The criterion of realism consists the following steps:

i. A coefficient of realism (alpha) is selected. The coefficient is between zero and one. When alpha is close to 1 the decision maker is optimistic about the future and when is it close to zero he is pessimistic about the future.

ii.   Determine the max as well as min of each alternative and obtain the following:

iii.   P = α (maximum) + 1-α (minimum)

iv.   Choose the alternative that yields the maximum value P using example above

Table 6.5

Alternative	high	medium	low	No Action	min. of row	max	p=α(max)+(1-α)min
Expand	50	25	-25	-45	-45	50	31
Contract	70	30	-40	-80	-80	70	40
Subcontract	30	15	-10	-10	-10	30	22

NB: Units in '000'

Assume that coefficient of realism, α = 0.8

The best decision under Hurwicz criterion would be to construct since it has the highest weighted payoff of Ksh 40,000.

Minimax regret criterion (Savage criterion)

Was developed by L.J Savage. He pointed out that the decision has been made and state of nature has occurred. Thus, the decision maker should attempt to minimise regret by minimising the maximum opportunity loss of each alternative.

Steps

i.   Develop opportunity loss table and then find the max opportunity loss within each alternative.

ii.   Select the alternative with the min or smallest opportunity loss

Table 6.6

Alternative	High	Moderate	Low	Nil	Max
Expand	20,000	5,000	24,000	35,000	35,000
Construct	0	0	39,000	70,000	70,000
Subcontract	40,000	15,000	0	0	40,000

The company should expand because this decision will minimise its regret which is Sh. 35,000.

Decision making under risk

In this environment, it is possible to attach some probabilities in the various states of nature. Several states of nature may occur each at a given probability. The decision maker would either choose the expected monetary value or the expected opportunity loss (EOL)

These two are similar in the sense that the alternatives that maximise the EMV will minimise EOL at the same time.

Expected Monetary Value

Given a decision payoff table with conditional value (payoffs) and probability assessments for all states of nature, it is possible to determine the expected monetary value for each alternative if the decision would be expected a large number of times. The EMV of an alternative is just the sum of possible payoffs of the alternative each weighted by the probability of that pay off occurring so that EMV (alternatives) =

Payoff of 1ˢᵗ state of nature ×probability of 1ˢᵗ state of nature + Payoff of 2ⁿᵈ state of nature × probability of 2ⁿᵈ state of nature + payoff of last state of nature ×probability of last state of nature.

Steps

i. Construct a payoff table listing the alternative decisions and various states of nature. Enter conditional profit for each decision event combination along with the associated probability.
ii. Calculate the EMV for each alternative decision by multiplying the conditional profit by assigned probability and adding the resulting values
iii. Select the alternative that yields the highest EMV.

Example

Consider the following table for Thomson Manufacturing Company

States of nature	Favourable mkt	unfavourable mkt
Alternative	$	$
Construct a large plant	200,000	-180,000
Construct a small plant	100,000	-20,000
Do nothing	0	0
Probability	0.5	0.5

Which alternative will give the best EMV?

EMV (large) = 0.5 (200,000) + 0.5 (-180,000) = 10,000

EMV (small) = 0.5 (100,000) + 0.5 (-20,000) =40,000

EMV (Nothing) = 0

The largest expected monetary value results from the 2ⁿᵈ alternative which is to construct a small plant.

Expected opportunity loss

An alternative for maximising EMV is to minimise EOL. EOL represents the amounts by which the maximum possible profit will be reduced under various possible stock action. The cost of action that minimise these losses is the optimal decisions alternative.

Procedures

i. Prepare the conditional profit table for each decision taken and the associated profit
ii For each event determine the conditional opportunity loss by subtracting the pay off of the max pay off of that event.
iii. Calculate the EOL for each decision alternative by multiplying the conditional opportunity loss by the associated probabilities and then adding the values.
iv. Select the alternative that yields the lowest value

Alternative	favourable	unfavourable
Large	0	180,000
Small	100,000	20,000
Do nothing	200,000	0

EOL (large) = 0.5 0) +0.5(180,000) = 90,000

EOL (small) = 0.5 (100,00o) + 0.5(20,000) = 60,000

EOL (nothing) = 0.5 (200,000) + 0.5 (0) = 100,000

The decision to construct the small plant will be selected since it has the lowest EOL

Note: i) The minimum EOL will always result in the same decision as EMV

ii) The minimum EOL= Expected value of project information (EVPI)

Expected value of perfect information (EVPI)

EV with PI is the expected or average return in the long run. If we have perfect information before a decision has been made. To calculate this value we choose the best alternative for each state of nature and multiply its payoff by the probability of occurrence of that state of nature

The EVPI is the expected outcome with perfect information minus the expected outcome without perfect information i.e. the maximum EMV.

EVPI = EV, PI – EMU (max) EVPI = Min (EOL)

EVPI = (200,000 × 0.5) + (0 x 0.5) – 40,000

EVPI = $60,000 = MIN. EOL

EV PI = is the information which guarantees the future with 100% degree of accuracy.

Multistage decision making

Decision trees

It is a graphic representation of the decision alternatives, states of nature, probabilities attached to the state of nature and conditional losses.

It consists of a network of nodes. Two types of nodes are used; decision node represented by a square and states of nature (chance of event) node represented by a circle.

Alternative courses (strategies) originate from the decision node as main branches (decision branches)

At the end of each decision branch there's a state of nature mode from which emanates chance events in the form of sub-branches (chance branches)

The respective pay offs and probability associated with alternative course and chance events are shown alongside those branches. At the terminal of those branches are shown the expected value of outcomes.

Diagram 6.1
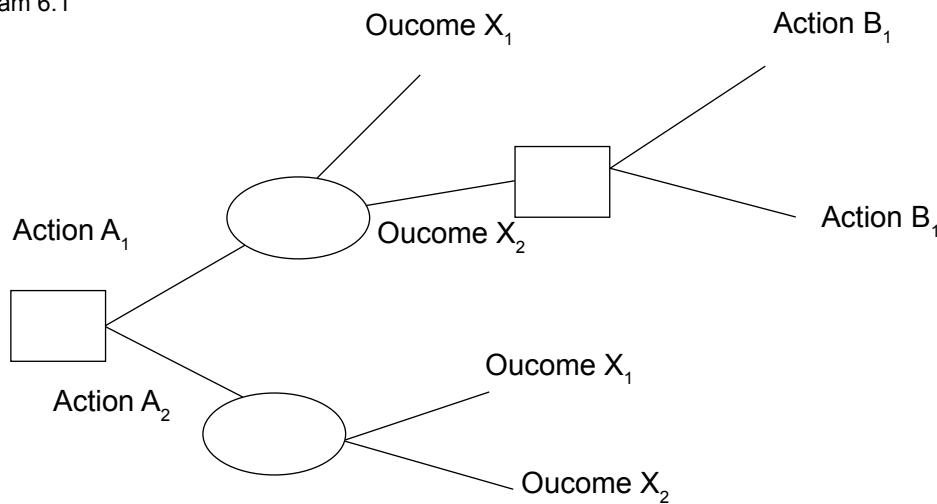
The general approach used in decision tree analysis is to work backward through the tree from right to left compiling the expected value of each chance node and then choosing the particular branch leaving a decision node which leads to the chance node with the highest expected value.

Steps in decision tree analysis

1. Identify the decision point and the alternative courses of action at each decision point systematically.
2. At each decision point determine the probability in payoff associated with each course of action.
3. Commencing from the extreme right, compute the expected pay off (EMV) from each course of action
4. Choose the course that gives the best payoff for each of the decision.
5. Proceed backwards to the next stage of decision point.
6. Repeat above steps until the 1st decision point is reached.
7. Identify the courses of action to be adopted from the beginning to the end under the different possible outcomes for the situation as a whole.

Advantages of Decision Tree Approach

i. It structures the decision process and helps decision making in orderly and systematic and consequent manner.
ii. It requires the decision maker to examine all possible outcomes whether desirable or undesirable.
iii. It communicates the decision making to others in an easy clear manner illustrating each assumption about the future.
iv. It displays the logical relationship between the parts of a complex decision and identifies the time sequence in which various actions and subsequent events occur.

Limitations

i. Decision tree diagram becomes more complex as the number of decision alternatives increases and more variables are introduced.
ii. It becomes highly complicated when interdependent alternatives and dependent variable are present in the problem.

iii.   It analyses the problem in terms of expected value and thus yields an average value solution

iv.   There is often inconsistency in assigning probabilities of different events

Perfect and imperfect information

Uncertainty about the future can sometimes be reduced by first obtaining more information about what is likely to occur. Such information can be obtained from various sources like market research, pilot testing, building a prototype model, etc. Information can be categorised into:

i.   Perfect information – is information which guarantees the future with 100% surety.

ii.   Imperfect information – is information which, although good and hence better than having no information at all, could be wrong in its prediction. Also referred to as sample information.

Expected value of sample information therefore is the maximum that the decision maker should be willing to spend in order to obtain sampled information.

Decision Trees and Bayes Theorem

There are many ways of getting probability data. The numbers can be assessed by a manager based on past experience and intuition. They can be derived from historical data or they can be computed from other available data using Bayes theorem approach. It recognises that a decision maker does not know with certainty what state of nature will occur. It allows the manager to revise his /her initial or prior probability assessment. These are called posterior probability.

This rule or theorem is given by

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

It is used frequently in decision making where information is given in the form of conditional probabilities and the reverse of these probabilities must be found.

Example 1

In a class of 100 students, 36 are male and studying accounting, 9 are male but not studying accounting, 42 are female and studying accounting, 13 are female and are not studying accounting.

Use these data to deduce probabilities concerning a student drawn at random.

Solution:

	Accounting A	Not accounting $\overline{A}$	Total
Male M	36	9	45
Female F	42	13	55
Total	78	22	100

$P(M) = \dfrac{45}{100} = 0.45$

$P(F) = \dfrac{55}{100} = 0.55$

$P(A) = \dfrac{78}{100} = 0.78$

$P(\bar{A}) = \dfrac{22}{100} = 0.22$

$P(M \text{ and } A) = P(A \text{ and } M) = \dfrac{36}{100} = 0.36$

$P(M \text{ and } \bar{A}) = 0.09$

$P(F \text{ and } \bar{A}) = 0.13$

These probabilities can be expressed differently as;

$P(M) = P(M \text{ and } A) \text{ or } P(M \text{ and } \bar{A})$

$= 0.36 + 0.09 = 0.45$

$P(F) = P(F \text{ and } A) \text{ or } P(F \text{ and } \bar{A})$

$= 0.42 + 0.13 = 0.55$

$P(A) = P(A \text{ and } M) + P(A \text{ and } F) = 0.36 + 0.42 = 0.78$

$P(\bar{A}) = P(\bar{A} \text{ and } M) + P(\bar{A} \text{ and } F) = 0.09 + 0.13 = 0.22$

Now calculate the probability that a student is studying accounting given that he is male.

This is a conditional probability given as P(A|M)

$P(A|M) = \dfrac{P(A \text{ and } M)}{P(M)} = \dfrac{0.36}{0.45} = 0.80$

From the formula above we get that,

$P(A \text{ and } M) = P(M)\, P(A|M)$ ……………….. (i)

Note that $P(A|M) \neq P(M|A)$

Since $P(M|A) = \dfrac{P(A \text{ and } M)}{P(A)}$ this is known as the Bayes' rule.

Bayes' rule/Theorem

This rule or theorem is given by

$P(A|B) = \dfrac{P(A) \times P(B|A)}{P(B)}$

It is used frequently in decision making where information is given in the form of conditional probabilities and the reverse of these probabilities must be found.

Example 2

Analysis of questionnaire completed by holiday makers showed that 0.75 classified their holiday as good at Malindi. The probability of hot weather in the resort is 0.6. If the probability of regarding holiday as good given hot weather is 0.9, what is the probability that there was hot weather if a holiday maker considers his holiday good?

Solution

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

Let H = hot weather

G = Good

P(G) = 0.75   P(H) = 0.6 and P(G|H) = 0.9 (Probability of regard holiday as good given hot weather)

Now the question requires us to get

P(H|G) = Probability of (there was) hot weather given that the holiday has been rated as good).

$$= \frac{P(H)P(G|H)}{P(G)} = \frac{(0.6)(0.9)}{0.75}$$

= 0.72.

Decision Making Under Competition

Game theory

Competition is an important economic factor. The strategy taken by individuals or organisations can dramatically affect the outcome of our decision. In the automobile industry, for example, the strategies of competitors to introduce certain models can dramatically affect the profitability of car makers.

Today's business cannot make important decisions without considering what other organisation or individuals are doing or might do. A game theory is one way to consider the impact of others on our strategies and outcome. A game is a context involving two or more decision makers each of whom wants to win.

Game theory is a study of how optimal studies are formulated in conflict situations. In the business context, the term game refers to conflicts through time. It is to do with business situations where usually the success of one person is at the expense of the other.

Competitive game

Properties

i. There's a finite number of participants. If the number of participants is two, the game is referred to as a two-person game. If we have M number of participants it is called an M-person game.

ii.   Each participant has a finite number of possible courses of action

iii.   Each participant must know all the courses of action available for the others but must not know which of these will be chosen.

iv.   A play of the game is said to occur when each player chooses one of its courses of action. The choices are assumed to be made simultaneously so that no participant knows the choice of the other until he has decided on his own.

v.   After all participants have chosen their courses of action their respective game options are finite.

vi   The game of participants depends on his action as well as those of others

Useful terminology

1.   Player – each participant (interested party) is called a player

2.   A play of the game results when each player has chosen a course of action. After each play of the game one player pays the other an amount determined by the courses of action chosen.

3.   Strategy – the decision/role which a player determines his/her course of action is called strategy. To reach a decision of which strategy to use neither player needs to know the other's strategy.

4.   Pure strategy – if a player decides to use only one particular course of action during every play he is said to use a pure strategy. A pure strategy is usually represented by a number with which a course of action is associated.

5.   Mixed strategy – if a player decides in advance to use all or some of his available courses of action in a fixed proportion he is said to use a mixed strategy. Thus a mixed strategy is a selection among pure strategies with some fixed proportions.

6.   Zero sum game – A game where a gain of one equals the loss of the other is known as a two person zero-sum game. In such a game, the interest of the two players are opposed so that the sum of their net gain (sum of the game) is zero. If there are n-players and the sum of the game is zero, it is an n-person zero sum game.

◼ Two person zero sum game

Characteristics of two-person zero – sum game

i.   Only two players participate

ii.   Each player has finite number of strategies to use

iii.   Each specific strategy results in a payoff.

iv.   Total payoff to the two players at the end of each play is zero

v.   Payoff is the outcome of playing the game. A payoff (game matrix) is a table showing the amounts received by the players named at the left hand side after all possible plays of the game. The payment is made by the player at the top of the table.

Fast Forward: Game theory attempts to mathematically capture behaviour in *strategic situations*, in which an individual's success in making choices depends on the choices of others.

Assumptions of Game theory

i.   Economic theory – Each player is assumed to be economically rational; each player desires to win i.e. sometimes referred to as requirements

ii.   Information – Both players have the same information set i.e. each player knows the strategies of the opponent and the consequential payoffs.

iii.   Zero Sum of the game – The gain of a given player equals the loss of the other player so that the game is zero sum in nature
iv.   No indifference – Each player has to play even if they are losing
v.   Repeatedness – The game is played expectedly over a long period of time

Example

Consider the following:

Strategy	Y_1	strategy Y_2	min. of row
X_1	3	5	3
X_2	1	-2	-2
Max. of column		3	5

Determine the strategy that each player will employ.

Both players have dominant or pure strategies therefore the game has a saddle point. Numerical value of saddle point is the game outcome. Saddle point exists only in pure strategy game. In this example the saddle point is 3

Note:

•   In reality player X and Y may not see the saddle point.After the game is played for some time, however, each player will realise that there is only one strategy to be played. From then on these players will only play one strategy which corresponds to the saddle point.
•   The value of the game is the average of expected game outcome. If the game is played in an infinite number of times the value of this game is 3. If a game has a saddle point the value of the game is equal to its nominal value.
•   The saddle point in this example is the largest number in its column and the smallest number in its row. This is true for all saddle points.

The minimax procedure is accomplished as follows

1.   Find the smallest number in each row
2.   Pick the largest of them. The number is called the lower (maxmin) and the row is the X's maxmin strategy.
3.   If the upper value and lower value are the same there is a saddle point which is equal to the lower and upper value; if not equal, there is no saddle point and if the value of the game lies between two values such a strategy is called a mixed strategy game.

Note:

•   The game is said to be fair if the maxmin value = minmax value =0 and it is said to be strictly determinable if the maxmin value = minmax value
•   Always look for a saddle point before attempting to solve a game.

Mixed Strategy Game

When there is no saddle point, players will play each strategy for a certain percentage of the time. This is called a mixed strategy game.

Two methods

1.   Algebraic method
2.   Arithmetic method

Algebraic method.

For a 2×2 game an algebraic approach can be used to solve for the percentage of the time each strategy is played.

Principle of rational expectation

X wants to decide his/her time so that he wins as much when Y is playing Y_1 as when Y is playing Y_2. The following diagram can be used:

Table 6.7

		Y	
		Y_1 (p)	Y_2(1-p)
X	X_1(Q)		
	X_2(1-Q)		

Where Q, 1-Q equals the proportion time when X plays strategy X_1 and X_2 respectively P, 1-P equals the percentage of time when y plays Y_1 and Y_2 respectively. The overall objective of each player is to determine the fraction of time that each strategy is to be played to maximise winnings.

Steps

1. To find X`s best strategy multiply Q and 1-Q times the appropriate game outcome and solve for Q and 1-Q by setting column 1 equal column 2 in the game.
2. To find Y`s best strategy multiply P and 1-P times the appropriate outcome numbers and solve for p and 1-p by setting row 1 equals to tow 2 in the game

Arithmetic Method

Steps

1. Subtract the two digits in column 1 and write them under column 2 ignoring the sign
2. Subtract the two digits in column 2 and write them under column 1 sign
3. Similarly proceed to the two rows. These values are called oddments. They are the frequencies with which the players must use their courses of action in their optimum strategies.

Table 6.8

		Y		
		Y_1 (p)	Y_2(1-p)	
X	X_1(Q)	4	2	9/11
	X_2(1-Q)	1	10	2/11
		8/11	3/11	

Value of the game= 9/11×2 + 2/11×10

$$=3^5/_{11}$$

Note: The arithmetic and algebraic method can only be employed if we are dealing with a 2x2 game matrix i.e. this is a situation where each of the two players has two possible courses of action.

Principle of dominance

The principle of dominance can be used to reduce the size of the games by eliminating strategies that will never be played. A strategy or a play can be eliminated if the player can always do as will or better than another strategy. In other words, a strategy can be eliminated if all of its outcomes are the same or worse than the corresponding game outcomes of another strategy.

Note: The objective is to reduce the game to 2x2 so that to use algebraic or arithmetic method.

Example

Using the principle of dominance reduce the size of the following game

	y_1	y_2	y_3	y_4	min of row
x_1	-5	-4	6	-3	-5
x_2	-2	6	2	-20	-20
Max of col.	-2	6	6	-3	

It has no saddle point.

Y will never play Y_2 and Y_3 as he will lose. He will play Y_1 and Y_4.

Reduce the following by dominance and find the value of the game.

Non zero-sum game

An assumption of zero-sum game is that the gain of a given player equals the loss of the other player. This may be possible in practice for some situations e.g. many games or sports like football; athletics etc. Also a fixed make competition can be modeled as a zero-sum game. However, in the business world many situations cannot realistically be modeled as zero-sum. In many cases the players will all gain e.g. cartels in the oil industry. They may all lose e.g. fierce competition in the matatu industry. Some may gain while others lose, not necessarily by the same amount.

CHAPTER SUMMARY

There are many types of decision making

Decision making under uncertainty

This refers to situations where more than one outcome can result from any single decision.

Decision making under certainty

When only one outcome exists for a decision, we are dealing with this category e.g. linear programming, transportation assignment and sequencing.

Decision making using prior data

It occurs whenever it is possible to use past experience (prior data) to develop probabilities for the occurrence of each data.

Decision making without prior data

No past experience exists that can be used to derive outcome probabilities. In this case, the decision maker uses his/her subjective estimates of probabilities for various outcomes.

Decision making under uncertainty

Several methods are used to make decision in circumstances where only the payoffs are known and the likelihood of each state of nature are known:

- Maximin Method
- Maximax method
- The Laplace method
- The Hurwicz method
- Expected Opportunity Loss (EOL) method
- The Expected Monetary Value method
- Minimax regret method

CHAPTER QUIZ

1. …………..theorem recognises that a decision maker does not know with certainty what state of nature will occur.
2. …………. is also called criterion of rationality.
3. …………. is also called criterion of realism.
4. Payoff is the outcome of playing the game is one of the characteristics of two-persons zero-sum game.
   (a) True
   (b) False
5. An assumption of ……………. is that the gain of a given player exactly equals the loss of the other player.
6. Which one of the following is not a characteristic of game theory?
   (a) Economic theory
   (b) Information
   (c) Infinite players
   (d) Zero-sum of the game
   (e) No indifference
   (f) Repeatedness
7. Which principle can be used to reduce the size of the games by eliminating strategies that will never be played?

ANSWERS TO CHAPTER QUIZ

1. Bayes Theorem
2. Laplace
3. Hurcwicz criterion
4. (a) True
5. zero – sum game
6. (c) Infinite players
7. Principle of dominance

QUESTIONS FROM PREVIOUS EXAMS

DECEMBER 2000 QUESTION 7

Explain the following terms as used in decision analysis.

(a)	Decision making under risk versus uncertainty	(4 marks)
(b)	Decision trees versus probability trees	(4 marks)
(c)	Minimax versus maximax criterion	(4 marks)
(d)	Pure strategy versus mixed strategy games	(4 marks)
(e)	Games with more than two persons versus non-zero-sum games	(4 marks)

(Total 20 Marks)

DEC 2001 QUESTION 7

An urban cablevision company is investigating the installation of a cable TV system in urban areas. The engineering department estimates the cost of the system (in present worth Sh) to be Sh. 7 million. The sales department has investigated four pricing plans. For each pricing plan, the marketing division has estimated the revenue per household in present worth Sh. to be:

plan	evenue per household (Sh.)
I	150
II	180
III	200
IV	240

The sales department estimates that the number of household subscribers would be approximately either 10,000, 20,000, 30,000, 40,000, 50,000 or 60,000.

Required:

(a) Construct a payoff table for this problem. (5 marks)

(b) What would be the company's optimal decision under the optimistic approach and the minimax regret approach? (8 marks)

(c) Suppose that the sales department has determined that the number of subscribers will be a function of the pricing plan. The probability for the pricing plans are given below:

Probability under pricing plan				
Number of subscribers	I	II	III	IV
10,000	0	0.05	0.10	0.20
20,000	0.05	0.10	0.20	0.25
30,000	0.05	0.20	0.20	0.25
40,000	0.40	0.30	0.20	0.15
50,000	0.30	0.20	0.20	0.10
60,000	0.20	0.15	0.10	0.05

Which pricing plan is optimal?

(d) Briefly explain the main difference between the approaches used in part (b) and (c) above.

JUNE 2002 QUESTION 7

(a) Define the following terms used game theory:
  (i) Dominance (2 marks)
  (ii) Saddle point (2 marks)
  (iii) Mixed strategy (2 marks)
  (iv) Value of the game (2 marks)

(b) Consider the two-person zero-sum game between players A and B given by the following payoff table

Player B Strategies

		1	2	3	4
Player A	1	2	2	3	-1
strategies	2	4	3	2	6

Required:

(i) Using the maximin and minimax values, is it possible to determine the value of the game? Give reasons. (3 marks)

(ii) Use graphical methods to determine optimal mixed strategy for player A and determine the value of the game. (9 marks)

(Total: 20 marks)

DECEMBER 2002 QUESTION 7

(a) The probability distribution showing the number of hours a specially trained accounts clerk is required to work daily is as follows:

Hours	Frequency
0	0.4
1	0.3
2	0.2
3	0.1

The accounts clerk is assigned to this work daily based on the expected daily requirement. In addition, customers pay Sh. 2,500 per hour for this service. The accounts clerk contracted wage is Sh. 1,500 per hour. If more hours are required than are specified in the contract, the cost is Sh. 3,000 per hour.

Required:

(i) Determine the hours the accounts clerk should be contracted.   (4 marks)
(ii) Calculate the net gain or loss to the company for each possible number of hours if the contract is written for the expected number of hours.   (6 marks)
(b) A student team was planning strategy for a management game being played in one of their classes. They had to decide whether to price their firm's products high or low. They know that their subsequent profits in either case depended on whether the economy moved up or down – a variable controlled by the instructor. They estimated that a high price in an upward economy would net Sh. 5 million in profit but a low price would yield Sh. 3 million profit. If the economy went down, a high price would net Sh. 2 million and a low price Sh. 1 million in profit.

Required:

Formulate the above information as a game and determine the students' best strategy.

(4 marks)

(c) A charity is about to launch an appeal for a children's ward in a national hospital and has to choose between the following fund raising strategies. Strategy I involves an initial outlay of Sh. 40,000 and administrative costs of 5 cents for each Sh. 1.00 collected. Strategy II has no initial outlay but instead involves an extensive door-to-door campaign with administrative costs of 30cents for each Sh. 1.00 collected. It is estimated that, whichever strategy is used, the amount collected will be as follows:
Amount (Sh.) 50,000   100,000 150,000 200,000
Probability   0.2 0.4 0.3 0.1

Required:

(i) Construct a payoff table   (2 marks)
(ii) Determine which strategy the charity should adopt   (4 marks)

(Total: 20 marks)

DECEMBER 2003 QUESTION 7

a) In the context of game theory, explain the following statement "winning isn't everything; it is the only thing" (3 marks)

b) The optimal solution of a two-person zero-sum game always represents a saddle point regardless of whether the players use pure or mixed strategies. Explain. (5 marks)

c) Joseph Njau and Anne Wairimu can use one strategy some of the time and the other strategy the rest of the time. Consider the following case of player one (Joseph Njau) and the opponent (Ann Wairimu). Joseph Njau has two strategies A and B whereas the opponent, Ann Wairimu has strategies X and Y. Utilities have been assigned as indicated below:

Ann Wairimu

	X	Y
A	1	-2

Joseph Njau

	X	Y
B	-15	2

Required:

Determine the percentage time Joseph Njau plays strategy A and strategy B. Also determine the percentage time that Ann Wairimu should play strategy X and Strategy Y. (12 marks)

(Total: 20 marks)

CHAPTER SEVEN

Mathematical
Programming

CHAPTER SEVEN
Mathematical Programming

▶ **OBJECTIVES**

    i.   At the end of this chapter, you should be able to:
    ii.   Define terms such as decision variables, objective and constraint function.
    iii.   Formulate a linear programming problem.
    iv.   Interpret a computer assisted solution.
    v.   Cite applications to transport and assignment problems.

Fast Forward: A business model is a framework for creating economic, social, and/or other forms of value.

▶ **INTRODUCTION**

There are many activities in an organisation which involve the allocation of resources. These resources include labour, raw materials, machinery and money. The allocation of these resources is sometimes referred to as programming. Problems arise because the resources are usually in limited supply. If a company makes several products, using the same machine and labour force, management must decide how many of each product to produce. The decision will be made so that management's objective is satisfied. Management may wish to plan production in such a way as to maximise the total contribution made each month, or to maximise the utilisation of the machinery each week, or to minimise the cost of labour each week. The decision variable in this case is the amount of each product to be made in a given time period.

Similarly, if the company has an amount of capital to invest in a number of projects, the money allocated to each project will be governed by some objective. It may be necessary to minimise risk, or to maximise capital growth. The decision variable in this case is the amount of money allocated to each project.

In general, the objective is to determine the most efficient method of allocating these resources to the variables so that some measure of performance is optimised. Modelling methods can often be used to aid in this allocation process.

Mathematical programming is the use of mathematical models and techniques to solve programming problems. There are a number of techniques which fall under mathematical programming but we will only consider the one which is most commonly used: linear programming.

7.1 LINEAR PROGRAMMING

Linear programming is a suitable method for modelling an allocation problem if the objective and the constraints on the resources can all be expressed as linear relationships of the variables. The techniques have a number of distinct steps.

The linear programme must first be formulated mathematically. This means that the variables over which we have control and the objective must be identified. The objective and the constraints on the resources are then written down as linear relationships in terms of the variables.

Once the linear programme is complete, all of the feasible combinations of the variables are identified. The combination which optimises the objective may then be selected. If only two variables are involved, a graphical solution is possible. If, however, we have a multivariable problem, we must resort to an algebraic method for which a computer package will be used.

Once the optimum solution has been identified, it must be evaluated. This will include a sensitivity analysis.

As with any other mathematical aid to decision making, the solution from the linear programme is just one piece of management information which contributes to the final decision.

▶ DEFINITION OF KEY TERMS

1. Linear programming - A branch of mathematics that uses linear inequalities to solve decision-making problems involving maximums and minimums; or, a mathematical procedure for minimising or maximising a linear function of several variables, subject to a finite number of linear restrictions.

2. Slack variable - In *linear programming* a slack variable is a variable which is added to a constraint to turn the inequality into an equation. This is required to turn an inequality into an equality where a linear combination of variables is less than or equal to a given constant in the former.

3. Surplus variable - In *linear programming* a surplus variable is a variable which is subtracted from a constraint to turn the inequality into an equation.

This is required to turn an inequality into an equality where a linear combination of variables is greater than or equal to a given constant in the former.

4. Basic variables – These are variables associated with unit columns of the Simplex Tableau. All other variables are called non-basic variables.

5. Feasible region - The set of all values which satisfy all constraints; otherwise it is an infeasible region

6. Sensitivity analysis - A technique of assessing the extent to which changes in assumptions or input variables will affect the ranking of alternatives.

7. Shadow price - is the maximum price that management is willing to pay for an extra unit of a given limited resource.

▶ INDUSTRY CONTEXT

Linear programming models are used to allocate scarce resources in a way which meets a business objective. The objective might be to maximize weekly profit or to minimize daily costs. The steps in formulating a linear-programming model are:

Step 1: Identify the decision variables

Step 2: Identify the linear objective function and the constraints

Step 3: Express the objective function in terms of the variables

Step 4: Express the constraints in terms of the variables

▶ EXAM CONTEXT

Student should be very prepared to tackle business modeling questions. In most cases the questions are not straightforward and this will require in-depth analytical skills. The past papers have, however, been outlined below for the student to practice adequately:

Past Paper Analysis

12/06, 6/06, 12/04, 6/04, 12/03, 6/03, 12/02, 6/02, 12/02, 6/02, 12/01, 12/00, 6/00

Problem formulation

The basic procedure is the same for the formulation of all linear programmes:

Step 1: Identify the variables in the problem for which the values can be chosen, within the limits of the constraints.

Step 2: Identify the objective and the constraints on the allocation.

Step 3: Write down the objective in terms of variables.

Step 4: Write down the constraints in terms of the variables.

The same procedure is used no matter how many variables there are, but we will look first at two variable problems.

▮ To formulate a two variable linear programme

Example 1

A small family firm produces two old-fashioned non-alcoholic drinks. 'Pink fizz' and 'Mint pop'. The firm can sell all that it produces but production is limited by the supply of a major ingredient and by the amount of the machine capacity available. The production of 1 litre of 'Pink fizz' requires 0.02 hours of machine time, whereas the production of 1 litre of 'Mint pop' requires 0.04 hours of machine time. 0.01 kg of a special ingredient is required for 1 litre of 'Pink fizz'. 'Mint pop' requires 0.04 kg of this ingredient per litre. Each day the firm has 24 machine hours available and 16kg of the special ingredient. The contribution is £0.10 on 1 litre of 'Pink fizz' and £0.30 on 1 litre of 'Mint pop'. How much of each product should be made each day, if the firm wishes to maximise the daily contribution?

Solution

Step 1: Identify the variables within the limits of the constraints. The firm can decide how much of each type of drink to make. Let p be the number of litres of 'Pink Fizz' produced per day. Let m be the number of litres of 'mint pop' produced per day.

Step 2: Identify the objective and the constraints. The objective is to maximise the daily contribution

Let £P per day be the contribution. This is maximised within constraints on the amounts of machine time and the special ingredients available.

Step 3: Express the objectives in terms of the variables:

P = 0.01 + 0.30m   (£/day)

This is the objective function – the quantity we wish to optimise.

Step 4: Express the constraints in terms of the variables. The contribution is maximised subject to the following constraints on production:

Machine time: to produce p litres of 'Pint fizz' and m litres of 'Mint pop' requires (0.02p + 0.04m) hours of machine time each day. There is a maximum of 24 machine hours available each day, therefore the production must be such that the number of machine hours required is less than or equal to 24 hours per day. Therefore:

0.02p + 0.04 m < 16 24 hours/day

Special ingredient: to produce p litres of 'pink fizz' and m litres of 'Mint pop' requires (0.01p + 0.04m) kg of the ingredient each day. There is a maximum of 16 kg available each day, therefore the production must be such that the amount of the special ingredient required is at most 16kg per day. Therefore:

0.01p + 0.04m < 16 kg/day

There are no further constraints on the production, but it is sensible to assume that the firm cannot make negative amounts of drink, therefore.

Non negativity:

P >0, m >0

The complete linear programme is as follows.

Maximise:

P = 0.10p + 0.30 m (£/day)

Subject to:

Machine time: 0.02 p + 0.04m < 24 hours/day

Special ingredient: 0.01p + 0.04m < 16 kg/day

 p,m >0

Example 2

A manufacturer of high precision machined components produces two different types, X and Y. In any given week, there are 4,000 man-hours of skilled labour available. Each component X requires one man-hour for its production and each component Y requires 2 man-hours. The manufacturing plant has the capacity to produce a maximum of 2,250 components of type X each week as well as 1,750 components of type Y. Each component X requires 2kg of plate. Each week there are 10,000 kg each of rod and plate available. The company supplies a car manufacturer with the unions that at least 1,500 components will be produced each week in total.

If the unit contribution for component X is £30 and for component Y is £40, how many of each type should be made in order to maximise the total contribution per week?

Solution:

First the linear programme must be formulated.

Step 1: Choose the variables: produce X components of type X per week and Y components of type Y per week.

Step 2: What is the objective? What are the constraints on the production? The objective is to maximise the total weekly contribution. The production is constrained by the amount of:

   a)   Labour – maximum available is 4000 hours per week
   b)   Machine capacity – there is a separate limit for each product. The machines can make at most 2,250 components of type X each week and at most 1750 components of type Y each week.
   c)   Rod – maximum available is 10,000kg per week
   d)   Plate – maximum available is 10,000kg per week

In addition, minimum amounts of each product are required:

   a)   Regular orders – the number of component X made must be at least enough to satisfy the regular orders
   b)   Union agreement – the total number of components (x +y) must at least satisfy the agreement.

Step 3: The objective function. Let £P be the total contribution per week, where:

P = 30 x + 40 y   (V/week)

Step 4: The constraints on production. For each resource, the amount of resource required each week to produce X components of type X and Y components of type Y is given below, together with the maximum amount of the resource available.

   Labour required: $1x + 2y \leq 4000$ hours/week

Machine capacity required: $x \leq 2250$ components/week

   $Y \leq 1750$ components /week

   Rod required: $2x + 5y \leq 10{,}000$ kg/week

   Plate required: $5x + 2y \leq 10{,}000$ kg/week

In addition

   Regular orders: $x \geq 600$ components/week

   Union agreement: $x + y \geq 1500$ components/week

   Non-negativity: $x, y \geq 0$

The complete linear programme is:

Produce x components of type X and Y components of type Y each week. Maximise:

   P = 30 x + 40 y   (£ /week)

Subject to the constraints:

Labour: $1x + 2y \leq 4000$ hours/week

   Machine capacity: $x \leq 2250$ components/week

Y ≤1750 components/week

Rod: 2x + 5y ≤ 10,000 kg/week

Plate: 5x + 2y ≤ 10,000 kg/week

Regular orders: x ≥ 600 components/week

Union agreement: x + y ≥1500 components/week

Non-negativity: x, y ≥ 0

We will look next at a problem involving more than two variables. The procedure is identical.

To formulate a multivariable linear programme

Example 1

Electra Plc produces personal computers and word processors. Currently four models are being produced.

The Jupiter – 512K memory, single disk drive;

The Venus – 512K memory, double disk drive;

The Mars – 640K memory, double disk drive;

The Saturn – 640K memory, hard disk

Each computer passes through three departments in the factory – sub, assembly, and test. The details of the times required in each department for each model are given in Table 1 together with the maximum capacities of the departments. The marketing department has assessed the demand of each model. The maximum forecast demands are also given in the table, together with the unit contribution for each model:

Table 7.1 Times required in each department for each model

Unit production time, hours Maximum available hours/month

Department	Jupiter	Venus	Mars	Saturn	
Sub-assembly	5	8	20	25	800
Assembly	2	3	8	14	420
Test	0.1	0.2	2	4	150
Maximum forecast Demand/month	100	45	25	20	
Contribution	15	30	120	130	

Construct the linear programme for this product mix problem, if the objective is to maximise the total contribution per month.

Solution

Step 1: Choose the variables produce:

Let

J units of the Jupiter per month

V units of the Venus per month

M units of the Mars per month and

S units of the Saturn per month

Step 2: What is the objective? What are the constraints on the production? The objective is to maximise the total contribution each month. The production is constrained by the number of the labour hours available in the three departments and by the number of each model which can be sold.

Step 3: The objective function. Let £p be the total contribution per month, where:

P = 15j + 30v + 120m + 130s   (£/month

Step 4: The constraints on production. For each department, the amount of time required to produce j, v, m and s units of the respective models is linked to the maximum time available.

Sub – assembly: $5j + 8v + 20m + 24s \leq 800$ hours/month

Assembly: $2j + 3v + 8m + 14s \leq 420$ hours/month

Test: $0.1j + 0.2v + 2m + 4s \leq 150$ hours/month

Demand for Jupiter: j ≤ 100 units/ month

Demand for Venus: v ≤ 45 units/month

Demand for Mars: m ≤ 25 units/month

Demand for Saturn: s ≤ 20 units/month

Non-negativity: j, .v, m, .s ≥ 0

The complete linear programme is each month produce j, v, m and s models, respectively of the Jupiter , Venus, Mars and Saturn. Maximize:

P= 15 j + 30 v + 120m + 130 s (£/month)

Subject to the constraints given above.

Example 2:

A portfolio manager wishes to invest up to £100,000 to maximise the total annual interest income. She has narrowed her choices to four possible investments, A, B, C and D. Investment A yields 6% per annum, investment B yields 8% per annum, investment C 10% per annum and investment D 9% per annum. The four investments have varying risks and conditions attached to them. For liquidity, at least 25% of the funds must be placed in investment D. Volatile government policies indicate that no more than 20% of the investments should be in C, whereas tax considerations require at least 30% of the funds to be placed in A. Formulate a linear programming model for this investment problem.

Solution:

Invest £a in investments, A, £b in investment B, £c in investment C and £d in investment D. The objective is to maximise the total interest income per year. The investment is constrained by the considerations of safety, liquidity, government policy and taxation. Let £R be the total interest income per year, where:

R = 0.06a + 0.08b + 0.10 c+ 0.09d (£/year)

Maximise subject to:

Total investment: a +b + c + d ≤ 100,000 £ available

Safety: a +b ≥ 0.5 (a +b + c +d) £

Liquidity: d ≥ 0.25 (a + b + c + d) £

Government policy: c≤ 0.2 (a + b +c +d) £

Tax: a≥ 0.3 (a + b + c + d) £

Non-negativity: a, b, c, d   ≥ 0

To solve a linear programme, it is conventional to arrange the constraints so that the variables appear only on the left hand side of the inequalities. This is done below. The complete linear programme is:

Invest £a in investment A,

£b in investment B

£c in investment C

And £d in investment D.

Maximise the total interest income per year, where:

R = 0.06a + 0.08 b + 0.10 c + 0.09 d   (£ /year)

Subject to the constraints:

Total investment: a + b + c + d ≤ 100.000 £

 Safety: 0.5 a + 0.5 b – 0.5c + 0.5 d ≥ 0 £

Liquidity: - 0.25 a – 0.25b – 0.5 c – 0.5 d ≥ 0 £

Government policy: - 0.2 a – 0.2 b + 0.8 c – 0.2 d ≤ 0 £

Tax: 0.7 a – 0.3 b – 0.3 c- 0.3 d   ≥0 £

Non-negativity: a, b, c, d ≥ 0

The objectives in the four examples above have all required a quantity to be maximised. The procedure is identical at this stage if the objective is to minimise some measure.

Solving the linear programme

We are now at the stage of considering how to find the values of the variables which will satisfy all of the constraints simultaneously and optimise the objective. We are more accustomed to dealing with equations rather than inequalities. It is a simple matter to change the inequalities into equations. We add an additional variable to the left hand side of the inequality.

To demonstrate this procedure, let us refer to example 2 which concerned the manufacturer of machine components X and Y. We will include an additional variable in each constraint to produce a set of equations. This variable is denoted by s hence s1 is included in the first constraint, s2 in the second constraint and so on. We will also impose the condition that the values of these variables cannot be negative i.e. $s1 \geq 0$ all $\geq$. This means that the variable is added to the left hand side for all $\leq$ constraints and subtracted from all $\geq$ constraints. The linear programme becomes: produce x components of type x and y components of type Y each week. The objective is to maximise the total weekly contribution.

Maximise

$P = 30x + 40y$ (£/week)

Subject to the constraints:

Labour: $1x + 2y + s_1 = 4000$ hours/week

Machine capacity: $x + s_2 = 2250$ components/week

$Y + s_3 = 1750$ components/week

Rod: $2x + 5y + s_4 = 10,000$ kg/week

Plate: $5x + 2y + s_5 = 10,000$ kg/week

Regular orders: $x - s_6 = 600$ components/week

Union agreement: $x + y - s_7 = 1500$ components/week

Non-negativity: $x, y \geq 0$

These additional variables are called **Slack variables**. In the $\leq$ constraints. They represent the amount of the resource not used, that is, the difference between the resource used and the maximum available. For example, look at the labour constraint above. Suppose 1,000 components of hours. Since 4,000 hours are available, the spare capacity, or slack is (4,000-3,000) = 1,000 hours. For this combination, the negative slack variables are referred to as **Surplus variables** since they represent the amount of resource being used over and above the minimum requirement. For example, look at the 'regular orders' constraint, when 1,000 components of type X are being produced. The minimum number of type X required by this constraint is 600, hence a production level of 1,000 gives a surplus of 400 components above the minimum. Therefore, s6 takes the value 400.

Start

We now have a set of simultaneous equations. However, the number of variables is greater than the number of equations. A unique set of solutions will arise only if the number of variables and the number of equations are the same. The best we can do is to identify a set of feasible solutions to the equations. This set of feasible solutions gives all combinations of the variables which satisfy all of the constraints. We will then select from this set the particular solution or solutions which optimise the objective.

How do we set about identifying the set of feasible solutions? This can be done graphically if the problem involves only two variables. However, we must resort to an algebraic method if the problem is multivariate.

The algebraic solution of a linear programme

A linear equation represents a set of points which lie on a straight line. A linear inequality represents an area of a graph. For example, $x \le 7$ says that X takes a value which is less than 7 or is equal to 7. The situation can be illustrated graphically as follows. Draw the line $x = 7$, see left hand graph in figure 12.1. This divides the graph into three sets of points for which $x = 7$, the line itself; those for which $x < 7$, the area to the left of the line; and those for which $x > 7$, the area to the right of the line. We do not require this last set. It is usual to shade the area not required. See, the right hand graph in Diagram 7.1 below

Diagram 7.1 Graphical representation of the inequality $x \le 7$
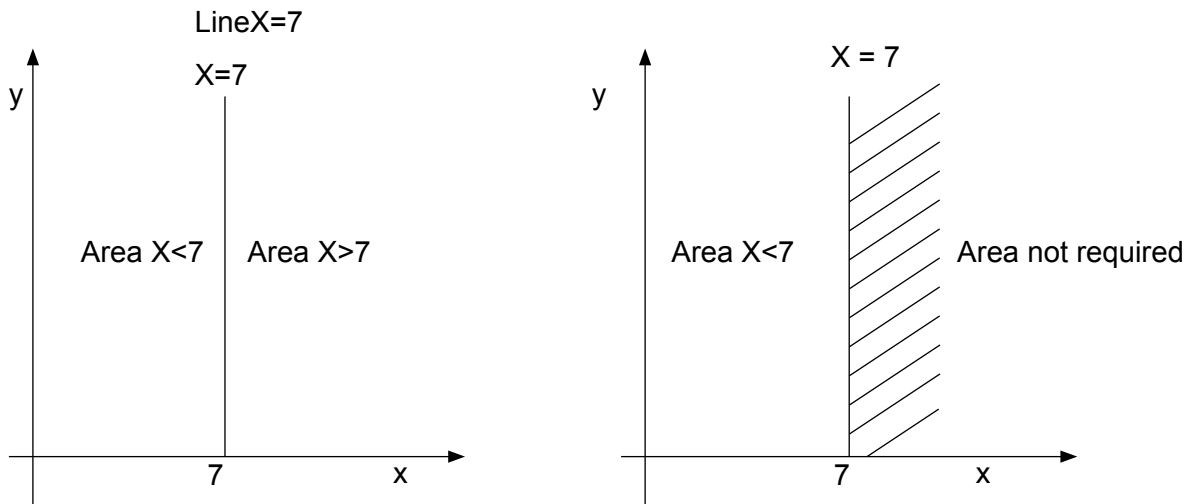
Suppose $x + y \le 10$, which area does this represent? The procedure is the same as in the previous example. First of all we draw the line $x + y = 10$. See the left hand graph below. Again the line divides the graph into three sets of points: those for which $x + y = 10$, the line; those for which $x + y < 10$, the area below the line; and those for which $x + y > 10$, the area above the line.

A useful technique for deciding which is the rejected area on the graph is to take any point on the graph away from the line and substitute its values into the inequality. If the inequality still makes sense, then that point is feasible solution. If the inequality is untrue, then the point is infeasible and lies in the rejected region. The origin is a convenient point to use. Substitute $x = y = 0$ into the inequality $x + y \le 10$, we have $0 + 0 \le 10$ which is a true statement, therefore the origin is a feasible solution and we should reject the other side of the line. See the right hand graph I figure 2

Diagram 7.2 Graphical representation of the inequality $x + y \le 10$
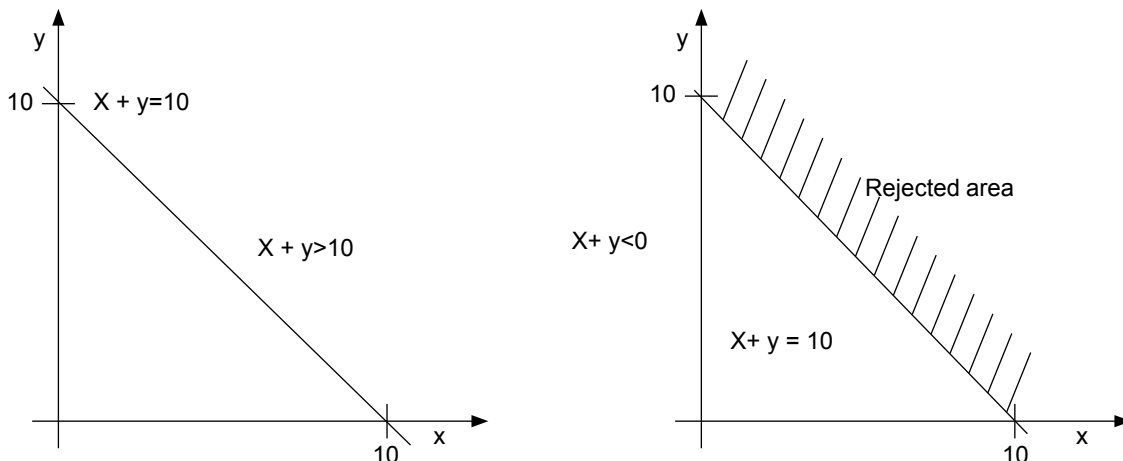
Each constraint in the linear programme can be drawn in this way and the rejected area shaded. If all of the constraints are drawn on the same graph, the area which remains unshaded is the set of points which satisfies all of the constraints simultaneously. This area is called the feasible region. For a linear programme, it does not matter which variable is plotted on which axis. The origin should always be included on the graph. False zeros must not be used. Let us now apply this procedure to the linear programme for example about the production of the two types of soft drink. We can illustrate the constraints graphically.

Machine time; $0.02p + 0.04 m \leq 24$ hours/day.

Plot the line $0.2p + 0.04 m = 24$. An easy way of plotting the line is to find the points where the line crosses the p and the m axes. Put p = 0 into the equation and calculate p, ie when m = 0, p = 1200. Plot these two points and join them to give the line. This method always works unless the line passes through the origin. In that case revert to the alternative procedure of substituting any other value of p and finding the equivalent value of m.

To find which side of the line to shade put p = 0 and m = 0 in the inequality:

$0.02 \times 0 + 0.04 \times 0 < 24$

This statement is true, so the origin is included in the feasible area

Special ingredient: $0.01p + 0.04m \leq 16$

Then you plot the line:

$0.01p + 0.04 m = 16$

Again the origin is included in the feasible region so we shade out the area above the line

Shade out negative values of each variable

Putting these four constraints together on one graph gives:

Diagram 7.3 Graphical representations of the constraints for example 1
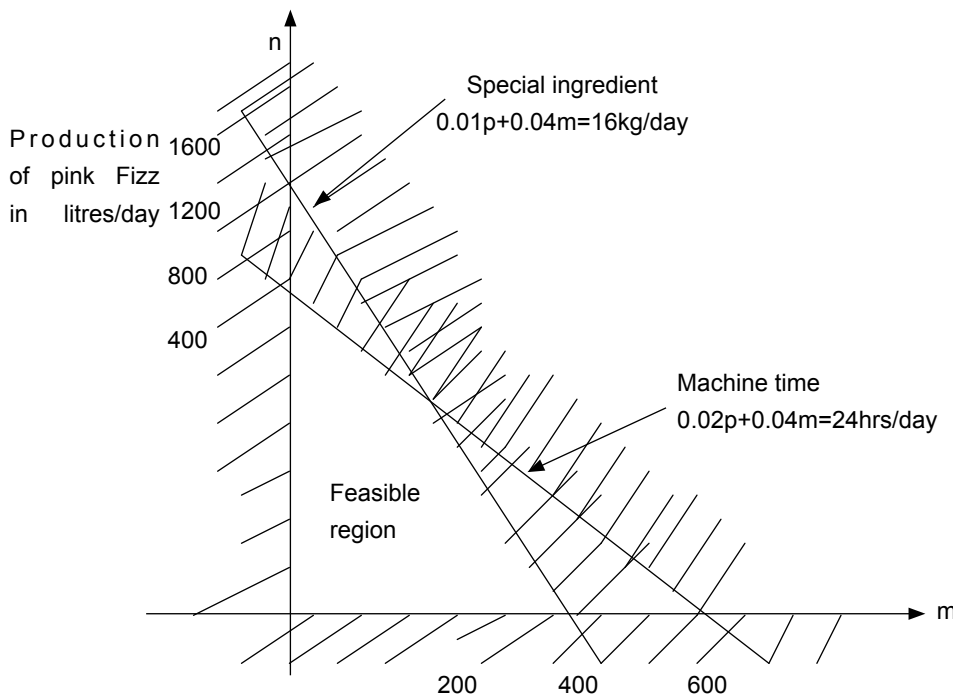
The area left unshaded by all of the constraints is the feasible region and this contains all of the possible combinations of productions which will satisfy the given constraints. The co-ordination of any point within the feasible region represents a possible combination of soft drink production for this firm.

We must now consider how to choose the production which will maximise the firm's daily contribution. The objective function is:

P= 0.01p + 0.30m (£/day)

If we like p = 100 £ per day, then we can illustrate the objective function graphically. If we then give p another value, the new line will be parallel to the one for p = 100 £ per day.

We can generate the entire family of possible contribution lines by drawing one in particular, then moving across the feasible region parallel to it. The further from the origin we move, the larger is the contribution.

If we draw a contribution line on the graph of the linear programme, as in figure 4 we can move parallel to this line across the feasible region in the direction of increasing contribution until we reach the last feasible solution(s), before the line moves into the infeasible region.

Diagram 7.4 Linear Programme For Example 1
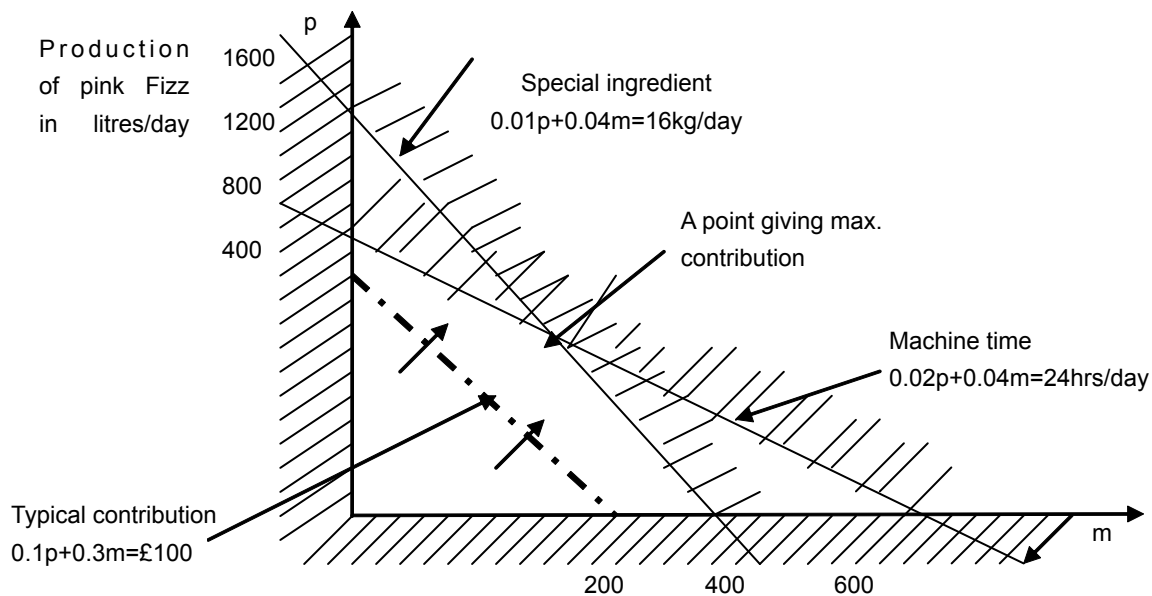
We can see the point A is the last feasible solution. The co-ordinates of point A give the optimum combination of production for the two drinks. The approximate co-ordinates of point A can be read from the graph, but, for precision, the co-ordinates are calculated by solving simultaneously the equations of the two constraints which form point A.

These two constraints are called the **binding** or **limiting constraints**. They are the resources which are being used fully and therefore prevent the daily contributions from increasing further. The optimum solution is the intersection of:

$$0.02p + 0.04 m = 24 \quad (1)$$

$$0.01p + 0.04m = 16 \quad (2)$$

Subtract (2) from (1)

$$0.01p = 8$$

Therefore:

P = 800 litres/day'

Substitute into (2) and find m:

x 800 + 0.04 m = 16

Therefore: m = 200 litres/day

To maximise the daily contribution, the firm should produce 800 litres of 'Pink Fizz' and 200 litres of 'Mint Pop' each day. This will yield a maximum contribution of:

0.10 x 800 + 0.30 x 200 = £ 140/day

This combination utilises all of the machine time and special ingredient available each day. There is no spare capacity or slack on either of the constraints.

This method of identifying the optimum corner depends on a suitable profit line being drawn.

The following is a practical note which will help to obtain a suitable profit line from which to identify the optimum corner. Choose any inconvenient point near the middle of the feasible region. Suppose in the above example the point m = 200, p = 200 is chosen. The daily contribution from this product mix is:

P = 0.10 p + 0.30 m = 0.10 × 200 + 0.30 x 200 = £80 /day

The other entire product mix which give daily contribution of £80 lie on the line:

80 = 0.10p + 0.30 m (£/day)

One point on this line is already known, i.e. m = 200, p = 200. A second point might be m = 0, hence, p = 800. This particular daily contribution line is now drawn on the graph and the procedure described above is followed to identify the optimum solution (s). It is clear from the procedure that the optimum will always be at a corner of the feasible region, or, if the objective function is parallel to one of the constraints, at any point on the line joining two corners.

We have assumed that the variables in the linear programme are continuous or, if not, then fractions are acceptable. It will often be the case that part units are allowed the time period of the problem. For example, if two models of a car are being produced and the objective of the linear programme is to maximise the machine usage per week. For such a product 'work in progress' is allowable on weekly basis.

If, however, we are allocating workers to task, part workers are not acceptable. In this case the optimum solution must produce integer values. The feasible solutions are all the points in the feasible region where the variables are integers. The last point within the feasible region which has integer coordinates is selected and this may no longer be at a corner of the feasible region.

For two variable linear programmes, it does not make much difference to the solution procedure if the variables must be integers. The feasible region is placed by the set of feasible points within the constraint boundaries. The typical objectives function is moved through these points, rather than through the feasible region as a whole. In the multivariable case, however, the method of integer programming is used.

Refer to Example 2 which is concerned with the production of two machined components. We wish to know the product mix which will achieve maximum total contribution per week.

Solution:

The feasible region for each constraint is as follows:

The feasible region, containing all of the possible product mixes for this problem, is shown unshaded.

We wish to identify the optimum product mix which will maximise the weekly contribution. The objective function is:

P = 30 x + 40 y   (£/week)

To plot this function for a typical value of the weekly contribution, we select the point x = 1000, y = 1000 which is in the feasible region. The weekly contribution for this product mix is:

P = 30 x 1000 + 40 x 1000 = £ 70,000/week

We will use the contribution line:

70,000 = 30x + 40y (£/week) as the trial line. This line also passes through the point x = 0,y = 1750. It is shown by the broken line on figure 5. In the direction of increasing contribution leads us to point A as the last feasible solution.

Diagram 7.5 A linear programme for the weekly production of machined components of type x and y
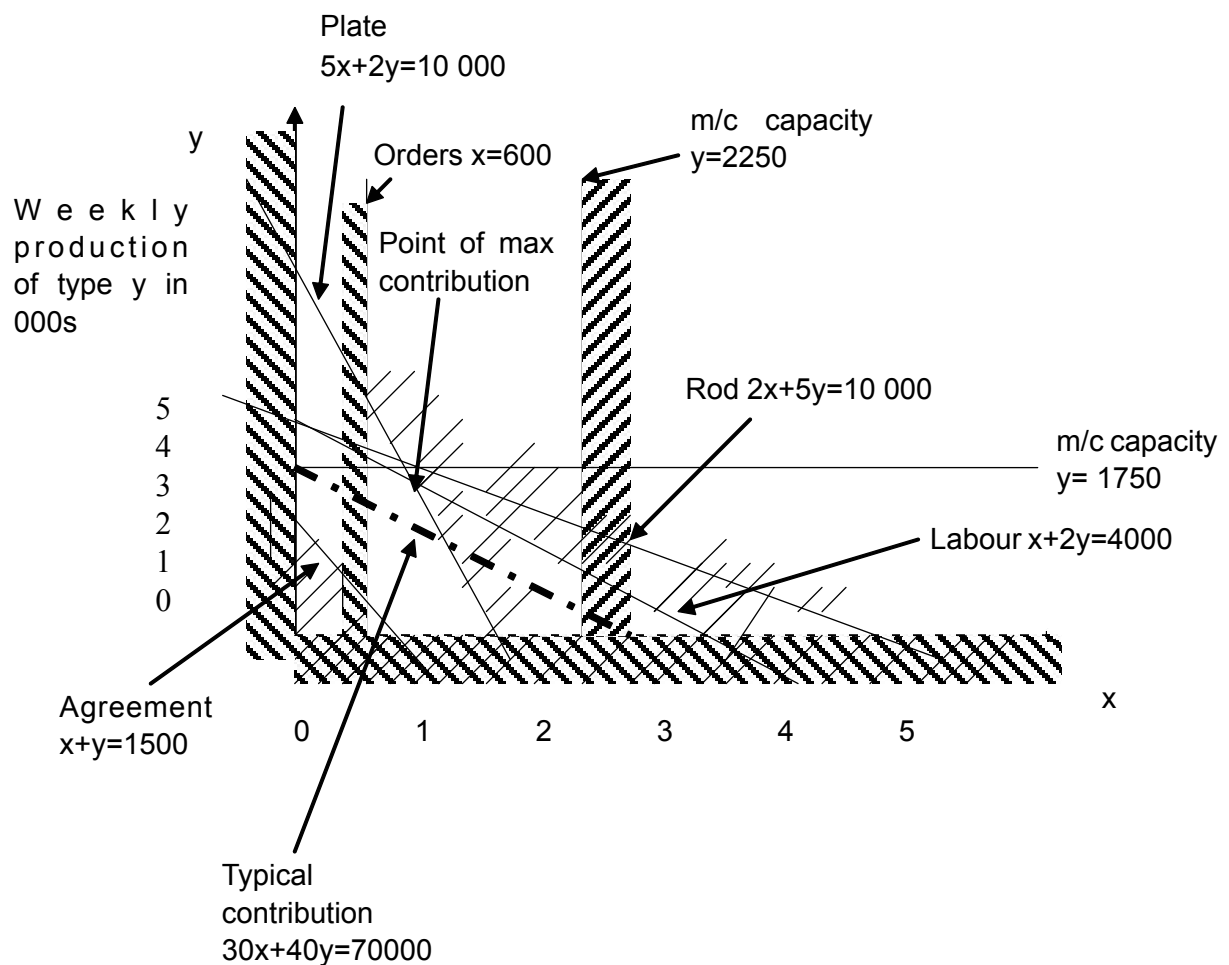
The binding constraints are therefore:

Labour: x + 2y ≤ 4000 hours/week

Plate: 5x + 2y ≤ 10,000 kg/week

Solving the corresponding equations simultaneously gives

X + 2y = 4000   (1)

5x + 2y = 10,000   (2)

(2) – (1) 4x   = 6,000

Therefore x = 1250, and y = 1500 by substitution.

The optimum product mix is 1,500 of component x and 1,250 of component y each week. The maximum contribution per week will then be:

P max = 30 x 1500 + 40 x 1250 = £95,000/week

This product mix uses all of the labour hours available and all plate. These are the binding constraints. However, there will be spare capacity on machine time for both components and spare rod capacity. The production will also exceed the minimum required by the regular orders and the minimum required by the union agreement.

We find the value of the slack variables in the machine time constraints are 750 of component x and 500 of machine tool y, i.e.

1500 + s2 = 2250, therefore s2 = 750 components/week, and

1250 + s3= 1750, therefore s3 = 500 components/week

The slack in the rod constraint is:

2x 1500 + 5 x 1250 + s = 10,000

Therefore s = 750 kg/week

The surplus on the regular orders constraints is:

1500 – s6 = 600

Therefore: s6 = 900 components/week

Above the minimum needed for the regular orders. The surplus on the agreement is:

1500 + 1250 – s7 = 1500

Therefore s7 = 1250 components/week

Above the minimum required by the union agreement.

As we have already said, the optimum solution will normally be at a corner of the feasible region. It is possible, therefore, once the graph has been drawn to identify the optimum corner by evaluating the objective function at each corner of the feasible region. A **basic solution** is the name given to the set of variable values at a corner of the feasible region. The basic variables are those variables which have non-zero values at a particular corner.

Occasional problems arise when solving a linear programme. The problem may be infeasible. In this case, there is no feasible region. No combination of the variables satisfies all of the constraints simultaneously and the linear programme is unbounded. In this case, the solution can be increased indefinitely without violating constraint. This usually means that the linear programme is formulated incorrectly, with some constraints missing.

The issue of non-unique solutions was mentioned earlier. These arise when objective function is parallel to a binding constraint. Any point on that constraint between the two optimum corners, will give the optimum value of the objective function. Any one of these points forms an optimum solution to the model. This can be useful situation since it gives the decision maker some flexibility.

Sensitivity Analysis

In most decision making activities it is prudent to maximise the preferred course of action to see what effect changes in the problem will have on the decision. Linear programming is no exception. There are three aspects of the problem which we need to consider.

The effect of additional supplies of the limiting resources;

The effect of changes in the non-limiting resources;

The effect of changes in the coefficients of the objective function.

How do changes in the non-limiting resource affect the optimum solution?

In example 2.6 we considered the two limiting constraints of labour and plate. The other constraints are not binding at the original optimum solution. These constraints are:

-Machine time to produce component X

-Machine time to produce component Y

-Rod

-Regular orders

-Union agreement

What happens as each of these constraints is charged?

The first three are less-than-or – equal – to constraints. Any of their availabilities will not affect the optimum solution. However, any decrease in these three constraints can affect the solution. The tightening of one of the non-limiting constraints will cause it to move towards the origin. At first, the only change will be a reduction in the size of the feasible region. When, however, the particular constraint passes through the original optimum corner. It will itself become limiting and a new optimum solution will emerge.

It is useful to know what the lower limits are on these constraints. The machine capacity for X can be reduced by 750 hours, from 2,250 to 1,500 hours, before it affects the solution. The machine capacity for Y can be reduced by 500, from 1,750 to 1,250 hours. The supply of rod can be reduced by 750 per week, from 10,000kg to 9,250kg. These reductions are the values of the slack variables mentioned earlier. The greater- than- or-equal to constraints, for the regular orders and the union agreement, act in the opposite way. Any reduction in these requirements will increase the feasible region but will not affect the optimum solution.

Any increase in these constraints will first reduce the feasible region but will not affect the optimum solution. If the regular orders for X increase by at least 90 to 1,500, the optimum will begin to change. If the union's agreement was increased by at least 1,250, to more than 2,750, there would be no feasible region and no solution. These increases are the surplus referred to earlier.

How do changes in the coefficients of the objective function affect the optimum solution?

It is inevitable that the circumstances under which a linear programme is formulated will change. Major changes will probably mean that the work will have to be done again but may be possible to identify the effect of minor changes from the solution to the original problem. In this section, we consider changes to the objective function. If the objective is to maximise weekly contribution, a change in the cost of raw material will alter the coefficients in the objective function.

In an investment portfolio problem, if the objective is to maximise the annual return on the investments, a change in the interest rate earned on one of the investment will change that coefficient in the objective function.

We will consider situations when the coefficient change is one at a time.

Suppose:

P = ax + 4y (£/week) represents the objective function for a profit maximising linear programme, where £4 per unit is the profit on product Y and £a per unit is the profit on product X. The profit on product X is liable to change. Suppose this linear programme has been graphed with X and Y in the conventional directions. It is helpful to re-arrange the objective function so that y is the subject:

Y = p/4 – (a/4) x

The profit line cuts the Y axis at p/4 and the slope of the profit line is – (a/4). The intercept on the Y axis is independent of the value of a, but the slope of the line increases as an increase, and decreases as a decrease. In other words, as the value of slope change, the profit lines rotate. Small rotations in either direction will not usually alter the optimum corner. However, larger rotations will result in different corners emerging as the optimum. It is useful to know the range which 'a' can take before a particular corner ceases to be the optimum. A similar argument would apply if the coefficient of x was fixed and the coefficient of y was liable to change.

The simplex solution of multi-variable linear programmes.

An algebraic solution method is required if a linear programme contains more than two variables. The basic principle of solution of multi-variable model is very simple. It is assumed that the optimum solution is at one of the 'corner' of the feasible region. Therefore, we systematically evaluate the objective function for each corner until we find the one which gives the optimum value of the objective function. We employ the techniques of matrix algebra and an algorithm for moving from corner to corner of the feasible region in such a way that a move is made only if it improves the value of the objective function. If, at a particular basic solution, no further move is recommended then we know that the optimum solution has been reached. This algorithm is called the **Simplex method**. A detailed explanation of the simplex method is not necessary, since multi-variable linear programming models are normally solved by using one of the many computer packages which are readily available for this purpose. However, an understanding of the basic principles of the method is helpful for fully interpreting and evaluating the solution to a linear programme which has been obtained by computer package.

The basic simplex method assumes that the linear programming model is a maximising one, subject to a set of ≤ constraints. This means that the algorithm can take the origin as the initial corner. The search for the optimum always starts from a zero value for the objective function.

The simplex method can be adopted for minimising problems and for problems with ≥or = constraints. This involves the introduction of artificial, as well as slack and surplus, variables. We have omitted these complications, since the problems are usually solved by a computer package which automatically introduces these variables into the model.

The basic model with which we will work may be formally written as:

Maximise Z = c1 x1 + c2 x 2 + …… + c n x n

The C1 are constants. This is maximised subject to a set of m linear constraints:

a_{11} x_1 + a_{12} x_2 +a_{13} x_3 + …… a_{1n} x_n ≤ b_1

a_{21} x_1 + a_{22} x_2 +a_{23} x_3 + …… a_{2n} x_n ≤ b_2

a_{31} x_1 + a_{32} x_2 +a_{33} x_3 + …… a_{3n} x_n ≤ b_3

a_{m1} x_1 + a_{m2} x_2 +a_{m3} x_3 + …… a_{mn} x_n ≤ b_m

 x ≥ 0

There are n variables and m constraints. The double subscripts for the coefficients in the left hand side of the constraints refer first to the constraints, then to the variable. For example, + a32 is in constraint 3 and is the coefficient of the variable x2. We will illustrate the use of the simplex method by considering a simple two variable problem which we will first solve graphically. This will enable us to compare the graphical and algebraic solutions.

Interpretation of computer generated solution

Example

Maximise $25x_1 + 20x_2 + 24x_2$

where: x_1 = Xtrgrow, x_2 = Youngrow, x_3 = Zupergrow

Subject $0.3x_1$ $+0.2x_3$ ≤ 500

$0.5x_2 + 0.4x_3$ ≤ 1000

$0.2x_1 + 0.1x_2 +0.1x_3$ ≤ 800

$0.4x_2\ 0.3x_3$ ≤ 600

x1 ≥ 1500

x_1 ≥ 0, x_2 ≥ 0, x_3 ≥ 0

The computer generated solution for this problem is as follows;

Objective value = **71666.7**

Variable	Value	Obj. Coeff	Obj Value Contribution
X1: Xtragrow	1666.7	25	4166.7
X2: Youngrow	1500	20	30000
X3: Zupergrow	0	24	0

Constraint	RHS	Slack-/ Surplus+
1(<)	500	0
2(<)	1000	250-
3(<)	800	316.67-
4(<)	600	0
5(>)	1500	166.7+

Sensitivity Analysis

Variable	Current obj coeff	Min obj coeff	Max Obj Coeff	Reduced cost
X1: Xtragrow	25	13.50	Infinity	0
X2: Youngrow	20	9.78	Infinity	0
X3: Zupergrow	24	-Infinity	31.67	7.67

Constraint	Current RHS	Min RHS	Max RHS	Dual price
1(<)	500	450	975	83.33
2(<)	1000	750	Infinity	0
3(<)	800	483.3	Infinity	0
4(<)	600	0	800	50
5(>)	1500	-Infinity	1666.7	0

Required

Interpret the data generated by the the computer.

Solution.

Table 1:

Objective value, is the solution to objective function (e.g the solution to this example is 71,436)

The four columns of table 1 are to be interpreted as follows;

- Variable: these are the variables of the model. In our example we have x_1 = Xtragrow, x_2 = Youngrow and x_3 = Zupergrow

- Value: this is value that the variables assume at optimal solution (to optimise the objective function one needs to produce this amounts of the variables). In our example we are required to produce 1,666.67 of x1 and 1,750 of x2 and none of x3
- Objective coefficients: these are the coefficients of the objective function
- Objective value contribution: this is the value contributed by each variable to the objective function (for x1=25×1,666.67), the total of this is equal to our objective value (i.e 41,666.67+35,000=76,666.67).

The 3 columns of the second part of table1 can be interpreted as follows;

- Constraints: this is constraints of the model representing the limited resources.
- RHS: the Right hand side value is the limiting value of the constraint. e.g for the first constraint the maximum amount of material A is 500 tons.
- Slack/surplus: at optimal production not all the materials for some of the constraints will be fully utilised, slack is the amount of material that is left over after production. For constraint 1 and 4 no material remained; this also implies that these are the binding constraints i.e their adjustment directly affects the objective solution

To illustrate the use of the simplex method.

A firm manufactures two products, X and Y, subject to constraint on three raw materials, RM1, RM2, and RM3. The objective of the firm is to select a product mix which will maximise weekly profit.

The linear programme for the problem is:

Produce x units of product x per week and y units of product y per week.

Maximise the weekly profit, £P, where P = 2x =Y (£/week)

Maximise subject to:

RM1: 3x≤ 27 kg/week

RM2: 2y≤ 30 kg/week

RM3: x +y≤ 20 0kg/week

x, y ≥ 0

Determine the optimum product mix and the maximum value of the weekly profit. State the spare capacity on each resource.

Solution

Simplex method arranges the coefficients in the left hand side of the constraints equations in a matrix format. Label the columns with the name of the variables to which they refer. Put the right hand side values of the constraints in a separate column on the right of the matrix label the rows with the names of the variables which are basic (have --- values) at the initial corner (the origin). Finally, add the objective function as an addition row to the table. There are several slightly different of the objective function to be entered as negative values. The resulting matrix is called coefficients of the objective function to be entered as negative values. The resulting matrix is called the first simplex tableau. The above procedure is step 1 in the logarithm

Table First simplex Tableau

Basic variables   variables RHS

	X	y	S_1	S_2	S_3	b
S_1	3	0	1	0	0	27
S_2	0	2	0	1	0	30
S_3	1	1	0	0	1	20
Objective Function, p	-2	-1	0	0	0	0

Step 2: Find the largest negative value in the objective function row (-2). The corresponding column is called the pivot column x. Divide the right hand side values (in the b column) by the corresponding numbers in the pivotal column. This produces a set of ratios.

Basic variables			variables			RHS	ratios
X	y	S_1	S_2	S_3	b		b/pivotal column element
S_1 3	0	1	0	0	27		27/3=9 ◄——— pivotal row
S_2 0	2	0	1	0	30		30/0 = ∞
S_3 1	1	0	0	1	20		20/1 = 2<u>0</u>
Objective	-2	-1	0	0	0		0

Function, p

Step 3: Choose the smallest positive ratio, 9. The corresponding row, s, is the pivotal row. The intersection of the pivotal column, x and the pivotal row s1, is the pivotal element 3, marked by in table in step 2 above.

Step 4: Divide all of the elements in the pivotal row by the element, 3 replace the pivotal row by this new row in Table below. Replace the row label, s1 by the label from the pivotal column, x. the new row labels are the basic variables for the second basic solution.

Step 5: Using arithmetic operations on the rows (row operations in matrix algebra), reduce all of the other elements in the pivotal column, x, to zero. These arithmetic operations must use only the pivotal row as the basis.

R1 denotes the ith with the notation new R3 = Old R3 – New R1 means that the new row 3 is obtained by subtracting the new pivotal row (row 1) from the old row 3. The operations used are listed at the right hand side of Table below.

Table Second Simples Tableau

Basic variables	Variables					RHS	Ratios
	X	y	S_1	S_2	S_3	b	b/ pivot column element
S_1	1	0	1/3	0	0	9	New R_1 = Old R_1 pivotal element(3)
S_2	0	2	0	1	0	30	New R_2 = Old R_2 – 0 X New R_1
S_3	0	1	-1/3	0	1	11	New R_3 = Old R_3 – 1 New R_1
Objective function, p	0	-1	2/3	0	0	0	New P = Old P – (-2) x New R

Step 6; repeat steps 2 to 5 until all of the elements in the objective function row are zero or positive.

Table: Second simplex tableau with ratios

Basic variables	Variables					RHS	Ratios
	X	y	s_1	s_2	s_3	b	b/ pivot column element
s_1	1	0	1/3	0	0	9	9/0 = 00
s_2	0	2	0	1	0	30	30/2 = 15
s_3	0	1	-1/3	0	1	11	11/1 = 11 ⟵ pivotal row
Objective function, p	0	-1	2/3	0	0	18	New P = Old P – (-2) x New R
		Pivotal column y					

Table Third and final simplex tableau

Basic variables	Variables					RHS	Ratios
	X	y	s_1	s_2	s_3	b	b/ pivot column element
s_1	1	0	1/3	0	0	9	New R_1 = Old R_1 pivotal element(3)
s_2	0	0	2/3	1	-2	8	New R_2 = Old R_2 – 0 X New R_1
s_3	0	1	-1/3	0	1	11	New R_3 = Old R_3 – 1 New R_1
Objective function, p	-0	0	1/3	0	0	29	New P = Old P – (-1) x New R_3

All values in the objective function row are now positive or zero, therefore this tableau represents the optimum solution.

To interpret the final simplex tableau, we will look at the values around the edge of the table first.

Table Partial final simplex tableau

Basic variables	Variables					Right hand side , b
	X	y	s_1	s_2	s_3	
X						9 = value of x
s_2						8 = value of slack or RM2
Y						11 = Value of y
Objective function, p	0	0	1/3	0	1	0

Cost of forcing 1 unit of a non basic variable into the solution

Shadow prices

The basic variables are those which are non-zero at the optimum corner. The values of the basic variables are in the corresponding row of the B column. Therefore:

X = 9 units/week

Y = 11 units/week

And the slack for raw material 2 is 8kg per week.

All other variables are zero that is the slack on constraint 1, s1 and constraint 3, s3   are zero. This means that these constraints are binding and the available raw material 1 and 3 are fully used. The optimum value of the objective functions is in the 'b' column of the objective function row. The maximum value of weekly profit is £ 29. This solution corresponds exactly with the graphical solution. The figures in the objective function row and the slack variable columns shown in Table above give the shadow prices. The shadow price on constraint 1, RMI, is £ 1/3 per kg and the shadow price for constraint 3 is £1 per kilogramme of RM1 becomes available; the weekly profit will increase by 33 pence (less any additional costs above the normal cost of RM1). Similarly if an extra kilogram of RM3 becomes available, the weekly profit will increase by £ 1 (less any additional costs). The figures for the shadow prices may be checked from the graphical solution.

We will look at constraint 1 only to illustrate the point.

Constraint 1, for raw material 1 is 3x = 27 kg per week, if this constraint is relaxed by one kilogramme, 3x = 28. The optimum corner will still be the intersection of constrains 1 and 3. Look back at the graph to check this. The new optimum corner has the co-ordinates:

X = 28/3 = 9 1/3

And 28/3 + y = 20

Giving:

Y = 32/3 = 10 2/3

The new value of the maximum weekly profit is:

2 x (28/3) + (32/3) = 88/3 = £ 29.33/ week

This is again 33 pence for one kilogramme

The shadow price for RM1 is 33 pence kilogram. The remaining values on the page on the edge of the final tableau, are those in the objective function row and the variable columns. In this example, the value in both the x and the y columns are zero. These numbers will be non- zero if any of the variables is non-basic in the optimum solution. For example, if the optimum solution had said that we should produce only product X, then Y would be non-basic, is y = 0. In that case, the figure in the Y column of the objective function row tells us by how much the maximum value of the objective function would decrease if we insisted on producing one unit of Y.

Suppose the following had been the final tableau for the current problem:

Table 7.3 Modified final tableau

Basic variables	Variables					Right hand side , b
	X	y	s_1	s_2	s_3	
x						9
s_2						8
s_3						11
Objective function, p	0	0.5	1/3	0	0	18 = value of maximum profit

In this solution, the optimum produces 9 units of x and none of y. If we feel that we must produce some units of y, the value of the objective function will decrease by £0.5 for each unit of y which is produced.

Shadow or dual prices

Definition: A shadow price or a dual price is the amount increase (or decrease) of the objective function when one more (or one less) of the binding constraints is made available.

Maximise $4X_1 + 3X_2$

Subject to: $0.5X_1 + 0.33X_2 \leq 12$ (Machine hours)

$0.5X_1 + 0.5X_2 \leq 14$ (labor hours)

Starting with machine hours; let's assume that one more machine hour is available (with labour hours remaining constant)

We get:

$0.5X_1 + 0.33X_2 = 13$

$0.5X_1 + 0.5X_2 = 14$

Solving this simultaneously we get the values of X_1 and X_2 as

$0.17 X_2 = 1$

$X_2 = 5.88$

$X_1 = 22.12$

Thus the contribution is

$4(22.12) + 3(5.88) = Sh.106.12$

Comparing this with its original contribution of Sh100.24 (see example 1) we see increasing machine hours by one unit has increased contribution by Sh5.88, which is the shadow price per machine hour.

Note: This figure is also arrived at if we assume that machine hours are reduced by 1 unit ie 12-1.

Similarly assuming that one more labour hour is made available, then contribution change is:

$0.5X_1 + 0.33X_2 = 12$

$0.5X_1 + 0.5X_2 = 15$

Solving this simultaneously gives:

0.17 X2 =3

X2 = 17.65

X1 = 12.35

Which give a contribution of:

4(12.35) + 3(17.65) = Sh102.35

The contribution change is Sh2.11 which is the shadow price per labour hour.

Note:

The shadow prices apply in so far as the constraint is binding; for example, if more and more labour hours are available, it will reach a point where labour hours are no longer scarce thus labour hours cease to be a binding constraint and its shadow price becomes a zero. (All non-binding constraints have zero shadow price). Logically, it is senseless to pay more to increase a resource, which is already abundant.

Interpretation of shadow prices

A shadow price of a binding constraint indicates to management how much extra contribution will be gained by increasing a unit of the scarce resource.

In the example above, Sh2.11 is the shadow price for labour hours. This implies that management is ready to pay up to Sh2.11 extra per hour for the extra hours i.e. say an employee is paid Sh5 per hour and one day he works for two hours extra (overtime), the management is prepared to pay up to Sh7.11 per hour for the two hours overtime worked.

▧ Sensitivity analysis and the simplex method

The final tableau of the simplex algorithm can be used to carry out a sensitivity analysis of the solution of a linear programming model. For limiting constraints, the values in the corresponding slack variable column represent the change in the values of the basic variables if one additional unit of the limiting resource is available.

Example

To illustrate the use of the final simplex tableau for a sensitivity analysis of the limiting constraints, we will use the final simplex tableau of Example above to determine:

The effect on the optimum solution if one additional kilogramme of RM1 becomes available

The effect on the optimum solution if two additional kilogrammes of RM1 become available

The effect on the optimum solution if five additional kilogrammes of RM3 become available

The maximum number of additional kilogrammes of RM3 which can be used without spare capacity being created

The effect on the optimum solution if two fewer kilogrammes of RM1 are available

Solution

The linear programme and the final simplex tableau are now produced

Maximise the weekly profit, £P, where P = 2x + y   (£/week)

Subject to RM1: 3x   ≤ 27 kg/week

    RM2: 2y ≤ 30 kg/week

    RM3: x +y   ≤ 20 kg/week

    x, y ≥ 0

Table 7.4 Final simplex tableau

Basic variables	Variables					Right hand side , b
	X	y	s1	s2	s3	
x	1	0	1/3	0	0	9
s2	0	0	2/3	1	-2	8
y	0	1	-1/3	0	1	11
Objective function, p	0	0	1/3	0	1	29

If one additional kilogramme of RM1 is available, this limiting constraint is relaxed by one kilogramme. The values in the s1 column are the changes in the basic variables which result from this relaxation. The final tableau is re-written below with only the relevant values and calculations shown.

Table 7.5 partial modified final simplex tableau (1kg extra of RM1)

Basic variables	Variables					Right hand side , b
	X	y	s1	s2	s3	
X			1/3			9 + (1/3) = 9 1/3
s2			2/3			8 + (2/3) = 8 2/3
y			- 1/3			11 – (1/3) = 10 2/3
Objective function, p			1/3			29 + (1/3) = 29 1/3

One additional kilogramme of RM1 causes the value of x to increase by 1/3 of a unit, the slack for RM2 to increase by 2/3 of a kilogramme, the value of y to decrease by 1/3 of a unit and the maximum value of the weekly profit to increase by £1/3 i.e. by the shadow price. The new optimum solution requires 9 1/3 of product x and 10 2/3 of product y to be produced each week. The slack on constraint 2, the amount of raw materials not used, is 82/3 kg. The other variables are zero. The slack on constraints 1 and 3 is zero; therefore all RM1 and RM3 are used.

If two additional kilogrammes of RM1 are available, this limiting constraint is relaxed by two kilogrammes. The values in the s1 column are multiplied by two. The resulting values are then the changes in the values of the basic variables which arise from the additional two kilogrammes. The final tableau is shown in table below with only the relevant values and calculations shown.

Table 7.5 Partial modified final simplex tableau (2kg extra of RM1)

Basic variables	Variables						Right hand side , b
	X	y	C	s1	s2	s3	
x			1/3 x2				9 + (2/3) = 9 2/3
s2			2/3 x2				8 + (4/3) = 8 1/3
y			- 1/3 x2				11 – (2/3) = 10 1/3
Objective function, p			1/3×2				29+ (2/3) = 29 2/3

The new optimum solution requires 9 2/3 of product x and 10 1/3 of product y to be produced each week. The slack on constraint 2 is 9 1/2kg. The other variables s2 and s3 are zero. This means that these constraints are binding. The maximum value of weekly profit is £29.67. This solution can be illustrated in the same way as 1 above.

If five additional kilogrammes of RM3 are available, this limiting constraint is relaxed by five kilogrammes. The values in the s3 column multiplied by five kilogrammes are shown in the modified final tableau below

Table 7.6

Basic variables	Variables					Right hand side , b
	X	y	s1	s2	s3	
x			0 x5			9 + (0) = 9
s2			-2 x5			8 + (-10) = - 2
y			- 1 x5			11 + (5) = 16
Objective function, P			1×5			29 + (5) = 34

A problem has now arisen. The value of the slack variable, s2 for raw materials 2, has become negative. This is not allowed, since variables must always be positive or zero. If we consult the graphical solution we can see at once what has happened. The RM3 constraint has been relaxed so far that it is no longer limiting. The tableau gives a point outside the feasible region. We are not able to use all the extra five kilogrammes of RM3. This problem is discussed further in section 4.

The RM3 constraint is represented by the s column of the final tableau. The only negative value in the s column is the marginal value of s2 which is –2. As RM3 is relaxed, the value of s2 decreases by 2, but it cannot be negative. When s2 reaches zero, the limiting position for the RM3 constraint will occur. Suppose this limiting position is reached when the RM3 constraint has been relaxed by r kg, the value of s2 will then be zero, therefore:

$8 + (-2 xr) = 0$

Which means that r = 4kg. The RM3 constraint may be relaxed by 4kg from 20 to 24kg before it ceases to be binding. This takes it to intersection of the RM1 and RM2 constraints where x = 9, y =15 and the RM3 constraint is x +y =24

If fewer kilogrammes of RM1 are available, this limiting constraint is tightened by two kilogrammes. The values in the s column are multiplied by two. The resulting values are deducted from the values of the basic variables. The final tableau is re-written below.

Table 7.7

Basic variables	Variables					Right hand side , b
	X	y	s_1	s_2	s_3	
X			0 x 5			9 - (2/3) = 8 1/3
s_2			-2 x 5			8 - (4/3) = 6 2/3
y			- 1 x 5			11 – (- 2/3) = 11 2/3
Objective function, p			1/3×2			29 - (2/3) = 28 1/3

The new optimum solution requires 8 1/3 of product x and 11 2/3 of products y to be produced each week. The slack on constraint 2 is 6 2/3kg while the slack on constraints 1 and 3 is zero. This means that these constraints are binding. The maximum value of weekly profit is £ 28.33.

This analysis is tedious to do by hand using the simplex method, even with the simplest two variable models. All commercial standard linear programming computer packages will provide the information. This is how multivariable sensitivity analysis is done in practice. The principles are exactly the same as those for the two variable models we have just completed.

The dual linear programming model

The dual linear programming is used to investigate a problem from a different perspective to the one obtained from the usual primal model. The primal and dual models give the same solution and the same sensitivity information. The only reason for using one rather than the other is that computationally one may be easier to solve. With increasingly powerful computer packages the need for the primal/dual switch is becoming less relevant. The variables in the dual model are the shadow prices of the original, or primal, model. The structures of dual and primal models are similar. If the primal model has been built, the corresponding dual model is derived from it. In general, a linear programming problem can be described by:

Maximise $Z = c_1 x_1 + c x + \ldots + c_n x_n$

Subjects to a set of m linear constraints

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots a_{1n} x_n \geq b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots a_{2n} x_n \geq b_2$

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \ldots A_{3n} x_n \geq b_3$

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots a_{mn} x_n \geq b_m$

$x_i \geq 0$

The above linear programme maximises and has all $\leq$ constraints. Any linear programming model can be put into this form and converted into its dual, as we show below. The dual model is:

Minimise G = b1 y1 + b2 y2 + …. bm ym

Subject to a set of n linear constraints

$a_{11}\, y_1 + a_{21} y_2 + a_{31}\, y_3 + \dots a_{m1}\, x_m \leq c_1$

$a_{12}\, y_1 + a_{22}\, y_2 + a_{32}\, y_3 + \dots a_{m2}\, x_m \leq c_2$

$a_{13}\, y_1 + a_{23}\, y_2 + a_{33}\, y_3 \dots a_{m3}\, x_m \leq c_3$

$a_{1n}\, y_1 + a_{2n}\, y_2 + a_{3n}\, y_3 + \dots a_{mn}\, x_m \leq c_n$

   $y_i \geq 0$

There are m dual variables y, one for each of the m primal constraints, and n constraints, one for each of the x variables in the primal. The coefficients c, in the primal objective function and the right hand side values, b, of the constraints in the primal constraints are interchanged row to column on the dual. The dual variables, y are the shadow prices in the minimal problem and vice versa. In this case the dual minimises, the primal maximises. The primal has $\leq$ constraints and the dual $\geq$ constraints.

To set and interpret the dual linear programme

A firm makes two products, R and Q both of which require two raw materials RM1 and RM2. Each kilogramme of product R requires 2kg of RM1 and 3.5kg of RM2. Each kilogramme of product Q requires 3kg of RM1 and 1.5kg of RM2. Each week 10kg of RM1 and 12kg of RM2 are available. There is an unlimited supply of labour and machine time and the firm can sell all its production. The unit profit on product R is £5 and on product Q is £8.

Required

1. Set up a profit maximising linear programming model for this problem
2. Set up the dual linear programming model
3. Explain the relationship between the two models in 1 and 2
4. Find the optimum solution for the two models graphically.

Solution:

Produce x1 kg of product R and x2 kg of product Q each week. Maximise weekly profit, £p where:

P = 5x1 + 8x2 (£/week)

Subject to;   RM1: 2x1 + 3x2≤10

     3.5x1+1.5x2≤12

Using the primal model, the dual model is:

minimise : G = 10 y1 + 12y2

subject to:

product R: 3y1 + 1.5 y2 ≥ 8 £/ unit

Primal model: The variables are the amount of each product to be produced each week. The objective function is the total profit per week from the production of R and Q. Each constraint refers to one raw material. The left hand side of the constraint gives the total requirement for that raw material. The left hand side of the constraint gives the total requirement for that raw material by both of the products. The right hand side gives the total raw material available each week.

Dual model: the variables are the primal shadow price, that is, the amount which would be added to the value of the objective function if one more unit of the raw material was available. The shadow prices represent the value of one unit of the raw material. The objective function is the total value per week of the raw materials used in the production of R and Q. each constraint refers to one product. The left hand side of the constraint gives the total value of both raw materials used to make one kilogramme of that product. The right hand side gives the unit profit generated by that product. Let us look at the dual model again and try to identify the individual components

Minimise G:

$10y_1 + 12y_2$

Product R:

$2y_1 + 3.5 y_2 \geq £5$ profit per R

Product Q:

$3y_1 + 1.5y_2 \geq £8$ profit per Q

Each constraint says that the total value of the raw materials used in that product must be more than or equal to the unit profit on that product. The solution of either the primal or the dual model enables us to solve the other model.

The graphical solution for the primal problem is given below.

Diagram 7.6 Primal model

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + ... a_{1n} x_n \leq b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + ... a_{2n} x_n \leq b_2$

$a_{31} x_1 + a_{32} x_2 + a_{33}x_3 .... A_{3n} x_n \leq b_3$

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + ... a_{mn} x_n \leq b_m$

Primal model
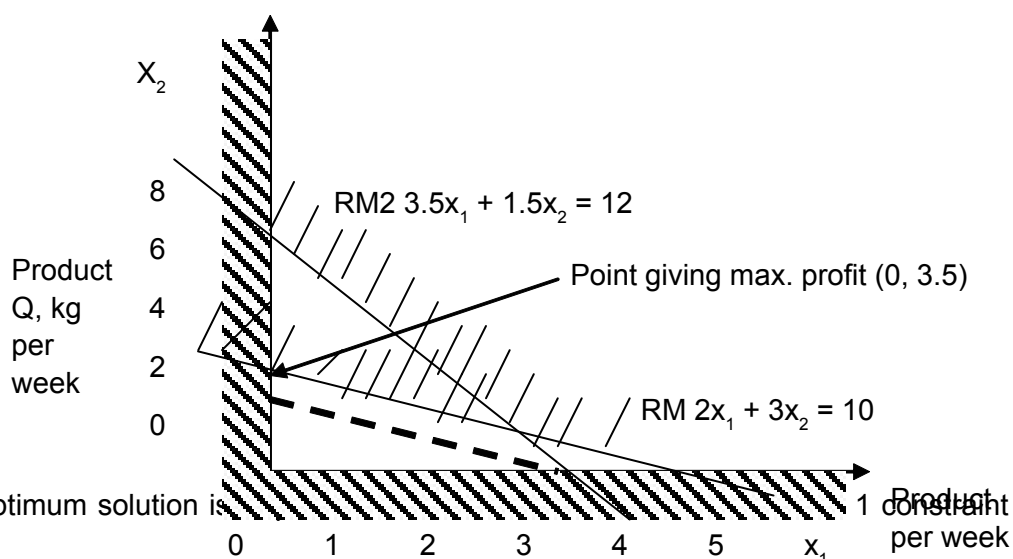
The optimum solution is

axis. To maximise profit we should produce 31/2 kg of Q and zero kilogrammes of R. all raw material 1 will be used, but not all of RM2. The maximum profit is 31/2 x 8 = £ 26.67 per week. The graphical solution for the dual is shown below.

Dual model

This is a minimisation problem. We wish to make the objective function as small as possible, hence we move towards the origin, parallel to trial objective function. The last point in the feasible region is given by Z. this is the optimum solution for the dual. Z is the intersection of the product Q constraint and the y-axis i.e.

Y2 =0 and $3y_1 + 1.5y_2 = 8$

Therefore y2 = 0 and y1 = 2 2/3

The minimum value for the dual problem is:

G = 10 x 2 2/3 + 12 x 0 = £26.67

Diagram 7.7
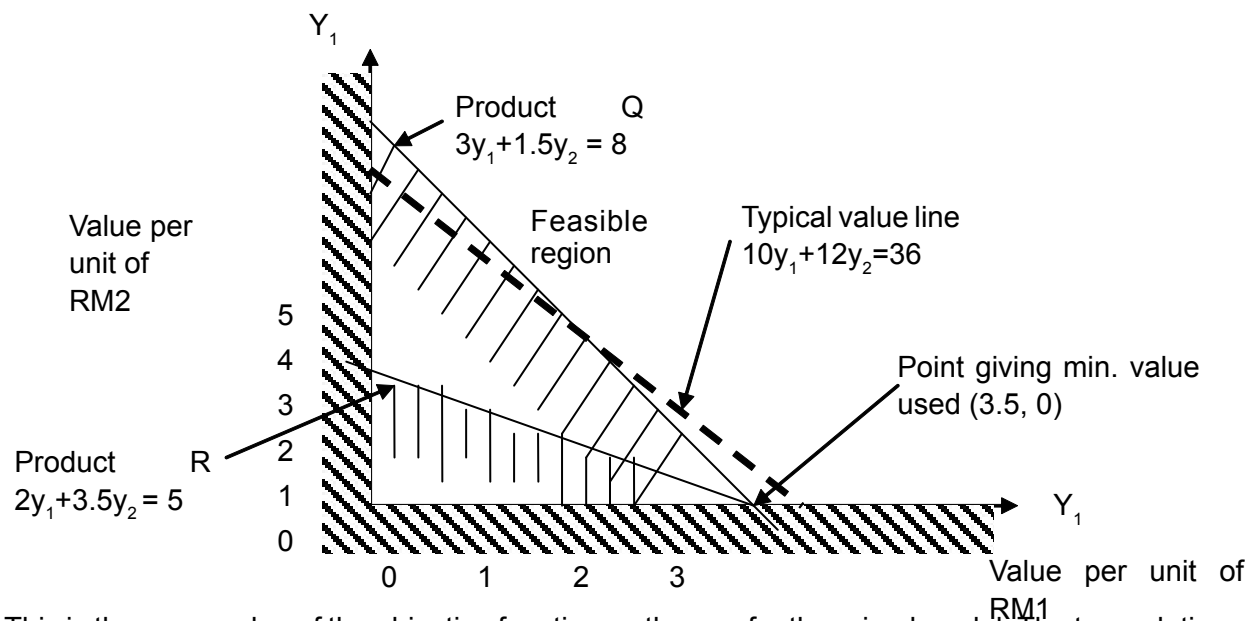
This is the same value of the objective function as the one for the primal model. The two solutions combined tell us that the maximum profit is £26.67 per week, when we produce 31/2 kg of Q and none of R, the value of the raw materials, the shadow price, is £2.67 per kg of RM1 and zero for RM2. This is the same as the information which we would derive from a full analysis of the primal problem alone.

CHAPTER SUMMARY

The same method of formulation applies to two variable and multivariable models. The two variable models, however, can be solved graphically. The constraints, which are usually inequalities either ≤ or ≥ are presented on the graph by lines and areas. Each constraint divides the graph into a rejected area and an acceptable area. The area in which all of the constraints are satisfied is called the feasible region. This feasible region contains all possible solutions to the problem.

The optimum point which is always at a corner of the feasible region is found by plotting typical objective function on the graph. The objective function is moved, parallel to this trial line away from the origin if the objective is to maximise, or towards the origin if the objective is to minimise. The last point that this line touches before it completely leaves the feasible region gives the values of the variables which will optimize the objective function.

Sensitivity analysis is very important in linear programming since the values used in the model may be subject to uncertainty. The procedure allows us to consider variation and uncertainty in the objective function coefficients and in the right hand side values of the constraints.

Multivariable linear programming models are solved by computer using the simplex algorithm. It provides the optimum value of the objective function, the values of the decision variables and the values of the slack or surplus variables. In addition, it gives the shadow prices of the resources. The final tableau can also be used in sensitivity analysis to show the full effect of variation in the scarce resources, on the objective function, and on each of the constraints.

Each primal linear programme has dual formulation. The solutions to the primal and the dual are identical. The dual may be derived from the primal model by interchanging the role of the coefficients in the model. There are sometimes advantages in solving a simpler dual rather than a complex primal, formulation.

CHAPTER QUIZ

1. In *Linear programming* a ………… is a variable which is added to a constraint to turn the inequality into an equation
2. In *Linear programming* a ………….. is a variable which is subtracted from a constraint to turn the inequality into an equation.
3. The area which remains unshaded is the set of points which satisfies all of the constraints simultaneously. This area is called?
4. A …………. is the name given to the set of variable values at a corner of the feasible region.
5. Each …….. linear programme has dual formulation. The solutions to the ……… and the dual are identical. The dual may be derived from the ………… model by interchanging the role of the coefficients in the model.

ANSWERS TO CHAPTER QUIZ

1. Slack variable
2. Surplus variable
3. Feasible region
4. Basic solution
5. Primal

QUESTIONS FROM PREVIOUS EXAMS

JUNE 2000 QUESTION 6

Regal Investments has just received instructions from a client to invest in three stocks: airline, insurance and information technology. The client has Sh10,000 available for investment. She has instructed that her money be invested in the three stocks so that no more than Sh5,000 is invested in any one stock but at least 1,000 is invested in each stock. She further instructed Regal Investments to use its current data and invest in a manner that maximises her expected overall gain during a one-year period. The stocks, the current price per share, and the projected stock price, a year from now, are summarized as follows:

Stock	Current price (Sh.)	Projected price (Sh) 1 year hence
Airline	25	35
Insurance	50	60
Information Technology (IT)	100	125

The problem was solved by the Management Scientist software giing the following output:

Variable	Variable value (Sh.)	Coefficient sensitivity (reduced cost)
Airline stock	200.00	-
Insurance stock	20.00	-
IT stock	40.00	-

Constraint	Slack/Surplus	Shadow price (Sh)
Amounts available	-	0.25
Maximum-airline stock	-	0.15
Maximum-insurance stock	-	0.00
Maximum –IT stock	-	0.00
Minimum-airline stock	-	0.00
Minimum-insurance stock	-	-0.05
Minimum-IT stock	-	0.00

Objective coefficient ranges

Variable	Lower limit	Current value	Upper limit
Airline stock	6.25	-	No limit
Insurance stock	No limit	-	12.50
IT stock	20.00	-	40.00

Right Hand side Ranges

Constraint	Lower limit	Current value	Upper limit
Amounts available	7,000	-	11,000
Maximum-airline stock	4,000	-	8,000
Maximum-insurance stock	1,000	-	No limit
Maximum –IT stock	4,000	-	Ni limit
Minimum-airline stock	No limit	-	5,000
Minimum-insurance stock	0	-	4,000
Minimum-IT stock	No limit	-	4,000

Required:

a) Formulate the above problem   (6½ marks)

b) Determine the values of the columns with blanks, that is, coefficient sensitivity, slack/surplus, objective coefficient ranges current value and right hand side ranges current value columns. For slack/surplus column indicate whether it is a slack or a surplus.

c) What is the optimal solution including the optimal value of the objective function?
                                                                                                  1½ marks)

d) If the client had an additional Sh1,000 available for investing, how much would the expected over-all one-year gain increase?   (1½ marks)

e) If the client increased the allowed maximum investment amount to Sh6,000 for just one stock, should it be IT stock? Why?   (1/2 mark)

f) Based on your stock choice in (e) above, how much would the objective function increase?   (1/2 mark)

g) For your choice in (e) above, how much could be allowed maximum investment amount be raised before optimal investment mix might change?   (1 ½ marks)
                                                                                          (Total: 20 marks)

JUNE 2002 QUESTION 6

a) Define the following terms as used in linear programming:
   (i)   Feasible solution   (2 marks)
   (ii) Transportation problem   (2 marks)
   (iii) Assignment problem   (2 marks)

b) The Tamu-Tamu products Company Ltd. is considering an expansion into five new sales districts. The company has been able to hire four new experienced salespersons. Upon analysing the new salesperson's past experience in combination with a personality test which was given to them, the company assigned a rating to each of the salespersons for each of the districts. These rating are as follows:

	District				
	1	2	3	4	5
Salesperson					
A	92	90	94	91	83
B	84	88	96	82	81
C	90	90	93	86	93
D	78	94	89	84	88

The company knows that with four salespersons, only four of the five potential districts can be covered.

Required:

(i) The four districts that the salespersons should be assigned to in order to maximize the total of the ratings.   (6 marks)

(ii) Maximum total rating.   (8 marks)
                                                                                          (Total: 20 marks)

JUNE 2003 QUESTION 6

A small company produces only two sizes of frames for stereo receivers: standard size and slim-line. The accounting department has provided the following analysis of the unit profit:

	Standard	Slim-line
Selling price	Sh6.00	Sh 4.25
Raw materials	Sh0.75 (1.5 units @ Sh. 0.50/unit)	Sh0.50 (1 units @ Sh. 0.50/unit)
Packaging	Sh0.25	Sh0.25
Labour	0.40 hours	0.25 hours

Labour is considered a fixed cost as it is by salaried workers at the plant.

There are 350 units of the raw materials and 300 packing boxes available daily. Both products utilise the same packing boxes. At most 10 workers (at 8 hours/day) will be assigned to this project. This problem was solved by the management scientists software with part of the output shown below.

Variable	Value	Reduced cost
Standard	33.333	0.000
Slim-line	266.667	0.000

Constraint	Slack/surplus	Dual prices
Raw materials	33.333	0.000
Packing boxes	0.000	1.000
Labour	0.000	10.000

Objective coefficient ranges

Variable	Lower limit	Upper limit
Standard (C1)	3,500	5,600
Slim-line (C2)	3,125	5,000

Right side ranges

Constraint	Lower limit	Upper limit
Raw materials	316.667	320.000
Packing boxes	200.000	80.000
Labour	75.000	

Required

(a) Formulate the above problem   (6 marks)
(b) What is the optimal daily production plan   (2 marks)
(c) What is the maximum selling price for the standard model that will keep the same optimal solution as in (b) above?   (1 mark)
(d) Suppose C1 was changed from its current value of Sh5.50. Would the optimal solution change? Why?   (1 mark)
(e) Suppose C2 was changed from its current value to Sh4.00. Would the optimal solution change? Why?   (1 mark)
(f) Suppose simultaneously C1 changed to Sh5.50 and C2 changed to Sh.4.00. Would the optimal solution change? Why?   (2 marks)
(g) What is the shadow price for man-hours? Interpret.   (2 marks)
(h) What is the shadow price of the packing? Interpret.   (2 marks)
(i) Suppose simultaneously the amount of material available increased from 350 to 500, the number of boxes available increased from 300 to 310 and the number of man-hours increased from 80 to 84. What conclusion can be drawn regarding the shadow prices? Why?   (3 marks)
(Total: 20 marks)

DECEMBER 2003 QUESTION 6

a) One of the properties (assumptions) of Linear programming is that fractional values of the decision variables are permitted that is divisibility. However, fractional values might sometimes be meaningless.
Explain one case where fractional values might be meaningless. What approach should be used to over-come this assumption?   (3 marks)
b) Explain the connection between the following:
   (i)   Reduced costs column and the range of optimality   (2 marks)
   (ii) Dual prices and the range feasibility   (2 marks)
c) Lake Naivasha Floatway Tours has Sh4 million that may be used to purchase new rental boats for hire during the coming December holidays. The boats can be purchased from the two different manufacturers. Pertinent data concerning the boats are summarised below:

Boat type	Manufacturer	Cost (Sh.)	Maximum seating capacity	Expected Daily profit (Sh.)
Speedhawk	Sleekboat	60,000	3	7,000
Silverbird	Sleekboat	70,000	5	8,000
Chui	Racer	50,000	2	5,000
Simba	Racer	90,000	6	11,000

Floatway Tours would like to purchase at least 50 boats and would like to purchase the same number from Sleekboat as from Racer to maintain goodwill. At the same time, Floatway Tours wishes to have a total seating capacity of at least 200.

Required:

Formulate the above problem   (13 marks)

(Total: 20 marks)

DECEMBER 2004 QUESTION 6

The linear programming model and output model below represent a problem whose solution will tell a road kiosk owner how many of the four different types of household goods stock in order to maximise profits. It is assumed that every item stocked will be sold. The variables measure the packets of Unga, spaghetti, rice and sugar to stock respectively. The constraints measure storage space in units, special display racks, demand and a marketing restriction, respectively.

Maximise $Z = 4x_1 + 6x_2 + 5x_3 + 3.5x_4$

Subject to:

$2x_1 + 3x_2 + 3x_3 + x_4 < 120$

$1.5x_1 + 2x_2 < 54$

$2x_2 + x_3 + x_4 < 72$

$x_2 + x_3 > 12$

Where x_1 = Packets of Unga

X_2 = Packets of Spaghetti

X_3 = Packets of Rice

X_4 = Packets of sugar

Optimal solution

Variable	Value	Reduced cost
X1	12.00	-
X2	0.00	0.50
X3	12.00	-
X4	60.00	-

Constraint	Stock/surplus	Dual/shadow price
1	-	2.00
2	-	-
3	-	1.50
4	-	-2.50

Objective coefficient ranges

Variables	Lower limit	Current value	Upper limit
X1	1.50	-	5.00
X2	No limit	-	6.50
X3	5.50	-	7.50
X4	3.00	-	No limit

Required

(a) Determine the retailer's optimal profit level.   (2 marks)

(b) Determine and interpret the missing values under:
    (i)   Reduced cost column   (2 marks)
    (ii)   Slack/surplus column, indicate whether the value is a slack or surplus.
            (2 marks)
    (iii)   Dual/shadow price column   (2 marks)
    (iv)   Current value column under objective coefficient ranges   (2 marks)
    (v)   Current value column under right hand side ranges (2 marks)

(c) Interpret the value 0.50 under the reduced cost column and the values; 2.00, 1.50 and -2.50 under the dual/shadow price column.   (2 marks)

(d) Determine whether the current optimal solution would change if the current profit of packets of Unga is increased by Sh2.00   (2 marks)

(e) Determine by how much the amount of space would increase before there is a change in the dual/shadow price.   (2 marks)

(f) The above problem could have been solved manually. Explain how the optimal solution can be determined using the manual approach.   (2 marks)

                                                  (Total: 20 marks)

CHAPTER EIGHT

Network Planning and Analysis

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Network Planning and Analysis

▶ OBJECTIVES

i.   At the end of this chapter, you should be able to:
ii. Define network analysis
iii. Cite importance and application of network analysis in real business world
iv. Crash a project
v.   Find an optimum solution for transportation and assignment models
vi. Discuss various special cases of the assignment problem

▶ INTRODUCTION

Network analysis is a family of related techniques developed to aid management in the planning, co-ordination and controlling of large complex projects using limited resources like personnel, material, money, time etc in order to achieve some objective.

> **Fast Forward:** A project is a finite endeavo*u*r (having specific start and completion dates) undertaken to create a unique product or service which brings about beneficial change or added value.

▶ DEFINITION OF KEY TERMS

Project: It is a human undertaking which consists of a series of interrelated tasks geared toward a definite objective. It involves a considerable amount of time, human and financial resources for completion.

Task/activity: This is an individual identifiable activity in a project which has a definite beginning and end. A task requires time and other resources for its completion. An activity or task is considered non divisible. An activity is represented in a network by an arrow.

Event: This denotes the beginning or the ending of an activity. It is just a moment in time hence does not require any resources or time. An event is represented in a network by a circle or node.

▶ INDUSTRY CONTEXT

Business modelling has traditionally been done by operations research (OR) and management science professionals. It involves the use of quantitative and computer methods for planning the efficient allocation of resources in business, industry and the agencies of government. It overlaps economics as a discipline but it primarily focuses on the internal operations of organisations emphasises planning rather than market mechanisms for allocating resources makes more extensive use of quantitative methods.

Operations management often presents complex problems that can be modelled by linear functions. The mathematical technique of linear programming is instrumental in solving a wide range of operations management problems.

Linear programming is used to solve problems in many aspects of business administration including:

- product mix planning
- distribution networks
- truck routing
- staff scheduling
- financial portfolios
- corporate restructuring

▶ EXAM CONTEXT

The most recognisable feature of PERT is the "PERT Networks", a chart of interconnecting timelines. PERT is intended for very large-scale, one-time, complex, non-routine projects. Students should be keen on the flow of scheduling as well as flow of events while coming up with the ultimate allocation pattern to avoid misallocation.

A student will be required to tackle questions from network analysis step by step. Questions are easily raised as outlined below:

Past Paper Analysis

12/06, 6/06, 12/05, 6/05, 12/04, 6/04, 12/03, 6/03, 12/02, 6/02, 12/01, 6/01, 12/00, 6/00

8.1 NETWORK ANALYSIS

This is a system of interrelationship*s* between jobs and tasks for planning and control of resources of a project by identifying critical path of the project.

In Network analysis attention should be directed to some activities. These include:

Dummy activity

This is used in a network for

i.   improving clarity of the network
ii. To facilitate a logical flow of activities in the network.

A dummy activity consumes no time or resources. Its duration is zero. A dummy activity is represented by a dotted arrow/line.

■ Activity floats

Floats give a measure of the flexibility which exists for either individual activities or a path of activities in a network in relation to whether we can extend project completion time as measured by the critical path. These are the total, free and independent floats.

a)   Total float(TFij)

This indicates the amount of time by which a non-critical activity can be delayed without affecting the project duration dates. It is equal to the difference between the total time allowed on a performance of an activity and the actual time required for its performance. For critical activities, the total float is zero. For non-critical activities, the total float can be computed using the following formula:

$TF_{ij} = LC_j – ES_i – D_{ij}$

Where

TFij = total float

LCj= latest completion time at the head of the activity

ESi = earliest start time at the tail of the activity

Dij = means the duration of the activity.

b)   Free float (FFij)

This indicates how far a non-critical activity can be delayed beyond its earliest start time without affecting the earliest start time of the activities immediately following it. Critical activities have zero free float. For non-critical activities, free float can be determined as follows:

$FF_{ij} = ES_j – ES_i – D_{ij}$

Where

FFij = free float

ESj = earliest start time at the head of the activity.

ESi = earliest start time at the tail of the activity.

Dij = duration of the activity.

c)   Independent float (IF)

This is the amount of time an activity can be delayed without delaying the project completion time if all preceding activities are completed as early as possible and subsequent activities are started as soon as possible.

It is given as follows:

$IF_{ij} = ES_j – LC_i – D_{ij}$

Where

IFij = independent float

ESj = earliest start time at the head of the activity.

LCi = latest completion time at the tail of the activity

Dij = duration of the activity

Event slack

An event slack is the maximum time an event can be delayed without delaying the overall project completion time. For an event i ,it given as LCi – Esi

Note:

Total float ≥ free float ≥ independent float

When asked to calculate float or slack without specifying, take that to mean the total float.

Role of network analysis

Network analysis in project management will help:

Define the job to be done, breaking it into individual activities;

Integrate them in a logical time sequence;

Afford a system of dynamic control over the progress of the plan

How would you use network analysis to solve a management problem?

Phase I: Formulation

Identify the project

Determine and list all the activities in the project.

Estimate the duration and cost of all activities listed above listing all resources required.

Establish a logical flow of all the activities – first, second, and so on.

Phase II: Scheduling

Draw the network diagram of the project indicating various events and their duration.

Determine the earliest and the latest occurrence of each event.

Calculate activity floats.

Determine the **critical path** for network – the critical path is defined as an unbroken continuity of activities sequence from start to finish which is the longest path (and) which represents the minimum time the project can be completed. A delay of an activity on the critical path inevitably delays the whole project.

Note: *T*here can be more than one critical path in one network.

Phase III: Implementation and control

  i)   Analyse the effects of delays in the project time, reduction of costs (activity crashing).
  ii) Check off the progress of implementation with the plans.

iii)   Reassign or reschedule the resources that are used by the project as appropriate.

iv)   Revise the network and set a new schedule where necessary.

There are two approaches to pictorially represent a network:

Activity on arrow approach

Activity on node approach

Rules to construct network diagrams

1.   As a rule, no two activities can begin and end at the same event node as they would not be uniquely identified. (Use a dummy activity to ensure uniqueness).

Diagram 8.1
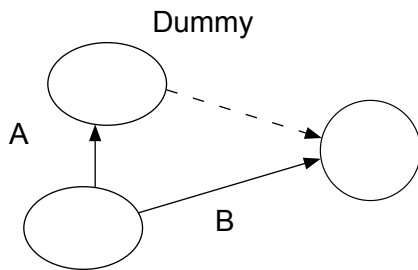
A complete network can have only one beginning event node and one ending event node.

Diagram 8.2
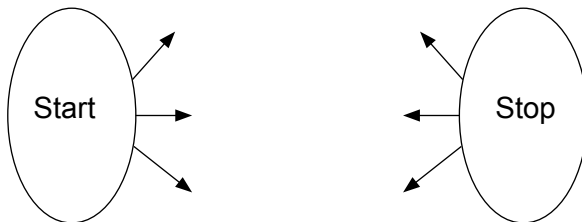
2.   Every activity must have one preceding or "tail" event and one succeeding or "head" event. Note that many activities may use the same tail event and many may use the head event.

Diagram 8.3
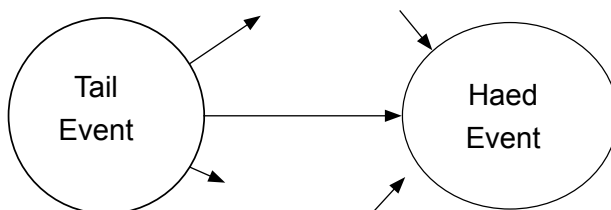
3.   "Loops" are a series of activities which lead back to the same event and are NOT allowed because *a* network is a progression of activities always moving onwards in time.

Diagram 8.4
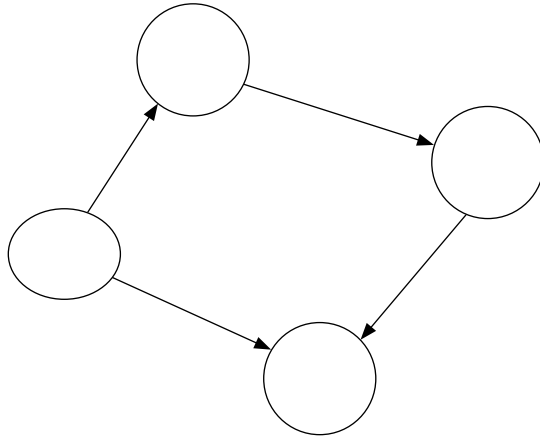
4. All activities must be tied into the network i.e. they must contribute to the progression or be discarded as irrelevant. Activities which do not link into the overall project are termed "**danglers**". Danglers should not be used.

Diagram 8.5
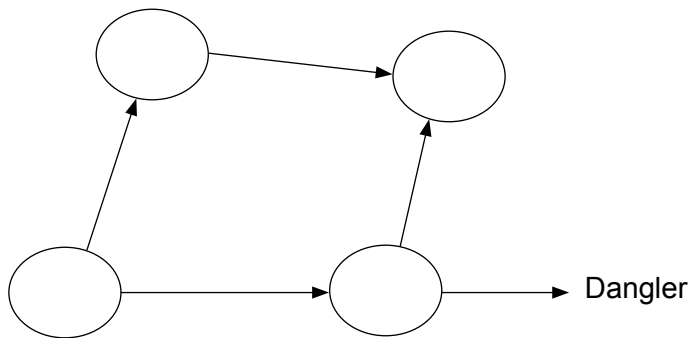
5. Networks proceed from left to right.

6. Networks are not drawn to scale

7. Use straight lines, not bent or curved ones.

8. The arrows should not cross each other unless it is completely necessary.

9. The length of the arrow is not proportional to its duration.

10. When the drawing is complete, use code number to number the events. The code number at the beginning of an activity must be smaller than the code number at the end of an activity. These code numbers ensure that each activity can be uniquely identified in the diagram

Two techniques were developed almost simultaneously in the period 1956-1958 by different firms for different reasons and independently:

    (a)   The critical path method (CPM)
    (b)   The program*me* evaluation and review technique (PERT)

The main difference is that CPM is a certainty (deterministic) model while PERT is a stochastic (uncertainty) model with respect to the project completion time.

Other initial differences:

i)   Mechanic of drawing network

PERT used AOA approach where AOA = Activity-on-Arrow approach.

Diagram 8.6
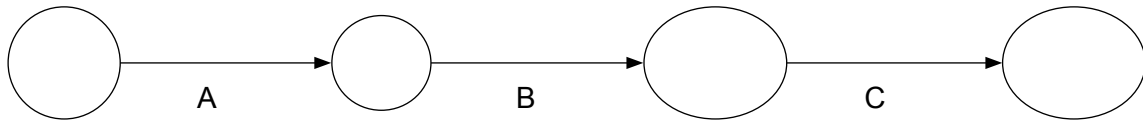
CPM used AON approach where AON = Activity-on–Node approach.

Diagram 8.7
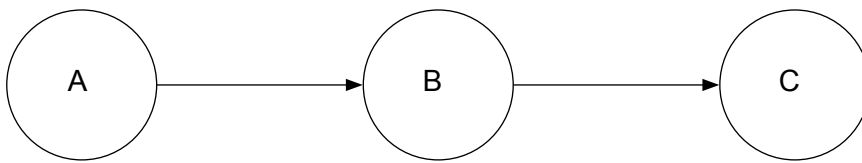
ii)   How to estimate activity time

CPM used a single time estimate therefore it was deterministic in the estimate.

PERT on the other hand used multiple time estimates allowing for any doubt to be included in the estimates therefore probabilistic.

These estimates were reduced to three for examination purposes.

a)   Optimistic time estimate = a

This is the shortest time an activity can take to be complete. It represents uder real estimate such that the probability is small that the activity can be completed in less time.

b)   Pessimistic time estimate = b

This is the longest time an activity can take to be completed. It is the worst time estimate representing bad luck, such that the probability is small that the activity will take longer.

Most likely time estimate = m

This refers to the time that would be expected if you work under normal conditions. Used to give the activity expected (mean) time using the following formula:

Expected activity time, te = $\dfrac{a + 4m + b}{6}$

Critical activity

1.   It must be critical for the completion of the project.

2.   It must be started and completed as originally scheduled otherwise the project will be late.

3.   It must have the first priority in the use of resources.

4.   Requires keen management attention and control.

Three conditions to be satisfied simultaneously

1. $ES_i = LC_i$

2. $ES_j = LC_j$

3. $ES_j - Es_i = D_{ij}$ or $LC_j - Lc_i = D_{ij}$

Critical path calculations

Fast Forward: Today, it is commonly used with all forms of projects, including construction, software development, research projects, product development, engineering, and plant maintenance, among others.

Two phases are computed:

a) Phase I is the Forward Pass computations. It is used to determine the ES_i for each activity.

b) Phase II is the Backward pass computations. It used to determine the LC_j for each activity.

Phase I – Forward Pass = ES_i

Computation begins from the initial event. Move forward until you reach the terminal event.

The initial event time is zero which is base time.

Earliest start time of the activity beginning from the immediate next event which is base time plus the duration of the activity.

If there *is* more than one activity going into an event, for the ES_i from that event, take the maximum allowable time using the following formual:-

$ES_j = $ max $ES_i + D_{ij}$ for all (ij) acitivities.

Phase II – Backward pass = LC_j

Start from the terminal event *and* move backwards to reach the initial event.

Terminal event $LC_{ij} = ES_i$

For the next event it is the difference between the maximum allowable time and the duration of the activity.

Take minimum allowable time, $LC_i = $ min $LC_j - D_{ij}$ for (ij) all activities

Crashing the project

Crashing means trying to perform an activity in a time shorter than the activity required i.e. putting in extra effort to deliberately reduce the activity time. Consequently, the duration of some activities can be reduced if some additional resources are employed but by introducing such additional resources the cost of performing such activities increases and the normal time by which an activity is required to be reduced the greater the amount of resources required to be employed.

Normal duration (Dn)

This is the time required to perform an activity under normal circumstances and with minimum direct costs.

Crash duration (Dc)

This is the time taken to perform an activity if the duration is reduced or shortened.

Normal costs (Cn)

This is the absolute minimum direct costs required to perform an activity within the normal duration.

Crash costs (Cc)

This is the cost incurred to achieve the reduced performance time. It is generally more than the normal costs because of introduction of additional resources.

Incremental costs (Ic)

This is the increase in cost incurred per unit of time that is used. It is determined by the following formula:

$$\text{Incremental costs (Ic)} = \frac{crashcost - normalcost}{\text{Normalduration} - \text{Crashduration}} = \frac{\Delta\ cost}{\Delta\ duration}$$

■ Guideline for crashing

Identify the critical activities and the critical path(s)

Crash the durations of the critical activities only because they are the ones that determine the project duration.

Determine the incremental cost of all the critical activities.

If there are several critical activities to be crashed select the critical activity with the least incremental cost to be crashed first.

For each crashing step a unit of time is the maximum amount that can be used because there is a possibility of new critical activities and critical path emerging.

Check to be sure that the critical path being crashed is still critical. This is because a reduction in activity time along the critical path may cause non-critical activities and paths to become critical.

If a project has more than one critical path then all critical paths must be reduced simultaneously by an equal amount.

When to stop crashing

For technical reasons the duration of an activity cannot be reduced indefinitely. If crash time represents the minimum duration that is allowable consequently stop crashing activity time or duration when:

The management wish has been achieved.

When all crashable activities have reached their climax crash limits.

When it is no longer economical to continue crashing.

Resource scheduling and profiling

So far we have assumed that an activity can begin as soon as all the activities which must precede it have been completed. This assumes that there are sufficient resources to perform all work defined by the activity even if concurrently. Such an assumption may not always be true in practice. Further, resource utilisation needs to be planned so that it is as steady (smooth/even) as possible.

Planning resource requirements is done through the use of a time (Gantt) chart levelling of resources usage which will lead to the construction of a resource profile. A resource profile is used as a calendar of time schedule by the personnel who will implement the project.

We perform resource scheduling and profiling because:

We accept that concurrent activities may not be possible due to resource constraints.

For other resources such as manpower, the application should be planned for in advance to minimize losses due to shortages at certain time or idleness at other times.

Smoothing a profile

This is the process of attempting to reduce the peaks and troughs in the resource allocation so that we have a more even usage of personnel. Smoothing requires that we make use of activity floats since we generally cannot extend completion time. We cannot reschedule critical activities. We can normally reschedule only non-critical activities because they have floats. Such rescheduling should be for as little as possible since it is prudent to delay an activity for as short a time as possible. Smoothing is conducted on the basis of trial and error because there is no analytical method for it.

Illustration

Consider the following project network and activity times (in weeks).
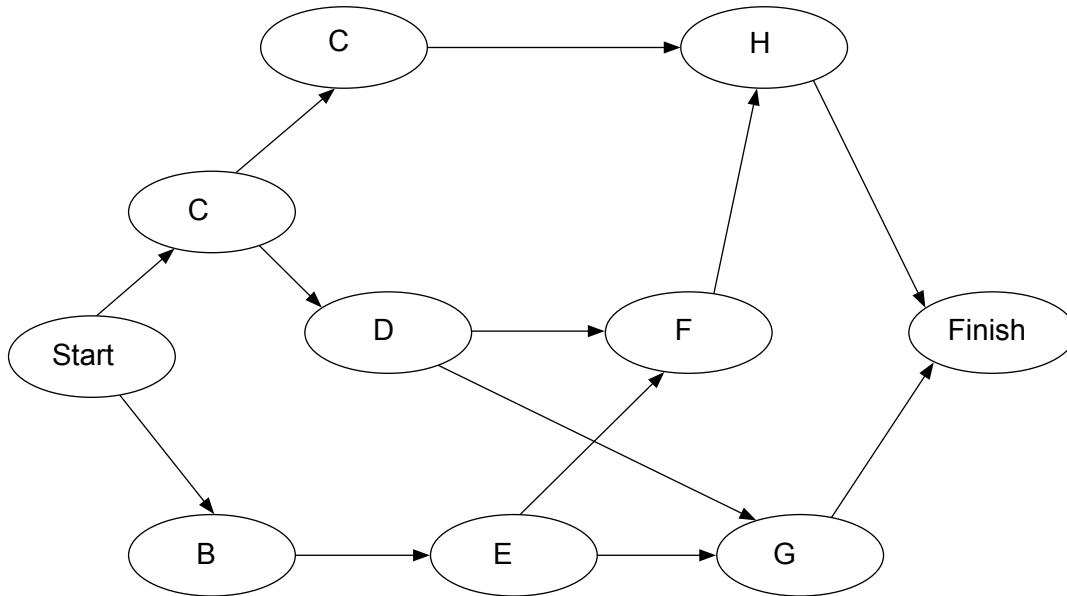
Activity	Activity Times (weeks)
A	5
B	3
C	7
D	6
E	7
F	3
G	10
H	8

Required:

(i) How long will it take to finish this project?

(ii) Can activity D be delayed without delaying the entire project? If so, by how many weeks?

(iii) What is the schedule for activity E?

Solution

i) We can determine the project duration by drawing a schedule to ascertain the critical activities.

Activity	Duration	ES	EF	LS	LF	Slack	Critical Activity
A	5	0	5	0	5	0	Yes
B	3	0	3	1	4	3	No
C	7	5	12	7	14	7	No
D	6	5	11	5	11	0	Yes
E	7	3	10	4	11	7	Yes
F	3	11	14	11	14	0	Yes
G	10	11	21	12	22	11	No
H	8	14	22	14	22	0	Yes
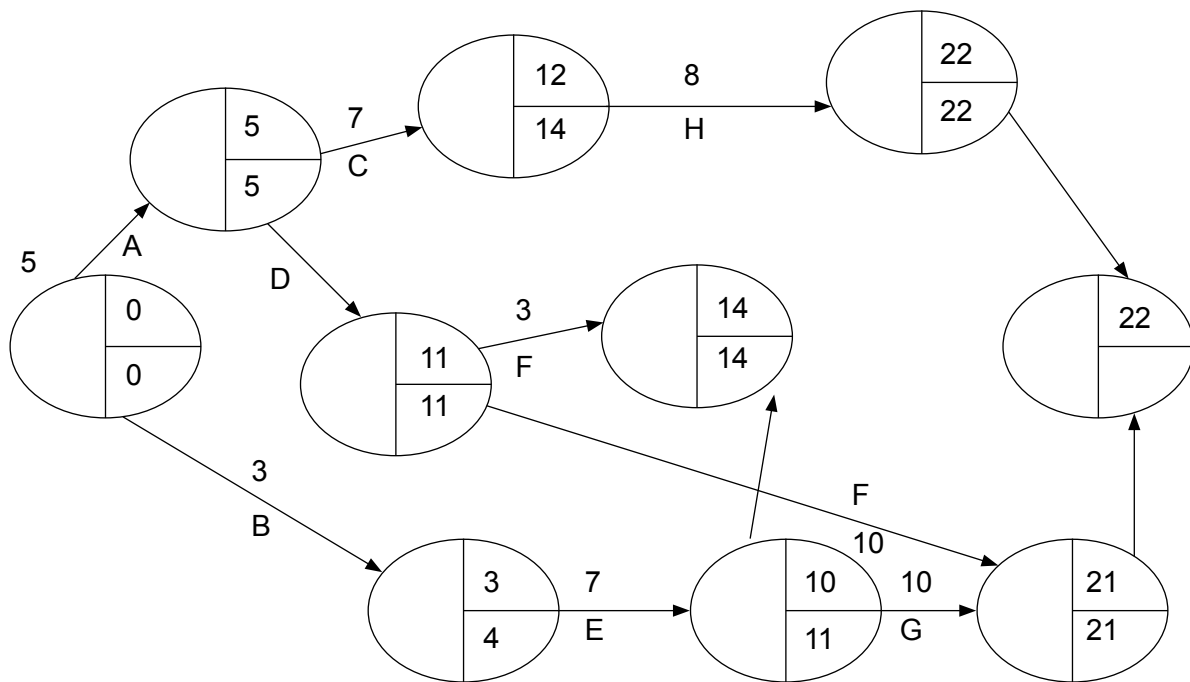
The critical path as identified using the float method (see table) is ADFH.

It thus takes 22 days to complete the project.

ii) Activity D can't be delayed without delaying the entire project because it is a critical activity.

iii) The schedule for activity E is as below:
EST 3rd week
LST 4th week
EFT 10th week
LFT 11th week.

Note:

The PERT *di*agram, can also be drawn as below:
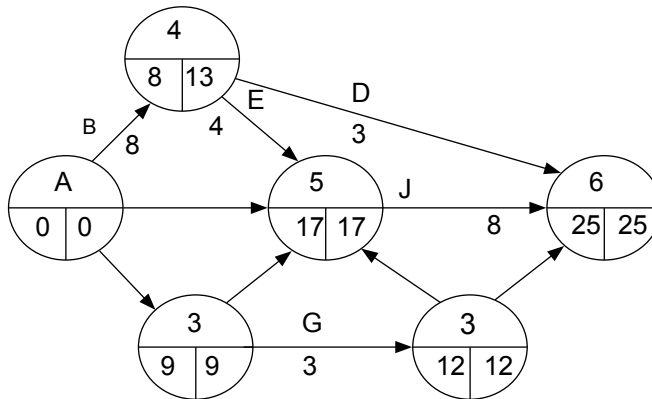
The critical path is A – G – H – J.

Clearly, the project cannot be completed within the remaining time of 1 week. There is need to increase the duration of critical path by ½ week.

The assumption of PERT *a*s used in network analysis is that the activity time duration can be best described by a beta probability distribution.

Illustration

Lilian Wambugu is the project manager of Jokete Construction Company. The *c*ompany is bidding on a contract to install telephone lines in a small town. It has identified the following activities along with their predecessor restrictions, expected times and worker requirements.

Activity	Predecessors	Duration Weeks	Crew Size Workers
A	-	4	4
B	-	7	2
C	A	3	2
D	A	3	4
E	B	2	6
F	B	2	3
G	D, E	2	3
H	F, G	3	4

Lilian Wambugu has agreed with the client that the project should be completed in the shortest duration.

Required:

(i) Draw a network for the project.
(ii) Determine the critical path and the shortest project duration.
(iii) Lilian Wambugu will assign a fixed number of workers to the project for its entire duration and so she would like to ensure that the minimum number of workers is assigned and that the project will be completed in 14 weeks.
Draw a schedule showing how the project will be completed in 14 weeks.

(iv) Comment on the schedule you have drawn in (iii) above.

Solution

(ii) Critical path is B – E – G - H
     Shortest project duration = 14 weeks.

(iii) Time float   :   TF = LFT – EST – D

TF (A) = 6 – 0 – 4 = 2

TF (B) = 7 – 0 – 7 = 0

TF (C) = 14 – 4 – 3 = 7

TF (D) = 9 – 4 – 3 = 2

TF (E) = 0

TF (F) = 11 – 7 – 2 = 2

TF (G) = 0

TF (H) = 0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
H													4	
G									3					
F							3		3					
E							6							
D				4										
C				2								2		
B				2										
A		4												
Time (wk)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Workers	6	6	6	6	8	8	8	9	9	3	3	4	4	4
Reschedule F by 2 wks	6	6	6	6	8	8	8	6	6	6	6	4	4	4
Reschedule C by 7 wks	6	6	6	6	6	6	6	6	6	6	6	6	6	6

(iv) Comment

Minimum number of workers needed throughout the duration is 6 workers.

C and F are pushed to the end of their slack times to ensure that there is a smooth engagement of the resources.

Illustration

James Mutiso is a computer engineer in an information technology firm. The firm has decided to install a new computer system to be used by the firm's help desk. James Mutiso has identified the activities required to complete the installation.

The table below provides a summary of the activities' durations and the required number of technicians:

Activity	Duration (weeks)	Required number of technicians
1 – 2	3	2
1 – 3	1	4
2 – 4	3	4
2 – 5	2	2
3 – 4	2	4
3 – 6	4	4
4 – 5	2	2
5 – 6	2	2
6 – 7	2	2

Required:

(i)   Draw a gantt chart for the project.
(ii) Mr. Mutiso would like to reschedule the activities so that not more than 6 technicians are required each week.

Determine if this is possible and how it can be achieved by rescheduling the activities.

Solution
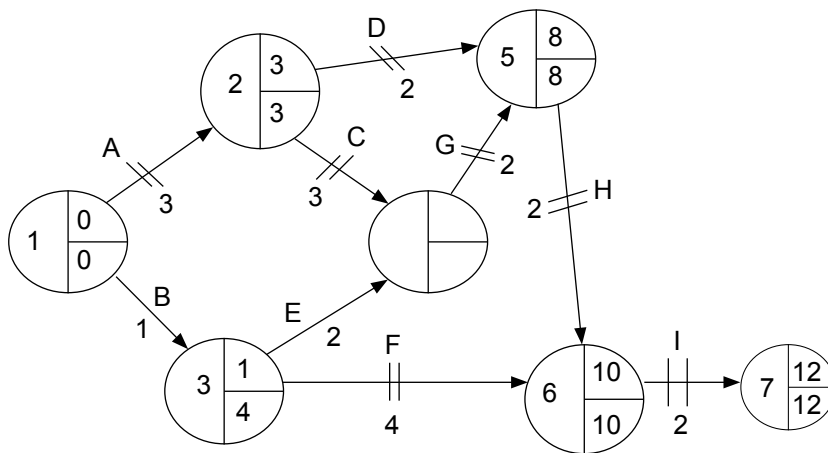
Critical path = A – C – G – H - I
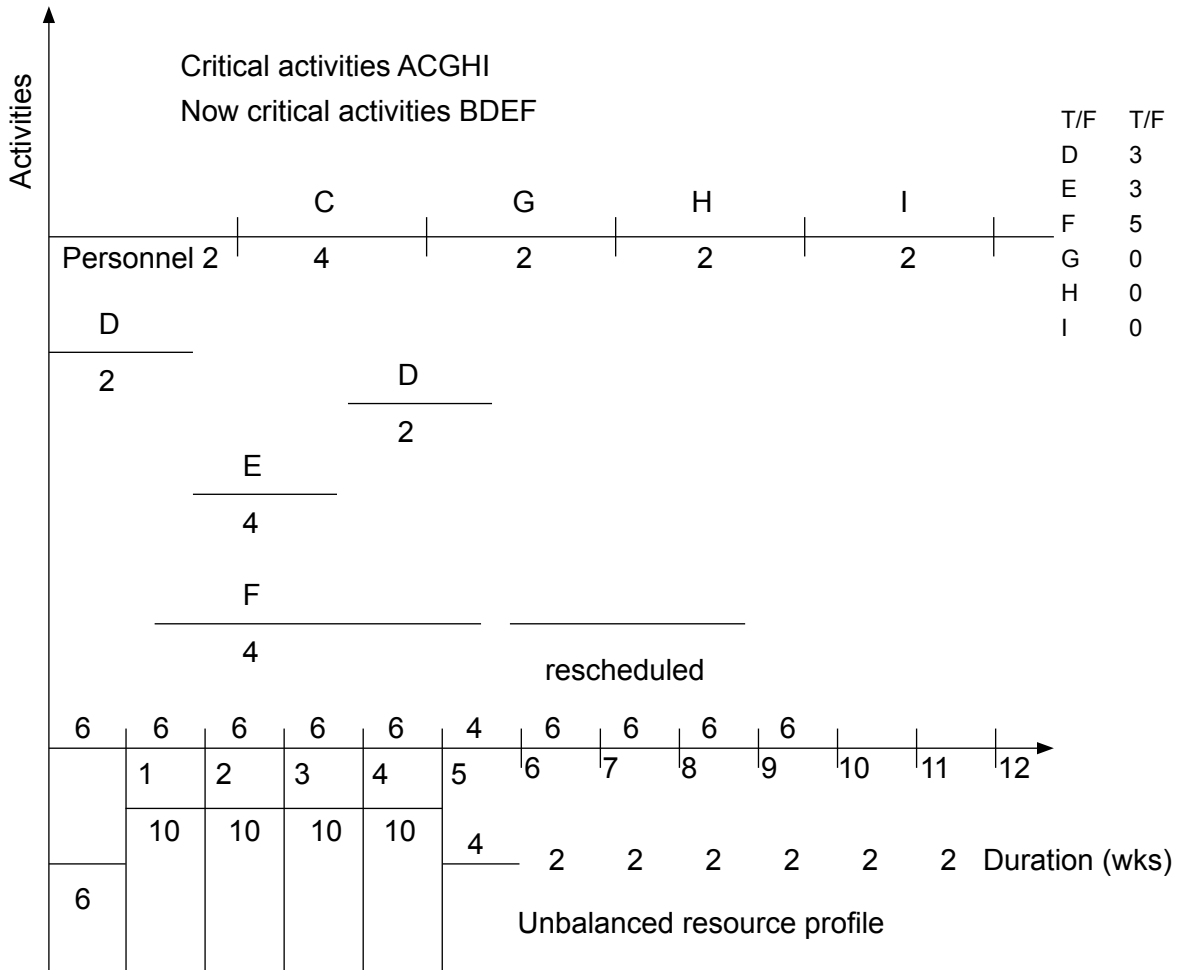
To have at most 6 technicians Mr. Mutiso should reschedule activity F to start on the 6th week i.e. delay it by 5 weeks.

8.2 TRANSPORTATION AND ASSIGNMENT MODELS

The linear programming methods discussed in Chapter *Seven* are suitable for a range of allocation problems. The work involved in solving the model can be drastically reduced by use of a computer package. This leaves the decision maker free to concentrate on the interpretation and evaluation of final solution. However, the *p*ackage still requires the formulation of the linear programming model. This can be a major task for large problems. The variables must be identified and the constraints formulated.

For certain types of allocation problem*s*, the use of specially designed a*l*gorithm simplifies the building of the initial model. In this chapter*,* we will look at two related examples of this type of algorithm which are suitable for solving transportation and assignment problems.

In both cases the allocation concerns items which are transferred from a number of origins to a set of destinations according to a particular objective. The objective is often one which minimises the total cost of the transfer. For example, a company has three factories and five

regional distribution centres. The management requires the transfer of the finished goods from the factories to the distribution centres to be achieved at minimum cost. This is a situation in which the transportation method would be appropriate.

Assignment is a particular case of the transportation problem. For each combination of origin and destination, the transfer involves one item only. For example, a machine shop has six lathes of varying ages and design. On a given morning, the machine shop manager has six jobs to allocate. The jobs will take different lengths of time on different lathes. The manager wishes to allocate one job to each lathe to minimise the total working time. The assignment algorithm may be used to solve problems such as this.

In this chapter we will describe the application of these two algorithms using small problems. It should be borne in mind that, in practice, the problems will he much larger and are solved using computer packages. In addition, transportation models often involve several stages, for example, factory-to depot-to-retail outlet. In these cases the basic algorithm has to be modified and more sophisticated methods used.

Transportation Problem and Algorithm

This problem is concerned with the allocation of items between suppliers (called origins) and consumers (called destinations) so that the total cost of the allocation is minimised. The problem can he solved using either linear programming methods or the special transportation algorithm. The linear programming method is illustrated in Example 1.

The transportation problem

Example 1: To Illustrate a basic transportation problem

Ace Foods Ltd manufacture soft drinks at two plants, A and B. Bottles for the two plants are supplied by two firms, P and Q. For the month of November, plant A requires 5,000 bottles and plant B requires 3,500 bottles. Firm P is able to supply a maximum of 7,500 bottles and firm Q is able to supply a maximum of 4,000 bottles. The cost per bottle of transport between each supplier and each plant is shown in the

Table 1: costs, requirements and availabilities of bottle supply

Transport cost, pence/bottle to plant available bottles

		A	B	
From supplier	P	4	4	7500
	Q	3	2	4000
Bottles required		5000	3500	

How should the bottles be supplied to the plants to minimise the total transport cost?

Solution

It is always useful with transportation to see if there is an obvious solution. Ideally, we would like to use only the cheapest routes. Supplier Q will be preferred by both plants, since it is cheaper than P. Unfortunately Q has only 4,000 bottles, compared with the total requirement *of* 8,500. The cheapest solution will probably be to use the 2p per bottle route from Q to plant B, to supply all the requirement at B (3,500). The balance from Q (500) should be sent to A at 3p per bottle. The rest of the demand at plant A will come from P at a cost of 4p per bottle. The total cost of this allocation is:

0.02 x 3500 + 0.03 x 500 + 0.04 x 4500 = £265/mon*t*h

We have no proof that this is the most economic allocation. One of the important aspects of the model which we are about to consider, is that it provides a solution, demonstrates that it is the optimum solution and shows the effect on the solution of any changes which arise in the problem.

We will now solve the above problem using a conventional linear programming model with a graphical solution.

Suppose firm P supplies x bottles to plant A and y bottles to plant B. Firm Q must supply the remaining (5000 — x) bottles to A and the remaining (3500 — y) bottles to B. The objective is to minimise the total transport cost, C, pence, where:

$C = 4x + 4y + 3(5000 — x) + 2(3500 — y)$

Therefore:

$C = x + 2y + 22,000$

and:

$Z = C - 22,000 = x + 2y$

Z will take its minimum value, when C takes its minimum value. The values of x and y which minimise Z = x + 2y, will also minimise C. The objective function is minimised subject to:

Requirement at A: x   <5000 bottles

Requirement at B: y   <3500 bottles

Supply from P: x + y   <7500 bottles

Supply from Q: (5000 — x) + (3500y) <4000 bottles

i.e.:

x + y >4500 bottles

x, y> 0

These constraints are illustrated in Figure 1.
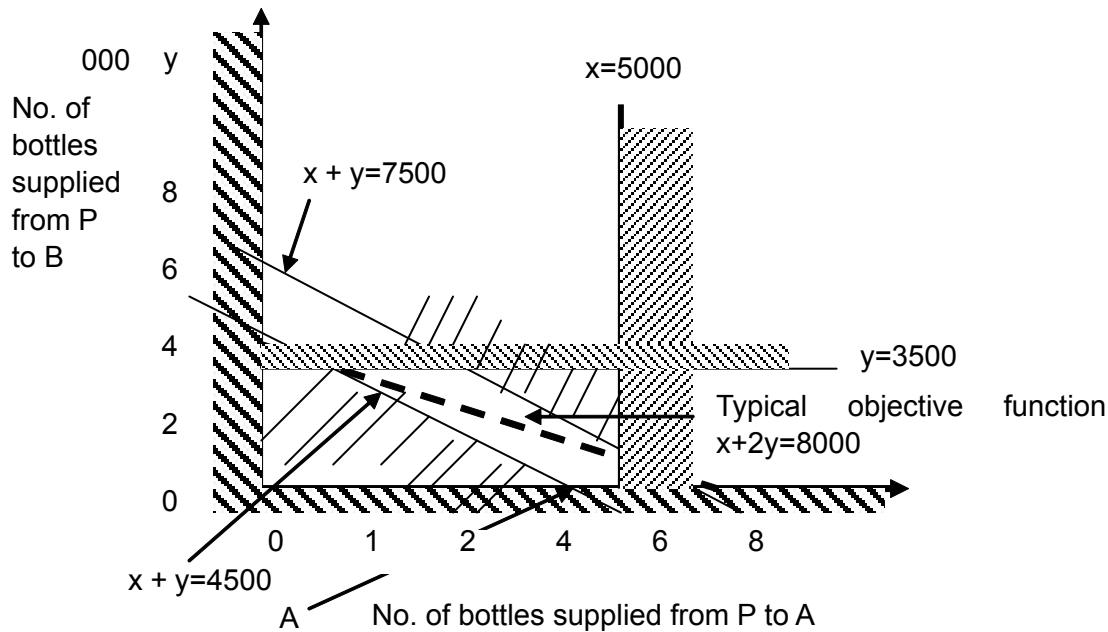
The point x = 4000, y = 2000 is a feasible solution. At this point:

Z = 4000 + 2 x 2000 = 8000 p

The trial objective function is 8000=x+2y. This is shown in figure.1. Moving in the direction of decreasing values of Z, corner A is the optimum. At this corner, x = 4500 and y = 0. Therefore the optimum solution is for P to supply 4500 bottles to A and none to B, while 0 supplies 500 bottles to A and 3500 bottles B. At this solution, the minimum cost is:

Cmin = 4500 + 2 x 0 + 22,000 26,500 p = £265

The only spare capacity is at firm P which retains 3,000 bottles. This is the solution that we thought would be the minimum. We have now shown that this is the case.

The transportation algorithm

The problem in section1 may be solved using the transportation a1gorithm. To use this algorithm, a number of conditions must be satisfied:

1. The cost per item for each combination of origin and destination must be specified.
2. The supply of items at each origin must be known.
3. The requirement of items at each destination must be known.
4. The total supply must equal the total demand.

Example.1 satisfies the first three conditions but not the last one. However, a dummy plant can be included for which the requirement is the difference between the total available and the total required. In Example 1 the dummy plant would have a requirement for (11,500 - 8500) =3,000 bottles.

Any items allocated to a dummy destination represent items which do not leave the supplier. In a similar way, if the total supply is less than the total demand, a dummy supplier is included to supply the shortfall. Any items allocated from this dummy represent items not supplied.

The transportation algorithm has four stages:

Stage 1: Arrange the data in tableau format and find any feasible allocation. A feasible allocation is one in which all demand at the destinations is satisfied and all supply at the origins is allocated.

Stage 2: Test the allocation to see if it is the optimal solution.

Stage 3: If the first allocation is not optimal, re-allocate in order to move to a better, lower cost, solution.

Stage 4: Test again for optimality.

Repeat this iterative process until the optimum allocation is found.

Finding an initial allocation

The initial allocation can be made using any method which will produce a feasible solution. However, a systematic approach tends to produce more useful solutions. We will look at two methods of finding an initial allocation, the minimum cost method and Vogel's method. The procedure is explained in Example 2.

Example 2: To Illustrate methods of finding an Initial allocation

Three warehouses P, Q and R can supply 9, 4 and 8 items respectively. Three stores at A, B and C require 3, 5 and 6 items respectively. What is the minimum cost of allocating the items from the warehouses to the stores if the unit transportation costs are as shown in table below?

Table.2 Costs, requirements and availabilities for Example 2

	Transport costs £/item, to stores			Total available
	A	B	C	
From warehouses P	10	20	5	9
Q	2	10	8	4
R	1	20	7	8
Total required	3	5	6	

Solution

The information on costs, availability and requirements are given but total supply is bigger than demand. The warehouses have 21 items available but the stores require only 14, a dummy store is needed to absorb the 7 items which are surplus to requirements. These 7 items will never actually leave the warehouse, and therefore their transportation costs are assumed to be zero. The first transportation tableau is given below:

Table.3 Balanced transportation tableau

	Transport costs £/item, to stores				Total available
	A	B	C	dummy	
From warehouses P	10	20	5	0	9
Q	2	10	8	0	4
R	1	20	7	0	8
Total required	3	5	6		21

To find the first feasible allocation we will use the minimum cost method and Vogel's method in turn. However, it should be remembered that only one method is actually required.

Method 1: Minimum Cost Method

1. We allocate as much as possible to the cell with the minimum unit cost.
    2. We adjust the remaining availabilities and requirements.
3. We choose the next smallest cost and allocate as much as possible to this cell, *and so on, until supply and demand are all zero.*
4. If more than one cell has the smallest value of unit cost, then we choose at random.

In table.4, the transport costs are placed in the separate boxes in the upper right of each cell. The subscripts indicate the order in which the allocations are made and should help you to follow the explanation. The dash in a cell indicates that this cell is no longer available.

Table 4 First allocation using minimum cost method

	To store					Total available			
	A	B	C		Dummy				
P	10	20	5		0	9	2	0	
	-	-	23	71					
From warehouse Q	2	10	8		0	4		0	
	-	45	-	-					
R	1	20	7		0	8	5	1	0
	32	16	44		-				
Total required	3 0	5 1 0	6 4 0		7 0	21			

Key:

Unit transport cost

Units
allocated

The smallest cost is zero. We choose any one of cells (P, Dummy), (0, Dummy) or (R, Dummy). Cell (P, Dummy) is chosen and we allocate the maximum amount, 7 units, to the cell. We reduce by 7 the amount available at P and the amount required by the dummy. Cross out the cells which now cannot be used, that is, (0, Dummy) and (R, Dummy).

Neither of the other zero costs are available, so the next smallest cost is 1 in cell (RA). We allocate as many units as possible, 3, to this cell. Adjust the row and column totals and cross out cells which are no longer available, that is (P, A) and (Q, A).

The smallest cost, still available, is 5 in cell (P, C). The remaining 2 items at P are allocated to this cell. The row and column totals are adjusted and we cross out the remaining cell in row P.

The final allocations are, in order, (R, C), (Q, B) and (RB).

If the allocation is feasible, the total available at each warehouse and the total required at each store should now be zero. The above allocation is feasible.

Cost £ ((3 x 1) + (4 x 10) + (1 x 20) + (2 x 5) + (4 x 7) + (7 x 0) = £101

We do not know yet whether this allocation is the cheapest but it should give a reasonable cost.

Method 2: Vogel's Method

This method uses penalty costs. For each row and column the penalty cost is the difference between the cheapest available route and the next cheapest. We try to minimise these penalties.

To calculate the penalty cost for each row and column, we look at the least cost cell and next smallest cost cell. For each row and column subtract the smallest cost from the next smallest. This gives the penalty cost of not allocating into the cell with the cheapest cost.

We choose the row or column with the largest penalty cost and allocate as much as possible to the cell with the smallest cost in that row or column. In this way, the high penalty costs are avoided as far as possible.

As with the previous method, we adjust the row and column totals.

We cross out the remaining cells in any row or column for which the supply or demand is now zero, since these cells are no longer available.

Return to 1 and re-calculate the penalty costs, ignoring the cells which have been used or crossed out.

We repeat these steps until all demand is satisfied. The subscripts in table.5 show the order of choosing the penalty costs and making the allocations.

Table 5 First allocation using Vogel's method

	To store					Penalty		
					Total available			
	A　B　C		Dummy			1	2	3
P	10	20	5	0	9,8,2,0	5	5	5
	-	1	6	2				
From warehouse Q	2	10	8	0	4,0	2	-	-
	-	41	-	-				
	1	20	7	0	8,5,0	1	1	73
R	32	-	-	53				
Total required	3	5	6	7	21			
	0	1,0	0	2,0				
1st penalty	1	101	2	0				
2nd penalty	92	0	2	0				
3rd penalty	-	0	2	0				

After the third allocation, there is only one way of completing the solution. The remaining items are allocated as follows—(P, B), (P, C) and (P, Dummy).

Cost = (1 x20+6x5+2x0+4x 10+3x 1+5x0)'93

Again, we do not yet know whether this is optimal but we do know that it *is* a cheaper allocation than the £101 for the minimum cost method.

▮ Testing for optimality

To test for optimality, we must first determine whether the initial allocation is basic, that is, whether it is a solution at a corner of the feasible region. The tableau shown in table 4 gives a feasible solution, that is, a solution inside or on the edge of the feasible region. If the allocation is basic, there should be one basic variable for every constraint. In a problem with m warehouses and n stores (including the dummy), there are (m + n - 1) independent constraints. A basic solution will therefore have (m + n - 1) allocated cells. These (m + n - 1) variables must be in independent positions. It is not necessary to worry about independence at this stage since any problems will emerge during the test for optimality.

If the allocation has (m + n - I) independent variables, the methods for testing for optimality may be applied directly. If there are fewer variables, the tests have to be modified, as will be illustrated in table 6. However, if there are more than (m + n - 1) variables, then the allocation procedure has been used incorrectly. It should be possible to modify the allocation to give a cheaper cost with the correct number of variables.

Refer to Example.2. We will test each of the allocations for basicness. The tableau has 3 rows and 4 columns, therefore a basic solution will have (3 + 4 -1) = 6 allocated cells. We can see that this is the case for both allocation methods. The two methods have given solutions at different corners of the feasible region. The testing procedures may be used without modification.

The initial allocation is tested to determine whether it is the cheapest solution and, if it is not, it should be changed. We will illustrate two methods of testing for optimality.

In the **stepping stone method**, the costs of using the unallocated cells - the shadow costs - are calculated. The procedure is long and rather clumsy but the physical meaning is clear.

The **MODI (modified distribution) method** is a mathematical procedure which gives the same shadow costs much more quickly, although the physical meaning is not so obvious.

In both methods, if the allocation is not optimal, a stepping stone procedure is used to move to the next basic allocation. Once a basic solution has been found, the algorithm enables us to move from corner to corner of the feasible region, until the optimum solution is found.

Example.3: To illustrate the test for optimality using the stepping stone method

We will use the allocation produced by the minimum cost method to illustrate the procedure. The allocation is repeated in table 6 below

Table 6 First allocation using minimum cost method

From warehouse	To store A	B	C	Dummy	Total available
P	10 -	20 -	5 / 2	0 / 7	9
Q	2 -	10 / 4	8 -	0 -	4
R	1 / 3	20 / 1	7 / 4	0 -	8
Total required	3	5	6	7	21

Key:

Units allocated / Unit transport cost

The stepping stones are the cells which have allocations in them - (P, C), (P, dummy), (0 B), (A, A), (R, B) and (A, C). We take one of the empty cells and pretend that we move one item into it. This move upsets the totals for the row or column in which the empty cell lies. The amounts in some of the allocated cells are then adjusted to restore the balance. We use these allocated cells, the stepping stones, to calculate the cost of the transfer of this one item into the empty cell. If the cost is positive, using the empty cell will increase total costs and we do not want to do this. If the cost is negative, using the cell will reduce costs. This means that the present allocation is not optimal and we can find a better solution using this cell.

It does not matter which empty cell is chosen as the start. We will choose (P, A). We add 1 item to (P, A). The allocation is no longer correct. Store A is receiving 4 items, when it wants only 3. Warehouse P is supplying 10 when it has only 9 items. We must adjust the A column and the P row. To balance the A column, we must deduct 1 item from the stepping stone (R, A). This corrects column A, but unbalances row R, reducing its supply from 8 to 7.

We can re-balance row P by subtracting 1 item from either (P, C) or (P, Dummy). If we choose (P, Dummy), there is no other allocated cell in the dummy column which could be used to re-adjust the dummy column, therefore we do not make this choice. Adjustments can be made using only those cells which have allocations already. We must use (P, C). We deduct 1 item from (P, C). This corrects row P, but unbalances column C. We now have problems with row A and column C. Both can be adjusted simultaneously by adding 1 item into (R, C). The physical effect of using the empty cell (P, A) and returning to a balanced allocation is shown in table 7 below.

The net cost effect of moving 1 item into (P, A) is:

+1 x (P, A) cost—1 x (R, A) cost +1 x (R, C) cost—i x (P, C) cost

= + (1 x 10) – (1 x 1) + (1 x 7) – (1 x 5) = +£11/item

Table 7 Testing empty cell (P, A)

Physical change- items

	A	C
P	Test cell +1	Allocated cell -1
R	Allocated cell -1	Allocated cell +1

Using (P, A) would cost an extra £11 for each item sent from P to A. The shadow price is positive therefore we do not choose to use this empty cell.

We return to the original allocation and repeat the procedure for the other empty cells in turn. Look next at the cell (R, Dummy) and use the stepping stones (P, Dummy), (P. C), and (A, C) to show the physical and cost changes of moving 1 unit into (P, Dummy):

Table 8 Testing empty cell (R, Dummy)   Table9 testing empty cell (R,dummy)

Physical   change—Items

cost change, £

	Test cell	Allocated cell			Test cell	Allocated cell
	+1	-1	P		+1	-1
	Allocated cell -1	Allocated cell +1	R		Allocated cell +1	Allocated cell +1

The net cost change of adding 1 item to (R, Dummy) is:

+0—0 + 5—7 = —2 per item

By allocating into cell (R, Dummy), it is possible to reduce the costs, therefore, the present allocation is not optimal. We can find a cheaper allocation, saving £2 per item, by using (R, Dummy) and this stepping stone route. We must, however, complete the testing of all the empty cells, since there may be a cell which gives an even better saving.

Let us next construct the stepping stone path for the empty cell (0, Dummy). We must remember that to re-balance rows and columns we can step on allocated cells only. A four step circuit is not possible this time. We must look for a more complex route. We allocate 1 item to cell (0, Dummy). There is only 1 allocated cell in row Q and only 1 allocated cell in the Dummy column. Suppose we choose to move from (Q, Dummy) to (Q, B). We deduct 1 item from this cell which balances row Q. Column B can be balanced by (R, B) only, therefore we add 1 item to this cell. We can balance row R via (R, A) or (R, C) but (R, A) is the only allocation in column A, so we do not use this cell. If we did, we would not be able to balance column A. We deduct 1 item from (R, C). The route home should now be clear. We balance column C by adding 1 item to (P, C) and balance row P by deducting one item from (P, Dummy). This last move also balances the Dummy column and the circuit is complete. We should remember that as long as the initial allocation is basic, it is possible to find a suitable route round the tableau which starts and ends with the chosen empty cell. The physical effects and cost changes are summarised in Tables.10 and 11.

Table 10 testing empty cell (Q, dummy)

Physical change - items

B	C	Dummy
Empty	Allocated +1	allocated -1
Allocated -1	empty	test +1
Allocated +1	Allocated -1	Empty

Table 11 testing empty cell (Q, dummy)

Cost change £

B	C	Dummy
Empty	Allocated +5	allocated 0
Allocated -10	empty	Test +0
Allocated +20	Allocated -7	empty

The net cost affect of adding 1 item to the empty cell (Q, Dummy) is:

    +0—0+5--7+20—10= +8/item

The total cost of allocation will increase by £8 per item if we allocate into this empty cell. We do not choose to make this change. The shadow costs for the remaining empty cells are calculated in a similar way to that described above. The full set of values is shown in table 12:

Table 12

	To store				Total available
P	10 +11	20 +2	5 2	0 7	9
From warehouse **Q**	2 +11	10 4	8 +11	0 +8	4
R	1 3	20 1	7 4	0 -2	8
Total required	3	5	6	7	21

Key:

Unit transport cost

Units allocated

Shadow cost of empty cost
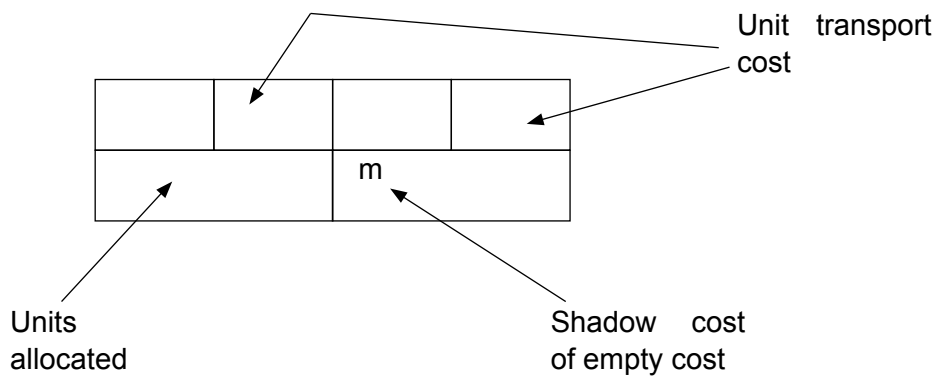
This is not the optimal allocation because cell (R, Dummy) has a negative shadow cost of -.E2. The present cost of the allocation, £101, can be reduced by using this cell and the stepping stone circuit which gave the net saving of £2 per item.

We will continue with this example, to find the optimum allocation, in section.5, but first we will discuss the MOD1 method of calculating the shadow costs. The stepping stone procedure is a clumsy method and it is easy to make errors. It is more sensible to use the mathematical approach of the MODI method to test for optimality. The procedure does not give an insight into the physical problem but it does produce the same shadow prices with much less effort.

*T*o begin with, consider only the allocated cells. Each unit cost, Cjj, for these cells is split into two components, ui for the row and vj for the column. For example, cell (R,B), which is in row 3 and column 2, has a unit cost c32 = £20. This is split into the row component u3 and the column component v2, i.e.

   c32 = 20 = U3 + v2

For each empty (non-basic) cell, we calculate the shadow cost from:

   Sij = cij – (ui + vj)

This shadow cost is the extra cost of transporting one item by the route i to j. If all of the shadow costs are positive or zero, that is, Sij > 0, then the solution is optimal. In this case, if an allocation is moved into an empty cell for which the shadow cost is positive, the total costs will increase, if the shadow cost is zero, the total costs remains unchanged.

Example.4: To test a basic allocation (or optimality using the MODI method

Refer again to the initial allocation obtained using the minimum cost method. We will test this allocation for optimality using the MODI method. The initial allocation is repeated below:

Table 13 First allocation using minimum cost method

		A	B	C	Dummy		Total available
			10		20	5	0	
P			-		-	2	7	9
From warehouse			2		10	8	0	
Q			-		4	-	-	4
			1		20	7	0	
R		3		1		4	-	8
Total required		3		5	6	7	21

The row components, u and the column components, v,are calculated using the allocated cells. The allocated cells are (P, C), (P, Dummy), (0, B), (R, A), (A, B), and (A, C), which give the following six simultaneous equations. These six equations contain seven variables, hence there is no unique solution. The actual values given to the components are not important as long as the set of values is consistent,

c13 = 5 = U1 + v3   for   allocated   cell   P.   C)
c14 = 0 = U1 + v4   for   allocated   cell   (P, Dummy)

c33 = 7 = u3 + v3   for   allocated   cell   (A,   C)
c31 = 1 = u3 + v1   for   allocated   cell   (A, A)

c32 = 20 = u3 = v2   for   allocated   cell   (R,   B)

 c22 = 10= u2 + v2   for   allocated   cell   (0,   B)

A value is assigned to any one of the components, then the values of the other components are found from the equations. We will choose to set u1 = 0. It follows that v3 = 5, v4 = 0, U3 2, v1 = -1, v2 18, and u2 = -8. We can now calculate the shadow costs for the unallocated cells, from the equation:

s c,1 — (u, + v)

Substituting gives the following shadow costs:

s11 = 10 – (0 – (-1)) = + 11   for empty cell (PA)

s12 = 20 – (0 + 18) = +2   for empty cell (P,B)

s21 = 2 – (-8 – 1) = + 11   for empty cell (0, A)

s23 = 8 – (-8 + 5) = + 11   for empty cell (Q, C)

s24 = 0 – (-8 + 0) = + 8   for empty cell(Q, Dummy)

s34 = 0 - (2 + 0) = - 2 for empty cell (R, Dummy)

These values are entered on the tableau in table 14:

Table 14 Testing the Initial allocation - MODI method

	To store				Total available	
	A B C			Dummy		
P	10	20	5	0	9	u1 = 0
	+11	+2	2	7		
From warehouse	2	10	8	0	4	u2 = -8
Q	+11	4	+11	+8		
	1	20	7	0	8	u3 = 2
R	3	1	4	-2		
Total required	3	5	6	7	21	
	V1 = -1	V2 = 18	V3 = 5	V4 = 0		

The shadow costs are the same as those found using the stepping stone method in table 12. Route (R, Dummy) has a negative shadow cost of - £2 per item; therefore the solution is not optimal. Items must be re-allocated using this cell and the associated stepping stone circuit to reduce the costs.

Finding the optimum solution

The iterative procedure for finding the optimum allocation is as follows:

If there is more than one empty cell with a negative shadow cost, choose the cell with the largest negative value.

Find the stepping stone circuit for this empty cell, as described above.

Identify those cells from which items are to *be* deducted and determine the amount which could be deducted from each cell, without any of the allocations becoming negative. The minimum value of these figures gives the maximum amount that can be allocated to the chosen cell. Reallocate around the circuit.

There is no guarantee that further improvements cannot be made. The new allocation must be checked for optimality using the MODI method. The minimum cost is found when all of the shadow costs are positive or zero.

We will continue with Example.4.

Example.4 continued: To test a basic allocation for optimality using the MODI method

Cell (R, Dummy) is the only one with a negative shadow cost, -—2. We wish to allocate as much as possible into this cell.

The stepping stone circuit for (R, Dummy), which gives the —2, is shown below with the existing allocations and Unit costs.

Table 15 Stepping stone circuit for (R, Dummy)

	C	Dummy
P	+ 5	- 0
	2	7
R	- 7	+ 0
	4	-2

+ denotes items are to be added to this cell. – denotes that items are to be deducted from this cell

The - cells are (P, Dummy) and (R, C), which contain allocations of 7 and 4 items. The minimum value in the — cells is 4, which means that 4 items can be moved round the circuit, into the + cells and out of the — cells. The total saving in cost is (2 x 4) = £8. The revised tableau is given in table 16.

Table 16 Revised allocations

	To store A B C Dummy				Total available
P	10 -	20 -	5 2+4	0 7-4	9
From warehouse Q	2 -	10 4	8 -	0 -	4
R	1 3	20 1	7 4-4	0 0+4	8
Total required	3	5	6	7	21

The solution is still basic since there are 6 allocations. We will re-check for optimality, using the MODI method. Using the allocated cells, (P, C), (P, Dummy), (Q, B), (R, A), (R, B) and (R, Dummy):

c13 = 5 = u1 + v3 Choose u1 = 01 then v3 = 5

c14 = 0 = u1 + v4 v4 = 0

c34 = 0 = u3 + v4 u3 = 0

c31 = 1= u3 + v1   v1 = 1

C32 = 20 = u3 + v2   v2 = 20

c22 = 10 = u2 + v2   u2 = -10

The shadow costs for the empty cells are:

Sij = cij – (ui + vj)

S11 = 10 – (0 + 1) = + 9

S12 = 20 – (0 + 20) = 0

S21 = 2 – (-10 + 1) = +11

S23 = 8 – (- 10 + 5) = + 13

S24 = 0 – (- 10 + 0) = + 10

S33 = 7 – (0 + 5) = + 2

None of the shadow costs is negative, therefore the allocation is optimal.

Table 17 Testing the optimum allocation using the MODI method

	To store A B C Dummy				Total available
P	10 +9	20 0	5 6	0 3	9 u1 = 0
From warehouse Q	2 +11	10 4	8 +13	0 +10	4 u2 = -10
R	1 3	20 1	7 +2	0 4	8 u3 = 0
Total required	3	5	6	7	21
	v1 = 1	v2 = 20	v3 = 5	v4 = 0	

The minimum cost is:

£101 + (4 x (— 2)) = £93

Solution:

- Warehouse P sends 6 items to store C and retains 3 items.
- Warehouse Q sends 4 items to store B.
- Warehouse R sends 3 items to store A, 1 to store B and retains 4 items.

If this second allocation is not optimal, the re-allocation procedure is repeated as many times as is necessary.

It should be noted that the minimum cost was achieved on the first allocation using Vogel's method. This will frequently happen for small scale problems. Vogel's method tends to produce a better first tableau but there is no guarantee that it will give the optimum immediately. It should also be noted that the allocation produced by Vogel's method is different to the one above (see Example.2). There is an alternative optimum solution:

    • Warehouse P sends 1 item to Store B, 6 items to Store C and retains 2 items.
    • Warehouse Q sends 4 items to Store B.
    • Warehouse R sends 3 items to Store A and retains 5 items.

The existence of an alternative optimum solution was signaled to us by the zero shadow cost for cell (P, B). These zero shadow costs are associated with alternative allocations which generate the same total cost.

Sensitivity analysis

The final allocation, together with the shadow costs of the empty cells, can be used for sensitivity analysis. The shadow cost tells us by how much the total cost will increase, if we are forced to allocate one item to that empty cell. If we are forced to send one item from warehouse Q to store C, the extra cost will be £13, much more than the unit cost of £18 for the (Q C) route itself. The extra cost arises because we have to re-balance the allocation using the following stepping stone circuit.

Table 18 Stepping stone circuits for (Q, C)

Physical change - items

	B	C	Dummy	
P	empty	Allocated -1	Allocated +1	
Q	Allocated -1	Test +1	empty	
R	Allocated +1	empty	Allocated -1	

Table 19 Stepping stone circuit for (Q, C)

	B	C	Dummy	
P	empty	Allocated -5	Allocated +0	
Q	Allocated -10	Test +8	Empty	
R	Allocated +20	empty	Allocated -0	

The net change in cost is:

$$+8 - 5 + 0 + 0 + 20 - 10 = +£13/item$$

The maximum number of items which would be moved round this circuit is the minimum quantity in the - cell that is:

(P, C) = 6, (R, Dummy) = 4 or (Q, B) = 4

Four items is the maximum number which could be moved.

The zero shadow cost in cell (P, B) was mentioned in the previous section. The stepping stone circuit for this empty cell is:

Table.20 stepping stone Circuit for (P, B)   Table.21 stepping stone circuit for (P, B)

test +1	allocated -1
allocated —1	allocated +1

test +20	allocated -0
allocated -20	allocated +0

Items can be allocated to cell (P, B) and the net effect on the cost is zero. This means that there is another allocation which will give the same minimum cost of £93. The maximum number of items which can be added to (P, B) is the minimum quantity in the — cells, (R, B) = 1 and (P, Dummy) = 3. Only one item can, therefore, be moved around the circuit into (P, B).

The shadow costs may also be used to indicate how the cost for an empty cell must change before the optimum allocation is affected; the shadow cost for the empty cell (R, C) is + £2, and the actual cost of transfer is £7 per item. The actual cost would have to reduce to at most (7 - 2) = £5 per item, before we would use this cell to reduce the overall costs.

It is more difficult to determine the effect of cost changes in the allocated cells. If the costs reduce, we are encouraged to put more items into that cell. If the costs of an allocated cell increase, at some point we will wish to stop using that cell and transfer to another.

If we look at the allocated cell, (P, C), it has an actual cost of £5 per item. If this cost is reduced, it will not affect the physical allocation since this cell already has the full requirement for Store C allocated to it.

If the cell cost is increased from £5, we must look at the stepping stone circuits which use (P, C). These are the circuits which give the shadow costs of £3 for (Q, C) and £2 for (R, C). In both of these circuits (P, C) is a — cell and any increase in the £5 cost will reduce the shadow cost of these empty cells. The physical allocation will change when the unit transfer cost of (P, C) increases by more than £2, from £5 to more than £7. The shadow cost of (R, C) will then become negative. At this point, it will be advantageous to use this empty cell, changing the (P, C) allocation.

For the current optimum allocation, the cost of (P, C) has an upper limit of £7 and a lower limit of £0. Between these limits the physical allocation is the same but the total costs will change.

Variations in the transportation problem

Forbidden Allocations

If an allocation from a particular origin to a particular destination is impossible for some reason, the algorithm can be forced to avoid this allocation by assigning a large cost to the cell. The exact value is unimportant but it must be much larger than the other costs in the tableau. The algorithm will then automatically avoid this cell.

Example 5: Forbidden allocations

This example also illustrates how the transportation algorithm can be used to solve problems which are not the straight forward transfer of goods from origins to destinations. In this example, we will deal with the transfer of goods through time. We have a four-month production schedule to meet. The demand and production capacities are given below.

Table 22 Demand and production capacity

Month	Production capacity	Demand items
1	300	300
2	350	275
3	325	400
4	375	300

There is an initial stock of 50 items held at the beginning of month 1. Items can be made to meet immediate demand or for stock to meet future demand. If orders are not met during the required month, the sales are lost. The variable costs are £100 per item. The stockholding costs are £2 per item per month. What is the optimum production schedule?

Solution

The situation can be modelled using a transportation tableau with the rows representing the initial stock and monthly production, and the columns representing monthly requirements. The forbidden cells are those which involve meeting past orders from future production. These cells are given an infinite cost in table.23.

Table 23   Data for production schedule for months 1 - 4

	Cost £ per item months M1 M2 M3 M4	Total Available			
stock M1	2	4	6	8	50
M1 Production	100	102	104	106	300
M2	∞	100	102	104	350
M3	∞	∞	100	102	325
M4	∞	∞	∞	100	375
Total demand	300	275	400	300	

The normal procedure is now followed to solve this transportation problem to minimise the cost of meeting the schedule (See Example 8).

Degeneracy

A solution is degenerate when there are fewer than (m + n - 1) allocations in the tableau. This problem can be overcome by allocating a very small amount, essentially zero, to an independent cell. The number of allocations is then increased to (m + n — 1). The procedure in the MODI test for optimality will show us which empty cells to use.

Example 6: Degenerate solutions

Three warehouses (X, Y and Z) can supply 6, 3 and 4 items to 3 shops (L, M and N), which require 4, 5 and 1 items respectively. The unit costs of transport are given in the tableau.

Table 24 Data for example.6

	To shop (£/item) L M N	Total Available	
X	6	4	9	6
From Warehouse Y	5	3	2	3
Z	2	3	6	4
Total Required	4	5	1	

How should the items be allocated in order to minimise the total cost of transport?

Solution

There is a total of 13 items available which is more than the total requirement for 10 items, therefore, we include a dummy shop which absorbs the surplus from the warehouses. We will use Vogel's method to find an initial allocation:

Table.25 initial allocation for example 8 using Vogel's method

	To shop					Penalty cost		
	L	M	N	Dummy	Total available	1	2	3
X	6 -	4 3	9 -	0 3₁	6 3 0	4₁	2	2
From warehouse Y	5 -	3 2	3 1₂	2 -	3 2 0	2	1	2
Z	2 4₃	3 -	3 -	6 -	4 0	2	1	1
Total required	4 0	5 3 3	1 0 0	3 0	13			
1st penalty	3	0	4	0				
2nd penalty	3	0	4₂	-				
3rd penalty	3₃	0	-	-				

The cost of the allocation is:

4 x 3+0 x 3+3 x 2+2 x I +2 x4=28

For a basic solution, there should be $(3 + 4 — 1) = 6$ allocations, but here there are only 5. The allocation is degenerate. As the MODI method proceeds, we will have to make one zero allocation, to convert an empty cell into a pretend allocated cell. This gives the required total of 6 allocated cells. It will then be possible to calculate all of the u and v components and hence the shadow costs.

We begin the MODI procedure using the initial 5 allocated cells. The extra zero allocation is made when we can go no further. See table.26.

The allocated cells are used to find the row and column components from $c_{ij} = u_i + v_j$ with $= 0$. We can calculate v_2, v_4, u_2 and v_3 without any problem but we cannot find u_3 or v_1. We need an additional allocated cell. We can put the zero allocation into any empty cell in the v_1 column or the u_3 row. It does not matter which of these cells is chosen. Cell (Z, N) is used. The procedure can be completed and the shadow costs found for the empty cells from s_{ij} c_{ij} $- (u_i + v_i)$. The figures are given in the table:

Table 26 Testing a degenerate allocation—MODI method

	To shop					
	L	M	N	Dummy	Total available	
	6		4	9	0	$u_1=0$
X	+7	3	+6	3	6	
From warehouse	5		3	2	0	$u_2=-1$
Y	+7	2	1	+1	3	
	2		3	6	0	$u_3=3$
Z	4	-4	0	-3	4	
Total required	4	5	1	3	13	
	$v_1=-1$	$v_2=4$	$v_3=3$	$v_4=0$		

Two of the shadow costs are negative. The allocation is not optimal. It is necessary to re-allocate into cell (Z, M) or (Z, Dummy). We will start with (Z, M), since this has the larger negative shadow cost. The stepping stone circuit for (Z, M), showing the items allocated to each cell is:

Table 27 Stepping stone circuit for (Z, M)

M	N
Allocated -2	Allocated +1
Test +	Zero allocation 0

To find the number of items to move round the circuit, we look at the — cells, (Y, M) and (Z, N), which contain 2 and zero units. This means we must move the zero allocation round the circuit so that cell (Z, N) becomes empty again, cell (Z, M) takes the zero allocation and becomes the pretend allocated cell. The other allocations remain unchanged. When we test for optimality this time, we find that all of the shadow costs are positive. The allocation is optimal. This means that the initial solution with the 5 allocations is actually the optimal solution. See the tableau in table 28:

Table 28   Testing the optimal allocation – MODI method

| | To shop | | | | | |
	L	M	N	Dummy	Total available	
X	6	4	9	0		u1=0
	+3	3	+6	3	6	
From warehouse Y	5	3	2	0		u2=-1
	+3	2	1	+1	3	
Z	2	3	6	0		u3=-1
	4	0	+4	+1	4	
Total required	4	5	1	3	13	
	v1=3	v2=4	v3=3	v4=0		

This type of result does occur in some problems in which then an ordinary allocation is degenerate. There will be some problems in which the re-allocation process adds items to the zero allocation cells and it becomes an ordinary allocated cell. The degeneracy of the solution then disappears. The usual procedures will then lead to a better allocation.

Maximisation

The transportation algorithm assumes that the objective function is to be minimised. However, if a suitable problem requires the objective function to be maximised, the algorithm can be modified slightly to deal with this. For example, we may wish to transfer the items in Example 5 in such a way that the total contribution is maximised. In such a case the data required are the unit contributions between each origin and destination. The procedure is to multiply all of the unit contributions by (-1_ and then to proceed in the usual way.

The Assignment Problem

The assignment problem is a special case of the transportation problem, in which the number of origins must equal the number of destinations, that is, the tableau is square. Also, at each destination, the 'demand' = 1 and at each origin the 'supply' = 1. Any assignment problem may be solved using either linear programming or the transportation algorithm. However, the particular structure of this problem has resulted in the development of a specially designed solution procedure called the **Hungarian algorithm**

The assignment algorithm

The algorithm has three stages to it.

Stage 1:

1. Set out the problem in tableau format, as in the transportation algorithm.
2. For each row in the tableau, find the smallest row element and subtract it from every element in the row
3. Repeat for the columns.

There is now at least one zero in every row and every column. The assignment problem represented by this 'reduced' tableau is equivalent to the original problem and the optimum allocation will be the same for both, the objective of the Hungarian algorithm is to continue to reduce the matrix until all of the items to be assigned, can be allocated to a cell with a zero value. This means that the total value of the reduced objective function will be zero. Since negative values are not allowed, an objective function values of zero is the optimum.

Stage 2: For a feasible solution, there must be exactly one assignment in every row and every column.

1. If the assignments are made only to cells with zero values, this will give us the minimum value of the objective function.
2. Find a row with only one zero in it, and make an assignment to this zero. If no such row exists, begin with any zero.
3. Cross out all other zeros in the same column.
   Repeat 1 and 2 until no further progress can be made.
   If, at this stage, there are still zeros which are not either assigned or deleted, then:
4. Find a column with only one zero and assign to it.
5. Cross out all other zeros in the same row.
6. Repeat 4 and 5 until no further movement is possible.

If all zeros are still not accounted for, repeat I to 6. If the solution is feasible, that is, all of the allocations have been made to zeros, and then the solution must also be optimal. If the solution is not feasible, go on to Stage 3.

Stage 3:

1. Draw the minimum number of straight lines through the rows and columns (not diagonals) that all zeros in the tableau are covered.
2. Find the smallest element without a line through it.
3. Subtract this number from every element without a line through it.
4. Add the chosen number to every element with two lines through it.
5. Leave alone all elements with one line through them.

This procedure has now created at least one new zero. Return to Stage 2 and repeat the procedure until the optimum solution is reached.

Example 7: To illustrate the application of the assignment algorithm

A company has 4 distribution depots and 4 orders to be delivered to separate customers. Each depot has one lorry available which is large enough to carry one of these orders. The distances between each depot and each customer are given in table.29:

Table 29 Distance from depots to customers

	Distance in miles Customers			
	I	II	III	IV
A	68	72	75	83
Depot　B	56	60	58	63
C	38	40	35	45
D	47	42	40	45

How should the orders be assigned to the depots in order to minimise the total distance travelled?

Solution

It will help you to understand the problem if you try to find a solution to it using a familiar technique, before applying the mechanics of the Hungarian algorithm. Try using Vogel's penalty cost method. See how close you are to the optimum given at the end of the section. The availabilities and requirements are one for each row and column.

Stage 1 of the Hungarian algorithm: Find the smallest row elements.

Table 30　To find the smallest row elements

	Customers				Smallest row element
	I	II	III	IV	
A	68	72	75	83	68
Depot　B	56	60	58	63	56
C	38	40	35	45	35
D	47	42	40	45	40

Subtract the smallest element from each element in its row

Table 31

0	4	7	15	
0	4	2	7	
3	5	0	10	
7	2	0	5	
0	2	0	5	Smallest column element

Subtract the smallest column element from each element in its column

Table 32

0	2	7	10
0	2	2	2
3	3	0	5
7	0	0	0

Make assignments as described in Stage 2 above; (0) denotes an assignment

Table 33

(0)	2	7	10
0	2	2	2
3	3	(0)	5
7	(0)	0	0

We can make only three zero assignments and we require four. This is not a feasible solution. Go on to Stage 3. We draw the least number of lines to cover all of the zeros.

Table 34

(0)	2	7	10
0	2	2	2
3	3	(0)	5
7	(0)	0	0

The smallest number without a line through it is 2. Adjust the tableau as described in Stage 3 above, that is, deduct 2 from any number without a line through it, add 2 to any number at the intersection of two lines and leave all the remaining numbers, which are cut by one line. We then re-allocate depots to customers.

Table 35 Adjusted tableaus with assignments to zeros

	I	II	III	IV
A	(0)	0	7	8
B	0	(0)	2	0
C	3	1	(0)	3
D	9	0	2	(0)

We have now made the required four allocations to zeros and therefore the solution is optimal. We allocate depot A to customer I, depot B to customer II, depot C to customer III and depot D to customer IV. The solution, though optimal, is not unique. (C, III) must always be allocated because it is the only zero in the C row. There are two other optimum allocations.

Table 36 First alternative optimum allocation

	I	II	III	IV
A	(0)	0	7	8
B	0	0	2	(0)
C	3	1	(0)	3
D	9	(0)	2	0

Table 37 Second alternative optimum allocation

	I	II	III	IV
A	0	(0)	7	8
B	(0)	0	2	0
C	3	1	(0)	3
D	9	0	2	(0)

The minimum mileage for each of these three allocations can be calculated from the original tableau:

Allocation 1 = 68 + 60 + 35 + 45 = 208 miles

Allocation 2 = 68 + 63 + 35 + 42 = 208 miles

Allocation 3 = 72 + 56 + 35 + 45 = 208 miles

All three solutions give the same total mileage.

Note: For larger problems than the one in Example.7, it may be more difficult to be sure that, at Stage 3, step 1, the minimum number of lines has been drawn to cover all of the zeros. The following rule of thumb' may be helpful:

    i.   Choose any row or column which has a single zero in it.
    ii.   If a row has been chosen, draw a line through the column in which the zero lies.
    iii. If a column is chosen, draw a line through the row in which the zero lies.
    iv. Repeat steps 1 to 3 until all of the zeros have been covered.

Special cases of the assignment problem

Maximise the objective function

The assignment algorithm is designed to minimise the objective function. If we have an assignment problem, but we wish to maximise the objective function, we deal with this as we would for the transportation algorithm. We set up the first tableau and multiply all the values in the cells by -1.

Example 8: To use the assignment algorithm to maximise an objective

A company has 6 sales areas and 6 salesmen. From past experience it is known that the salesmen perform differently in the different areas. The company's sales director has estimated sales for each person in each area. These are given in table 38:

Table 38 Sales by area and salesmen

| | Sales, £ 000 Area | | | | | |
	I	II	III	IV	V	VI
A	68	72	75	83	75	69
B	56	60	58	63	61	59
salesman C	35	38	40	45	25	27
D	40	42	47	45	53	36
E	62	70	68	67	69	70
F	65	63	69	70	72	68

How should the sales director assign the salesmen to the areas to maximise total sales?

Solution

We multiply all of the values in the tableau by (-1):

Table 39 Modify data and find smallest row element.

| | Area | | | | | | Smallest element |
	I	II	III	IV	V	VI	
A	-68	-72	-75	-83	-75	-69	-83
B	-56	-60	-58	-63	-61	-59	-63
C	-35	-38	-40	-45	-25	-27	-45
D	-40	-42	-47	-45	-53	-36	-53
E	-62	-70	-68	-67	-69	-70	-70
F	-65	-63	-69	-70	-72	-68	-72

We subtract the smallest (most negative) element from each element in the row.

Table 40 Subtract row elements and find smallest column elements

	Area					
	I	II	III	IV	V	VI
A	15	11	8	0	8	14
B	7	3	5	0	2	4
C	10	7	5	0	20	18
D	13	11	6	8	0	17
E	8	0	2	3	1	0
F	7	9	3	2	0	4
	7	0	2	0	0	0

Smallest element

We subtract the smallest column element from each element in the column.

Table 41 Subtract smallest column element

	Area					
	I	II	III	IV	V	VI
A	8	11	6	0	8	14
B	0	3	3	0	2	4
C	3	7	3	0	20	18
D	6	11	4	8	0	17
E	1	0	0	3	1	0
F	0	9	1	2	0	4

Forbidden Allocations

Again this problem is solved in the same way as it was in the transportation algorithm. If a particular assignment is impossible for some reason, we insert a value into the relevant cell which is much bigger than any other value. The algorithm will then automatically avoid this allocation.

Unequal numbers of origins and destinations

If the tableau is not square, then additional dummy rows or columns must be included to make it square. The values assigned to these dummy cells will usually be zero. Destinations which receive allocations from dummy rows (origins) are the ones which, in practice, will not receive an allocation. Allocations which are made to dummy columns represent items which are not allocated.

Exercise 1

In the Kingdom of the Republic of Idion there are five coal mines which have the following outputs and production costs:

Mine	output (tonnes/day	production cost (£/tonne)
1	120	25
2	150	29
3	80	34
4	160	26
5	140	28

Before the coal can be sold, it must be 'cleaned' and graded at one of three coal preparation plants. The capacities and operating costs of these three plants are as follows:

Plant	capacity (tones/day)	operating cost (£/tonne)
A	300	2
B	200	3
C	200	3

All coal is transported by rail at cost of £0.5 per tonne kilometer, and the distances (in kilometers) from each mine to the three preparation plants are:

Preparation plant	Mines (Distance)				
	1	2	3	4	5
A	22	44	26	52	24
B	18	16	24	42	48
C	44	32	16	16	22

Required

a) Using a transportation model, determine how the output of each mine should be allocated to the three preparation plants.

b) Following the installation of new equipment at coal mine number 3, the production cost is expected to fall to £30 per tonne. What effect, if any, will this have on the allocation of coal to the preparation plants?

c) It is planned to increase the output of coal mine number 5 to 180 tonnes per day which can be achieved without any increase in production cost per tonne. How will this affect the allocation of coal to the preparation plants?

(ACCA, June 1986)

Answer

a)   Five mines supply three preparation plants

The total mines output per day   = 650 tonnes/day

The total preparation plant capacity   = 700 tonnes/day

Therefore introduce a dummy mine to indicate which plant will not be fully used.

The unit costs for each combination of mine and preparation plant comprise:

Unit variable production cost at the time

+ unit operating cost at the plant

+ unit transport cost

The values shown in the following tableau. Vogel's penalty cost method is used to find the first allocation and the MODI method to test for optimality.

	To prep plant A	B	C	Tons/day available	Penalty costs	MODI
1	38 — 70_5	37 — 50_4	50 — $+16_-$	120,70,0	1 12_5	$u_1=0$
2	53 — $+12_-$	40 — 150_3	48 — $+11_-$	150 0	8_3	$u_2=3$
3 (From mine)	49 — 40	49 — $+1_-$	45 — 40	80 0	4	$u_3=11$
4	54 — **+13-**	50 — $+10_-$	37 — 160_2	160 0	13_2	$u_4=3$
5	42 — 140	55 — $+14_-$	42 — $+4_-$	140 0	0	$u_5=4$
Dummy	0 — 50_1	0 — $+1_-$	0 — $+4_-$	50 0	0	$u_6=-38$
Tones/day required	300,250, 180,40, 0	200 50 0	200 40 0			
Penalty costs	38_1 4 7	37 3 12_4	37 5 3			
	$V_1=38$	$V_2=37$	$V_3=34$			

There must be (m – n - 1) = 8 entries for a basic solution. There are 8 entries. All the shadow costs are positives, therefore, this is the optimum allocation.

Mine 1 supplies 70 tonnes per day to A and 50 to B; mine 2 supplies 150 tonnes per day to B; mine 3 supplies 40 tonnes per day to A and 40 to C; mine 4 supplies 160 tonnes per day to C; and mine 5 supplies 140 tonnes per day to A.

Preparation plant A has 50 tonnes per day spare capacity even though it has the cheapest operating costs. The total costs of the above allocation are:

70 x 38 + 50 x 37 + 150 x 40 + 40 x 49 + 40 x 45 + 160 x 37 + 140 x 42 = $26,070/day

a)   Production costs at Mine 3 fall from $34 to $30 per tonne. All mine output is already taken by the plants and production costs are like a fixed cost and do not affect the allocation, therefore total cost will be reduced by 80 x 4 = $320 per day

b)   Mine 5 plans to increase output by 40 tonnes per day from 140 to 180. All of Mine 5's output is allocated to Plant A which has 50 tonnes per day spare capacity. The extra 40 tonnes per day output will go from Mine 5 TO Plant A, increasing costs by 40 x 42 = $1,680 per day

Exercise 2

a)   Briefly describe and contrast two methods of finding an initial feasible solution to a transportation problem

b)   The Braintree Electronics Company produces video cassette tapes for purchase by the general public. The demand (in hundreds) and production capacities (in hundreds) for the three months in the forth quarter of the year are shown below

Month	Normal time	Overtime Demand	capacity
October	300	400	150
November	450	400	150
December	800	400	150

Note that capacity exists to produce the video cassette tapes by working normal and overtime hours, where capacity remains constant over time but demand increases for the Christmas sales. The company does not have any initial inventory and does not wish to have any inventory on hand after December.

The costs for production of the video cassette tapes are £150 (per hundred) if produced during normal working hours and £180 (per hundred) if produced during overtime. It has been determined that inventory costs are £20 (per hundred) per month. You should assume that all orders are satisfied on time and that all demands and outputs occur at the midpoint of each month.

Required

i*)*   Formulate this production scheduling situation as a transportation problem with six 'sources' and three 'destinations', showing the unit cost associated with each source destination combination

ii*)*   Use the transportation algorithm to find the optimum production schedule over this period. State the total production cost of your solution

(ACCA, June 1988)

a)   Describe two methods of finding an initial feasible solution. We describe two methods these are the minimum cost and Vogel's penalty cost methods

b*)*   Braintree Electronics – video tapes. There are six sources of production: October normal (400) and December overtime (150). Total capacity is 1650 ('00 tapes) and total demand is 1550 ('00 tapes). Therefore, we need a dummy demand column. The costs in the tableau are production and inventory costs.

	(£/`00 video tapes0 To demand				Total capacity	
	Oct	Nov	Dec	Dummy		MODI
Oct: normal	150 300	170 50	190 50	0 **+30_**	400 100 50 0	u1=0
Overtime	180 **0_**	200 **0_**	220 50	0 100	150 100 0	u2=30
Nov: normal	∞ _	150 400	170 **0_**	0 **+50_**	400 0	u3=-20
Overtime	∞ _	180 _	200 150	0 +20_	150 0	u4=10
Dec: normal	∞ _	∞ _	150 400	0 +70_	400 0	u5=-40
Overtime	∞ _	∞ _	180 150	0 **+40_**	150 0	u6=-10
Total demand	300 0	450 50 0	800 400 200 50 0	100 0	1650	
MODI	v1=150	v2=170	v3=190	v4=-30		

The initial allocation was done using the minimum cost method. There are 9 entries in the table giving a basic solution. The MODI method is used to test for optimality.

All the shadow costs are positive or zero, therefore, this allocation will give the minimum cost. The total cost is:

300 x 150 + 50 x 170 etc = £251,000

The zero shadow costs mean that there are other allocations (found using the stepping stone method) that will give this same cost.

Exercise 3

    a) Explain the terms
       i) Degeneracy
       ii) Inequality of supply and demand
           Non-unique solution,
       iv) in transportation problems, explain how the transportation algorithm is adapted to overcome these difficulties
    b) The total Wedgetown Pottery Company has orders to be completed next week for three of its products – mugs, cups, bowls – as given in the table below

Product	Order units
Mugs	4000
Cups	2400
Bowls	1000

There are three machines available for the manufacturing operations, and all three can produce each of the products at the same production rate. However, the unit costs of these products vary depending upon the machine used. The unit costs (in £) of each machine are given in the following table

		Mugs	cups	bowls
	A	1.20	1.30	1.10
Machine	B	1.40	1.30	1.50
	C	1.10	1.00	1.30

Furthermore, it is known that capacity for next week for machines B and C is 3000 units and for machine A is 2000 units

Required

i) Use the transportation model to find the minimum cost production schedule for the products and machines. Determine this minimum cost

ii) If this optimal solution is not unique, describe all other production schedule with the minimum cost. If the production manager would like the minimum cost schedule to have the smallest number of changeovers of production on machines, recommend the optimal solution

Answers

(a) (i) Degeneracy:

In the transportation algorithm, an allocation is degenerate if the number of routes used is less than (number of rows + number of columns - I). The difficulty may be overcome by the use of dummy routes. For example, if the actual number of routes used is one too few, then an empty cell is selected and treated as if it were an allocated cell, if we wish, we can allocate a very small amount to this cell. The amount allocated should be so small that the associated transport costs can be ignored. If the transport allocation is degenerate by two routes then we use two empty cells in this way, and so on.

(ii) Inequality of supply and demand:

The transportation algorithm requires the total amount demanded at all of the destinations to be equal to the total amount available at all of the origins. If this is not the case, then the algorithm cannot be used until a dummy origin or destination has been added to satisfy this condition.

If the total supply exceeds the total demand, then a dummy destination is added which has a demand equal to the excess supply. If the total supply is less than the total demand, then a dummy origin added which has a supply equal to the shortfall. The transportation costs for all routes involving the dummy are zero.

(iii) Non-unique optimal solution:

The optimal solution is non-unique if there is more than one allocation of routes which produce the minimum cost. The existence of other optimal allocations can be identified by examining the shadow cost. If any of the shadow costs are zero, then alternative optima exist.

A zero shadow cost means that items may be moved into that cell without increasing the total cost of the allocation. Non-unique optimal solutions are not a difficulty in the transportation problem, but it does mean that the decision maker must use some criterion, other than cost to choose the allocation of routes to be used.

(b) (i) Find the minimum cost production schedule using the transportation method:
Total capacity of the machines is 8,000 units.
Total requirement is 7,400 units.
Therefore (here is excess capacity of 600 units. A dummy product must be included in the transportation tableau, for which the demand is 600 units next week.
Set up a transportation tableau and use Vogel's method to make the initial allocation.

	Product				capacity	penalty
	mugs	cups	bowls	dummy		
A	1.20 10004	1.30 _	1.10 10003	0.00 _	2000 1000 0	1.10,0.10 0.10
B Machine	1.40 24004	1.30 _	1.50 _	0.00 6001	3000 2400 0	1.30,0.10 0.10
C	1.10 6004	1.00 24002	1.30 _	0.00 _	3000 600 0	1.00,0.10 0.20
Demand	4000 0	2400 0	1000 0	600 0		
Penalty	0.10 0.10 0.10	0.30 0.302	0.20 0.20 0.203	0.00		

For a basic allocation we require:

No of allocated cells = no of rows + no of columns -1

In this case, no of rows + no of columns 3 + 4-1= 6, which is the same as the number of allocated cells. The allocation is basic.

Test for optimality

For each allocated cell, we split the unit cost, cij, into a row component, ui, and a column component, vj

For each empty cell, we calculate the shadow cost, sij, where suj = cij – (ui + vj)

Allocated cells:

C11 = 1.20 = u1 + v1   let u1 = 0.00, then v3 = 1.20

C13 = 1.10 = u1 + v3   v3 = 1.10

C21 = 1.40 = u2 + v1   u2 = -0.20

C24 = 0.00 = u2 + v4   v4 = -0.20

C31 = 1.10 = u3 + v1   u3 = -0.10

C31 = 1.10 = u3 + v1 u3 = -0.10

C32 = 0 = u3 + v2 v2 = -1.10

For the empty cells:

S12 = 1.30 – (0.00 + 1.10) = 0.20

S14 = 0.00 – (0.00 + 0.20) = 0.20

S22 = 1.30 – (0.30 + 1.00) = 0.00

S23 = 1.50 – (0.20 + 1.10) = 0.20

S33 = 1.30 – (-0.10 + 1.10) = 0.30

S34 = 0.00 – (-0.10 – 0.20) = 0.30

All of the shadow costs are positive or zero, therefore the allocation is optimal. Since there is a zero shadow cost, we know that it is a non-unique optimum. The minimum cost production schedule is.

Machine A makes 1,000 mugs and 1,000 bowls and is fully used;

Machine B makes 2,400 mug and is under-used by 600 units;

Machine C makes 600 mop and 2,400 cups and is fully used.

The minimum cost is:

1.20 x 1000 + 1.10 x 1000 + 1.40 x 2400 + 1.10 x 600 + 1.00 x 2400 = £8720

(ii)   Select the optimum allocation which has the least number of machine changeovers:
The current optimum schedule requires 2 machine changeovers (1 on each of machines A and C)

Using the stepping-stone method we can allocate N items to cell (B cups) and

Adjust the row and column allocations. We find that N 2400 and that the solution has become degenerate. However, the solution is still optimum. The total cost is £8720. The alternative optimum is:

Machine	Product mugs		cups		bowls		dummy	
A	1.20		1.30		1.10		0.00	
	1000		–		1000		–	
B	1.40		1.30		1.50		0.00	
			2400		–		600	
C	1.10		1.00		1.30		0.00	
	3000		–		–		–	

This solution requires only one changeover (machine A) and therefore is the preferred choice.

The preferred minimum cost production schedule is:

Machine A makes 1,000 mugs and l000 bowls and is filly used;

Machine B makes 2,400 cups and is under-used by 600 units;

Machine C makes 3,000 mugs and is fully used.

Exercise 4

a) Explain briefly how the transportation algorithm can be modified for profit maximisation rather than the minimisation of costs.

b) The Orange Computer Company manufactures one product, a dot-matrix printer, which is currently in short supply. Four of Orange's main outlets, large specialty computer shops at Abbotstown, Beswich, Carlic and Denstone, already have requirements which in total exceed the combined capacity of its three production plants at Rexford, Eadon and Tristron. The company needs to know how to allocate its production capacity to maximize profits

Distribution costs (£) per unit from each production plant to each speciality shop are given in the following table.

		To Abbotstown	To Beswich	To Carlic	To Denstone
		£	£	£	£
	Rexford	22	24	22	30
From	Seadon	24	20	18	28
	Tristron	26	20	26	24

Since the four specialty shops are in different parts of the country, and as there are differing transportation costs between the production plants and the specialty shops, along with slightly three of its different production costs at each of the production plants, there is a pricing structure that enables different prices to be charged at the four shops. Currently the price per unit charged is £230 at Abbotstown, £235 at Beswich, £225 at Carlic, and £240 at Denstone. The variable unit production costs arc £150 at plants Rexford and Tristron, and £155 at plant Seadon.

Required:

i) Set up a matrix showing the unit contribution to profit associated with each production plant/speciality shop allocation.

ii) The demands at Abbotstown, Beswich, Carlic, and Denstone are 850, 640, 380, and 230 to produce respectively. The plant capacity at Rexford is 625, at Seadon is 825, and at Tristron is 450. Denstone vary. Use the transportation algorithm to determine the optimal allocation.

iii) Determine the contribution to profit for the optimal allocation.

(ACCA, June 1990)

Answers

(a) How can the transportation algorithm be modified to maximise rather than minimise? Instead of minimising the positive unit costs of all of the cells, calculate the unit profits, make them negative and put these in each cell. Use the transportation algorithm as usual to minimise these negative profits.

Alternatively, load the cells with the largest profits (instead of smallest costs) to give an initial allocation. Test the empty cells as usual, but use any cell which has a price. If all the shadow prices are negative or zero, then allocation give the maximum profit.

(b) Orange produces printers in three factories and sells them to four main outlets. How many of the available printers should be supplied from each factory to each outlet in order to maximise profit?

Factories R, S and T supply shops A, B, C and D.

The contribution per printer = selling price at the shop for a particular cell

- variable cost at the factory

- Factory to shop transport cost

For example the contribution per printer supplied A = 230 – 50 = £180 per printer.

The matrix of contributions is given below:

		Contribution £ per printer			
		To Shop			
		A	B	C	D
From Factory	R	58	61	53	60
	S	51	60	52	57
	T	54	65	49	66

The total demand from the four shops is:

850 - 640+380+ 230 = 2100 printers

The total supply t the three Factories is:

625+825+450=1900 printers

There is a shortfall of 200 printers and a dummy factory is needed in the transportation algorithm to take up the 200 shortfall. In the transportation algorithm, printers supplied by the dummy factory represent printers which are not supplied to that particular shop.

	To shop				Total Available	penalty
	A	B	C	D		
R	58	61	53	60		1 3
	6253	—	—	—	625 0	
S	51	60	52	57	825 405	3 8
From factory	256	4204	3805	—	25 0	8 1
T	54	65	49	66	450 220	1 ... 1 12
	—	2202	—	2301	0	
Dummy	0	0	0	0		0 0
	2007	—	—	—	200 0	0 0
Demand	850 225 200 0	640 420 0	380 0	230 0	2100	
Penalty	4 4 73 51 516	4 4 1 604	1 1 1 52 525	61		

The penalty cost method will be used to give a first allocation, except that the difference between the largest profit and the next largest is taken to give the penalty instead of the smallest cost - the next smallest cost.

This m allocation is b where there are (r + c-1 =) seven allocated cells. There are seven entries therefore the MODI method may be used to check for optimality without any problems. We use each allocated cell to calculate each row component, v, and each column component, v. For each allocated cell:

cij = ui + vj

We choose to sit u = 0 and the other values follow. For each empty cell we calculate

the shadow profit using:

sij = cij – (ui + vj)

This gives the following table:

		To shop					U
		A	B	C	D		
R		58	61	53	60		u1=0
		625	*-6*	-6	*-8*		
S		51	60	52	57		u2=-7
	From factory	25	420	380	-4		
T		54	65	49	66		u3=-2
		-2	220	*-8*	230		
Dummy		0	0	0	0		u4=-58
		200	*-9*	*-1*	*-10*		
V		v1=58	v2=67	v3=59	v4=68		

All of the shadow prices are negative, therefore any change would reduce the contribution, consequently this is the optimum allocation.

The Rexford factory supplies all of its 625 printers to Abbotstown.

Seadon supplies 25 to Abbotstown, 420 to Beswich and 380 to Garlic.

Triston supplies 220 to Beswich and 230 to Ocustonc.

Abbotstown does not receive 200 oldie printers it asked for.

(iii) The contribution to profit of this allocation is:

58 x 625 +51 x 25 + 60 x 420 + others

Exercise 5

(a) The assignment problem can be regarded as a special case of a transportation problem.
Describe these special features of the assignment problem and explain why the transportation algorithm tends not to be used to solve such problems.

(b) The Midland Research Association has recently been notified that it has received government grant for the research grants to undertake four major projects. The managing director has to assign a research officer to each of these projects. Currently there are five research officers - Adams, Brown, Carr, Day, Evans who are available to carry out these duties. The amount of time required to complete each of the research projects is dependent on the experience and ability of the research optimal officer who is assigned to the project. The managing director has been provided with an estimate of the project completion time (in days) for each officer and each project.

Research officer	Project			
	1	**2**	**3**	**4**
Adams	80	120	60	104
Brown	72	144	48	110
Carr	96	148	72	120
Day	60	108	52	92
Evans	64	140	60	96

As the four projects have equal priority, the managing director would like to assign research officers in a way that would minimise the total time (in days) necessary to complete all four shops at projects.

Required:

(i) Determine an optimal assignment of research officers to projects, and hence determine the minimum total number of days allocated to these four projects.

(ii) State any further allocations that would result in the same total number of days. If research officers Brown, Carr and Day express a preference for projects 2 or 3, whilst officers Adams and Evans express their preference for projects 1 or 4, which of the optimal allocations seems to be the most sensible for the managing director to make?

iii) What feature of this particular project duration matrix could be exploited to simplify the problem?

(ACCA, June 1989)

Answers

(a) The special features of the assignment problem:

The assignment problem must have an equal number of origins and destinations. The supply is one unit at each origin and the demand is one unit at each destination

The transportation algorithm can not to be used to solve transportation problems under the above conditions because of difficulties due to degenerate allocations. There will usually be several degenerate stages which increases the number of iterations required.

(b) (i) Determine the optimal allocation of people to projects and the minimum total time required for the four projects

Since there are 5 research officers and 4 projects, we must include a dummy project which takes zero time no matter which officer is assigned to it. However, if we look at the data in more detail, we can see that Carr is estimated to require the longest time to complete each of the projects. It follows, therefore, that the algorithm will allocate the dummy project to Carr. Knowing this we can reduce the problem to the allocation of 4 projects to 4 people.

Research Officer	1	2	3	4	Smallest value in the row
Adams	80	120	60	104	60
Brown	72	144	48	110	48
Day	60	108	52	92	52
Evans	64	140	60	96	60

Step 1: For each row, deduct the smallest value in the row from each value in the row.

Research Officer	1	2	3	4
Adams	20	60	0	44
Brown	24	96	0	62
Day	8	56	0	40
Evans	4	80	0	36
Smallest value in column	4	56	0	36

Step 2. For each column, deduct the smallest value in the column fro each value in the column.

Research Officer	Project time, days			
	1	2	3	4
Adams	16	4	0	8
Brown	20	40	0	26
Day	4	0	0	4
Evans	0	24	0	0

Step 3: Beginning at row 1, if there is only one available zero in a row, make an allocation to it and cross out all other zeros in the same column. When no further progress can be made with the rows, repeat the procedure for the columns

Research Officer	Project time, days			
	1	2	3	4
Adams	16	4	0	8
Brown	20	40	0	26
Day	4	0	0	4
Evans	0	24	0	0

There are three allocations only, therefore we know we have not reached the optimum solution.

Step 4: Draw the minimum number of lines through the cells with zeros. Find the smallest number in a cell without a line through it. Subtract this number from the values in the unlined' cells and add it to the values in the cell, with two lines through them.

Research Officer	Project time, days			
	1	2	3	4
Adams	16	4	0	8
Brown	20	40	0	26
Day	4	0	0	4
Evans	0	24	0	0

The smallest number in a cell, not covered by a line is 4.

Research Officer	Project time, days			
	1	2	3	4
Adams	12	0	0	4
Brown	16	36	0	22
Day	4	0	4	4
Evans	0	24	4	0

Step 5: Repeat the allocation procedure with the new tableau

Research Officer	Project time, days			
	1	2	3	4
Adams	12	0	0	4
Brown	16	36	0	22
Day	4	0	4	4
Evans	0	24	4	0

The allocation is still not optimal. Repeat steps 4 and 5 above

Research officer	Project time, days			
	1	2	3	4
Adams	12	0	0	4
Brown	16	36	0	22
Day	4	0	4	4
Evans	0	24	4	0

The smallest Number in an Uncovered cell Is 4.

Research	Project time, days			
Officer	1	2	3	4
Adams	8	0	0	0
Brown	12	36	0	18
Day	0	0	4	0
Evans	0	28	8	0

Repeat the allocation procedure

Research	Project time, days			
Officer	1	2	3	4
Adams	8	0	0	0
Brown	12	36	0	18
Day	0	0	4	0
Evans	0	28	8	0

We now find that the solution is optimal but not unique. Brown must be allocated to project 3, but Adams, Day and Evans can be allocated to more than one of the other projects. The total time used will not be affected.

We will take the option in which Adams does project 2, Day does project I and Evans does project 4. The total time used is 120+48+60-,-96=324 days.

(ii) State the alternative optimum and select the most sensible one:

The alternative optimum allocations are:

Officer	Project	Project	Project
Adams	2	2	4
Brown	3	3	3
Day	1	4	2
Evans	4	1	1

Officer Brown must have his preference for project 3 which means that Officer can have his preference for project 2. We must, therefore, choose alternaiivenumba3, which gives project 4 to Adams and project I to Evans.

(iii) How can the problem be simplified?
The elimination of Officer Carr from the project has already been exploited in

part (b)(i).

CHAPTER SUMMARY

Two techniques were developed almost simultaneously in the period 1956-1958 by different firms for different reasons and independently:

(a) The critical path method (CPM)

(b) The programme evaluation and review technique (PERT)

The main difference is that CPM is a certainty (deterministic) model while PERT is a stochastic (uncertainty) model with respect to the project completion time.

Time Estimates

(a) Optimistic time estimate = a

This is the shortest time an activity can take to be complete. It represents real estimate such that the probability is small that the activity can be completed in less time.

(b) Pessimistic time estimate = b

This is the longest time an activity can take to be completed. It is the worst time estimate representing bad luck. Such that the probability is small that the activity will take longer.

(c) Most likely time estimate = m

This refers to the time that would be expected if you work under normal conditions. Used to give the activity expected (mean) time using the following formula:

Expected activity time, te $= \dfrac{a + 4m + b}{6}$

CHAPTER QUIZ

1. What denotes the beginning or the ending of an activity?
2. These desriptions relate to a certain activity. Which activity is it?
   (a) used to improve clarity of the network
   b) Used to facilitate a logical flow of activities in the network.
3. This is the longest time an activity can take to be completed
4. This is the process of attempting to reduce the peaks and troughs in the resource allocation so that we have a more even usage of personnel.
5. A solution is ……….. when there are fewer than (m + n - 1) allocations in the tableau.

ANSWERS TO CHAPTER QUIZ

1. Event
2. Dummy activity
3. Pessimistic time estimate
4. Smoothing a profile
5. Degenerate

QUESTIONS FROM PREVIOUS EXAMS

JUNE 2000 QUESTION 8

a) Explain the following terms as used in network analysis:
  (i) Backward pass   (2 marks)
  (ii) Crashing   (2 marks)
  (iii) Stock   (2 marks)
  (iv) Earliest start times   (2 marks)
  (v) Critical-path activities   (2 marks)
b) XYZ Construction Company is building a 250-unit apartment complex in Embakasi, Kenya. The project consists of hundreds of activities involving excavating, framing, wiring, plastering, painting, landscaping and more. Some of the activities must be done sequentially and others can be done simultaneously. Also, some of the activities can be completed faster than normal by acquiring additional resources.

Required:

  (i) How would Quantitative Techniques be used to solve this problem?   (2 marks)
  (ii) What would be the uncontrollable inputs?   (2 marks)
  (iii) What would be the decision variables of the model? The objective function? The constraints?   (3 marks)
  (iv) Is the model deterministic or stochastic?   (1 mark)
  v) Suggest assumptions that could be made to simplify the model.   (2 marks)
  (Total: 20 marks)

DECEMBER 2000 QUESTION 8

a) A small construction project involves the following activities:

	Activity	Normal		Crash	
		Time (days)	Cost (Sh.)	Time (days)	Cost (Sh.)
1 – 2	Clear ground	6	60,000	5	70,000
1 – 3	Lay foundation	5	30,000	3	50,000
2 – 4	Build walls	3	10,000	2	15,000
3 – 4	Roofing & piping	7	40,000	4	55,000
3 – 5	Painting	4	20,000	3	30,000
4 – 5	Landscaping	2	10,000	1	17,500

Required

(i) Determine the shortest time and the associated cost to finish this project (8 marks)
(ii) If a penalty of Sh4,500 must be charged for every day beyond 12 days, what is the most economical time for competing the project? (4 marks)
b) Explain the four different methods or approaches for organising and displaying project information. (8 marks)
(Total: 20 marks)

DECEMBER 2001 QUESTION 8

Consider a project which has been modelled as follows:

Activity	Immediate Predecessor(s)	Completion Time (hours)
A	-	7
B	-	10
C	A	4
D	A	30
E	A	7
F	B,C	12
G	B,C	15
H	E,F	11
I	E,F	25
J	E,F	6
K	D,H	21
L	G,J	25

Required:

a) Determine the project's expected completion time and its critical path.   (12 marks)
b) Can activities E and G be performed at the same time without delaying the completion of the project?   (2 marks)
c) Can one person perform A, G and I without delaying project?   (2 marks)
d) By how much time can activities G and L be delayed without delaying the entire project?   (2 marks)
e) By how much time would the project be delayed if activity G were delayed by 3 hours and activity L by 4 hours?   (2 marks)
                                                                        (Total: 20 marks)

JUNE 2004 QUESTION 8

Mrs. Mwangi wants to open a cafeteria in Kisumu. A small business enterprise adviser whom she approached, listed for her six major activities to be carried out. The table below gives a summary of the normal time estimates of each activity, crash time and the cost reduction per day.

	Activity	Predecessor	Normal time (weeks)	Crash time (weeks)
A	Procurement of materials	-	3	3
B	Plumbing	A	6	4
C	Masonry	-	5	3
D	Electrical works	C	8	7
E	Carpentry	C	6	4
F	Finishing	B,D,E	4	2

Activity	Cost Slope (Sh.)
A	-
B	45,000
C	30,000
D	60,000
E	22,500
F	75,000

Required

(a) The normal completion time of the project and the critical activities   (6 marks)
(b) (i) The shortest time the project can be completed.   (8 marks)
    (ii) The additional cost to be incurred if the project is crashed.   (2 marks)
(c) Explain the meaning of the cost slope and how it is computed.   (2 marks)
(d) Assumption made when crashing   (2 marks)

DECEMBER 2004 QUESTION 8

The following data relate to activities in the development of a two-stage audit software by Mwangi and Onyango associates for Mwitu Ltd.

Activity	Description	Immediate predecessor	Expected time (days)
A	Evaluate the current system	-	7
B	Adjust the current system	A	3
C	Develop a new system	-	6
D	Dry-run the new system	C	3
E	Test the system	B,D	2

The recommended crashed activity times and total costs are shown below:

Activity	Expected time (days)	Cost (Sh.'000')
A	6	600
B	3	200
C	6	500
D	3	200
E	1	550

At the start of day 8, the following activity status report is received:

Activity	Actual cost (AC, sh. '000)	Percent completion
A	800	100
B	100	67
C	450	100
D	250	50
E	0	0

Required:

a) (i) In terms of cost, is the project on schedule? Make a recommendation on the action to be taken. (10 marks)

(ii) Clearly explain the assumptions that are made in calculating the 'crash' costs for the activities. (2 marks)

(b) Explain how the crash cost per unit is computed. Why is it important to compute them? (3 marks)

(c) Explain the purpose of a cost status report. (5 marks)

(Total: 20 marks)

CHAPTER NINE

Simulation and Queuing Theory

CHAPTER NINE
Simulation and Queuing Theory

▶ **OBJECTIVES**

At the end of this chapter, you should be able to:

- Cite various applications of simulation in the business world
- Discuss merits and demerits of simulation
- Cite various features of queuing theory
- Mention various problems associated with queuing systems
- Define optimum order quantity, re-order level and re-order interval
- Interpret the total cost equation
- Explain the effect of different quantity discount levels

▶ **INTRODUCTION**

Simulation is a technique used to make decisions under conditions of uncertainty whereby a model of the real system is used and then a chain of repeated trial and error experiments are conducted to forecast the behaviour of the system over a period. It is an imitation of the real system; for instance, model of an aircraft, a windmill.

Fast Forward: The act of simulating something generally entails representing certain key characteristics or behaviours of a selected physical or abstract system.

▶ **DEFINITION OF KEY TERMS**

Random numbers – A set of numbers arranged in random order. Used in statistics to refer to numbers drawn without bias or conscious choice to ensure equal chances for each item drawn.

Queue – A group of customers waiting for service in a system rendering some service.

Service facility – this is the facility that provides service to customers on the queue.

Waiting time – this is time taken in a queue before being provided with a service.

Time in the system – is equal to waiting time plus service time.

Balking – this is a case when a member of the population refuses to join the queue due to the size of the queue.

FIFO queue discipline – this system that gives priority to those who arrive first.

Priority queue discipline – this system allows different priorities entitling them to get service on a pre-emptive basis or a non-pre-emptive basis.

Pre-emptive basis allows some customers to interrupt customers already receiving service while non-pre-emptive basis does not allow interruption of those already receiving the service.

Single phase system – service is received from only one station.

Multi-phase system – service must be received from more than one station sequentially.

▶ INDUSTRY CONTEXT

Most practical inventory systems deal with hundreds or thousands of items. A large manufacturer or a supermarket are examples. Not all of the stock items should be dealt with in the same way. It is sensible to concentrate efforts on the items which have high annual value, rather than on those which have a small annual value. Hence, the use of Pareto effect. For instance the airline uses a large amount of fuel costing pence per litre but with a high annual value due to volume.

It is important to note that most of the questions from simulation are run by computers and the student will be expected to analyse and provide solutions. It is of essence for a student to understand the various formulae for queuing to avoid interchanging them during exam.

▶ EXAM CONTEXT

The student will be expected to apply the imaginary theory of what happens in a manufacturing industry or wholesale businesses and retail businesses and apply to exam questions.

Past Paper Analysis

12/05, 6/06, 6/05, 6/01, 6/00, 12/04

9.1 SIMULATION

Reasons for adopting Simulation

Simulation may be the only method available because it is difficult to observe the actual environment.

It may be infeasible to develop a mathematical solution.

Actual observation of a real system may be too expensive.

Time may be insufficient to allow the real system operate extensively, for instance, study of sales in a firm for a number of years.

It may be disruptive to adopt actual operation and observation of a real system.

Monte Carlo method is one of the most common simulation techniques. It uses random numbers and yields a solution that is close to the optimal but not necessarily exact solution. It should be noted that only models under uncertainty could be studied with the help of Monte Carlo method because Monte Carlo simulation method is based on the continued observation of the system over a long period and experience.

Generating Random Numbers

To simulate a sample will involve the use of random numbers. Random numbers refers to numbers selected in such a way that each number has an equal chance of selection. Once it is selected, it is transformed into an observation drawn from the probability distribution specified in the model under study.

The following are the methods used to generate random numbers:

1. They may be obtained from random number table in the memory of the computer.
2. An electronic device may be constructed as part of a digital computer to generate true random numbers.
3. Mid-square method:involves the following procedures:
   a) Take a four digited number and square it e.g $1537^2 = 2362369$
   b) From the result above take the four digits starting with the third from the right e.g 2**3623**69
   c) To get the 2nd random start with the new four-digited number obtained in (b) above and repeat the procedure.

Advantages of Simulation

It is less costly and less risky than if the actual system was put in experiment.

Currently, simulation is done with computers hence less time is required.

Additionally, computer simulation can be repeated making users have control over the development of the model.

Simulation does not require simplifications and assumptions as is the case in analytical solutions.

It is easier to explain a simulation model to managers since it describes characteristic of a real system process.

Simulation may still be used to check the correctness of the analytical solution.

In situations where it is difficult to predict bottlenecks, simulation may be employed.

Since simulation calls for breaking down of a system into subsystems, it allows one to gain increased knowledge of the operating system.

Simulation is of magnitude importance in cases where complex relationships of a predictable nature exist.

Since simulation keeps mathematics aside, it is easily understood by the operating personnel.

Simulation models are flexible hence modifiable to accommodate the changing environments of the real situation.

It is easier to use than mathematical models and it is quite superior to mathematical models.

Simulation involves personnel into the exercise making the trainee gain confidence and familiarise himself with data processing.

Disadvantages of Simulation

It does not produce optimum solutions since each simulation is run like a single experiment.

It is time consuming since it involves repetition of the experiment especially when done manually.

The difficulty in finding the optimum number of parameters increases as the number of parameters increase.

There is a high chance of over reliance on simulation technique even in cases where mathematical models would be appropriate.

Business applications of Simulation

    i.   Solution of queuing problems.
    ii.   To study inventory systems under uncertainties.
    iii.   Planning for the new product introduction into the market.
    iv.   Developing a number of network simulation models.
    v.   Testing of decision rules for Hospital operating policies.

Example

    1.   Describe the advantages and disadvantages of using simulation to investigate queuing situations compared with the use of queuing theory formulae.
    2.   The time between arrivals at a complaints counter in a large department store has been observed to follow the distribution shown below:

Time between arrivals (minutes)	Probability
0 – 4	0.25
4 – 8	0.45
8 – 12	0.20
12 – 16	0.10

Customers' complaints are handled by a single complaints officer but all customers who consider their complaints to be 'serious' or who have to wait 5 minutes or more before being seen by the complains officer demand to see the store manager, who deals with them separately. The time to deal with complaints by the complaints officer has a normal distribution with a mean of 7 minutes and a standard deviation of 2 minutes. It is estimated that 20% of customers with complaints consider that their complaint is 'serious'.

Required:

(a) Using the above information, a table of random digits and the following table derived from the standard normal cumulative distribution table, describe how you would simulate the arrival and service flows of this system.

Random number	Number of deviations from mean	Random number	Number of deviations from mean
00 – 01	- 2.5	61 – 77	+ 0.5
02 – 04	- 2.0	78 – 88	+ 1.0
05 – 10	- 1.5	89 – 94	+ 1.5
11 – 21	- 1.0	95 – 97	+ 2.0
22 – 38	- 0.5	98 – 99	+ 2.5
39 – 60	0		

(b) Use the following random digits to simulate the handling of 10 complaining customers, some of whom may have 'serious' complaints.

Inter-arrival time	09	06	51	62	83	61	59	20	82	68
Serious complaint	5	0	7	3	8	2	9	8	1	6
Service time	39	60	50	31	02	02	83	90	71	16

(c) Use your simulation to estimate the proportion of customers who eventually see the store manager. Hence estimate the total amount of time that the store manager spends dealing with complaints, assuming an 8 hour day and that the time he spends per complaint has the same distribution as that of the complaints officer.

(d) Explain briefly how to use simulation to decide if it would be worthwhile employing an additional complaints officer.

Solution

Advantages:

a)   Simulation can be used to investigate the behaviour of problems which are too complex to be modeled mathematically.

b)   The technique can also be used when the variables in the problem, e.g. arrival time, service time, do not follow the standard distributions required for the mathematical models, i.e. Poisson distribution, negative exponential distribution.

c)   The basic principles of the simulation technique are fairly simple and it is, therefore, more attractive to people who are not expert in quantitative techniques.

Disadvantages

a)   Simulation is not an optimising technique. It simply allows us to select the best of the alternative systems examined.

b)   Reliable results are possible only if the simulation is continued for a long period.

c)   A computer is essential to cope with the amount of calculation required in (b).

2.   (a) There are three stochastic variables in the problem:

i.   Time between arrivals;

ii.   Time to deal with the complaint;

iii. Whether the complaint is 'serious' or not.

For each variable, a range of random numbers is allocated to each value. The size of the range is determined by the probability of the value occurring.

Since the time between arrivals is a grouped frequency distribution, we will use the mid-points of the groups to represent the group:

Time between arrivals, mins	Mid-point time, mins	Probability	Cumulative Probability	Random Numbers
0 but < 4	2	0.25	0.25	00 - 24
4 but < 8	6	0.45	0.70	25 – 69
8 but < 12	10	0.20	0.90	70 – 89
12 but < 16	14	0.10	1.00	90 - 99

The time taken to deal with complaints follows a normal distribution with a mean of 7 minutes and a standard deviation of 2 minutes. If each service time, t, is z standard deviation from the mean, then:

$$z = \frac{t - \mu}{\sigma}$$

therefore: $t = \mu + \sigma z = 7 + 2z$ (minutes)

We can use the table of random numbers and associated z values, given in the question, to generate a random series of values for z and use these to generate a random series of values for t.

Serious complaint.

	Probability	Random number
Serious	0.2	0 – 1
Not serious	0.8	2 - 9

(b) Assume that the simulation clock begins at time zero.

Customer number	iat. mins		Arrival time, mins	Serious complaint		Service time, mins				See Manager
	RN	Time	mins	RN	Y/No	RN	Time	Start	End	Yes/No
1	09	2	2	5	No	39	7	2	9	No
2	06	2	4	0	Yes	60	7	-	-	Yes
3	51	6	10	7	No	50	7	10	17	No
4	62	6	16	3	No	31	6	17	23	No
5	83	10	26	8	No	02	3	26	29	No
6	61	6	32	2	No	02	3	32	35	No
7	59	6	38	9	No	83	9	38	47	No
8	20	2	40	8	No	90	10	Waiting>	5 mins	Yes
9	82	10	50	1	Yes	71	8	-	-	Yes
10	68	6	56	6	No	16	5	56	61	No

(c) Out of the 10 customers in the simulation, 3 saw the manager. We estimate that 3/10 = 30% of all customers see the manager. The time he spends each day dealing with complaints may be estimated by:

30% × average number complaints per hour × average time per complaint × 8 hours

= 0.3 × number complaints per hour × 7 minutes × 8 hours

To determine the average number of complaints per hour, find the average time between the arrival of complaining customers.

Average time between arrivals = Σ (time × probability)

$$= 2 \times 0.25 + 6 \times 0.45 + 10 \times 0.2 + 14 \times 0.1$$

$$= 6.6 \text{ minutes}$$

Therefore, average number of customers per hour = 60/6.6 = 9.09. Time spent by manager dealing with complaints is:

0.3 × 9.09 × 7 × 8 = 152.71 minutes = 2 hours 33 minutes per day

(d) Repeat the simulation but with a second complaints officer. The proportion of complaints going to the manager can then be calculated and hence, the time he spends each day can be found as in (c). The value of the reduction in the manager's time must be balanced against the cost of providing a second complaints officer.

To simulate a stock control problem

Kyang Plc manufactures cars. The batteries for their Lunar model are bought from an outside supplier. From past experience, Kyang find that the weekly demand for the batteries can be approximated by a normal distribution with a mean of 500 and a standard deviation of 10, over the range 470 to 530. There is an initial stock of 2,000 batteries and the company has decided to order in the batches of 2,500 whenever the stock level falls below 1,500 batteries. Again, past experience indicates that the time between the order being placed and the delivery, varies as follows:

Lead time distribution, Kyang Plc

Lead time, weeks	1	2	3	4
Probability	0.20	0.50	0.25	0.05

The unit cost of holding is Sh0.5 per week, applied to the total stock held at each week ending. The cost associated with placing an order is Sh50 and the unit cost of being out of stock is put at sh 20 per week.

Use a simulation over a period of 20 weeks to estimate the average cost per week of the above policy. Assume that all accounting is done at the end of the week and that all ordering and delivery occurs at the beginning of the week.

Solution

The variables in the problem are the demand and the lead time. Since the demand is approximated by the continuous normal distribution, we will consider demand in steps of 5 batteries. For example, the probability of a demand for 510 batteries will be estimated by P(507.5<demand<512.5).

Allocate random number ranges to the lead time

Lead time, weeks	Probability	Cumulative probability	Random Number
1	0.20	0.20	00-19
2	0.50	0.70	20-69
3	0.25	0.95	70-94
4	0.05	1.00	95-99

Allocate random number ranges to weekly demand

Demand/Week	Probability	Cumulative probability	Random Number
470	0.003	0.003	000-002
475	0.009	0.012	003-0011
480	0.028	0.040	012-039
485	0.066	0.106	040-105
490	0.121	0.227	106-226
495	0.175	0.402	227-401
500	0.197	0.599	402-598
505	0.175	0.774	599-773
510	0.121	0.895	774-894
515	0.066	0.961	895-960
520	0.028	0.989	961-988
525	0.009	0.998	989-997
530	*0.003	1.00	998-999*

*slight discrepancy due to rounding

We can now carry out the simulation.

Stock control simulation, Kyang Plc

Week No.	Opening stock	Demand		Closing stock	Reorder Yes/No	Lead time		Shortage
		RN	Amount			RN	Weeks	
1	2000	034	480	1520				
2	1520	743	505	1015				
3	1015	738	505	510	Yes	95	4	
4	510	636	505	5				
5	5	964	520	0				515
6	0	736	505	0				505
7	2500	614	505	1995				
8	1995	698	505	1490				
9	1490	637	505	985	Yes	73	3	
10	985	162	490	495				
11	495	332	495	0				
12	2500	616	505	1995				
13	1995	804	510	1485				
14	1485	560	500	985	Yes	10	1	
15	3485	111	490	2995				
16	2995	410	500	2495				
17	2495	959	515	1980				
18	1980	774	510	1470				
19	1470	246	495	975	Yes	76	3	
20	975	762	505	470				
		Total	10050	22865				1020

Mean demand = 10,050/20 = 502.5 batteries/week

Mean closing stock = 22,865/20 = 1143.25 batteries/week

Mean shortage = 1020/20 = 51.0 batteries/week

Number of orders placed during the 20 week period = 4, therefore, mean number orders/week = 4/20 = 0.2.

The expected weekly cost = $1143.25 \times 0.50 + 51 \times 20 \times 50$ = sh 1,602.

9.2 QUEUING

Fast Forward: A queuing discipline determines the manner in which the exchange handles calls from customers.

Queuing theory is the mathematical study of waiting lines (or *queues*). The theory enables mathematical analysis of several related processes, including arriving at the (back of the) queue, waiting in the queue (essentially a storage process), and being served by the server(s) at the front of the queue. The theory permits the derivation and calculation of several performance measures including the average waiting time in the queue or the system, the expected number waiting or receiving service and the probability of encountering the system in certain states, such as empty, full, having an available server or having to wait a certain time to be served.

Note: The Arrival of customers follows a Poisson probability distribution with its mean denoted by λ while service time follows an exponential probability distribution with mean denoted by μ.

1. Case of one service facility

P_n = Probability that there are n customers in the system, n>0

$$= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

P_o = Probability that the servicing facility is idle.

$$= 1 - \frac{\lambda}{\mu}$$

$E_{(n)}$ = Average number of customers in the system.

$$= \frac{\lambda}{\mu - \lambda}$$

$E_{(m)}$ = Average number of customers waiting in the queue.

$$= \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$E(v)$ = Average time spent in the system

$$= \frac{1}{\mu - \lambda}$$

$E(w)$ = Average time spent waiting in the queue.

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

Utilisation factor (ρ) : Probability that the servicing facility is busy.

$$\rho = \frac{\lambda}{\mu}$$

2. Queuing system with more than one parallel servicing facilities.

Assumptions

(1). There is no limit to the queue length.
(2). No customers leave without service.
(3). Arrivals assume a poisson process that is random.
(4). Service time follows a –ve exponential distribution with the same mean.

Utilisation factor $\rho k = \dfrac{\lambda}{k\mu}$

$P_{o} = $ probability that there are no customers in the system

$$= \cfrac{1}{\displaystyle\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^{k}\frac{k\mu}{k\mu - \lambda}}$$

$Pn = \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^{n} P_{o}$ (when n<k) n = No. of customers
                                              k = servicing facilities

$= \dfrac{1}{k!\,k^{n-k}}\left(\dfrac{\lambda}{\mu}\right)^{n} P_{o}$ (when n≥k)

Prob (n≥k) = probability that a customer has to wait since the number of customers exceed the servicing facilities.

$$= \cfrac{\lambda\,\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)(k\mu - \lambda)^{2}} P_{o}$$

$E_{(n)}$ = Average no. of customers in the system.

$$= \cfrac{\lambda\,\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)(k\mu - \lambda)^{2}} P_{o} + \dfrac{\lambda}{\mu}$$

$E_{(m)}$ = Queue length

$$= \cfrac{\lambda\,\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)(k\mu - \lambda)^{2}} P_{o} + \dfrac{\lambda}{\mu}$$

$E_{(w)}$ = Average time spent waiting in the system.

$$= \frac{\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)(k\mu-\lambda)^{2}}\, P_{o}$$

Example:

The number of shoppers queuing at any given time in a certain supermarket downtown Nairobi can be approximately represented by the equation: $y = x^3 – 14x^2 + 50x$ over the range $0 \le x \le 8.5$, where y is the number queuing and x is the time in hours after the store opens at 9.00 a.m. (so that, for example 10:30 a.m. is x = 1.5, and 5:30 p.m. – when the store closes – is x = 8.5).

Required:

a)   The management wants to know when they should deploy more cashiers and the number queuing at that time.

b   Determine the number of man-hours spent per day by shoppers queuing.

Solution

The classical optimisation technique of calculus can be used to find the time when management is to deploy more cashiers and the number queuing at that time. Since the number of shoppers queuing is given as $y = x^3 – 14x^2 + 50x$, taking the first derivative gives the rate of change in our queue as follow:

Recall: $y = x^3 – 14x^2 + 50x$, …………………………………… differentiating.

$\Rightarrow \quad \dfrac{\partial y}{\partial x} = 3x^2 – 28x + 50$ ………… setting this equal to zero.

$\dfrac{\partial y}{\partial x} = 0$

$\Rightarrow \; 3x^2 – 28x + 50 = 0$ …………… solving using quadratic formula.

$$X = 28 \pm \frac{\sqrt{28^2 - 4\text{x}3\text{x}50}}{6}$$

$\Rightarrow \; x = 28 +_-$ or $\dfrac{\sqrt{28}^2 - 4x3x50}{6}$

$$X = 28 - \frac{\sqrt{28^2 - 4x3x50}}{6}$$

$\Rightarrow \; x = 28 + \dfrac{\sqrt{184}}{6}$

$= \dfrac{28 + 13.565}{6}$

$= \dfrac{41.56}{6}$

$= \underline{6.93}$

Or

$$\Rightarrow \quad x = \frac{28 - \sqrt{184}}{6}$$

$$= \frac{28 - 13.565}{6}$$

$$= \frac{14.435}{6}$$

$$= 2.405 \approx \underline{2.41}$$

Thus X is either 6.93 or 2.41

Performing the second derivative test to establish the minimum or maximum value as follows:

Recall

$$\frac{\partial y}{\partial x} = 3x^2 - 28x + 50$$

Hence, $\frac{\partial^2 y}{\partial x^2} = 6x - 28$ ……………… setting this equal to zero

$\Rightarrow \quad 6x - 28 = 0$ ……………… Solving for x

$\Rightarrow \quad 6x = 28$

$\Rightarrow \quad x \quad = \frac{28}{6}$

= 4.7 Positive.

Thus at the stationary point, the values of x are either 2.41 or 6.93. This shows that when x = 2.41 i.e. < 0 the point is maximum while at x – 6.93 the point is minimum.

The value of the function can now be obtained.

Since y = x³ – 14x² + 50x…………………substituting for x = 2.41

Y = 2.41³ – 14 (2.41)² + 50 (2.41)…………solving

=53.184121 ≈ 53

Notice that management ought to deploy more clerks at the maximum point i.e. at x = 2.41 or 11.25 am. The number of clerks to employ at this time is 53.

(ii) The number of man-hours spent per day by shoppers queuing can be obtained using integration as follows:

Given the number of shoppers going as:

y = x³ – 14x² + 50x over the range 0 ≤ x ≤ 8.5

Taking the integral we get:

$$\Rightarrow \quad \int_{0}^{8.5} (x^3 - 14x^3 + 50x)x\partial x$$ ……………… Opening the branches

$$\Rightarrow \int\limits_{0}^{8.5} x^4 - 14x^3 + 50x^2 \partial x$$

$$\Rightarrow \int\limits_{0}^{8.5} \frac{x^5}{5} - \frac{7x^4}{2} + \frac{50x^2}{x}$$

$$= \int\limits_{0}^{8.5} \frac{x^5}{5} - \frac{7x^4}{2} + \frac{50x^2}{x}$$

= <u>839.3</u> man hours per day.

Example

Charles Nzioka, who is a barber, has found out that he can shave on average 4 customers per hour. The arrival rate of customers averages 3 customers per hour.

Required:

    (i)   The proportion of time that Charles Nzioka is idle.
    (ii) The probability that a customer receives immediate service upon arrival.
    (iii) Average number of customers in the queuing system.
    (iv) Average time a customer spends in the queuing system.

Solution

Service Rate = 4 customers per hour,

Arrival Rate = 3 customers per hour,

Time for shaving 1 person = ¼ of an hour.

∴ With 3 customers coming in per hour then the proportion of time he is busy is;

3 × ¼ = ¾ of an hour.

    The proportion of the time he is idle is 1-¾

                       = ¼ of an hour

ii) Probability of receiving immediate service is the probability that Nzioka is idle = ¼

iii) Average number of customers in the queuing system.

$$L_s = \frac{ArrivalRate}{ServiceRate - ArrivalRate}$$

$$L_s = \frac{3}{4 - 3} = 3 Customers$$

iv) Average time a customer spends in the queuing system.

$$W_q = \frac{ArrivalRate}{ServiceRate(ServiceRate - ArrivalRate)}$$

$$= \frac{3}{4(4-3)} = 45 \text{ minutes}$$

9.3 INVENTORY CONTROL DECISIONS

Introduction

In most cases a company will hold stocks of goods; hence there is capital tied up in these goods. This unusable capital represents a cost to the company in the form of lost interest payments, or investment opportunities. The stock will incur costs since storage must be provided to house it; personnel must be employed to manage it; it must be insured, transported to the warehouse, offloading charges will be incurred and so on.

It is also important to take into consideration the fact that demand for the inventory will probably be uncertain. The smaller the stock level, the more likely it is that shortages will arise. These shortages will themselves represent a cost to the company, either in lost production or lost custom.

Fast Forward: Stock control is used to evaluate how much stock is used. It is also used to know what is needed to be ordered. Stock control can only happen if a stock take has taken place.

Definition of key terms

1. Lead Time – A lead time is the period of time between the initiation of any process of production and the completion of that process. Thus the lead time for ordering a new equipment from a manufacturer may be anywhere from 2 weeks to 4 months.

2. Economic Order Quantity (EOQ) – EOQ constitutes the quantity purchased of either stocks or raw materials that is considered most optimum. This is the quantity that minimises both holding costs and ordering costs.

3. Economic Batch Quantity (EBQ) – Also called Optimal Batch Quantity or Economic Production Quantity, is a measure used to determine the quantity of units that can be produced at minimum average costs in a given batch or production run.

4. ABC System or Pareto analysis – is a statistical technique in decision making that is used for selection of a limited number of tasks that produce significant overall effect. It uses thePareto principle.

Management must make decisions about the control of stock levels with a view to minimising the cost to the company while achieving more efficiency in the availability of material to fulfill planned usage requirements. Consideration should be given to the following control levels:

a) Minimum stock level
b) Maximum stock level
c) Re-order level
d) Re order quantity (Note the re-order quantity is not necessary the EOQ)

Minimum stock level

This is the level below which stocks should not be allowed to fall. It is essentially a base (buffer) stock level. If stock falls below this point, there is a danger of stock-out and firms will incur shortage costs. This may also be referred to as safety stock. It can be expressed as:

Minimum stock level = Reorder level − (Normal consumption × minimum reorder period)

Stock-out may be caused by various factors such as delay on the part of the supplier, an increase in material usage due to a change in the pattern of production and increase in scrap levels in the production process and delays in placing orders due to scarcity of suppliers.

Note: Reorder period or lead time is the period of time in days, weeks or months that elapse before an order made is received and ready for use.

Maximum stock level

This is the upper limit above which stock should not be allowed to exceed. Each material to be kept in store must have a maximum level and stock should not be allowed to go beyond this level. If stock level goes beyond this point then the firm will be overstocking hence incur high holding costs. It is computed as follows.

Maximum Stock Kevel = Reroder Level + Reoder Quantity −(Minimu Consuotion × minimum reoder operiod)

In setting the maximum stock level, the cost accountant must take into account various other factors that may act as a constraint. This may include the nature of the materials being stocked, rate of consumption of materials, lead time or reorder period, availability of adequate storage space and the cost of storing versus the benefits derived from advantageous purchasing.

Re-order level

It is a point that lies between minimum and maximum stock levels at which purchase orders must be placed to ensure that goods ordered are received before the minimum stock level is reached. It is the level of stocks if and when approached; orders for stock replenishment must be made to cater for the unused stocks. This level is normally higher than the minimum stock level to cover for emergencies such as abnormal usage or unexpected delay in the delivery of new supplies. It can be expressed as follows:

Reorder level = maximum consumption × maximum reorder period

Re-order quantity

This is the quantity of stock ordered once the reorder point is reached. The quantity is such as to minimise stock costs taking into consideration the cost of holding stocks and making an order.

The EOQ is an example of a reorder quantity. However, reorder quantity must not be the EOQ. Given the maximum stock level, the reorder level, minimum usage and the minimum reorder level, it may be computed as follows.

Reorder quantity = maximum stock−reorder level + (Minimum usage × maximum reorder period

Illustration

The following information was extracted from the books of Danex Holdings regarding its stocks:

Reorder quantity	1,800
Reorder period	4 weeks
Maximum consumption	450 units/week
Normal consumption	300 units/week
Minimum consumption	150 Units/week
Maximum reorder period	5 weeks
Minimum reorder period	3 weeks

Required

Determine the following stock levels for Danex Holdings:

    i.   Re-order level
    ii.   Maximum stock level
    iii. Minimum stock level

Solution

(i)   Reorder level

Reorder level = maximum consumption × maximum reorder period
        =450 units/week × 5 weeks

        =2,250 Units

(ii)   Maximum stock level

Maximum Stock Level = Reroder Level + Reoder Quantity −(Minimu Consuotion × minimum reoder operiod)
Maximum stock level = 2,250 Units + 1,800 Units − (150 units/week × 3 Weeks)
              = (2,250 +1,800 − 450) Units
              = 3,600 Units

iii) Minimum stock level

Minimum stock level = Reorder level − (Normal consumption × minimum reorder period)
Minimum stock level = 2,250 Units - 1,200 Units
           1,050 Units

Costs associated with materials

Purchase cost

This is the price charged by the supplier on an item of inventory. Purchase price will remain irrelevant, where prices are fixed and no discounts are offered or no advantageous purchasing exists. However, if there exists discounts associated with quantity purchased, they remain relevant for decision making.

Purchase cost = Acquisition price per unit x Number of units

◼ Holding or carrying cost

These are costs incurred because firms own or maintain inventories. They are associated with high stock levels and include opportunity cost of funds tied up in stock, incremental insurance costs, incremental warehousing and storage costs, incremental material handling costs and cost of obsolescence and theft of stock. The relevant holding cost should include those items which vary with the level of stock. Costs unaffected by changes in the inventory levels are irrelevant in decision making and thus not included in carrying costs, for instance rent, depreciation of equipment and salaries for storekeepers. Costs such as insurance costs should be included only when premiums are charged on the fluctuating value of stocks. Therefore, fixed annual insurance cost is irrelevant and thus should not be included in the ordering cost.

Carrying costs = Holding cost per unit per annum x Average stock

◼ Ordering and procurement costs

This is the cost of getting an item into the firm's inventory. It usually consists of clerical costs of preparing a purchase order, receiving deliveries and paying invoices. Ordering costs that are common to all stock decisions are irrelevant, and only incremental ordering costs are used. Note that ordering costs are incurred each time an order is made and are associated with low stock levels.

Stock-out costs are incurred as a result of an item not being in stock. They include loss of future sales due to disappointed customers, loss of goodwill, lost contribution or profit from lost sales, extra costs of speeding up orders etc.

Ordering costs = number of orders made per year x cost per order

$$\text{Where number of orders per year} = \frac{\text{Annual demand}}{\text{number of units odered each time}}$$

The Economic Order Quantity (EOQ) Model

The Economic Order Quantity (EOQ) Model is a simple model that helps the manager to determine the optimum quantity of stock to order so as to keep total costs at a minimum. The main costs of inventory are: Holding or carrying costs, Ordering or set up costs, Shortage costs

This model is based on various assumptions:

1. It assumes that the annual demand is certain, constant and continuous over time.
2. Holding costs are known and constant
3. Ordering costs are known and constant
4. The same quantity is ordered every time an order is made since demand as assumed is not subject to fluctuate significantly.
5. The supply lead time is known and constant
6. Price and cost per unit is constant
7. No stock outs are permitted and delivery is instantaneous
8. Customers' orders cannot be held while fresh orders are awaited.

Economic Order Quantity (EOQ) as determined by the model

EOQ constitutes the quantity purchased of either stocks or raw materials that is considered most optimum. This is the quantity that minimises both holding costs and ordering costs. The EOQ will change if either the cost of placing an order or the cost of carrying inventory in stock (holding cost) changes. (*Since total cost constitutes of ordering cost and Holding cost which the EOQ model targets to minimize.*)

As the quantity of purchase increases there is a reduction in ordering costs, but an increase in holding costs as illustrated in the graph below:

GRAPHICAL ILLUSTRATION OF THE EOQ
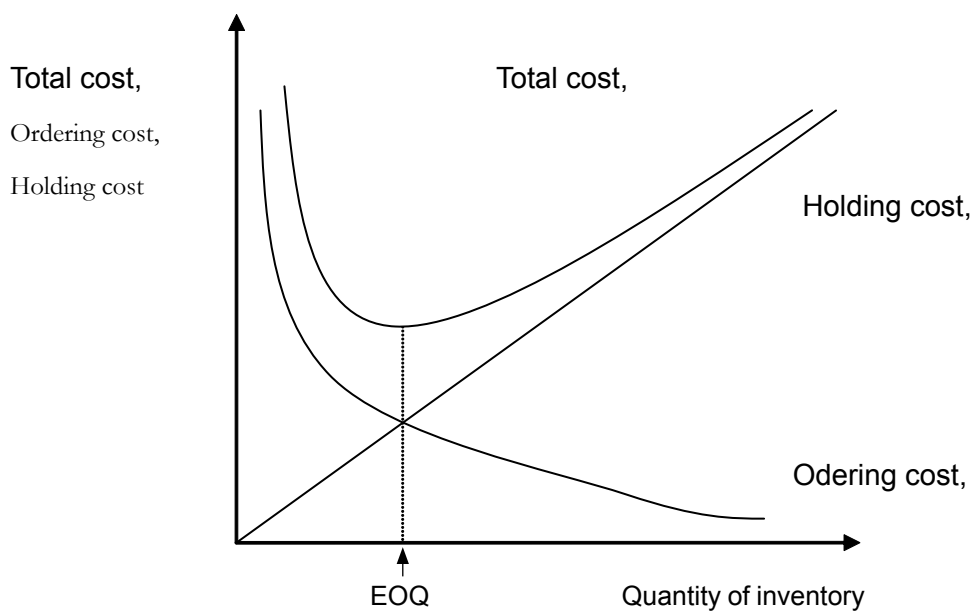
EOQ graph 1

The aggregate stock cost is lowest at the EOQ; at this point, the total cost is at minimum. Note that the total cost in this case comprises of the holding and ordering costs only.

The various costs are determined as follows:

Total Cost = Tota Odering Cost + Total Holding Cost

Total Odering Cost = Cost per Order × No. of Oders in a Period

Where No. of Oders in a Period = $\dfrac{\text{Annual Demand}}{\text{Quantity per Order}}$

Total Holding Cost = Avarage Stock Quantity × Holding Cost per Unit

Where Avarage Stock Quantity = $\dfrac{\text{Begining Inventory + Ending Inventory}}{2}$

The cost of the goods procured is not taken into account while determining the EOQ where the price quoted is fixed and no discount are offered.

Mathematically, the EOQ can be determined by the following formula.

$$EOQ = \frac{\sqrt{2DC_0}}{C_h}$$

Where   D   is the annual demand

           C_0 is the cost of making one order

           C_h is the holding cost per unit per annum

To derive this formula, you need to understand the relationship between the various elements.

Total Cost = Total Ordering Cost $(\frac{D}{Q} C_0)$ + Total Holding Cost (½ QC_h)

Total Odering Cost = Cost Per Order C_0) × No of Oders in a Period (D/Q)
Total Holding Cost = Avarage Stock Quantity (½ Q) × Holdind cost per Unit(C_h)

The number of orders in a period is equal to the total annual demand (D) divided by the quantity purchased per order (Q).

The average stock is equivalent to half the quantity procured (beginning inventory) since the ending inventory is zero (we assume that the firm exhausts the entire stock before reordering and that there is no safety stock) as illustrated in the graph below.

To illustrate stock level variation with
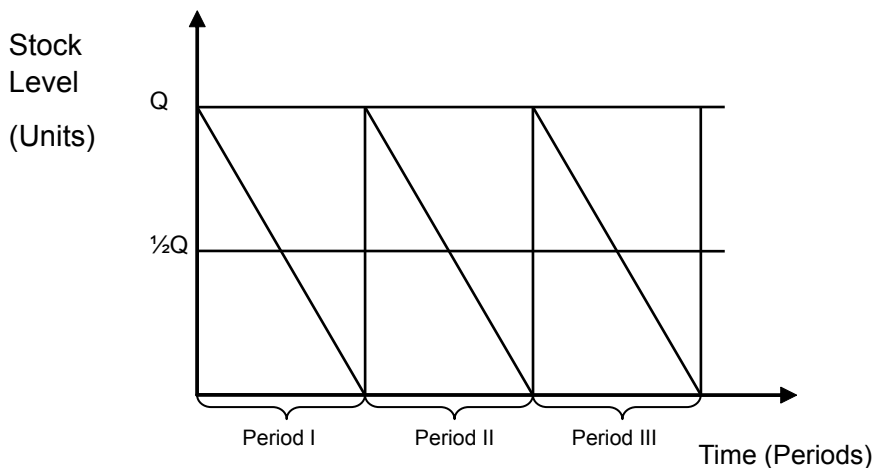
EOQ graph 2

Stock purchased is sold at a constant rate until it is exhausted. An instantaneous replenishment is done to bring the stock back to the initial amount. There is certainty in the behaviour of consumption.

At the optimum stock level, the holding and the ordering costs are equal. Take a look at the EOQ Graph 1 above; optimum stock level, EOQ, is at the point where

i.e.

$$\frac{1}{2}Q_h = D/Q\ C_0$$

Make Q the subject of the formula.

Multiply both sides of the equation by Q to get

$$\frac{1}{2}Q_h = D_0$$

Multiply both sides of the equation by 2 to get

$$Q^2C_h = D_0$$

Divide both sides of the equation by C_h to get

$$Q^2 = \frac{2DC_0}{C_h}$$

Obtain Q by taking the square root of both sides of the equation

$$Q = \sqrt{\frac{2DC}{C_h}}$$

therfore $EOQ = \sqrt{\dfrac{2DC}{C_h}}$

Illustration

ABC Ltd has an aggregate demand of 1.2 Million units. Each time they place an order there is an ordering cost of Shs.1, 000, holding cost is Shs.100 per unit. Determine:

    i.   EOQ
    ii.   No. of orders to be made based EOQ
    iii. Total cost of stocks based on the EOQ

Solution

Data provided; D=1.2 million units, C_0 =Shs.1, 000, C_h =Shs.100

(i) $EOQ = \sqrt{\dfrac{2DC}{C_h}}$

$$EOQ = \sqrt{\frac{2 \times 1,200,000 \times 1,000}{100}} = \sqrt{\frac{2,400,000,000}{100}} = \frac{48989}{10}$$

$$= 4,899\ \text{Units}$$

(ii)

$$\text{Number of oders} = \frac{\text{Annual demand}}{\text{Quantity Per order}} = \frac{1,200,000}{489}$$

$$= 244.94$$

$$(\text{Approx})\ 245\ \text{Orders}$$

(iii)

$$\text{Total cost} = (\tfrac{1}{2}QC_h) + (\frac{D}{Q}C_0)$$

$$= (\tfrac{1}{2} \times 4{,}899 \times 100) + (\frac{1{,}200{,}000}{4{,}899} \times 1{,}000)$$

$$= sh \quad 489{,}900$$

Illustration

Aries' Jewelers Inc. purchases 15,000 one-quarter-carat diamonds each year for various mountings. The following information relating to the diamonds is available.

Purchase cost per diamond	Shs.200
Cost to carry one diamond in inventory for one year	5
Cost of placing one order to the Company's supplier	40

The maximum order that the insurance company will permit is 750 diamonds. The minimum order that the supplier will permit is 150 diamonds, with all orders required to be in multiples of 150 diamonds. The company has been purchasing in the maximum allowable volume of 750 diamonds per order.

Required:

i.   Determine the volume the company should be placing its orders.

$$\text{EOQ} = \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{\frac{2 \times 60{,}000 \times 40}{5}} = \sqrt{960{,}000}$$

$$= 979.79 \text{ Units (Approx. 980 Units)}$$

The orders must be in multiples of 150 units. Therefore, total number of diamonds to be ordered should be either 900 units or 1,050 Units. By computing the total cost of holding and ordering the diamonds, we obtain the following summaries.

Ordering 900 Units   Ordering 1050 Units

$$\text{Total cost} = (\tfrac{1}{2} \times 900 \times 5) + (\frac{15{,}000}{900} \times 40) \qquad \text{Total cost} = (\tfrac{1}{2} \times 1050 \times 5) + (\frac{15{,}000}{1050} \times 40)$$

$$\begin{array}{ll} = 2{,}250 + 667 & \qquad\qquad = 2{,}625 + 571 \\ = 2{,}917 & \qquad\qquad = 3{,}196 \end{array}$$

Therefore, the Cost Accountant should recommend the purchase of 900 units as it is more economical.

Illustration

A company uses 50,000 widgets per annum which are acquired at Sh100 each. The ordering and handling costs are Sh1500 per order and carrying costs are 15% of the cost of inventory per annum.

Required:

Calculate the economic order quantity.

Solution

Annual demand, D = 50,000, Cost of ordering C_o = Shs1,500,

Holding costs per unit per year, C_h = Shs (15% x 100) = Shs15

$$Q = \sqrt{\frac{2DC}{C_h}} = \sqrt{\frac{2 \times 50,000 \times 150}{15}} = \sqrt{\frac{15,000,000}{15}} = \sqrt{1,000,000} = 1,000 \text{ units}$$

Economic batch quantity (EBQ)

This applies to manufacturing industries. Most companies producing a number of different products organise their production on a batch basis rather than on a continuous one. Stock replenishment is gradual rather than instantaneous as for EOQ. There are no ordering costs but set up costs such as incremental labour, incremental overheads, machine down time and other costs for setting up facilities for production e.g. heating time are required.

The challenge here lies in determining the optimum amount to produce per batch. The EOQ formula can be modified in order to determine the optimum length of a production run and more so determine the optimum number of units that should be manufactured in each production run. This involves balancing set up costs and stock holding costs.

The EBQ formula for a manufacturing firm is a modification of the EOQ model formula for a merchandising firm. Set up costs substitute ordering costs in the EBQ formula.

The formula can be expressed as follows;

$$Q = \sqrt{\frac{2DS}{C_h\left(1 - \frac{d}{p}\right)}}$$

Where, D = Annual demand for the product

S = Set up costs per batch (costs incurred in making preparation for production run)

C_h = Cost of holding a unit per year

Q = Economic batch quantity

d = Consumption rate, for instance daily demand

p = Production rate i.e. quantity produced per unit of time say, a day, month or year. It is the quantity which would be produced in one time period by continuous production. In our case

$\left(1 - \frac{d}{p}\right)$ = An adjustment factor to the Holding cost since during production, consumption of the product is still on. This only applies where there is gradual replenishment. Where the replenishment is instantaneous, the factor $\left(1 - \frac{d}{p}\right)$ is not included.

The EBQ formula can be derived on the same principles as the EOQ formula. It aims at minimising the total costs. The total costs are minimised at the point where set up cost equals the holding cost.

It takes time for the entire Q to be produced. During this time $\left(\frac{d}{p} \times D\right)$ units have been used. Hence the units available at the highest point (maximum stock level) will be equal to $\left(Q - \frac{Q}{P} D\right)$ Therefore; the holding cost per unit shall be based on the average inventory

$$\frac{\left(Q - \frac{d}{P} Q\right)}{2} = \frac{Q\left(1 - \frac{d}{P}\right)}{2}$$

A graphical representation of the gradual replenishment is as illustrated below;

To illustrate EBQ stock level variation with time
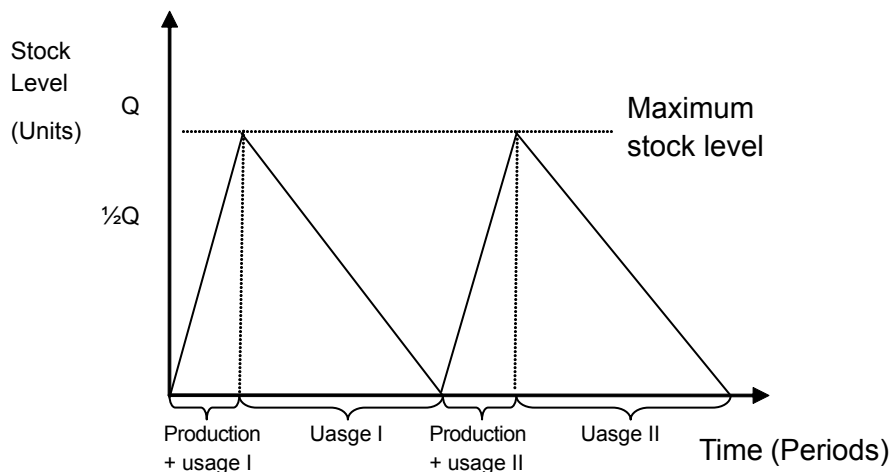
The change from C_h to $C_h\left(1 - \frac{d}{p}\right)$ reflect the fact that average stock will be $\frac{Q}{2}\left(1 - \frac{d}{p}\right)$ instead of $\frac{Q}{2}$. The reduction is caused by the fact that some units will be taken out of stock in for use even as stock is being replenished by fresh production. If the production is very fast, so **p** is very high in relation to **d**, periods of production will be very short so this will have little effect on $\frac{p}{d}$. $\frac{p}{d}$ will be small and $\left(1 - \frac{d}{p}\right)$ will be nearly 1.

Illustration

Assume the constant annual sales demand for a product is 4,500 units; set up costs amount to Sh450. The holding cost is Shs20 per unit per year. Assume 250 working days throughout the course of the year. The company produces 200 units per day during the production period.

Required

Calculate the Economic Batch Quantity (EBQ)

Solution

Given: D=4500, C_h=Sh20, S=Sh450, p=200 units

$$d = \frac{\text{Annual demand}}{\text{Number of days}} = \frac{4500}{250} = 18 \text{ units per day}$$

$$Q = \sqrt{\frac{2 D S}{C_h \left(1 - \frac{d}{p}\right)}}$$

$$Q = \sqrt{\frac{2 \times 4500 \times 450}{20 \left(1 - \frac{18}{200}\right)}} = \sqrt{\frac{4,050,000}{18.2}}$$

$$= \sqrt{222,527.47}$$

$$= 471.72 \approx 472 \text{ units}$$

Illustration

A firm manufactures a product AC169 for sale in the market. The firm has a capacity of producing 250,000 units per annum. The annual demand for the product is 50,000 units, annual carrying costs per unit per annum is Sh15. Labour and other costs incurred every time in setting the machine for production equals Sh1,500.

Required

Calculate the Economic batch quantity.

Solution

Given: Annual demand, D =50,000 units; Set up costs, S= Sh150;

Holding costs, C_h = Sh15; Production rate, P = 250,000 per annum

$$Q = \sqrt{\frac{2 D S}{C_h \left(1 - \frac{d}{p}\right)}}$$

$$Q = \sqrt{\frac{2 \times 50,000 \times 150}{15 \left(1 - \frac{50,000}{250,000}\right)}} = \sqrt{\frac{15,000,000}{12}}$$

$$= \sqrt{1,250,000}$$

$$= 1,118 \text{ units}$$

9.4 STOCK CONTROL SYSTEMS

ABC System or Pareto analysis

It is also called the 80/20 rule, control by importance/ exception. It concentrates on high value items. Here, Items are categorised into three classes as follows:

Class A: These are high cost, fast moving and high usage items. They are few accounting for only 20 percent of the total number of items yet account for 80 percent of the total inventory budget. These items are worth being under highest control.

Class B: These are medium moving goods. They account for 15 percent of the total number of the budget. They are moderately controlled.

Class C: These are slow moving low value items. They are very many accounting for 65 percent of the total number of items and only 5 percent of the total inventory budget. These items might be under simple physical control.

Periodic order system

The firm receives a new order of the amount specified by the order quantity at equal intervals of time. The order quantity is based on the likely demand (factors that affect demand of the firm's product), and the current stock levels. The firm determines the maximum and minimum inventory, the safety stock and the reorder level. For instance, a firm may be supplied with fixed amount of stock every Monday of the week for the entire period under consideration. This is mostly the case where consumption is uniform throughout the period.

The stock levels are reviewed at fixed intervals and a replenishment order is issued where necessary. The review is deemed beneficial as obsolete stock can be identified and eliminated at the earliest possible instance. Spreading of purchasing department load may yield economies in placing orders. Furthermore, because orders will be sequential, there may be production economies due to more efficient production planning and lower set up costs.

However, periodic order system requires larger stocks as reorder quantity must take into account the period between the reviews and lead times too. More so, the reorder levels are not always the EOQ. Where there arises a change in consumption habits and demand goes up significantly, stock out costs may result. In other words, to come up with an appropriate period of review, the demands must be reasonably consistent.

Continuous Review System

The firm places orders at regular intervals but the order quantity varies according to how much a firm requires to bring the level to some predetermined size or value (to replenish the stock already consumed). This is common with most enterprises. This system exists where consumption fluctuates throughout the period.

Just In Time Inventory System

This concept advocates zero inventory and stockless production through just-in-time purchasing and just-in-time production. Organisations create a closer relationship with the suppliers and arrange for more frequent deliveries of small quantities. The objective of just-in-time purchasing is to purchase goods so that delivery is made immediately before their use.

JIT is considered economical since it eliminates the cost of carrying inventory and reduces the inefficiencies that the inventories create. JIT purchasing increases the number of orders as the enterprises order more frequently and in smaller quantities. Holding cost is reduced by a significant proportion as it only arises due to waste and inefficiency created by inventory. It calls for 100 per cent quality.

Some of the major features of JIT include:

a) Frequent and reliable deliveries to avoid inventory build-up. Penalties are imposed on those who do not meet the deadline.
b) Strategic location of firms. This may be closeness to suppliers and/or customers
c) Improved communication between companies and suppliers through the use of computerised purchasing systems that allows for online ordering.
d) Single sourcing and building long-term relations with a few trusted suppliers.
e) Increased supplier involvement in the design aspects of a product to ensure that they meet the company's quality requirements.
f) Maintenance of strict quality control by all parties. Suppliers guarantee the quality of stock items

Benefits of JIT inventory system

The benefits include lower inventory level, emphasis on strict quality control by all parties, faster market response, smaller manufacturing facilities and lower set up costs.

9.5 MATERIAL HANDLING

The objective is to ensure that goods are delivered to the right places at the right time and in the right manner to avoid delays, congestion and unnecessary handling. A big percentage of production cost is taken up by material handling activities. A good material handling system should minimise these costs.

The manager needs to determine the type of equipment to be used to handle the material be it cranes, lifts, trucks or conveyors.

Various factors influence the type of materials to be used in handling the materials. They include the type of materials being moved, volume of materials, rate or frequency of movement, route of movement speed required, method of storage employed and safety or hazards involved.

Storage of Material; stores location and layout

Fast Forward: A warehouse management system, or WMS, is a key part of the supply chain and primarily aims to control the movement and storage of materials within a warehouse and process the associated transactions, including shipping, receiving, putaway and picking.

Stores should be strategically located to minimise production costs. They should be located closest to the factory as far as possible and where possible. In some instances, materials in the same store may be needed at different locations, either in the same factory building or at different plants. This calls for a more strategic planning on the location of the store where a new one has to be constructed or looking for an alternative to minimise the costs e.g. contracting an external third party. This may be through renting a warehouse or hiring transporters. Instances such as hiring of transporters would only be economical where special storage equipment which necessitates an enormous initial capital outlay are used, for instance freezers.

The layout of stores should ensure:

    i.   Ease of access for movement of material in and out of stores.
    ii.   The issue of perishable materials on a first in first out (FIFO) basis.
    iii. The segregation of toxic and dangerous materials in a separate location.
    iv. Security of materials by restriction of access to authorised personnel only.

CHAPTER SUMMARY

Simulation is one of the techniques which modellers use when mathematical models become too complex or their assumptions do not match reality. It can be used in complex situations without making assumptions about the input data.

We consider Monte Carlo simulation, in which all variables are converted into sets of discrete values. Data are collected and probability distributions constructed for these variables. Random numbers are used to sample from the distributions to give the values used in the simulation. For each model we define the variables involved and the rules which connect them. In small hand simulations, the results are presented in tables which can be analysed easily.

The model may be modified, re-run and the new results compared with the original ones. Simulation does not give an optimum solution in the way that linear programming does but it can guide us to a better solution. Before any output can be used, the model must be validated and the runs must be long enough to give representative results. Simulations are normally are normally carried out using computer packages.

The stock model is based on the equation for stock taking costs per time period:

    TC = ordering costs + stockholding costs (sh/time unit)

    The model assumes complete control over all aspects of the stocking system. The economic order that minimizes TC is:

$$\mathbf{EOQ} = \frac{\sqrt{2DC_0}}{C_h}$$

This basic model is modified in a variety of complex ways to deal with different situations. Quantity discounts can be evaluated by including the total purchase cost of one shortage per time unit. The shortages may be treated as either lost orders or as being supplied from the next order.

The model can be applied to batch production of stock within a company, rather than ordering from outside. Most companies producing a number of different products organise their production on a batch basis rather than on a continuous one. Stock replenishment is gradual rather than instantaneous as for EOQ. There are no ordering costs but set up costs such as incremental labour, incremental overheads, machine down time and other costs for setting up facilities for production e.g. heating time are required.

Quantity discounts

When quantity discounts are offered, the larger batches will increase the costs of stocking the item (ordering plus holding costs) but this will be offset to some extent by a reduction in the purchase price. If the cost of purchase is also included, the total cost equation becomes:

$$\text{Total cost of purchase and stocking} = \frac{C_o D}{q} + \frac{C_h q}{2} + CD$$

Where C is the unit purchase cost.

Example

A shopkeeper purchases soap powder in batches of 158 packets for Sh2 each. The supplier now offers the following discounts:

Order Quantity	Discount	Cost per packet
0-199	0	Sh 2.00
200-499	2%	Sh 1.96
500 or more	4%	Sh 1.92

Should the shopkeeper accept one of the discounts?

Solution

When price is Sh2.00 minimum annual total cost of purchasing 500 packets is Sh1,063.20.

At price sh 1.96 holding cost is:

   C_h=20% of sh 1.96 = sh 0.392/packet/year

$$\text{EOQ} = \frac{\sqrt{2DC_0}}{C_h} = \sqrt{2 \times 10 \times 500 / 0.392} = 159.72$$

The minimum achievable cost will occur when q = 200 packets.

The minimum total annual cost at Sh1.96

$$= \frac{10 \times 500}{200} + 0.392 \times \frac{200}{2} + 1.96 \times 500$$
$$= sh1,044.20 / year$$

It's below the range for the first discount, 200-499.

At price Sh1.92 holding cost is:

   C_h=20% of sh 1.92 = sh 0.384/packet/year

$$EOQ = \frac{\sqrt{2DC_0}}{C_h} = \sqrt{2 \times 10 \times 500 / 0.384} = 161.37$$

It's below the range for the second discount 500 and above.

The minimum achievable cost will occur when q = 500 packets.

The minimum total annual cost (at sh 1.92 per packet)

$$= \frac{10 \times 500}{500} + 0.384 \times \frac{500}{2} + 1.92 \times 500$$
$$= sh1,066 / year$$

Comparing the three solutions:

Cost/packet, sh	Order quantity	Min. total cost, sh/yr
2.00	158	1,063.20
1.96	200	1,044.20
1.92	500	1,066.00

The shopkeeper should take advantage of the first discount offered and place for 200 packets. This will reduce costs by sh 19.00 per year.

1. In a single phase system - service must be received from more than one station sequentially.
   a) True
   b) False
2. ………… this is a case when a member of the population refuses to join then queue due to the size of the queue.
3. Priority queue discipline is a system that gives priority to those who arrive first.
   (a) True
   (b) False
4. ………… method is one of the most common simulation techniques.
5. What is the probability that the servicing facility is idle?
6 A ………..is the period of time between the initiation of any process of production and the completion of that process.
7. It is a point that lies between minimum and maximum stock levels at which purchase orders must be placed to ensure that goods ordered are received before the minimum stock level is reached.
8. What is the formula for economic batch quantity?
9. It is also called the 80/20 rule, control by importance/ exception?
10. List the methods most commonly used in valuing inventory include

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ANSWERS TO CHAPTER QUIZ

1. (b) False – this is Multi-phase system
2. Balking
3. (b) False – this is FIFO queue discipline
4. Monte Carlo
5. $= 1 - \dfrac{1}{m}$

6. Lead time
7. Reorder level

8. $Q = \sqrt{\dfrac{2\,D\,S}{C_h\left(1 - \dfrac{d}{p}\right)}}$

9. ABC System or Pareto analysis

10. (a) First In First Out (FIFO)
    (b) Last In first Out (LIFO)
    (c) Weighted Average method (WAM)

QUESTIONS FROM PREVIOUS EXAMS

QUESTION 1

Discuss the importance of simulation technique to a finance manager of a company.

QUESTION 2

Consider a situation where the mean arrival rate (l) is one customer every 4 minutes and the mean service time $\left(\dfrac{1}{m}\right)$ is $2\dfrac{1}{2}$ minutes. Calculate the average number of customers in the system, average queue length, the average time a customer spends in the system and the average time a customer waits before being solved.

QUESTION 3

Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

   a) What is the probability that a person arriving at the booth will have to wait?
   b) What is the average length of the queue that form from time to time?

c) The telephone department will install a second booth when convinced that arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

QUESTION 4

A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrivals average 80 persons in an 8-hour service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes.

Calculate the average number of customers in the system, average number of customers waiting to be serviced, average time a customer spends in the system, and average waiting time for a customer.

Calculate how many hours each week does a tax adviser spend performing his job. What is the probability that a customer has to wait before he gets service? What is the expected number of idle tax advises at any specified time?

QUESTION 5

Orion Products Ltd is a retailer of materials used in the building trade, operating for 50 weeks each year. One of its fastest selling products is bought in from an outside manufacture at a cost of £8.50 per unit. Forecasted weekly demand for this item is 4,800 units. A three week lead time is required to obtain the product from the manufacturer and Orion's current practice is to order 24,000 units at a time. This order is placed when the stock level falls to 20,000 units.

Orion's financial analysts have established a cost of capital of 15% per annum for the use of inventory decisions within the company. In addition, an analysis of the purchasing operation shows that approximately 15 hours are required to process and co-ordinate an order for the item regardless of the quantity ordered. Purchasing salaries average £10 per hour, including employee benefits. In addition, a detailed analysis of 40 orders showed that £1,425 was spent on paper, postage and telephone related to the ordering process. Also the cost of receiving delivery of an order is estimated to be £80.

Required:

a) Show that C, the cost per order, is £265.625.
b) Calculate the annual total cost of purchasing, ordering and storing the product using the current practice.
c) Calculate the order quantity that minimises annual total cost. Determine the annual saving in using this optimum quantity assuming that Orion Products still requires the same level of protection as currently given by the use of safety stock.
d) The manufacturer offers a discount of 5% if Orion Products order 100,000 or more units at a time. If they wanted to change to these terms it would require extra storage space which would cost £25,000 per annum. Orion Products would still maintain the same level of safety stock.

Write a short report to the finance director of Orion Products, giving advice regarding the decision of whether or not to accept the discount. Include both financial information and any further factors that might influence the decision.

QUESTION 6

a)   Describe the features of a continuous stocktaking and perpetual inventory system.

b)   A retailer sells a large variety of products in its chain of stores. The number of weeks forward stock cover for a sample of 40 products has been calculated. The sample mean and standard deviation are 3.64 and 0.68 weeks respectively.

Required:

  i)   Calculate the 99% confidence interval for the mean value of stockholding weeks for all products.
  ii)   Calculate the sample size required in order to establish the mean stockholding value to within ± 0.15 weeks with 95% confidence.
  iii)   One of the retailer's products has:
      Maximum daily sales   150 units
      Average daily sales   120 units
      Maximum lead time   10 days
      Average lead time 8 days
      Economic order quantity   2,900 units
      Note: The re-order level is established so as to ensure no stock-outs.
      Calculate the average stockholding (in units)

QUESTION 7

  (a)   Briefly explain just in time (JIT) technique.
  (b)   Many organisations do not maintain a perpetual inventory control system.
      What considerations should be taken into account in such organizations, to ensure accurate results during the annual stock taking?

QUESTION 8

A retail company has been reviewing the adequacy of its stock control systems and has identified three products for investigation. The relevant details for the three products are set out below:

Item code	EOQ	Stock in the warehoused stores		Weekly sales Shs. '000'		
	'000' units	'000' units	Shs/unit cost	Minimum	Normal	Maximum
A	25	32.5	2.25	26	28	30
B	500	422.7	0.36	130	143	160
C	250	190	0.87	60	96	128

The management accountant has provided you with the following additional information:

  i)   The gross margin of products A, B and C are 42, 46 and 37. The company policy is to express the gross margin as a percentage of sales.
  ii)   There are six trading days a week. A trading year has 52 weeks.

iii) All orders are delivered by suppliers into the retailer's central warehouse. The lead-time is one week from the placement of order. A further week is required by the retailer in order to transfer stock from the central warehouse to stores. Both of these lead times can be relied on.

iv) The information produced above represents the expected occurrence of demand and costs.

v) An order for item C for 250,000 units was placed 2 days ago.

Required:

a) Calculate for each product:
   i) The minimum and maximum weekly sales units.
   ii) The stock-re-order level.
   iii) The maximum stock level.

b) Comment on the adequacy of the existing stock control of the three products.

QUESTION 9

Bora Supermarket carries on its operations in Nakuru Town. On annual basis, it orders 480,000 pens from a Nairobi-based distributor. A packet of twenty-four pens delivered to Bora's warehouse costs Sh480 including transport charges. The supermarket borrows money from BCD Bank at an interest rate of 10% per annum to finance its inventories.

The supermarket also incurs Sh1,500 to place an order for the pens and sh.8 carrying cost for each pen.

Required:

(i) Economic order quantity (EOQ) for the pens.
(ii) Total cost at the economic order quantity.
iii) For orders of 72,000 pens and above, the distributor is offered a discount rate of 10% of the delivery price.

Advise the management of the supermarket on whether to take advantage of the discount offer.

SUGGESTED SOLUTIONS TO EXAM QUESTIONS

SUGGESTED SOLUTIONS TO EXAM QUESTIONS

June 2000 Question 1

a) (i) Stochastic process is a probabilistic system whereby the state of a given phenomenon in future can be predicted from the current state and a matrix of transition probabilities (transition matrix)

(ii) Transition matrix is a matrix that contains all transition probabilities for a certain process or system. Transition probability is the conditional probability that will be in a future state given the current or existing state.

(iii) Recurrent state refers to a state that can be left and re-entered infinitely many times. It is the state probability of an event occurring at a point in time e.g. probability that a person will be shopping at a given grocery store during a given day of every month. It is given by a cyclic matrix.

(iv) Steady state is the long term state of the system. Provided the assumptions of Markov process persist, the system finally reaches an equilibrium called steady state. At equilibrium the following holds:

(Equilibrium state vector) (Transition Matrix) = (Equilibrium state vector)

b) (i) Interpretation of P_{22} = 0.5
This represents the probability (P_{22} = 0.5) that an account classified as current in the first month will be classified as current in the second month.
Interpretation of P_{23} = 0.2
Represents probability of an account classified as current in the first month becoming overdue in the second month.

(ii) Interpretation of "paid" and "bad" debts states having values of 1.
Paid and bad debts are absorbing states. Once an account is classified here it never changes status.

(iii) Transition Matrix   =

	Paid	Bad debt
Current	0.95	0.05
Overdue	0.87	0.13

Current state vector   =   (70,000   30,000)

Projection for paid and bad debts

$$(70{,}000 \quad 30{,}000) \begin{bmatrix} 0.95 & 0.05 \\ 0.87 & 0.13 \end{bmatrix}$$

Current overdue

Paid Bad debt

      Paid   Bad debts

= (92,600 7,400)

Therefore, Amount expected to be paid back is Sh92,600

June 2002: Question 2

a) (i) Transition matrix is a matrix whose elements are probabilities that a process will change from one state to another state in a defined period of time.

   (ii) Initial probability vector is a vector that gives the initial (starting) probabilities at each state. When an initial probability vector is multiplied by the transition matrix we get future or predicted probability vector.

   (iii) Equilibrium state is the long term or steady state of a markov process. Provided the assumptions of markov process persist, the system finally reaches an equilibrium called steady or long term status i.e. a state where no further net change occurs. At equilibrium the following holds
(Equilibrium state vector)(Transition matrix) = (Equilibrium state vector)

   (iv) An absorbing state is a state which cannot be left once it has been entered.

b) (i) Transition Matrix,

$$M = \begin{array}{c}\\ A \\ B \\ C \\ D\end{array} \begin{array}{cccc} A & B & C & D \\ 0.95 & 0 & 0 & 0.05 \\ 0.20 & 0.70 & 0 & 0.10 \\ 0 & 0.20 & 0.65 & 0.15 \\ 0 & 0 & 0.20 & 0.80 \end{array}$$

   (ii) Number of employees in each class after one year

$$\begin{array}{cccc} A & B & C & D \\ (60 & 90 & 150 & 200) \end{array} \begin{array}{cccc} 0.95 & 0 & 0 & 0.05 \\ 0.20 & 0.70 & 0 & 0.10 \\ 0 & 0.20 & 0.65 & 0.15 \\ 0 & 0 & 0.20 & 0.80 \end{array}$$

$$= \begin{array}{cccc} A & B & C & D \\ (75 & 93 & 137.5 & 194.5) \end{array}$$

Number of employees in each class after two years

$$(75 \quad 93 \quad 137.5 \quad 194.5) \begin{array}{cccc} 0.95 & 0 & 0 & 0.05 \\ 0.20 & 0.70 & 0 & 0.10 \\ 0 & 0.20 & 0.65 & 0.15 \\ 0 & 0 & 0.20 & 0.80 \end{array}$$

$$= \begin{array}{cccc} A & B & C & D \\ (89.85 & 92.6 & 128.275 & 189.275) \end{array}$$

Class	Number of Employees
A	90
B	93
C	128
D	189
Total	500

June 2003 Question 1

a) (i)

	TO			
	P&M	D&T	BEST	EXCEL
P&M	0.7	0.1	0.1	0.1
D&T	0	0.7	0.2	0.1
BEST	0	0	0.9	0.1
EXCEL	0.2	0.1	0.1	0.6

Year 2002 market share =   P&M   D&T BEST   EXCEL
                (0.2 0.3 0.4 0.1)

$$\text{Year 2003 market share} = (0.2 \quad 0.3 \quad 0.4 \quad 0.1) \begin{matrix} 0.7 & 0.1 & 0.1 & 0.1 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.6 \end{matrix}$$

= (0.16   0.24   0.45   0.15)

Audit Firm	Market Share		Comment
	Year 2002	Year 2003	
P & M	20%	16%	LESS
D & T	30%	24%	LESS
BEST	40%	45%	MORE
EXCEL	10%	15%	MORE

(ii) Possible actions for the audit firms which experience reduction in market share
Advertise other services they offer apart from auditing
Reduce their audit fees to attract more market share
To enhance their quality of services
To discuss with their clients on benefit of retaining them as auditors.

b)   (i)   Information in tabular form

Sector	Production (X)	Technical coefficient (A)					Final demand (D)
1	X_1	a_{11}	a_{12}	a_{13}	…	a_{1n}	d_1
2	X_2	a_{21}	a_{22}	a_{23}	…	a_{2n}	d_2
3	X_3	a_{31}	a_{32}	a_{33}	…	a_{3n}	d_3
4	X_4	a_{41}	a_{42}	a_{43}	…	a_{4n}	d_4
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
n	X_n	a_{n1}	a_{n2}	a_{n3}	…	a_{nn}	d_n

   (ii) $X = AX + D$

$IX - AX = D$

$(I-A)X = D$

$(I - A)^{-1}(I - A)X = (I - A)^{-1}D$

$X = (I - A)^{-1}D$

December 2003: Question 2
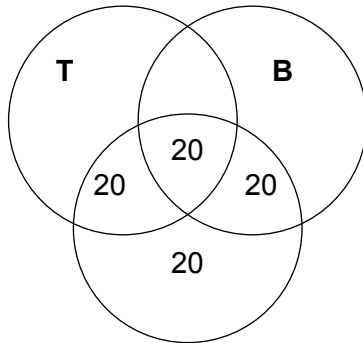
T = Tent
B = Sleeping Bag
S = Stove
Total = 200
T = 60
B = 100
S = 80
TnS = 40
BnS = 40
TnBnS = 20

20 people owned only a camping stone

It is not possible for 30 people to own both a tent and a sleeping bag but not a camping stove in this case. T = 60. If TnB = 30, then T = 20 + 20 +30 = 70 (more than 60)

(b)   1.   Family income

Y = Monthly income

This is a continuous random variable

2.   Family size

X = Number of dependants in the family

This is a discrete random variable

3.   Distance from home to the store site.

d = Distance in kilometres from home to the store site.

This is a continuous random variable.

4. Whether or not the family owns a car or uses public transport

> Z = owns a car and uses itself only
>
> = owns a car and sometimes uses public transport
>
> = Uses public transport (matatus) only
>
> This is a discrete random variable.

December 2004: Question 2

a) Profit, π = R – C

π = 400Q – 25Q² – 2Q³ + 10Q² – 250Q – 100

π = -2Q³ – 15Q² + 150Q – 100

For maximum or minimum profit, $\dfrac{d\pi}{dQ} = 0$

$\dfrac{d\pi}{dQ}$ = -6Q² 30Q + 150 = 0

Q² + 5Q – 25 = 0

$$Q = \frac{5 \pm \sqrt{25 + 4 \times 1 \times 25}}{2 \times 1} = \frac{-5 \pm 11.18}{2}$$

$$Q = 3.09$$

$$\frac{d^2\pi}{dQ^2} = -12Q - 30 = -67.08 < 0$$

= Maximum turning point

Therefore, Optimal number of unit = 309,000 units.

(b) C = 3Q³ – 30Q² + 50Q + 300

For maximum or minimum C, $= \dfrac{dC}{dQ} = 0$

$$\frac{dC}{dQ} = 9Q^2 - 60Q + 50 = 0$$

$$Q = \frac{60 \pm \sqrt{3600 - 4 \times 9 \times 50}}{2 \times 9} = \frac{60 \pm 42.43}{18}$$

Q = 0.98 or Q = 5.69

Second order condition

$\dfrac{d^2C}{dQ} =$ 18Q – 60

When Q = 0.98, $\dfrac{d^2C}{dQ^2} =$ -42.36 < 0 = > maximum turning point

When Q = 5.69, $\dfrac{d^2C}{dQ^2} =$ 42.42 > 0 => minimum turning point

Therefore, Q = 569,000 units

(c) Profit π   = Total Revenue – Total cost

$$= 400Q - 25Q^2 - \{3Q^3 - 30Q^2 + 50Q + 300 + 2Q^3 - 10Q^2 + 250Q + 100\}$$

$$\pi = -5Q^3 + 15Q^2 + 100Q - 400$$

First order condition $\dfrac{d\pi}{dQ} = 0$

$\dfrac{d\pi}{dQ} = -15Q^2 + 30Q + 100 = 0$

$$Q = \frac{-30 \pm \sqrt{900 + 4 \times 15 \times 100}}{2(-15)} = \frac{-30 + 83.07}{-30}$$

Q = 3.77

Second order condition, $\dfrac{d^2\pi}{dQ^2}$ = -30Q + 30 = -83.1 < 0 = > Maximum turning

point

Therefore Q = 377,000 units

(d)   Allowing Apex to charge the transfer fee would not help Ujumi Industries Ltd. to improve their optimal policy. This is because charging a transfer fee will help to reduce the total cost of Apex by Sh 100 per unit. On the other hand maxima cost will increase by Sh100 per unit. The net effect on Ujumi as a whole will be zero (0)

CHAPTER TWO: ANSWERS

$$SIR = \frac{Q3 - Q1}{2} = \frac{44 - 25}{2} = \frac{19}{2} = 8.5$$

The lower quartile (Q1) lies on position

= 29.5 + 6.63

= £36.13

The upper quartile (Q3) lies on position

$$= 3\left(\frac{N + 1}{4}\right)$$

$$= 3\left(\frac{382 + 1}{4}\right)$$

= 287.25

☐ the value of Q3 $= 59.5 + \frac{(287.25 - 279)}{52} \times 10$

= 61.08

The semi interquartile range $= \frac{Q3 - Q1}{2}$

= 12.475

   = £12,475

ii.   The top 10% is equivalent to the lower 90% of the retirees

The position corresponding to the lower 90%

$$= \frac{90}{100}(n + 1) = 0.9(382 + 1)$$

   = 0.9 x 383

      = 344.7

   ∴ the benefits (value) corresponding to the minimum value for top 10%

   $= 69.5 + \frac{(344.7 - 331)}{40} \times 10$

= 72.925

= £ 72925

iii. The lower 40% corresponds to position

   $= \frac{40}{100}(382 + 1)$

      = 153.20

$\therefore$ retirement benefits corresponding to its position

$$= 39.5 + \frac{(153.2 - 119)}{70} \times 10$$

$$= 39.5 + 4.88$$

$$= 44.38$$

$$= £\ 44380$$

Arithmetic mean = Assumed mean + $\dfrac{c\left(\sum fu\right)}{\sum f}$

$$= 63 + \frac{(428 \times 5)}{610}$$
$$= 66.51$$

The standard deviation $= c \times \sqrt{\dfrac{\sum fu^2}{\sum f} - \left(\dfrac{\sum fu}{\sum f}\right)^2}$

$$= 5 \times \sqrt{\frac{3086}{610} - \left(\frac{428}{610}\right)^2}$$

$$= 10.68$$

The position of the median lies m = $\dfrac{n+1}{2}$

$$= \frac{610 + 1}{2} = 305.5$$

$$= 60.5 + \frac{(305.5 - 191)}{120} \times 5$$

$$= 60.5 + \frac{(114.4)}{120} \times 5$$

Median = 65.27

Therefore the Pearsonian coefficient

$$= 3\frac{(66.51 - 65.27)}{10.68}$$
$$= 0.348$$

Comment

The coefficient of skewness obtained suggests that the frequency distribution of the loans given was positively skewed

This is because the coefficient itself is positive. But the skewness is not very high implying the degree of deviation of the frequency distribution from the normal distribution is small

4. a)   Discrete data have distinct values with no intermediate points, whereas continuous data can have any values over a range either a whole number or any fraction.
    b) Dispersion is the variation or scatter of a set of values.

Standard deviation is represented by;

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} \quad ; \quad s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

Where **s:** is for a sample and **σ:** is for a population

c) This is the *coefficient of dispersion* of a distribution that is used in comparing dispersion between distributions. It is given by;

$$coefficient\ of\ \text{variation} = \frac{s}{x} \times 100\%$$

.

CHAPTER THREE: ANSWERS

June 2007 Question 7:

a) Probability is the likelihood or chance that an event will occur.

Probability (P)= <u>Number of favourable chances</u>

Total number of possible chances

=   0 ≤P≤1

b) Let A and B be dependent events, then Bayes theorem states that:

$$P(A/B) = \frac{P(B/A) \times P(A)}{P(B)}$$

Bayes theorem is used to revise subjective probabilities when additional information becomes available.

c) (i) Let: N = No. shortage

S = Small shortage

L = Large shortage

P = Pass test

F = Fail test

Prior probabilities:

P(N) = 0.85, P(S)= 0.1, P(L) = 0.05

Revised probabilities

P(P/N)   =   0.9;   P(F/N)   =   0.1

P(P/S)   =   0.5;   P(F/S)   =   0.5

P(P/L)   =   0.2;   P(F/L)   =   0.8

P(F)   =   P(F/N) × P(N) + P(F/S) × P(S) + P(F/L) × P(L)

=   0.1 × 0.85 + 0.5 × 0.1 + 0.8 × 0.05

P(F)   =   0.175

P(L/F or S/F) =   P(L/F) + P(S/F)

$$= \frac{P(F/L) \times P(L)}{P(F)} + \frac{P(F/S) \times P(S)}{P(F)}$$

$$= \frac{0.8 \times 0.6}{0.175} + \frac{0.5 \times 0.1}{0.175}$$

=   0.514

(ii) P(P)= P(P/N) ×P(P/S) × P(S) + P(P/L) × P(L)

= 0.9 × 0.85 + 0.5 × 0.1 + 0.2 × 0.05

P (P) = 0.825

$$P(N/P) = \frac{P\left(\frac{P}{N}\right) \times P(N)}{P(P)}$$

$$= \frac{0.9 \times 0.85}{0.825} = 0.927$$

December 2001 Question 4

a(i) Past record, P = 0.4

Expected, P = 0.48, q = 0.52

n = 10 (from past record)

Probability of breaking previous record,
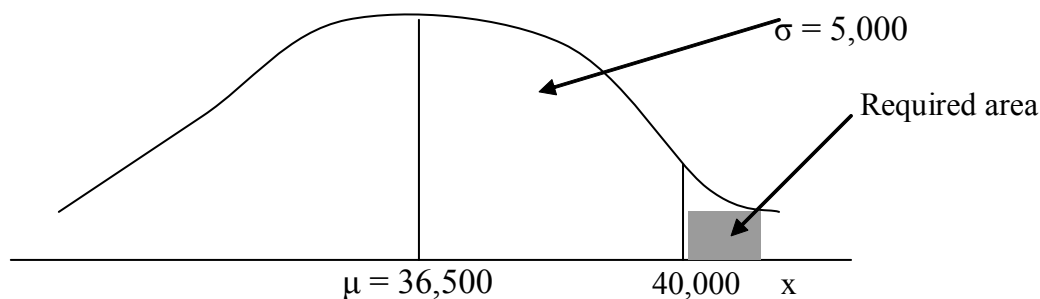
P(x>4) = 1 – {P(0) + P(1) + P(2) + P(3) + P(4)}

P(x>4) =1-{$10_{C0}(0.48)^0(0.52)^{10}$ + $10_{C1}(0.48)^1(0.52)^9$ + $10_{C2}(0.48)^2(0.52)^8$ + $10_{C3}(0.48)^3(0.52)^7$ + $10_{C4}(0.48)^4(0.52)^6$}

P(x>4) = 0.57

(ii) I used binomial distribution to solve part (i) because it is a two situation case for export or not for export. Expected probability of successes (exports) is given together with the total number (n) of past exports.

b) (i) μ = 36,500

σ = 5,000

$$Z = \frac{40,000 - 36,500}{5,000} = 0.7$$

P(36,500 <x<40,000) = 0.258

P(40,00<x) = 0.5 – 0.258 = 0.242 = 24.2%

Since 24.2% >20%, the company should distribute the tyres.

(ii) $P(X=x) = n\alpha P^x q^{n-x}$

$\mu = 36,500$

$\sigma = 5,000$
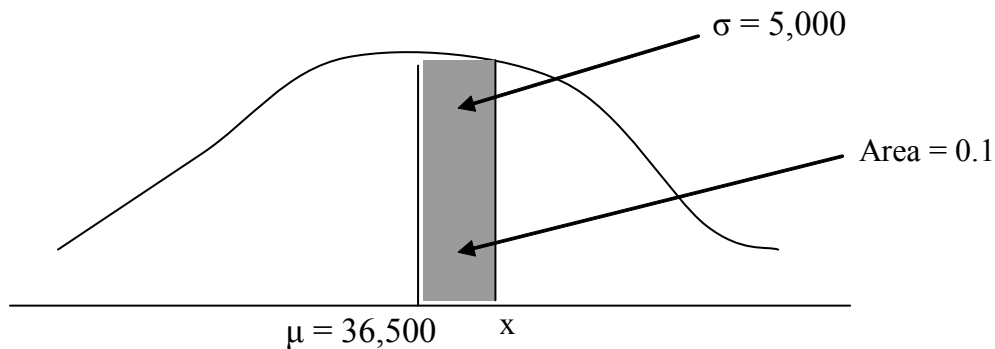
Z values are 0.1, is Z = 0.25

$$Z = \frac{x\ u}{d}$$

$$0.25 = \frac{x\ 36,500}{5,000}$$

x = 36,500 + 0.25 x 5,000

x = 37,750km

The company should set a mileage guarantee of 37,750km.

c) Advantages of normal distribution:
(i)   It is a continuous probability distribution which can take any value from -∞ to + ∞
(ii)   It can be used to calculate the probability between two points
(iii)   It is symmetrical about the mean = median = mode. It can be used to set confidence limits.
(iv)   It is the most widely used probability distribution because it can be used to describe many and diverse phenomenon in nature.
(v)   Can be used to calculate sample size required to minimise sampling error.
(vi)   It is used to set statistical control limits.

June 2002 Question 3

a)   Two events are said to be statistically independent if the occurrence of one event does not influence the occurrence of the other event.
b)   Men that had minor accident = 0.02 x 500 = 10
Men that had safety instructions given they had minor accident = 0.3 x 10 = 3
men that had no safety instructions given they had minor accident = 10 – 3 = 7
Overall No. of men that had safety instructions = 0.2 x 500 = 100
Overall No. of men that had no safety instructions = 0.8 x 500 = 400

	SI	NSI	TOTAL
A	3	7	10
NA	97	393	490
Total	100	400	

Let:
A   = Minor Accident
NA = No minor accident
SI   = Had safety instructions
NSI = Had no safety instructions

(i)   P(NA/NSI) $= \dfrac{393}{400}$ = 0.9825

(ii) P(NA/SI) = $\dfrac{97}{100} = 0.97$

c)   (i)   P(x = x) = $\dfrac{e^{-1} \, \mathrm{I}^{\,x}}{x!}$

P(x = 0) = $e^{-0.4}(0.4)^0$ = 0.6703

(ii) P(x≤2) = P(x = 0, 1 or 2) = P(x = 0)+ P(x = 1) + P(x =2)

$$= \frac{e^{-0.4}(0.4)^0}{0!} + \frac{e^{-0.4}(0.4)^1}{1!} + \frac{e^{-0.4}(0.4)^2}{2!}$$

= 0.6703 + 0.2681 + 0.0536

= 0.9920

June 2003 Question 7

a)   (i)   Expected profit for medium scale expansion
     =   50 x 0.2 + 150 x 0.5 + 200 x 0.3 = Sh. 145,000
Expected profit for large scale expansion
     =   0 x 0.2 + 100 x 0.5 + 300 x 0.3 = Sh. 140,000
Recommendation: Prefer medium scale expansion because it has a higher expected profit.
(ii) Variance, $\sigma^2 = \Sigma(x - \mu)^2 p_i$
     σ^2 (medium scale) = $(50–145)^2 \times 0.2 + (150 – 145)^2 \times 0.5 + (200 – 145)^2 \times 0.3$
     = 2725
σ medium = $\sqrt{2725}$ = 52.20
σ^2(Large scale) = $(0 – 140)^2 \times 0.2 + (100 – 140)^2 \times 0.5 + (300 – 140)^2 \times 0.3$
= 12400
σ Large = $\sqrt{12400}$ = 111.36

$\text{C.V} = \dfrac{\delta}{\mu}$

C.V Medium = $\dfrac{52.20}{145} \times 100$
$= 36\%$

C.V. Large = $\dfrac{111.36}{140} \times 100$ = 79.54%

Prefer the medium scale expansion because it is less riskier.

σ Medium < σ large

C.V. medium < C.V. Large

(iii) (i)   Let: S_1 = Supplier 1

    S_2 = Supplier 2

    G   = Good part

    B   = Bad part

Then:

$P(S_1) = 0.65$ $P(G/S_2) = 0.95$

$P(S_2) = 0.35$ $P(B/S_1) = 0.02$

$P(G/S_1) = 0.98$ $P(B/S_2) = 0.05$

$P(S_1/B) = P(B/S_1)$ x $P(S_1)$

$P(B)$

$P(B) = P(B/S_1)$ x $P(S_1)$ + $P(B/S_2)$ x $P(S_2)$

$P(B) = 0.02$ x $0.65 + 0.05$ x 0.35

$P(B) = 0.0305$

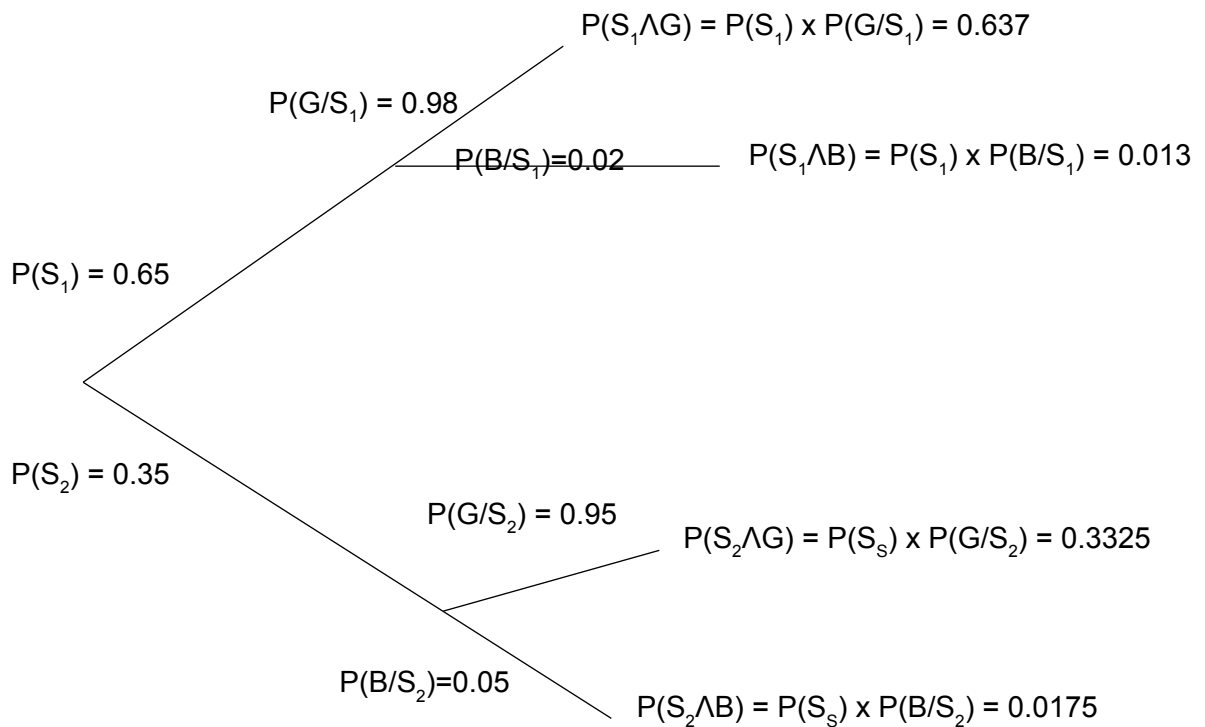$P(S_1/B)$) $\dfrac{0.02 \times 0.65}{0.0305}$ = 0.4262

$P(S_2/B)$ = $P(B/S_2)$ x $P(S_2)$

$P(B)$ =0.0305

$P(S_2 / B) = \dfrac{0.05 \times 0.35}{0.0305}$

= 0.5738

(ii)

$P(S_1 \wedge G) = P(S_1) \times P(G/S_1) = 0.637$

$P(G/S_1) = 0.98$

$P(B/S_1)=0.02$ $P(S_1 \wedge B) = P(S_1) \times P(B/S_1) = 0.013$

$P(S_1) = 0.65$

$P(S_2) = 0.35$

$P(G/S_2) = 0.95$

$P(S_2 \wedge G) = P(S_S) \times P(G/S_2) = 0.3325$

$P(B/S_2)=0.05$

$P(S_2 \wedge B) = P(S_S) \times P(B/S_2) = 0.0175$

December 2004 Question 7

a) (i) Prior probabilities are not necessarily subjective when performing a Bayesian decision analysis. Managers make judgements regardless of whether they have complete and flawless data in hand or not. Past data can be used to make decisions. Use of subjective probabilities is just persuasive with respect to decision analysis.

ii) Bayes action will be the same regardless of whether it is selected using expected monetary payoffs or expected utilities only if there is a linear relationship between monetary values and utility values. otherwise it will not be the same.

(iii) Maximax and maximin decision criteria point towards different actions.

Maximax criterion – Decision maker assumes that the best will occur and proceeds to maximise the best of the outcomes. It is a criterion of extreme optimism.

Maximin Criterion – Decision maker assumes the worst outcome will occur and selects that option that will give the best of the worst outcomes. It is a criterion on extreme pessimism.

b)

Prior Probabilities

Additional Information from sample data

Bayes Theorem

Bayes Theorem
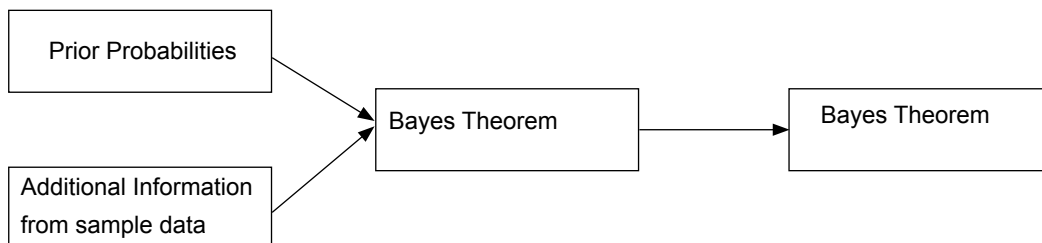
c) "There is only one thing certain and that is that nothing is certain." This statement means that we are sure that tomorrow will come, but we are not sure what the future holds for us. Probabilities are used to plan for the future since we cannot be 100% sure of what will happen.

CHAPTER FOUR: ANSWERS

December 2000 Question 4

a)   (i) Binomial distribution is a discrete distribution where the result of an experiment can take only one of the two possibilities. These two possibilities are usually referred to as "success" or "failure".

For a binomially distributed random variable, the probability of exactly X successes and (n-x) failures in an experiment performed n times is given by the formula:

$$P(x) = \begin{bmatrix} n \\ x \end{bmatrix} p^x q^{n-x}$$

Where: P = Probability of a success

         q = probability of a failure

ii)   Poisson distribution is a discrete distribution that is used to describe situations where occurrences are random and rare in space and time. To completely describe a Poisson distribution we only require to obtain the mean, $\mu = \lambda$

$$P(X=x) \quad = \quad \frac{e^{-\lambda}\lambda^x}{x!}$$

iii)   Normal distribution is a continuous sampling that is used to describe many and diverse phenomenon in nature.

It is completely defined by mean, μ and standard deviation, σ.

Normal probability density is given as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{(x-\mu)2\ 12\sigma-2}$$

iv) Chi-square distribution is a continuous sampling distribution that is used to test goodness of fit and independence between variables.

$$X^2 = \sum \frac{(f_o - f_e)^2}{f_e}$$

where f_o = observed frequency

         f_e = Expected frequency

v)   Fischer (F) distribution is a continuous sampling that is used to test the equality of variances of two normal populations.

If S_1^2 and S_2^2 represent sample variances of two normal populations, then

$$F = \frac{S_1^2}{S_2^2} \quad \text{where } S_1^2 > S_2^2$$

b) One example in the accounting profession where each of the above distributions can be applied:
   (i) Binomial distribution can be used to calculate the probability of accounts audited had procedural errors.
   ii) Poisson distribution can be used to calculate the probability of x bad debts given the mean λ. Can also be used to describe customers arriving for consultancy and audit service.
   iii) Normal distribution:
   To describe the distribution of invoices
   To construct confidence interval of mean invoice

iv) Chi-square distribution

To measure the extent of observed bad debts and expected bad debts agree.

v) F-Distribution

Can be used to test the variances between two sales populations.

December 2002 Question 2

(a) (i) Advantages of auditing a sample of the accounts
     It is less expensive
     It saves time. It takes lesser time than if the whole population of accounts were audited.
     Sampling error can be assessed.
     Leads to accurate results within a shorter time.
     Random sample is a representative of the whole population provided it is unbiased.
   ii) I would recommend systematic random sampling technique
     This is a technique where every nth element in an ordered population is chosen starting from a random point in the population frame. Sampling frequency,

$$n = \frac{\text{population size}}{\text{sample size}}$$

   iii) Advantages of systematic random sampling
     It has a lower sampling cost
     It is less time consuming
     Does not require tedious task of data collection
     It's quicker than simple random sampling

     Disadvantages

If there is periodicity (repeated patterns) in the population, then there is a danger of biased results.

Requires one to have access to all the elements in the population

It is not completely random as only the first element is selected randomly.

b) Qualities of a good point estimator
   (i) Should be unbiased – An estimator is said to be unbiased if its expected value is equal to the estimated parameter e.g. sample mean (X) is an unbiased estimator of the population mean (μ).
   ii) It should be consistent – As the sample size increases the estimator tends to the estimated parameter.
   iii) It should be sufficient – uses all the information in the sample in estimating a parameter.

iv)   It should be <u>efficient</u> – it has the minimum variance among competing estimators.

June 2003 Question 4

a) Error, $E = Z\sqrt{\dfrac{Pq}{n}}$

$$n = \dfrac{(Z\alpha/2)^2\, pq}{E^2}$$
$$= \dfrac{1.96 \times 0.5 \times 0.5}{E^2}$$

Recommended sample size for

i)   Early December survey

E = 0.02

n = $\underline{1.96^2 \text{ x } 0.5 \text{ x } 0.5}$ = 2401
          $(0.02)^2$

ii)   Pre-election day survey

E = 0.01

n = $\underline{1.96^2 \text{ x } 0.5 \text{ x } 0.5}$ = 9604
       $(0.01)^2$

b)   Hypothesis

$H_0: \mu_1 - \mu_2 = 0$ or $H_0 : (\mu_1 = \mu_2)$ (Mean completion time are equal)

$H_1: \mu_1 - \mu_2 > 0$ or $H_1: (\mu1 > \mu2)$ (New software have a smaller mean completion time)
Level of significance $\alpha = 0.05$

Degrees of freedom = $n_1 + n_2 - 2$ = 12 + 12 – 2 = 22
$\alpha = 0.05$ t-critical = 1.717 = 1.72
d.f = 22
Decision Rule: Reject H_0 if t-calculated > 1.72

      S^2 = $\dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}$

            = $\dfrac{(12 - 1) \text{ x } 40^2 + (12 - 1) \text{ x } 44^2}{12 + 12 - 2}$

    S2   = 1768

Calculated t

$$= \frac{(X_1 - X_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)S^2}}$$

$$t = \frac{(325 - 288) - (0)}{\sqrt{\left(\frac{1}{12} + \frac{1}{12}\right) \times 1768}} = \frac{37}{\sqrt{294\frac{2}{3}}}$$

$$= 2.16$$

Since t-calculated > 1.72, we reject H0 and conclude that the new software package provides a smaller mean completion time.

December 2004 Question 4

(a) (i)   Null hypothesis is a statement made concerning a population. Sample data is used to test its validity.

ii)   Parametric test is a test of hypothesis where an assumption is made about the distribution of a population.

b)   Uses of chi-square distribution

Test for independence

Test for goodness of fit

Test for variance

Test for equality of proportions

c)   (i) Null hypothesis, H0: Mode of study and exam results are independent

Alternative hypothesis, H1: Mode of study and exam results are not independent

(ii)   Computed test statistic X2 = 42.28

α = 0.05   Critical X^2 = 9.488

d.f = 2 x 2 = 4

Since X^2 calculated is greater than X^2 critical, we reject the null hypothesis and conclude that mode of study and exam results are not independent.

CHAPTER FIVE: ANSWERS

December 2000 Question Five

a)   Information provided by economic models include:

Functional relationship between costs and level of production
Functional relationship between price and units produced and sold
Functional relationship between profit, revenue and costs
Functional relationship between revenue, price and units produced and sold.
Sales revenue required to break even
Sales units to produce and sell to just break even.
Unit sales required to earn projected profit, T.
Marked equilibrium price.
Forecasted market shares of competitors
Predicted share prices in the stock exchange
Estimated provision for bad debts
Sale forecasts from experts (sales staff)
Consumer market surveys to help in preparing sales forecasts, improving product design
and planning for new products.

b) $F_{t+1} = (1 - \alpha)F_t + \alpha A_t$

Where F_t =   Current period forecast value
F_{t+1} =   Next period forecast value
A_t =   Current Actual value
α =   Smoothing index
α =   0.5 = 1 – a = 0.5
F_{t+1} = $0.5F_t + 0.5A_t$

Month	Amount, At (Sh.)	Ft+1 = 0.5Ft + 0.5At
December	30,660	30,660
January	27,190	30,660
February	30,570	28,925
March	30,640	29,747.50
April	29,730	30,193.75
May	31,530	29,961.88
June	29,720	30,745.94
July	33,070	30,232.97
August	30,010	31,651.49
September	27,550	30,830.75
October	30,130	29,190.38
November	27,940	29,660.19
December	-	28,800.10
January	-	28,800.00

Next January's forecast = Sh. 28,800.10

June 2002 Question 5

a) $\hat{Y} = a + b_1x_1 + b_2x_2 + b_3x_3$

   $\hat{Y} = 3.1 + 0.021x_1 + 0.075x_2 + 0.043x_3$

   $S_1 = 0.019$ $S_2 = 0.034$ $S_3 = 0.018$

   $\alpha = 5\%$

   $\alpha/2 = 0.025$ t = 2.09

   d.f = 24 – 4 = 20

Confidence interval = Point estimate ± tx standard error.

i)   t1 = $\dfrac{b_1}{S_1} = \dfrac{0.021}{0.019}$ = 1.1053

   95% confidence interval   = 0.021± 2.09 x 0.019

                = 0.021± 0.3971

                -0.3761 < β_1 < 0.06071

ii) $t_2 = \dfrac{0.075}{0.037}$ = 2.2059

   95% confidence interval   = 0.075± 2.09 x 0.034

                = 0.075± 0.07106

   0.00394 < β_1 < 0.14606

iii) $t_3 = \dfrac{0.043}{0.018} = 2.3889$

95% confidence interval $= 0.043 \pm 2.09 \times 0.018$

$= 0.043 \pm 0.03762$

$0.00538 < \beta_1 < 0.08062$

b)   Assumptions made:

sum of errors is zero

errors are normally distributed

errors are independent

Regression coefficient are small

Sample size is small and hence t-statistic is used

Predictor variables are independent.

The assumptions are reasonable because confidence interval estimate is symmetrical about the mean.

c)   Regression coefficient 0.043 gives the strongest evidence of being statistically discernible because it has the biggest t-ratio.

d) X_1 should be dropped because it is not statistically significant. Its ratio < 2.09. X_1 is therefore not necessary.

June 2003 Question 5

a) $Y = T + C + S + R$

$Y - T - S = C + R$ = Residual variation

Residual variation means other factors not explained by trend and seasonal variations. The two main constituents of residual variation are;

   i)   Random factors – These refer to variations on a time series that are caused by unpredictable events.

   ii)   Cycle factors – These refer to variations on a time series that are caused by underlying economic causes outside the scope of the immediate environment.

b)   Moving average centring and why it is employed

   When calculating moving averages with an even point period, the resulting moving average is placed in between two corresponding time points. These moving averages are centred by averaging successive overlapping pairs. This ensures that each calculated trend value (centred moving average) is in line with a time point.

c)   Procedure for calculating seasonal variation values (seasonal index)

   i)   Calculate the trend values e.g. using moving averages

   ii)   Calculate the difference between the original time series values and the trend i.e. y-t

   iii)   Compute the average of (y-t) for each season to obtain the average seasonal index.

d)   (i) Projecting "by eye"

   This is also called free hand method

   It involves fitting a trend line or curve by looking at the shape of the graph. This method is likely to give different answers.

   It may be used when calculated trend values are non-linear.

ii) Method of semi averages
In this method data is divided into two parts and then the average for each part is obtained. The two averages are then plotted and joined by a straight line representing the trend.
This method can be used when we have fluctuating linear trend values.

iii) Average change in trend per period from the range – used when we have fairly steady linear trend values.

e) How the management of an organisation might use seasonal variation figures and seasonally adjusted data.
Used to reveal seasonal influences on a time series data

Used to adjust trend forecasts

Used to predict future values

Seasonally adjusted data may be used to calculate the trend

Used feedback control.

December 2003 Question 5

a) The purpose of selecting participants from different functional fields is to:
Exchange ideas and experiences

Have a chance of hearing expert opinion from different fields

To compare how similar problems are handled in different fields

Arm participants with diverse knowledge so as to improve on the quality of their work.

Strategies acquired may backfire because future states of nature may change.

b) (i) The above output shows

Moving average – a three point (season) moving average is used to estimate the trend values
Seasonal – Irregular component – This is the difference between the sales and corresponding trend values.
Deseasonalised data – Time series data whose seasonal influences have been removed.

(ii) ii

	Seasons		
Year	1	2	3
1999	1.156	1.225	0.586
2000	1.162	1.236	0.599
2001	1.226	1.237	0.581
2002	1.176	1.219	0.557
Total	4.72	4.917	2.323
Average seasonal index	1.18	1.23	0.58

i) T_t = 1580.11 + 33.96t

1. When t = 13

   T_t = 1580.11 + 33.96(13)

   T_t = 2,021.59

2. When t = 14

   T_t = 1580.11 + 33.96 (14)

   = 2055.55

3. When T = 15

   T_t = 1580.11 + 33.96 (15)

   = 2089.51

June 2004 Question 5

a) Principal components of a time series are:

   Secular trend (T)

   Seasonal variations (S)

   Cyclic variation (C)

   Random variation (R)

b) (i) Differences between multiplicative and additive models:

   Multiplicative model expresses the time series model as a product of the four principal components
   That is Y = TCSR

   Additive model expresses the time series model as a sum of the four principal components.
   That is Y = T + C + R + S

   ii) Conditions under which each model is used

   Multiplicative model is used if the four principal components are not independent.

   Additive model is used when the four principal components are independent.

c) i) Purpose of seasonal index

   Used to evaluate seasonal effects on a time series

   Used to adjust trend forecasts

   Used to deseasonalise data

(ii)

Year	Quarter (Q)	Sales (Y)	Uncentred 4-Quarter M.A	Centred 4-Quarter M.A	Difference Y - MA
2001	1	55.0	-	-	-
2001	2	76.5	67.625	-	-
2001	3	61.2	67.475	67.55	-6.36
2001	4	77.8	64.825	66.15	11.65
2002	1	54.4	62.700	63.76	-9.36
2002	2	65.9	63.600	63.15	2.75
2002	3	52.7	64.825	64.21	-11.51
2002	4	81.4	69.150	66.99	14.41
2003	1	59.3	75.600	72.38	-13.08
2003	2	83.2	78.500	77.05	6.15
2003	3	78.5	-	-	-
2003	4	92.0	-	-	-

Subsidiary table

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2001	-	-	-6.35	11.65
2002	-9.36	2.75	-11.51	14.41
2003	-13.08	6.15	-	-
Total	-22.44	8.90	-17.86	26.06
Average Seasonal Index	-11.22	4.45	-8.93	13.03

CHAPTER SIX ANSWERS

December 2000 Question 7

a) Under the environment of risk it is not known exactly which state of nature will occur. However, there is sufficient information for us to estimate chances of occurrence of the various states of nature. Two decision criteria may be used:
Maximise expected monetary value

Minimise expected opportunity loss.

Under the environment of uncertainty it is not possible to obtain reliable or accurate data to assign probabilities to the various states of nature. Decision criteria used include:
Maximax

Maximin

Hurwicz

Minimax regret

Laplace

b) A decision tree is a graphical representation of the decision process indicating decision alternatives, the various states of nature, their probabilities and consequential pay offs.
A probability tree is a graphical representation of events or outcomes together with their corresponding probabilities.

c) Minimax Regret criterion is based on the concept of opportunity loss. The method seeks to minimise the maximum possible losses for each decision alternative.
Maximax criterion involves selecting the alternative with maximum payoff from choice of maximum payoffs for each decision alternative.

d) A pure strategy game is one in which each player knows exactly what the other player is going to do. Each player plays the same strategy throughout the game.
A mixed strategy game is one in which players do not know each other's strategies. Each player adopts one strategy part of the time and the other favourable option(s) the rest of the time.

e) n-person games involve more than two players. These represent real life conflict situations where each seeks to gain from others.
A non-zero sum game is one in which the gain of one player may not be equal to the loss of the other player.

December 2001 Question 7

a) Payoff Table (Sh. Million)

			No. of household subscribers (states of nature)					
			10,000	20,000	30,000	40,000	50,000	60,000
Plan &	I	150	-5.5	-4	-2.5	-1	0.5	2
Revenue	II	180	-5.2	-3.4	-1.6	0.2	2	3.8
	III	200	-5	-3	-1	1	3	5
	IV	240	-4.6	-2.2	0.2	2.6	5	7.4

Payoff = profit = Revenue/household x Number of households – cost of the system e.g. for (150, 10,000), payoff = 150 x 10,000 – 7,000,000 = -5,500,000 = -5.5 million
For (150, 20,000), payoff = 150 x 20,000 – 7,000,000 = -4 million

b) Optimistic approach is the max – max criterion

Plan	Max
I	2
II	3.8
III	5
IV	7.4

Adopt Plan IV since it gives the maximum of the maximums.

Minimax regret approach

			No. of household subscribers						
			10,000	20,000	30,000	40,000	50,000	60,000	Max
Plan	I	150	0.9	1.8	2.7	3.6	4.5	5.4	5.4
	II	180	0.6	1.2	1.8	2.4	3	3.6	3.6
	III	200	0.4	0.8	1.2	1.6	2	2.4	2.4
	IV	240	0	0	0	0	0	0	0

Adopt Plan IV because it has the minimum of the maximum regrets.

c EMV (I) = 0x – 5.5 + 0.05x – 4 + 0.05x – 2.5 + 0.4x – 1 + 0.3 x 0.5 + 0.2 x 2 = -0.175

EMV (II) = 0.05x -5.2 + 0.1 x -3.4 + 0.2x -1.6 + 0.3 x 0.2 + 0.2 x 2 + 0.15x 3.8 = 0.11

EMV (III) = 0.1 x -5 + 0.2 x -3 + 0.2 x -1 + 0.2 x 1 + 0.2 x 3 + 0.1 x 5 = 0

EMV (IV)= 0.2 x -4.6+ 0.25 x -2.2+0.25 x 0.2 + 0.15 x 2.6 + 0.1x5 + 0.05x 7.4 = -0.16

Plan II is optimal. It gives the highest expected monetary value of Sh110,000.

d)   Maximax and minimax are decision criteria under the environment of uncertainty. In this environment it is not possible to obtain reliable or accurate data to assign probabilities to the various states of nature.

Expected monetary value is a decision criterion under the environment of risks. In this environment it is not known exactly which state of nature will occur. However there is sufficient information for us to estimate chances of occurrence of the various states of nature.

June 2002 Question 7

a)   (i)   Dominance is a situation in which a certain strategy is better than the other(s). In a game situation a strategy is said to dominate the other(s) if it is superior.

(ii)   Saddle point is one that has the smallest value. of a pure strategy game as given by the saddle point of the payoff matrix.

(iii)   When a game has no saddle point, players result to mixed strategy game. A mixed strategy arises when a player decides to adopt one option part of the time and the other option(s) the rest of the time.

(iv)   Value of the game is the expected payoff to the winner when they play their best strategies. It can also be defined as the average payoff to a winner over a long series of plays.

b)   (i)

		Y1	Y2	Y3	Y4	M	IN
Player A	X1	2	2	3	-1	-1	
	X2	4	3	2	6	2	Max. min
Max		4	3	3	6		

Min max

It is not possible to determine the value of the game using maximum and minimax because there is no saddle point.

(ii) $X_1 + X_2 = 1$ => $X_2 = 1 = X_1$

$y_1 + y_2 + y_3 + y_4 = 1$

$x_1 \geq 0$

$y_1 \geq 0$

Use B's pure strategies to get A's expected payoffs.

1. $V1 = 2X_1 + 4X_2 = 2X_1 + 4(1 - X_1) = 4 - 2X_1$

X_1	0	2
V_1	4	0

2. $V2 = 2X_1 + 3X_2 = 2X_1 + 3(1 - X_1) = 3 - X_1$

X_1	0	3
V_1	3	0

3. $V3 = 3X_1 + 2X_2 = 3X_1 + 2(1 - X_1) = 2 + X_1$

X_1	0	1
V_1	2	3

4. $V4 = -X_1 + 6X_2 = -X_1 + 6(1 - X_1) = 6 - 7X_1$

X_1	0	6/7	1
V_1	6	0	-1

Plot the four lines as a function of X_1
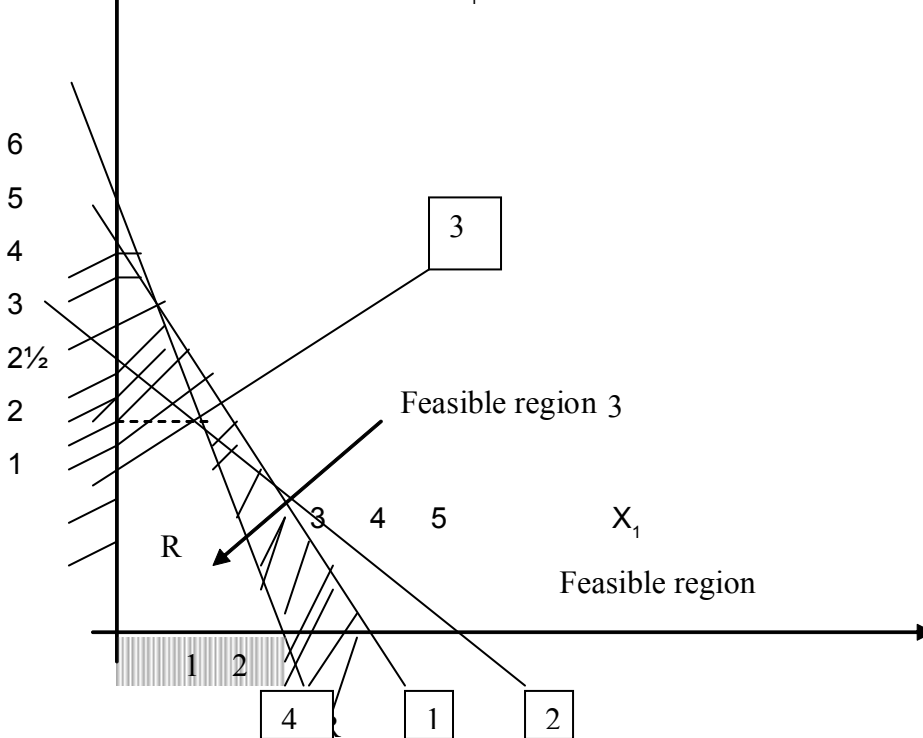
Maximum of the minimum occurs at $X_1 = ½$

This is the best intersection of lines 2, 3, and 4

Optimal solution

$X_1 = ½$, $X_2 = ½$

$V = 2 ½$

A to play strategy, 50% of his time and 50% of the time to play strategy 2; after which he will win 2 ½ points.

December 2002 Question 7

i)   Expected No. of hrs = 0 x 0.4 + 1 x 0.3 + 2 x 0.2 + 3 x 0.1 = 1 hour

(ii)

Hours	Revenue	Salary	Extra wages	Net gain/Loss
0	0	1,500	0	-1,500
1	2,500	1,500	0	1,000
2	5,000	1,500	3,000	500
3	7,500	1,500	6,000	0

a)

		ECONOMY	
		UP	DOWN
PRICE	HIGH	5	2
	LOW	3	1

The student's best strategy is to set a high price. This is because the high price strategy dominates over the low price strategy.

bExpected amount to be collected

= 50,000 x 0.2 + 100,000 x 0.4 + 150,000 x 0.3 + 200,000 x 0.1

= Sh.115,000

Cost for strategy 1:

= 40,000 + 0.05 x 115,000

= Sh. 45,750

Net expected amount for strategy I = 115,000 – 45,750 = Sh. 69,250

Costs for strategy II = 0.3 X 115,000 = Sh. 34,500

Net expected amount for strategy II = Sh. 115,000 – 34,500 = Sh. 80,500

    (i)   Payoff table

	Net payoff
Strategy I	69,250
Strategy II	80,500

(ii)   Strategy II should be adopted since it has the highest net expected amount.

    December 2003 Question 7a) "Winning isn't everything; it is the only thing"

This statement means that every player in a game desires to win than to lose. None of the players is charitable. Every player respects the rules of the game and uses them to maximise their payoffs.

b) Mathematically a saddle point is a point that is maximum with respect to one variable and a minimum with respect to another variable. If a game is pure strategy, the saddle point will be given by the element that is both a maximum in its column and a minimum in its row. A pure strategy game is one in which a player plays the same strategy throughout the game. The value of a strategy game is given by the saddle point.

c)

		ANN WAIRIMU	
		X	Y
JOSEPH	A	-1	-2
NJAU	B	-1.5	2

$$A = \frac{2 - (-1.5)}{(1+2) - (-1.5 - 2)} = \frac{3.5}{6.5} = \frac{7}{13} = 53.85\%$$

$$B = 1 - \frac{7}{13} = \frac{6}{13} = 46.15\%$$

$$X = \frac{2 - (-2)}{(1+2) - (-1.5 - 2)} = \frac{4}{6.5} = \frac{8}{13} = 61.54\%$$

$$Y = 1 - \frac{8}{13} = \frac{5}{13} = 38.46\%$$

CHAPTER SEVEN ANSWERS

June 2000 Question 6

(a) Let: X_1 = Number of shares to be invested in Airline

X_2 = Number of shares to be invested in Insurance

X_3 = Number of shares to be invested in Information

Technology

Stock	Current price (Sh.)	Projected price (Sh.)	Appreciation (Sh.)
Airline	25	35	10
Insurance	50	60	10
Information Tech.	100	125	25

Objective function: Max $Z = 10X_1 + 10X_2 + 25X_3$ (Stock appreciation or gain)

Constraints

1. $25X_1 + 50X_2 + 100X_3 \leq 10{,}000$ (Total investment)
2. $X_1 \leq 5{,}000$ (Investment in Airline)
3. $X_2 \leq 5{,}000$ (Investment in Insurance)
4. $X_3 \leq 5{,}000$ (Investment in I.T)
5. $X_1 \geq 1{,}000$ (Investment in Airline)
6. $X_2 \geq 1{,}000$ (Investment in Insurance)
7. $X_3 \geq 1{,}000$ (Investment in I.T)
8. $X_1, X_2, X_3 \geq 0$ (Non-negativity)

(b) Reduced costs of a basic variable is zero

Variable	Reduced cost
Airline stock	0
Insurance stock	0
IT stock	0

Slack   =   Unused amount of constraints

Surplus   =   Amount of constraints that exceed

Slack   =   Maximum capacity – used capacity

Surplus   =   Used capacity – Minimum capacity

NB: A constraint that has a shadow price is fully used up i.e. its slack/surplus = 0

Slack for insurance stock   =   5,000 – 50(20) =   4,000

Slack for IT stock   =   5,000 – 100(40) = 1,000

Surplus for Airline stock   =   25(200) – 1,000 = 4,000

Surplus for I.T stock   =   100(40) – 1,000 = 3,000

Constraint   Slack/surplus

Amount available 0

Maximum – airline 0

Maximum – insurance stock   4,000 (Slack)

Maximum – I.T stock   1,000 (Slack)

Minimum – airline 4,000 (Surplus)

Minimum – insurance stock 0

Minimum – I.T stock   3,000 (surplus)

Objective coefficient ranges:

Variable current value

Airline stock   10

Insurance stock   10

IT stock 25

Right hand side ranges

Variable Current value

Amount available 10,000

Maximum – airline 5,000

Maximum – insurance stock   5,000

Maximum – I.T stock   5,000

Minimum – airline 1,000

Minimum – insurance stock 1,000

Minimum – I.T stock   1,000

c)   Optimal solution

X_1 = 200 (Airline shares)

X_2 = 20 (insurance shares)

X_3 = 40 (IT shares)

Max Z   = $10X_1 + 10X_2 + 25X_3$

= 10 (200) + 10 (20) + 25 (50)

Maximum gain = Sh.3,200

i)   Overall one year gain increase = 1,000 x 0.25 = Sh. 250
ii)   It should not be IT stock because they have not exhausted the maximum amount allocated. The shadow price is zero (0).

iii) Increase allowed maximum investment to Sh.6,000 for airline stock because it has exhausted the maximum amount allocated (Slack = 0). Its shadow price = Sh.0.15. It has the highest shadow price. Objective function would increase by 1,000 x 0.15 = Sh.150.

iv) The allowed maximum investment amount (upper limit) for airline stock would be Sh.8,000.

June 2002 Question 6

(a) (i) Feasible solution is a solution that satisfies all the constraints in the problem

(ii) Transportation problem is a special case of linear programming that is concerned with the transportation or allocation of goods from various sources to various destinations. The purpose of transportation problem is to schedule the conveyance of goods between sources and destinations in such a way that costs are minimised and contributions are maximised.

iii) Assignment problem is a special case of a transportation problem that is concerned with the determination of the most economical (optimal) way of allocating tasks to facilities. Assignment problem assumes that a task can only be assigned one facility and one facility to one task. The number of tasks must be equal to the number of facilities.

b) Introduce a dummy sales person E, to make number of rows equal number of columns

	1	2	3	4	5
A	92	90	94	91	83
B	84	88	96	82	81
C	90	90	93	86	93
D	78	94	89	84	88
E	0	0	0	0	0

Reduce each row by the largest figure in the row and ignore the resulting negative sign because it is a maximisation problem.

	1	2	3	4	5
A	2	4	0	3	11
B	12	8	0	14	15
C	3	3	0	7	0
D	16	0	5	10	6
E	0	0	0	0	0

Minimum number of lines = 4 < 5 => Solution is not optimal

	1	2	3	4	5
A	0	1	0	1	11
B	10	8	0	12	15
C	1	8	0	5	0
D	14	0	5	8	6
E	2	2	2	0	2

Minimum number of lines = 5 = No. of rows on columns =>solution is optimal

Optimal solution

Sales person	District to be assigned	Ratings
A	1	92
B	3	96
C	5	93
D	2	94
E	4	0
Maximum total ratings		375

June 2003 Question 6

a) Let x = Number of standard size of frames
    y = Number of slim-line size of frames

    Z = Total contribution

        Contribution per unit of x = 6.00 – 0.75 – 0.25 = Sh.5

        Contribution per unit of y = 4.00 – 0.50 – 0.25 = Sh.3.50

        Objective function, max Z = 5x + 3.5y

        Subject to:

    $1.5x + y \leq 350$ (Raw materials)

    $x + y \leq 300$ (packaging boxes)

    $0.4x + 0.25y \leq 80$ (Labour hours)

    $x, y \geq 0$ (Non-negativity)

b) Optimal daily production plan

    produce 33.33 = 34 standard size of frames

    produce 266.667 = 267 slim-line size of frames

    optimal Z = 5(33.33) + 3.5(266.667) = Sh.1,100

c) Maximum selling price = Upper limit = Sh.5.60

d) The optimal solution would not change if C1 changed from its current value to Sh.5.50. Sh.5.50 is below the upper limit of Sh.5.60.

e) The optimal solution would not change because Sh.4 is less than Sh.5 (the upper limit of C_2)

Percentage change of C1 = $\dfrac{5.5 - 5}{5} \times 100$
$$= 10.0\%$$

Percentage change of C_2 = $\dfrac{4 - 3.5}{3.5} \times 100$
$$= 14.3\%$$

The optimal solution would not change if C_1 and C_2 were changed simultaneously to Sh.5.50 and Sh.4 respectively because the sum of percentage changes is less than 100%.

f) Shadow/dual price for man hours is Sh.10. If an extra one hour of labour is obtained, the profit would go up by Sh.10.

g) Shadow/dual price of packing is Sh.1. If we get an extra one packing box, profit will go up by Sh.1.

h) They will remain the same because the sum of percentage changes is less than 100%.

December 2003 Question 6

a) Examples where fractional values might be meaningless.

   (i)   Number of houses to be constructed e.g. 3¼ houses
   (ii) Number of machines to be used e.g. 3 ½ machines
   (iii) Number of employees required e.g. 5 ½ people etc.

To overcome the assumption of divisibility integer programming should be used.

b) (i)   Reduced costs column and the range of optimality in a linear programming output indicates how much the current costs or profits will be increased or decreased for a non-basic variable to become a basic variable.

   ii) Dual prices and the range of feasibility indicate how we can flex the utilisation of the resources. They also indicate the scarce and abundant resources.

c) Let X_1 = The number of speed hawk boats ordered

   X_2 = The number of silverbird boats ordered

   X_3 = The number of Chui boats ordered

   X_4 = The number of samba boats ordered

Objective function:

Maximise Z = $7000X_1$ + $8000X_2$ + $5000X_3$ + $11000X_4$

ubject to:

$60,000X_1$ + $70,000X_2$ + $50,000X_3$ + $90,000X_4 \leq 4,000,000$ (Budgeted amount)

X_1 + X_2 + X_3 + $X_4 \geq 50$ (Least number of boats to purchase)

X_1 + X_2 = X_3 + X_4 (Equal number of boats to be purchased from the two manufacturers)

$3X_1$ + $5X_2$ + $2X_3$ + $6X_4 \geq 200$ (Least capacity)

$X_1, X_2, X_3, X_4 \geq 0$ (Non-negativity)

December 2004 Question 6

a) $Z = 4X_1 + 6X_2 + 5X_3 + 3.5X_4$

Optimal profit:

$\quad$ = $\quad$ $4(12) + 6(0) + 5(12) + 3.5(60)$

$\quad$ = $\quad$ Sh.318

b) (i) $\quad X_1 = 0$

$\quad X_3 = 0$ $\qquad$ Because they are basic variables

$\quad X_4 = 0$

ii) $\quad$ 1 and 3 are slack variables ($S_1 = 0$, $S_3 = 0$)

$\qquad$ 4 is a surplus ($S_4 = 0$)

iii) $\quad$ Since $X_2 = 0$ and $X_1 = 12$

$1.5X_1 + 2X_2 + S_2 = 54$

$S_2 = 36$

Some special display racks have not been used

Dual or shadow price of unused resource is zero (0).

iv) Objective coefficient ranges

Variable	Current value
X_1	4
X_2	6
X_3	5
X_4	3.5

v) $\quad$ Right hand side ranges

Constraint	Current Value
1	120
2	54
3	72
4	12

c) $\quad$ Under reduced cost column

0.5 is the opportunity cost of spaghetti

If spaghetti is stocked its current profit margin of Sh.6 will be increased by Sh.0.50

Under dual/shadow prices

If one unit of storage is introduced profit will increase by Sh.2

If demand increases by one unit profit increases by Sh.1.50

Increase in market restriction by one unit decreased profit by Sh.2.50

d)   The current profit per unit of unga is Sh.4. An increase of Sh.2 makes the profit per unit to be Sh.6. Since Sh.6 is more than the upper limit (Sh.5), the current optimal solution will change.

e)   The current value of storage space is 120 units and its upper limit is 168 units. The amount of space would have to increase by more than 168 – 120 = 48 units before there is a change in the dual/shadow price.

f)   The above problem could have been solved manually using simplex method.
     Simplex method is a matrix algebra based method which gives an optimal solution in a finite number of identical steps. It starts with an initial solution which is subsequently improved in the succeeding steps until no further improvement is possible. At this point optimal solution is reached.

CHAPTER EIGHT ANSWERS

June 2000 Question 8

(a) (i) Backward pass is a time analysis technique that is used to determine the latest event times of the project. We start at the end of the network and work through the network back to the starting point cumulatively subtracting the duration of the activities. The figures so obtained are referred to as the latest event times. Whenever there are more than one possible paths to an event, the smaller of the difference obtained is used.

ii) Crashing is a technique that is used to analyse the cheapest way of reducing the overall project duration. The sooner a project is completed the better as time is a scarce resource. However, in order to be able to reduce the duration of an activity, additional resources may need to be employed resulting in an increase in costs. Since minimisation of cost is a desirable objective it becomes necessary to identify how best the project duration can be reduced while still achieving the objective of minimisation of cost.

iii) Slack refers to spare time in a network.
Activities on the critical path have no slack. If an activity has a slack, it means it can be delayed without having any impact on other activities or to the overall project duration.

iv) Earliest start times refers to the earliest time an activity can start.
The times obtained using the forward pass are referred to as the earliest start times.

v) Critical path activities are activities that have no float. Any interference with an activity on the critical path will result in a similar interference in the overall project.

(b) (i) Network analysis may be used to solve this problem. A project is broken down into

many smaller tasks known as activities. The analysis enables management to determine the most optimum sequence in which the activities will be performed in order to minimise overall cost and time. In network analysis four principal aids are used in managing the project. These are:

Project networks

Critical path analysis

Gantt charts

Project evaluation and review technique

ii) Uncontrollable inputs would include:
Materials required for each unit

Space required for each unit

Fixed costs

Plan costs

iii) The decision variables would be activity completion times and associated costs.
- Objective function would be to minimise both costs and completion times
- Constraints would be:
Availability of materials
Availability of space
Availability of labour etc.

iv) The model is stochastic because project completion time may not be known with certainty. Project costs are just estimates.

v) Assumptions that could be made to simplify the model

Activity durations are known

Project costs are known

Sequence of carrying out the activities in known

Critical activities can be identified

Project crash times are known

The cost slope of each activity is known or can be determined

Floats can be determined

Project can be easily planned, scheduled and controlled.

December 2000 Question 8

a) (i)
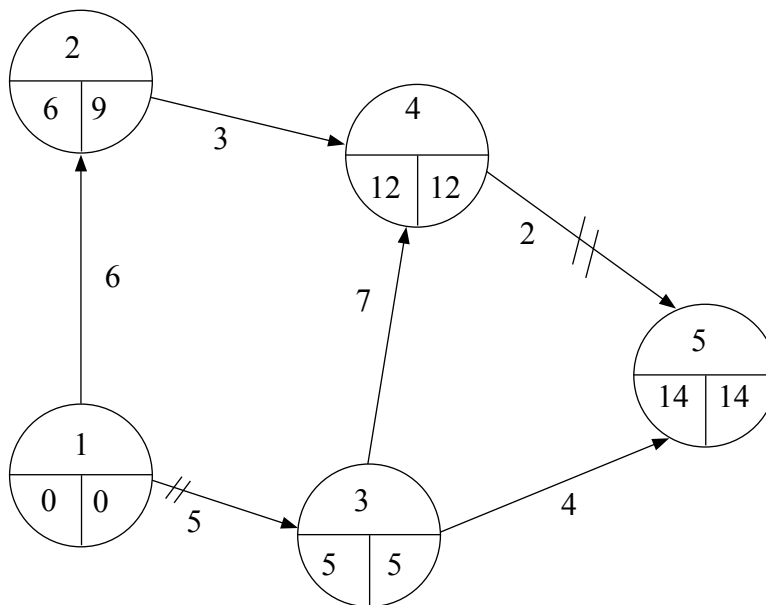
Critical path: 1-3, 3-4, 4-5

Normal project duration = 14 days

Normal project cost = Sh. 170,000

Cost slope = Crash cost – Normal cost

Normal time – crash time

Cost slope for (1-2)   =   70,000 – 60,000   =   Sh.10,000
                                          6-5

Activity	Cost slope (Sh.)	Crashable time (days)
1-2	10,000	1
1-3	10,000	2
2-4	5,000	1
3-4	5,000	3
3-5	10,000	1
4-5	7,500	1

Path	Time	Crash 3-4 by 3	Crash 4-5 by 1	Crash 1-3 & 2-4 by 1	Crash 1-3 & by 1
1-3,3-4,4-5	14	11			
1-2,2-4,4-5	11	11			
Project cost (Sh)	170,000	170,000 +3(5,000) = 185,000	185,000 +7,500 = 192,500	192,000+10,000 +5,000 =207,000	207,500+10,000 +10,000 = 227,500

The shortest project time is 8 days

Associated project cost = Sh.227,500

(ii) time (days)   Penalty   Total cost (Sh.)

        14   9000   170,000+9,000 = 179,000

        13   4500   175,000+4,500 = 179,500

12   0   180,000 + 0 = 180,000

11   0   185,000

10   0   192,500

9   0   207,500

8   0   227,500

The most economical time for completing the project is 14 days i.e. without crashing the activities.

b)   The four methods or approaches for organising and displaying project information.

   (i)   Project Network diagrams
          Show the various activities
          Show the sequence in which the activities are to be carried out
          Show the duration of each activity
   ii)   Critical path analysis (CPA)
          Critical path is the longest path through the network
          Critical activities in a network form the critical path
          Any interference with an activity on the critical path will result in a similar interference in the overall project.
   iii)   Gannt or progress charts
          A Gannt chart is a bar chart where activities are drawn against time scale
          The length of the bar is proportional to the length of an activity duration.
          It is used to schedule and balance resources. This helps to smooth the distribution of the resources available over the whole period of the project.
   iv)   Programme evaluation and review technique (PERT)
          This technique recognises the uncertainty involved in estimating the duration of an activity that has yet to take place.

STUDY TEXT

For this technique, three time estimates are required:

Optimistic time (t_o) – shortest time possible for every activity
Modal time T_M – most likely time for every activity
Pessimistic time (t_p) – longest possible time for every activity

Expected activity completion time = $\dfrac{\left(t_p + 4t_m + t_o\right)}{6}$

Standard deviation of activity completion time = $\dfrac{t_p - t_o}{6}$

December 2001 Question 8

a)   Network Diagram
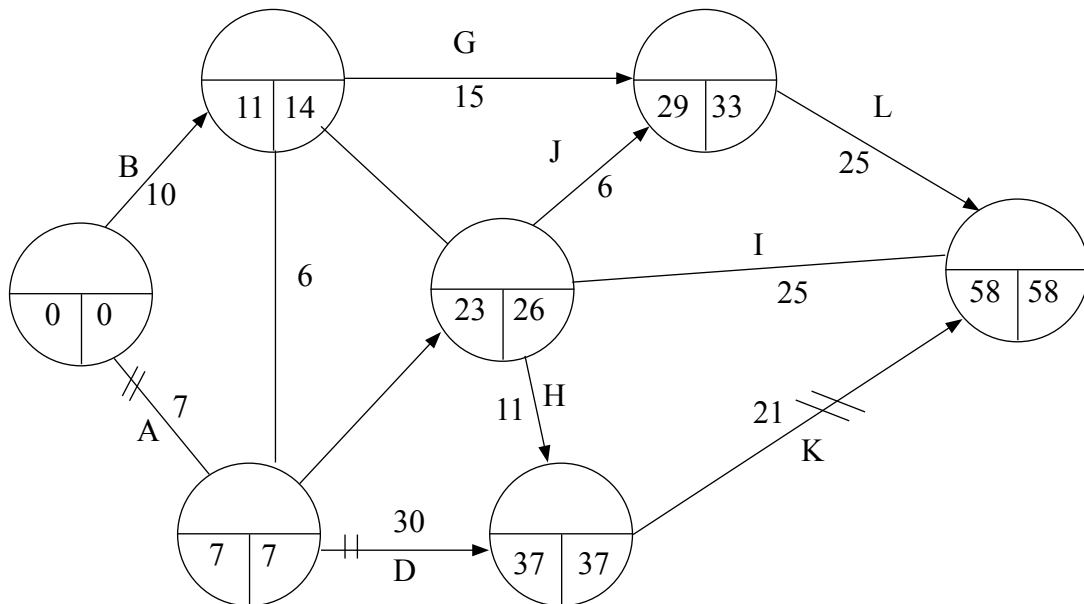
Project's expected completion time = 58 hours

Project's critical path is A-D-K

b)   Activities G and E are not critical activities

Activity G's earliest start time = 11 hours

Activity E's latest start time = 23 – 7 = 16 hours

Activity E's earliest time = 7 hours

Therefore Activities G and E can be performed at the same time without delaying the project.

c)   Activities A, G and I can be done without delaying the project. This is because G and I are not part of the critical path. G since its earliest start time is 11 hours which is well over the latest end time (7 hours) for activity A. Activity I can start after G and still be completed without delaying project time.

d)   TF(G) = 33 – 15 – 11 = 7

TF(L) = 58 – 25 – 29 = 4

Since G and L are not critical activities they can be delayed for 7 hours and 4 hours respectively without delaying the project time.

e) Delaying Activity G by 3 hours and activity L by 4 hours just makes the path critical. The project completion time is still 58 hours. The overall effect is that there is no delay in the project.

June 2004 Question 8

(a)
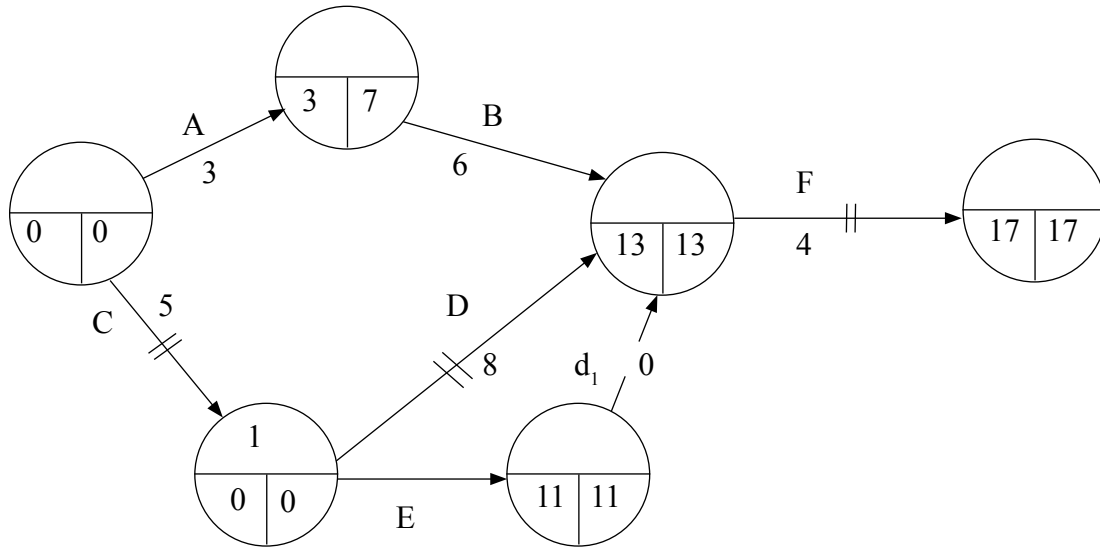
Normal completion time = 17 weeks

Critical activities are C, D and F.

(b) (i)

Activity	Time reduction	Cost slope (Sh)
A	0	-
B	2	45,000
C	2	30,000
D	1	60,000
E	2	22,500
F	2	75,000

Path	Time	Cash C by 2	Cash D by 1	Cash F by 2
C,D,F	17	15	14	
A,B,F	13	13	13	
C,E,F	15	13	13	
Additional cost Sh'000'		2 x 30 = 60	60+60x 1 = 120	120+75 x 2 = 270

17   17

(ii)   Additional cost if project is crashed = Sh. 270,000.

a)   Cost slope is the additional cost that is incurred when an activity time is reduced by one unit.

Cost slope   =   crash cost – Normal cost

Normal time – Crash time

b   Assumptions made when crashing

Cost slope is constant

There is a direct relationship between time and costs

December 2004 Question Eight

a)   (i)

Activity	Actual cost (a)	Budgeted cost (b)	Percent completion (p)	Expected budgeted cost $(p_i b_i)$	Overrun (0) OT underrun (u) = $a_i - p_i b_i$
A	800	600	100	600	200 (o)
B	100	200	67	134	-34 (u)
C	450	500	100	500	-50 (u)
D	250	200	50	100	150 (o)
E	0	550	0	0	0
				1334	266(o)

Percentage of over run cost = $\dfrac{266}{1334} \times 100 = 19.94\%$

Rule of the thumb = 5%

Percentage of over run cost is almost four times higher than the recommended 5%. In terms of cost, the project is not on schedule.

Recommendations: Monitor activities with over run costs. Nothing can be done on A since it is already complete. Activity D should be closely monitored to control its costs. The following measures may be necessary to undertake:

Lay off unnecessary staff to reduce labour costs.

Reschedule resources of D to lower their costs

Buy materials in bulk to get quantity discounts. This would reduce the cost of materials.

(ii)   Assumptions made in calculating "crash" costs for activities

The cost slope is constant

The relationship between cost and time is linear

b) Crash cost per unit   =   Crash cost – Normal Cost

Normal time – Cash time

Crash cost per unit computations are important because they provide a basis on which activity is to be crashed first. Activity with the smallest cost slope should be crashed first. Cost slope also helps us to compare additional costs incurred in crashing each activity by a unit time.

c) Purpose of a cost status report
Provide up-to-date information on the cost of each activity
Help to identify those resources that have cost over runs and cost under runs
Help to monitor and control resources and costs of each activity
Help to evaluate whether the project is on schedule in terms of cost and time.

CHAPTER NINE: ANSWERS

Question 1

Analyse the merits and demerits of simulation technique

Question 2

i)   Given λ =1/4 = 0.25 arrivals per minute or 15 arrivals per hour.
     μ = 1/2.5 = 0.4 service per minute or 24 services per hour.

ii)   The average number of customers in the system :-

$$E_{(n)} = \frac{\lambda}{\mu - \lambda} = \frac{0.25}{0.4 - 0.25} = \frac{0.25}{0.15}$$

     =1.66 customers

iii) Average queue length:-

$$E_{(m)} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{0.0625}{0.06}$$

$$= 1.04 \text{ customers}$$

iv) The average time a customer spends in the system.

$$E_{(v)} = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{0.4 - 0.25}$$

$$= \frac{1}{0.15}$$

= 6.66 minutes

v)   The average time a customer waits before being served:-

$$E_{(w)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{0.25}{0.4(0.4 - 0.25)}$$

$$= \frac{0.25}{0.06}$$

= 4.16 minutes

Question 3

Given λ   = 0.1 Arrival per minute
     μ   = 0.33 service per minute

(a). Prob. (an arrival has to wait) = $1 - P_o$

$$= \frac{\lambda}{\mu}$$

$$= \frac{0.1}{0.33}$$

$$= 0.3$$

(b). Average length of non-empty queues:

$$E_{(m/m > 0)} = \frac{\mu}{\mu - \lambda}$$

$$= \frac{0.33}{0.33 - 0.1}$$

$$= 1.43 \text{ persons}$$

(c). The average waiting time for an arrival before he gets service

$$E_{(w)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

If we fix $\mu = 0.33$, we want to find the new value of λ, say λ, for which $E_{(w)} = 3$ minutes. Then, we have

$$3 = \frac{\lambda'}{0.33(0.33 - \lambda')}$$

$\lambda' = 0.16$ arrival per minute.

Hence, we must increase the flow of arrivals from 6 per hour (present) to 10 per hour to justify the installation of the second booth.

Question 4

First it is necessary to find the value of P_o which express the probability of having no elements (customers) in the system.

$$P_0 = \frac{1}{\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3 + \frac{1}{24}\left(\frac{\lambda}{\mu}\right)^4 \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{1 + \frac{10}{3} + \frac{100}{18} + \frac{1000}{162} + \frac{10000}{1944}\left(\frac{12}{2}\right)}$$

$$= \frac{1}{46.91}$$

$$= 0.0213$$

Based upon the multi-channel model, the results are as follows (where $\lambda = 10$/hour, $\mu = 3$/hour, $k = 4$ service stations, and $P_0 = 0.0213$):

(i).   Average number of customers in the system.

$$E_{(n)} = \frac{\lambda\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)!(k\mu-\lambda)^{2}}P_0 + \frac{\lambda}{\mu}$$

$$= \frac{10\times3\left(\dfrac{10}{3}\right)^{4}}{3!(12-10)^{2}}\times0.0213 + \frac{10}{3}$$

=154.2×0.0213 + 3.33
= 6.61 customers

(ii).   Average number of customers waiting to be serviced or average queue length:

$$E_{(w)} = \frac{\lambda\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)!(k\mu-\lambda)^{2}}P_0$$

= 154.1×0.0213
= 3.28 customers

(iii).   Average time a customer spends in the system:

$$E_{(v)} = \frac{\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)!(k\mu-\lambda)^{2}}p_0 + \frac{1}{\mu}$$

$$= \frac{3\left(\dfrac{10}{3}\right)^{4}}{3!(12-10)^{2}}\times0.0213 + \frac{1}{3}$$

= 0.328+0.333
= 0.661 hour or about 40 minutes.

(iv).   Average time a customer waits before being served:

$$E_{(w)} = \frac{\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)!(k\mu-\lambda)^{2}}P_0$$

$$= \frac{3\times123.4}{24}\times0.0213$$

= 0.328 hour or about 20 minutes

(v).   Total time spent each week by a tax counselor:
The utilization factor (ρk) is given by

$$Pk = \frac{\lambda}{k\mu}$$

$$= \frac{10}{3 \times 4}$$
$$= 0.833$$

The expected time spent in servicing customers during an 8-hour day 8×0.833 = 6.66 hours.

Therefore, on an average, the tax adviser is busy 33.3 hours based on a 40 hour week.

(vi).   The probability that a customer has to wait:

$$P_{(n \geq k)} = \frac{\mu\left(\dfrac{\lambda}{\mu}\right)^{k}}{(k-1)!(k\mu - \lambda)}P_0$$

$$= \frac{3 \times 3\left(\frac{10}{3}\right)^{4}}{3!(12-10)} \times 0.0213$$

$$=0.6571$$

(vii).   The expected number of idle advisers at any specified time:
We know that the probability of no customers in the system is P_0, i.e., all the 4 counsellors are idle. We have to determine P_1, P_2 and P_3 i.e., the probability that 3 counsellors are idle, 2 counsellors are idle and one counselor is idle. Now,

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n}P_0 \quad \text{for } n < k$$

$$P_1 = \frac{1}{1!}\left(\frac{10}{3}\right) \times 0.0213$$
$$= 0.0709$$

$$P_2 = \frac{1}{2!}\left(\frac{10}{3}\right) \times 0.0213$$

$$=0.1182$$

$$P_3 = \frac{1}{3!}\left(\frac{10}{3}\right)^{3} \times 0.0213$$

$$=0.1314$$

Therefore, the expected number of idle adviser is
$$= 4P_0 + 3P_1 + 2P_2 + 1P_3$$
$$= 4(0.0213) + 3(0.0709) + 2(0.1182) + 0.1314$$
$$= 0.666$$

Thus, less than one (0.666) adviser is idle on an average at any time

Question 5

(a)   An order cost comprises:
Purchasing labour + stationery + delivery

$$\pounds\left(15(10)+\frac{1,425}{10}+80\right)=\pounds265.625 \text{ as stated.}$$

(b) Annual demand = 4,800×50   = 240,000
Unit storage cost = 0.15(8.5)   = £1.275
Buffer stock   = 20,000-3(4,800) = 5,600
Total annual stock = storage + replenishment + cost of goods

$$= \pounds\left(\frac{24,000}{2}+5,600\right)1.275+10(265.625)+240,000(8.5)$$

= **£2,065,096**

(c)Therefore, EOQ = $\sqrt{\dfrac{2 \times 265.625 \times 240,000}{1.275}}=10,000$

Annual cost using 10,000 order quantity

$$= \pounds\left(\frac{10,000}{2}+5,000\right)1.275+24(265.625)+240,000(8.5)$$

= £2,059,890
i.e. savings of £5,206

(d) The discount point is 100,000 units so there is no point in ordering in larger quantities. Therefore,
Annual cost = storage + replenishment + cost of goods + extra storage

$$= \pounds\left(\frac{10,000}{2}+5,000\right)1.21125+24(265.625)+240,000(8.5)+25,000$$

= £2,059,890
Therefore, worth taking discount

Question 6

(a) Can be taken from the notes.
(b) (i) 99% confidence interval

$$\mu = x \pm Z\frac{s}{\sqrt{n}}$$

$$= 3.64 \pm 2.58 \times \frac{0.68}{\sqrt{40}}$$

= 3.64±0.28
= 3.36 to 3.92 weeks

(ii)   Sample size

$$= \frac{(Zs)^2}{1}$$

$$= \frac{(1.96 \times 0.68)^2}{0.15} = 79$$

(c)   Re-order level = maximum usage × maximum lead time
=150×10 = 1,500
Safety stock = re-order level – (average usage × average lead time)
=1,500-(120×8) = 540
Average stock = safety stock + (EOQ × 0.5)
=540 + (2,900 ×0.5) = **1,990**

Question 7

(a) Just in time (JIT) is defined as the constant pursuit for the elimination of waste. It has the aim of eliminating as far as possible all manufacturing and finished goods inventories. It does this by ensuring that nothing is made or processed that is not needed. Thus it is a demand pull manufacturing system. The implication for an organisation using the JIT system involve total quality management, with a need to redesign plant layout, reschedule receipt of raw materials and the delivery of finished goods, as well as the redesign of the accounting system of the organization.(2 marks)

(b) In order to ensure accurate annual stock taking results in organizations that do not maintain a perpetual inventory control system, the following procedures need to be followed:
1. Stock should be properly stored in separate bins to facilitate easy identification.
2. Stores should be well arranged so that stock items are easily seen.
3. A sound internal control and internal check system regarding custody and issue of stores should be in place.
4. Written instructions should be given to the stock taker and these should be followed during stock-taking.
5. Stock should be taken by persons other than the storekeeper or his staff.
6. On stock taking day, the business should be closed to customers to avoid interruption during the stock taking.
7. Cut-off procedures should be strictly followed.
8. Stock taking should be undertaken in the presence of a senior official whose duties should have nothing to do with stores.
9. One person should count stocks, calling out the quantities with another person doing the recording.
10. Stock-sheets should be used and re-checked by another person for accuracy.
11. Stocks belonging to third parties should be counted and listed separately.

STUDY TEXT

Question 8

a)

	PRODUCT		
	14/363	11/175	14/243
Minimum Sales: (Units)			
Gross Margin $=\dfrac{Sales - \cos t}{Sales}$	$\dfrac{S-C}{S}=0.42$	$\dfrac{S-C}{S}=0.46$	$\dfrac{S-C}{S}=0.37$
	0.42S = S - C	0.54S = C	0.63S = C
	0.58S = C		
	C = 58%S		
	C = 58% x 26,000	C = 54% x 130,000	C = 63% x 60,000
	= Shs 15,080	= Shs 70,200	= Shs 37,800
	No of units $=\dfrac{Total \cos t}{unit \cos t}$	No of units	No of units
	$=\dfrac{15,080}{2.25}$	$=\dfrac{70,200}{0.36}$	$\dfrac{37,800}{0.87}$
	= 6,702 Units	=195,000 Units	= 43,448 units
Maximum weekly sales:	Cost = 58% x 30,000	Cost = 54% x 160,000	Cost = 63% x 128,000
	= 17,400	= 86,400	= 80,640
	Units = 17,400/2.25	Units = 86,400/0.36	Units = 80,640/0.87
	= 7,733 Units	= 240,000 Units	= 92,690 Units
(ii) Re-order level = Maximum Consumption x Maximum Re-order period			
NB: Maximum Re-order period = 2 weeks			
	7,733 x 2	240,000 x 2	92,690 x 2
	= 15,466 Units	= 480,000 Units	=185,380 Units
Maximum Stock Level = Re-order level + Re-order Quantity - Minimum x Minimum			
			Consumption
Re-order period			
	15,466 + 25,000 -	480,000 + 500,000 -	18,380 + 250,000 -
	(6,702 x 2)	(195,000 x 2)	(43,448 x 2)
	= 27,062 Units	= 590,000 Units	= 348,484 Units

b) Comments:
The reordering process for item 14/363 is satisfactory as no order is placed because stocks have not yet reached the re-order level.
The reordering process for product 14/243 is satisfactory as an order was placed immediately the stock hit the re-order level 2 days ago.
However, the re-ordering process for product 11/175 is inadequate, as an order should already have been placed as the stock level is below the re-order level.
The lead time under the current system is 2 weeks, both maximum and minimum. The stocks take long to reach the stores and a way should be found to reduce it to 1 week or less. This can be done by reducing the delivery process so that the stocks are delivered directly to the company stores.
The re-order level is adequately for maximum demand in the lead-time plus any random disturbance that may occur. However, the need to be analysed as the organization could be suffering unnecessary high carrying costs given that the re-order level is twice the maximum demand count.

Question 9

(i)

$$EOQ = \sqrt{\frac{2 \times 480,000 \times 1,500}{[(480/24) \times 0.1] + 8}}$$

$$= \sqrt{\frac{2 \times 480,000 \times 1,500}{(20 \times 0.1) + 8}}$$

$$= \ 12,000 \ pens$$

(ii) Total cost of EOQ
TC = Purchase Cost + Ordering Cost + Holding Cost

[Purchase price × Q] + [No. of orders × Ordering cost] + [Per unit holding cost × Q]

$$= [20 \times 480,000] + \left[\frac{480,000}{12,000} \times 1,500\right] + \left[\frac{1}{2} \times 12,000 \times 10\right]$$

= Sh. 9,720,000

(iii) Cost at discount option

$$= [480,000 \times 20 \times 90\%] + \left[\frac{480,000}{72,000} \times 1,500\right] + \left[72,000 \times \frac{1}{2} \times 9.8\right]$$

= 8,640,000 = 10,000 + 352,800

= Sh. 9,002,800

It is cheaper to take the discount offer as the total inventory costs are lower.

GLOSSARY

GLOSSARY

Constant – this is a quantity whose value remains unchanged throughout.

Variate – this is a quantity which takes various values in a particular problem

Function – it's a relationship in which values of a dependent variable are determined by the values of one or more independent variables

Exponential functions – These are functions which have at least one term as independent variable and are part of an exponent or power.

Logarithmic is a power to which a base must be raised in order to give a certain number i.e. a logarithm is an exponent

Statistics is the art and science of getting information from data or numbers to help in decision making.

Sample inference – The process of making conclusions about the population based on sample statistics is known as sample inference

Measurement is the process of allocating values to variables

Average measures give us values that may be considered to be typical of data being considered

Mean is the sum of all the values divided by the number of values

Median is the middle value in an ordered set of data values

Mode is the value with the highest frequency

Skewness describes the degree of non-symmetry in a distribution

Pearson product moment correlation coefficient, r is the square root of the coefficient of determination:

Permutation of n different objects taken r at a time is an arrangement of r out of n objects with attention given to the order of arrangement.

Probability is a measure of likelihood, the possibility or chance that an event in future will happen

Outcome is the result of an experiment

Sample space is the set of all possible outcomes in an experiment

An **event** of an experiment is a subset of a sample space

Mutually exclusive event – A set of events is said to be mutually exclusive if the occurrence of any one of the events precludes the occurrence of other, events

Favourable events – refers to the number of possible occurrences of a given event in an experiment

Independent events – events are independent if the happening or non-happening of one has no effect on the future happening of an event

Equally likely events – events are equally likely if the happening of one is not favoured over the happening of others

Decision - a decision is a commitment to irrevocable allocation of variable resources it is a

commitment to act and action is the irrevocable allocation of variable resources.

Objective - it's something that a decision maker seeks to accomplish or to obtain by means of his decision

Course of action – strategy or means available to decision maker by which the objectives may be attained.

Decision trees – it is a graphic representation of the decision alternative, states of nature, probabilities attached to the state of nature and conditional losses.

Strategy – the decision role which a player determines his/her course of action is called strategy.

Zero sum game is a game where a gain of one equals the loss of the other is known as a two person zero-sum game

Population consists of all the items with which a particular study is concerned

Sample is a much smaller number chosen from this population

Confidence limits are the outer limits to a confidence interval

The Null hypothesis is an assumption that nothing has changed

Type I error – If we reject a hypothesis when it should be accepted, we say that a type I error has been made.

Type II error – If we accept a hypothesis when it should be rejected, we say that a type II error has been made.

Programming – The allocation of resources is sometimes referred to as programming.

Basic solution is the name given to the set of variable values at a corner of the feasible region.

Simulation is a technique used to make decisions under conditions of uncertainty whereby a model of the real system is used and then a chain of repeated trial and error

Primal model - The variables are the amount of each product to be produced

Dual model - The variables are the primal shadow price, that is, the amount which would be added to the value of the objective function if one more unit of the raw material was available.

Queue – A group of customers waiting for service in a system rendering some service

Balking – this is a case when a member of the population refuses to join the queue due to the size of the queue.

FIFO queue discipline – this is a system that gives priority to those who arrive first.

Priority queue discipline – this is a system that allows different priorities entitling customers to get service on a pre-emptive basis or a non-pre-emptive basis.

Network analysis is a family of related techniques developed to aid management in the planning, co-ordination and controlling of large complex projects using limited resources

An event slack is the maximum time an event can be delayed without delaying the overall project completion time.

Crashing means trying to perform an activity in a time shorter than the activity requires.

Smoothing a profile – This is the process of attempting to reduce the peaks and troughs in the resource allocation so that we have a more even usage of personnel

Forbidden allocations – If an allocation from a particular origin to a particular destination is impossible for some reason, the algorithm can be forced to avoid this allocation by assigning a large cost to the cell

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